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Custom sandwich pairs

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ABSTRACT

For many equations arising in practice, the solutions are critical points of functionals. In previous papers we have shown that there are pairs of subsets, called sandwich pairs, that can produce critical points even though they do not separate the functional. All that is required is that the functional be bounded from above on one of the sets and bounded from below on the other, with no relationship needed between the bounds. This provides a distinct advantage in applications. The present paper discusses the situation in which one cannot find sandwich pairs for which the functional is bounded below on one set and bounded above on the other. We develop a method which can deal with such situations and apply it to problems in partial differential equations.

1. Introduction

Many problems arising in science and engineering call for the solving of the Euler equations or systems of equations for functionals, i.e., equations of the form

$$G'(u) = 0, \tag{1}$$

where $G(u)$ is a C^1-functional arising from the given data. However, when one wishes to solve Euler equations, one is not merely looking for extrema. One is interested in finding all critical points. In particular, how does one search for critical points when the functional is not semibounded? Is there anything that can be used to replace semiboundedness? There is an approach which works when one cannot obtain linking sets which separate the functional. In this approach, one looks for suitable sets A, B such that the functional is bounded from below on one set and bounded from above on the

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other with no requirement on the relationship between the bounds. In other words, one looks for suitable subsets A, B of a Banach space E, which are such that

$$a_0 := \sup_A G < \infty, \qquad b_0 := \inf_B G > -\infty \tag{2}$$

holding for a given C^1-functional G on E implies the existence of a PS sequence

$$G(u_k) \to c, \qquad G'(u_k) \to 0, \tag{3}$$

where $b_0 \leqslant c \leqslant a_0$. If A, B are such that (2) always implies (3), we say that A, B form a *sandwich pair*. It was shown in [7] that sandwich pairs are not rare. General criteria for sets to form sandwich pairs were given there.

In the present paper, we discuss the situation in which

(a) we cannot find linking sets which separate the functional

and

(b) we cannot find sandwich pairs such that the functional is bounded from below on one set and bounded from above on the other.

In some such situations, we have found sets A, B such that

$$a_0 := \sup_A G < \infty, \qquad \sup_{\hat{A}} G \leqslant b_0 := \inf_B G \tag{4}$$

holding for a given C^1-functional G on E implies the existence of constants $c, C \in \mathbb{R}$ and sequences $\{\nu_k\} \subset \mathbb{R}$, $\{u_k\} \subset E$ such that $\nu_k \to \infty$ and

$$G(u_k) \to c, \quad b_0 \leqslant c \leqslant a_0, \quad \big(\nu_k + \|u_k\|\big)\big\|G'(u_k)\big\| \leqslant C, \tag{5}$$

where $\hat{A}$ is a subset of A. We describe this result and present applications which take advantage of it. We call such sets *custom sandwich pairs*. We exhibit cases in which PS sequences will not work.

The sequence (5) is not quite a Cerami sequence, but it is just as effective in most applications. Cerami sequences were introduced in [2]. Variations were given in [1,4,5].

Our abstract theory will be presented in Section 2, and our applications are given in Section 3. In Section 4 we present some results concerning differential equations in Banach space which are used in our proofs given in the last section.

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2. Criteria

We now exhibit examples of custom sandwich pairs. We have

Theorem 1. *Let G be a C^1-functional on E, and let A be a continuous curve in E connecting 0 and ∞, and B the boundary of a bounded open set in E containing 0 such that*

$$a_0 := \sup_A G < \infty, \qquad G(0) \leqslant b_0 := \inf_B G. \tag{6}$$

Let $\psi(t)$ be a locally Lipschitz continuous nonincreasing positive function satisfying

$$\int_0^\infty \psi(t)\,dt = \infty. \tag{7}$$

Let $R > 0$ be such that $B \subset \mathbf{B}_R$, where

$$\mathbf{B}_R = \{u \in E\colon\ \|u\| \leqslant R\}.$$

Take $\nu > 0$ so large that

$$\int_R^\nu \psi(t)\,dt > a_0 - b_0 \tag{8}$$

and let $\tilde{A} = A \cap \mathbf{B}_\nu$. Then there are a constant $c \in \mathbb{R}$ and a sequence $\{u_k\} \subset E$ such that

$$G(u_k) \to c, \quad b_0 \leqslant c \leqslant a_0, \quad \|G'(u_k)\| \leqslant \psi\big(d(u_k, \tilde{A})\big). \tag{9}$$

Theorem 2. *Under the same hypotheses, there are constants $c, C \in \mathbb{R}$ and a sequence $\{u_k\} \subset E$ such that*

$$G(u_k) \to c, \quad b_0 \leqslant c \leqslant a_0, \quad \big(\nu + \|u_k\|\big)\|G'(u_k)\| \leqslant C. \tag{10}$$

Theorem 3. *Under the same hypotheses, for each sequence $\nu_k \to \infty$, there are constants $c, C \in \mathbb{R}$ and a sequence $\{u_k\} \subset E$ such that*

$$G(u_k) \to c, \quad b_0 \leqslant c \leqslant a_0, \quad \big(\nu_k + \|u_k\|\big)\|G'(u_k)\| \leqslant C. \tag{11}$$

3. Applications

In the present section we assume that Ω is a bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega$ sufficiently regular so that the Sobolev inequalities hold. Let $\mathcal{A}$ be a self-adjoint operator on $L^2(\Omega)$. We assume that $\mathcal{A} \geqslant \lambda_0 > 0$ and that

$$C_0^\infty(\Omega) \subset D := D\big(\mathcal{A}^{1/2}\big) \subset H^{m,2}(\Omega)$$

for some $m > 0$, where $C_0^\infty(\Omega)$ denotes the set of test functions in Ω (i.e., infinitely differentiable functions with compact supports in Ω) and $H^{m,2}(\Omega)$ denotes the Sobolev space. If m is an integer, the norm in $H^{m,2}(\Omega)$ is given by

$$\|u\|_{m,2} := \bigg(\sum_{|\mu| \leqslant m} \|D^\mu u\|^2\bigg)^{1/2}. \tag{12}$$

Here D^μ represents the generic derivative of order $|\mu|$ and the norm on the right-hand side of (12) is that of $L^2(\Omega)$. We shall not assume that m is an integer.

Let q be any number satisfying

$$2 \leqslant q \leqslant 2n/(n-2m), \quad 2m < n,$$

$$2 \leqslant q < \infty, \quad n \leqslant 2m,$$

and let $f(x,t)$ be a Carathéodory function on $\Omega \times \mathbb{R}$. This means that $f(x,t)$ is continuous in t for a.e. $x \in \Omega$ and measurable in x for every $t \in \mathbb{R}$. We make the following assumptions:

(A) The function $f(x,t)$ satisfies

$$\left|f(x,t)\right| \leqslant V(x)^q\left(|t|^{q-1} + W(x)\right)$$

and

$$f(x,t)/V(x)^q = o\left(|t|^{q-1}\right) \quad \text{as } |t| \to \infty,$$

where $V(x) > 0$ is a function in $L^q(\Omega)$ such that

$$\|Vu\|_q \leqslant C\|u\|_D, \quad u \in D, \tag{13}$$

and W is a function in $L^\infty(\Omega)$. Here

$$\|u\|_q := \left(\int_\Omega \left|u(x)\right|^q dx\right)^{1/q},$$

$$\|u\|_D := \left\|A^{1/2}u\right\| \tag{14}$$

and $q' = q/(q-1)$. (If $V(x)$ is bounded, then (13) will hold automatically by the Sobolev inequality. However, there are functions $V(x)$ which are unbounded and such that (13) holds.) With the norm (14), D becomes a Hilbert space. Define

$$F(x,t) := \int_0^t f(x,s)\,ds$$

and

$$G(u) := \|u\|_D^2 - 2\int_\Omega F(x,u)\,dx. \tag{15}$$

It is readily shown that G is a continuously differentiable functional on the whole of D (cf., e.g., [6]). We assume that λ_0 is an isolated eigenvalue of $\mathcal{A}$ having a finite-dimensional eigenspace $E(\lambda_0) \subset L^\infty(\Omega)$. (In many second-order elliptic problems, $E(\lambda_0)$ consists of a single eigenfunction which does not change sign.) In addition, we assume that for one eigenfunction $\varphi(x) \in E(\lambda_0)$ we have

$$\sup_{r>0} \int_\Omega \left[r^2\lambda_0\varphi^2 - 2F(x, r\varphi)\right] dx < \infty. \tag{16}$$

In addition,

$$2F(x,t) \leqslant \lambda_0 t^2, \quad |t| < \delta, \tag{17}$$

for some positive constant δ. Assume also that

$$H(x,t) = 2F(x,t) - tf(x,t) \leqslant W_1(x) \in L^1(\Omega), \quad x \in \Omega,\ t \in \mathbb{R}, \tag{18}$$

$$H(x,t) \to -\infty \quad \text{a.e. as } |t| \to \infty, \tag{19}$$

and

$$2F(x,t) \leqslant \gamma H(x,t) + W_2(x)\big(t^2+1\big) \tag{20}$$

for some $\gamma \in \mathbb{R}$ and $W_2(x) \in L^1(\Omega) \cap L^\infty(\Omega)$.

We have

Theorem 4. *Under the above hypotheses, the equation*

$$\mathcal{A}u = f(x,u), \quad u \in D, \tag{21}$$

has at least one nontrivial solution.

Remark 5. The significance of the nontrivial solution becomes important when $f(x,0) \equiv 0$. For then $u(x) \equiv 0$ is a solution of (21). Otherwise, we can come to the same conclusion assuming only (17).

Remark 6. It should be noted that the well-known hypothesis for superlinear problems

$$0 < \mu F(x,t) \leqslant t f(x,t), \quad |t| > R, \tag{22}$$

holding for some $\mu > 2$ implies (16), (18), (19) and (20) (i.e., all the hypotheses of Theorem 4 except (17)). To see this, note that it implies, in particular, that there exist positive constants c_3, c_4 such that

$$F(x,t) \geqslant c_3|t|^\mu - c_4, \quad x \in \Omega,\ t \in \mathbb{R}, \tag{23}$$

and, consequently,

$$F(x,t)/t^2 \to \infty \quad \text{as } |t| \to \infty,$$

which implies (16). It also implies

$$H(x,t) \leqslant (2-\mu)F(x,t), \quad |t| \geqslant R,$$

which implies both (18) and (19). If we write it as

$$(\mu-2)F(x,t) \leqslant -H(x,t),$$

we see that it implies (20) as well. Thus we have

Corollary 7. *Theorem* 4 *holds if we assume only* (17) *and* (22).

Remark 8. In the case that $E(\lambda_0)$ contains an eigenfunction which does not change sign, (16) is implied by

$$F(x,t)/t^2 \to \infty \quad \text{as } t \to \infty$$

or

$$F(x,t)/t^2 \to \infty \quad \text{as } t \to -\infty.$$

Remark 9. The following example is superlinear but does not satisfy (22). However, it does satisfy the hypotheses of Theorem 4 (if φ does not change sign). Let

$$f_1(t) = \lambda_0 t - \alpha_1 (\operatorname{sgn} t)|t|^{\alpha_1 - 1} \tag{24}$$

and

$$f_2(t) = \lambda_0 t - \alpha_2 (\operatorname{sgn} t)|t|^{\alpha_2 - 1}. \tag{25}$$

Choose $f(t) \in C(\mathbb{R})$ to satisfy

$$f(t) = \begin{cases} f_1(t), & -\infty < t < r, \\ f_2(t), & t > R, \end{cases} \tag{26}$$

where $1 < \alpha_1 < 2 < \alpha_2$ and $0 < r < R$. Note that $f(t)$ is superlinear for positive t and sublinear for negative t.

Proof of Theorem 4. We apply Theorem 3. We let N be the eigenspace $E(\lambda_0)$, and we take $M = N^{\perp}$. We note that (17) implies

$$G(w + y) \geqslant \varepsilon' \|w\|^2, \quad w \in M, \ y \in E(\lambda_0),$$

for $\|w + y\|$ sufficiently small. To see this, let $u = w + y$. Note that there is a $\rho > 0$ such that

$$\|y\|_D \leqslant \rho \quad \Rightarrow \quad \big|y(x)\big| \leqslant \delta/2, \quad y \in E(\lambda_0).$$

Now suppose u satisfies

$$\|u\|_D \leqslant \rho \quad \text{and} \quad \big|u(x)\big| \geqslant \delta \tag{27}$$

for some $x \in \Omega$. Then for those $x \in \Omega$ satisfying (27) we have

$$\delta \leqslant \big|u(x)\big| \leqslant \big|w(x)\big| + \big|y(x)\big| \leqslant \big|w(x)\big| + (\delta/2).$$

Hence

$$\big|y(x)\big| \leqslant \delta/2 \leqslant \big|w(x)\big|,$$

and consequently,

$$\big|u(x)\big| \leqslant 2\big|w(x)\big| \tag{28}$$

for all such x. Now we have by hypothesis (A) and (17)

$$\begin{aligned} G(u) &\geqslant \|u\|_D^2 - \lambda_0 \int\limits_{|u|<\delta} u^2\, dx - C \int\limits_{|u|>\delta} \big(|Vu|^q + \big|V^q u\big| W\big)\, dx \\ &\geqslant \|u\|_D^2 - \lambda_0 \|u\|^2 - C' \int\limits_{|u|>\delta} |Vu|^q\, dx \\ &\geqslant \|w\|_D^2 - \lambda_0 \|w\|^2 - C'' \int\limits_{2|w|>\delta} |Vw|^q\, dx \end{aligned}$$

$$\geqslant \left(1 - \frac{\lambda_0}{\lambda_1} - C''' \|w\|_D^{q-2}\right) \|w\|_D^2$$

in view of the fact that $\|y\|_D^2 = \lambda_0 \|y\|^2$,

$$\left|F(x,t)\right| \leqslant C\left(|Vt|^q + \left|V^q t\right|\right),$$

and (28) holds. Here, λ_1 is any number such that (λ_0, λ_1) is in the resolvent of $\mathcal{A}$. Moreover, in such a situation, the following alternative holds:

Either

(a) there is an infinite number of eigenfunctions $y \in E(\lambda_0) \setminus \{0\}$ such that

$$\mathcal{A} y = f(x, y) = \lambda_0 y, \tag{29}$$

or

(b) for each $\rho > 0$ sufficiently small, there is an $\varepsilon > 0$ such that

$$G(u) \geqslant \varepsilon, \quad \|u\|_D = \rho, \quad u \in D \tag{30}$$

(cf. [6]).

Since option (a) solves our problem, we may assume that option (b) holds. We let $A = \{r\varphi\colon r \geqslant 0\}$ and $B = \partial \mathbf{B}_\rho$ in Theorem 1. By (30) we see that (6) holds with $b_0 > 0$. By Theorem 3 we conclude that there are sequences $\nu_k \to \infty$ and $\{u_k\} \subset D$ such that

$$G(u_k) \to c, \quad b_0 \leqslant c \leqslant a_0, \quad \left(\nu_k + \|u_k\|_D\right)\left\|G'(u_k)\right\| \leqslant C. \tag{31}$$

In particular, we have

$$\|u_k\|_D^2 - 2\int_\Omega F(x, u_k)\,dx \to c \tag{32}$$

and

$$\left|\|u_k\|_D^2 - \left(f(\cdot, x_k), u_k\right)\right| \leqslant K. \tag{33}$$

Consequently

$$\left|\int_\Omega H(x, u_k)\,dx\right| \leqslant K'. \tag{34}$$

If $\rho_k = \|u_k\|_D \to \infty$, let $\tilde{u}_k = u_k/\rho_k$. Then $\|\tilde{u}_k\|_D = 1$. Since Ω is bounded, there is a renamed subsequence such that $\tilde{u}_k \to \tilde{u}$ weakly in D, strongly in $L^2(\Omega)$ and a.e. in Ω. By (32) we have

$$1 - \int_\Omega F(x, u_k)/\rho_k^2\,dx \to 0. \tag{35}$$

If (20) holds, we have in view of (34)

$$\limsup \int_{\Omega} 2F(x, u_k)/\rho_k^2\, dx \leqslant \limsup \int_{\Omega} W_2\big(u_k^2+1\big)/\rho_k^2\, dx$$
$$\leqslant \limsup \int_{\Omega} W_2\big(\tilde{u}_k^2+\rho_k^{-2}\big)\, dx$$
$$= \int_{\Omega} W_2 \tilde{u}^2\, dx.$$

In view of (35) this implies that $\tilde{u} \not\equiv 0$. Let Ω_0 be the subset of Ω on which $\tilde{u} \neq 0$. Then

$$\big|u_k(x)\big| = \rho_k \big|\tilde{u}_k(x)\big| \to \infty, \quad x \in \Omega_0. \tag{36}$$

If $\Omega_1 = \Omega \setminus \Omega_0$, then we have

$$\int_{\Omega} H(x, u_k)\, dx = \int_{\Omega_0} + \int_{\Omega_1} \leqslant \int_{\Omega_0} H(x, u_k)\, dx + \int_{\Omega_1} W_1(x)\, dx \to -\infty. \tag{37}$$

This contradicts (34), and we see that $\rho_k = \|u_k\|_D$ is bounded. Once we know that the ρ_k are bounded we can apply Theorem 3.4.1 of [6] to obtain the desired conclusion. □

Remark 10. It should be noted that the crucial element in the proof of Theorem 4 was (33). If we had been dealing with an ordinary Palais–Smale sequence, we could only conclude that

$$\|u_k\|_D^2 - \big(f(\cdot, u_k), u_k\big) = o(\rho_k),$$

which would imply only

$$\int_{\Omega} H(x, u_k)\, dx = o(\rho_k).$$

This would not contradict (41), and the argument would not go through.

We also have

Theorem 11. *The conclusion of Theorem* 4 *holds if in place of* (18), (19) *we assume*

$$H(x,t) \geqslant -W_1(x) \in L^1(\Omega), \quad x \in \Omega,\ t \in \mathbb{R}, \tag{38}$$

and

$$H(x,t) \to \infty \quad \text{a.e. as } |t| \to \infty, \tag{39}$$

and in place of (20) *we assume*

$$tf(x,t) \leqslant \gamma H(x,t) + W_2(x)\big(t^2+1\big). \tag{40}$$

Proof. If (40) holds, then

$$\limsup \int_\Omega u_k f(x,u_k)/\rho_k^2\,dx \leqslant \limsup \int_\Omega W_2\big(u_k^2+1\big)/\rho_k^2\,dx$$
$$\leqslant \limsup \int_\Omega W_2\big(\tilde u_k^2+\rho_k^{-2}\big)\,dx$$
$$= \int_\Omega W_2 \tilde u^2\,dx.$$

Now (35) implies $\tilde u \not\equiv 0$. We then proceed as before. We use (38) and (39) to replace (37) with

$$\int_\Omega H(x,u_k)\,dx = \int_{\Omega_0} + \int_{\Omega_1} \geqslant \int_{\Omega_0} H(x,u_k)\,dx - \int_{\Omega_1} W_1(x)\,dx \to \infty. \qquad \square \tag{41}$$

Remark 12. The following example is sublinear and satisfies the hypotheses of Theorem 11. Choose $f(t) \in C(\mathbb{R})$ to satisfy

$$f(t) = \begin{cases} f_1(t), & |t| < r, \\ f_2(t), & |t| > R, \end{cases} \tag{42}$$

where $1 < \alpha_1, \alpha_2 < 2$ and $0 < r < R$.

We also have

Theorem 13. *If we replace* (18) *and* (19) *with*

$$H(x,t) := 2F(x,t) - tf(x,t) \leqslant C\big(|t|^\alpha + 1\big) \tag{43}$$

and

$$h(x) := \limsup_{|t|\to\infty} H(x,t)/|t|^\alpha < 0 \quad a.e. \tag{44}$$

for some positive $\alpha < q$, then

$$\mathcal{A}u = f(x,u), \quad u \in D, \tag{45}$$

has at least one nontrivial solution.

Proof. As before, we can conclude that there are sequences $\{\nu_k\} \subset \mathbb{R}$, $\{u_k\} \subset D$ such that $\nu_k \to \infty$ and

$$G(u_k) \to c, \quad b_0 \leqslant c < a_0, \quad \big(\nu_k + \|u_k\|\big)\big\|G'(u_k)\big\| \leqslant C. \tag{46}$$

Let $\rho_k = \|u_k\|_D$. If $\rho_k \to \infty$, then

$$G(u_k) = \rho_k^2 - 2\int_\Omega F(x,u_k)\,dx \to c \tag{47}$$

and

$$\big(G'(u_k), u_k\big)/2 = \rho_k^2 - \int_\Omega f(x, u_k)u_k\,dx \to 0.$$

Hence,

$$\left|\int_\Omega H(x, u_k)\,dx\right| \leqslant C. \tag{48}$$

Let $\tilde{u_k} = u_k/\rho_k$. Then $\|\tilde{u}_k\|_D = 1$. Thus, there is a renamed subsequence such that $\tilde{u}_k \to \tilde{u}$ weakly in D, strongly in $L^2(\Omega) \cap L^\alpha(\Omega)$ and a.e. in Ω. By (43) and (44)

$$\begin{aligned}\limsup \int_\Omega H(x, u_k)\,dx/\rho_k^\alpha &\leqslant \int_\Omega \limsup\big[H(x, u_k)/|u_k|^\alpha\big]|\tilde{u}_k|^\alpha\,dx \\ &= \int_\Omega h(x)|\tilde{u}|^\alpha\,dx.\end{aligned}$$

Since $h(x) < 0$ a.e. in Ω, the last two statements imply that $\tilde{u} \equiv 0$. However, we see from (47) that in view of (32), we have

$$1 - 2\int_\Omega F(x, u_k)/\rho_k^2\,dx \to 0. \tag{49}$$

As before, this shows that $\tilde{u} \not\equiv 0$. This contradiction tells us that the ρ_k must be bounded. We can now apply Theorem 3.4.1 of [6] to conclude that there is a $u \in D$ satisfying

$$G(u) = c, \qquad G'(u) = 0. \tag{50}$$

The proof is complete. □

Theorem 14. *The conclusion of Theorem* 11 *holds if we replace* (18) *and* (19) *with*

$$H(x, t) := 2F(x, t) - tf(x, t) \geqslant -C\big(|t|^\alpha + 1\big) \tag{51}$$

and

$$h(x) := \limsup_{|t|\to\infty} H(x, t)/|t|^\alpha > 0 \quad \textit{a.e.} \tag{52}$$

for some positive $\alpha < q$.

The nonquadraticity condition (44) was first introduced by Costa and Magalhães [3].

4. Ordinary differential equations

In proving Theorem 1 we shall make use of various extensions of Picard's theorem in a Banach space. Some are well known.

Theorem 15. *Let X be a Banach space, and let*

$$B_0 = \{x \in X:\ \|x - x_0\| \leqslant R_0\}$$

and

$$I_0 = \{t \in \mathbb{R}:\ |t - t_0| \leqslant T_0\}.$$

Assume that $g(t,x)$ is a continuous map of $I_0 \times B_0$ into X such that

$$\|g(t,x) - g(t,y)\| \leqslant K_0\|x - y\|, \quad x, y \in B_0,\ t \in I_0, \tag{53}$$

and

$$\|g(t,x)\| \leqslant M_0, \quad x \in B_0,\ t \in I_0. \tag{54}$$

Let T_1 be such that

$$T_1 \leqslant \min(T_0, R_0/M_0), \qquad K_0 T_1 < 1. \tag{55}$$

Then there is a unique solution $x(t)$ of

$$\frac{dx(t)}{dt} = g(t, x(t)), \quad |t - t_0| \leqslant T_1, \quad x(t_0) = x_0. \tag{56}$$

Lemma 16. *Let $\gamma(t)$ and $\rho(t)$ be continuous functions on $[0,\infty)$, with $\gamma(t)$ nonnegative and $\rho(t)$ positive. Assume that*

$$\int_{u_0}^{\infty} \frac{d\tau}{\rho(\tau)} > \int_{t_0}^{T} \gamma(s)\, ds, \tag{57}$$

where $t_0 < T$ and $u_0 \geqslant 0$. Then there is a unique solution of

$$u'(t) = \gamma(t)\rho(u(t)), \quad t \in [t_0, T), \quad u(t_0) = u_0, \tag{58}$$

which is positive in (t_0, T) and depends continuously on u_0.

Proof. One can separate variables to obtain

$$W(u) = \int_{u_0}^{u} \frac{d\tau}{\rho(\tau)} = \int_{t_0}^{t} \gamma(s)\, ds.$$

The function $W(u)$ is differentiable and increasing in $\mathbb{R}$, positive in (u_0, ∞), depends continuously on u_0 and satisfies

$$W(u) \to L = \int_{u_0}^{\infty} \frac{d\tau}{\rho(\tau)} > \int_{t_0}^{T} \gamma(s)\, ds \quad \text{as } u \to \infty.$$

Thus, for each $t \in [t_0, T)$ there is a unique $u \in [u_0, \infty)$ such that

$$u = W^{-1}\left(\int_{t_0}^{t} \gamma(s)\, ds\right)$$

is the unique solution of (58), and it depends continuously on u_0. □

Lemma 17. *Let $\gamma(t)$ and $\rho(t)$ be continuous functions on $[0, \infty)$, with $\gamma(t)$ nonnegative and $\rho(t)$ positive. Assume that*

$$\int_{m}^{u_0} \frac{d\tau}{\rho(\tau)} > \int_{t_0}^{T} \gamma(s)\, ds, \tag{59}$$

where $t_0 < T$ and $m < u_0$ are given positive numbers. Then there is a unique solution of

$$u'(t) = -\gamma(t)\rho\big(u(t)\big), \quad t \in [t_0, T), \quad u(t_0) = u_0, \tag{60}$$

which is $\geqslant m$ in $[t_0, T)$ and depends continuously on u_0.

Proof. One can separate variables to obtain

$$W(u) = \int_{u}^{u_0} \frac{d\tau}{\rho(\tau)} = \int_{t_0}^{t} \gamma(s)\, ds.$$

The function $W(u)$ is differentiable and decreasing in $\mathbb{R}$, positive in $[m, u_0]$, depends continuously on u_0 and satisfies

$$W(u) \to L = \int_{m}^{u_0} \frac{d\tau}{\rho(\tau)} > \int_{t_0}^{T} \gamma(s)\, ds \quad \text{as } u \to m.$$

Thus, for each $t \in [t_0, T)$ there is a unique $u \in [m, u_0]$ such that

$$u = W^{-1}\left(\int_{t_0}^{t} \gamma(s)\, ds\right)$$

is the unique solution of (60), and it depends continuously on u_0. □

Theorem 18. *Assume, in addition to the hypotheses of Theorem* 15, *that*

$$\left\|g(t,x)\right\| \leqslant \gamma(t)\rho\big(\|x-v\|\big), \quad x \in B_0,\ t \in I_0, \tag{61}$$

where $\gamma(t)$ *and* $\rho(t)$ *satisfy the hypotheses of Lemma* 16 *with* $T = t_0 + T_1$ *and* v *is a fixed element of* X. *Let* $u(t)$ *be the positive solution of*

$$u'(t) = \gamma(t)\rho\big(u(t)\big), \quad t \in [t_0, T), \quad u(t_0) = u_0 \geqslant \|x_0 - v\|, \tag{62}$$

provided by that lemma. Then the unique solution of (56) *satisfies*

$$\left\|x(t) - v\right\| \leqslant u(t), \quad t \in [t_0, T). \tag{63}$$

Proof. Assume that there is a $t_1 \in [t_0, T)$ such that

$$u(t_1) < \left\|x(t_1) - v\right\|.$$

For $\varepsilon > 0$, let $u_\varepsilon(t)$ be the solution of

$$u'(t) = \big[\gamma(t) + \varepsilon\big]\rho\big(u(t)\big), \quad t \in [t_0, T), \quad u(t_0) = u_0. \tag{64}$$

By Lemma 16, a solution exists for $\varepsilon > 0$ sufficiently small. Let

$$w(t) = \left\|x(t) - v\right\| - u_\varepsilon(t).$$

Then, we may take ε sufficiently small so that

$$w(t_0) \leqslant 0, \qquad w(t_1) > 0.$$

Let t_2 be the largest number in $[t_0, t_1)$ such that $w(t_2) = 0$ and

$$w(t) > 0, \quad t \in (t_2, t_1].$$

For $h > 0$ sufficiently small, we have

$$\frac{w(t_2 + h) - w(t_2)}{h} > 0.$$

Consequently,

$$D^+ w(t_2) \geqslant 0.$$

But

$$\begin{aligned} D^+ w(t_2) &= D^+ \left\|x(t_2) - v\right\| - u'_\varepsilon(t_2) \\ &\leqslant \left\|x'(t_2)\right\| - u'_\varepsilon(t_2) \\ &= \left\|g\big(t_2, x(t_2)\big)\right\| - \big[\gamma(t_2) + \varepsilon\big]\rho\big(u_\varepsilon(t_2)\big) \\ &\leqslant \gamma(t_2)\rho\big(\left\|x(t_2) - v\right\|\big) - \big[\gamma(t_2) + \varepsilon\big]\rho\big(u_\varepsilon(t_2)\big) \\ &= -\varepsilon\rho\big(u_\varepsilon(t_2)\big) \\ &< 0. \end{aligned} \tag{65}$$

This contradiction proves the theorem. □

Theorem 19. *Let $g(t,x)$ be a continuous map from $\mathbb{R}\times H$ to H, where H is a Banach space. Assume that for each point $(t_0,x_0)\in\mathbb{R}\times H$, there are constants $K,b>0$ such that*

$$\big\|g(t,x)-g(t,y)\big\|\leqslant K\|x-y\|,\quad |t-t_0|<b,\quad \|x-x_0\|<b,\quad \|y-x_0\|<b. \tag{66}$$

Assume also that

$$\big\|g(t,x)\big\|\leqslant \gamma(t)\rho\big(\|x-v\|\big),\quad x\in H,\ t\in[t_0,T_M), \tag{67}$$

where $T_M\leqslant\infty$, and $\gamma(t)$, $\rho(t)$ satisfy the hypotheses of Lemma 16 with ρ nondecreasing and v a fixed element of H. Then for each $x_0\in H$ there is a unique solution $x(t)$ of the equation

$$\frac{dx(t)}{dt}=g\big(t,x(t)\big),\quad t\in[t_0,T_M),\quad x(t_0)=x_0. \tag{68}$$

Moreover, $x(t)$ depends continuously on x_0 and satisfies

$$\big\|x(t)-v\big\|\leqslant u(t),\quad t\in[t_0,T_M), \tag{69}$$

where $u(t)$ is the solution of (58) in that interval satisfying $u(t_0)=u_0\geqslant\|x_0-v\|$.

Proof. By Theorems 15 and 18 there is an interval $[t_0,t_0+m], m>0$, in which a unique solution of

$$\frac{dx(t)}{dt}=g\big(t,x(t)\big),\quad t\in[t_0,t_0+m],\quad x(t_0)=x_0 \tag{70}$$

exists and satisfies

$$\big\|x(t)-v\big\|\leqslant u(t),\quad t\in[t_0,t_0+m], \tag{71}$$

where $u(t)$ is the unique solution of

$$u'(t)=\gamma(t)\rho\big(u(t)\big),\quad t\in[t_0,T_M),\quad u(t_0)=u_0=\|x_0-v\|. \tag{72}$$

Let $T\leqslant T_M$ be the supremum of all numbers t_0+m for which this holds. If $t_1<t_2<T$, then the solution in $[t_0,t_2]$ coincides with that in $[t_0,t_1]$, since such solutions are unique. Thus a unique solution of (70) satisfying (71) exists for each $t_0<t<T$. Moreover, we have

$$x(t_2)-x(t_1)=\int_{t_1}^{t_2} g\big(t,x(t)\big)\,dt.$$

Consequently,

$$\begin{aligned}\big\|x(t_2)-x(t_1)\big\| &\leqslant \int_{t_1}^{t_2}\big\|g\big(t,x(t)\big)\big\|\,dt\\ &\leqslant \int_{t_1}^{t_2}\gamma(t)\rho\big(\big\|x(t)-v\big\|\big)\,dt\end{aligned}$$

$$\leqslant \int_{t_1}^{t_2} \gamma(t)\rho\big(u(t)\big)\,dt$$
$$= u(t_2) - u(t_1).$$

Assume that $T < T_M$. Let t_k be a sequence such that $t_0 < t_k < T$ and $t_k \to T$. Then

$$\big\|x(t_k) - x(t_j)\big\| \leqslant u(t_k) - u(t_j) \to 0.$$

Thus $\{x(t_k)\}$ is a Cauchy sequence in H. Since H is complete, $x(t_k)$ converges to an element $x_1 \in H$. Since $\|x(t_k) - v\| \leqslant u(t_k)$, we see that $\|x_1 - v\| \leqslant u(T)$. Moreover, we note that

$$x(t) \to x_1 \quad \text{as } t \to T.$$

To see this, let $\varepsilon > 0$ be given. Then there is a k such that

$$\big\|x(t_k) - x_1\big\| < \varepsilon, \qquad u(T) - u(t_k) < \varepsilon.$$

Then for $t_k \leqslant t < T$,

$$\big\|x(t) - x_1\big\| \leqslant \big\|x(t) - x(t_k)\big\| + \big\|x(t_k) - x_1\big\|$$
$$\leqslant u(t) - u(t_k) + \big\|x(t_k) - x_1\big\| < 2\varepsilon.$$

We define $x(T) = x_1$. Then, we have a solution of (70) satisfying (71) in $[0, T]$. By Theorem 15, there is a unique solution of

$$\frac{dy(t)}{dt} = g\big(t, y(t)\big), \qquad y(T) = x_1 \tag{73}$$

satisfying $\|y(t) - v\| \leqslant u(t)$ in some interval $|t - T| < \delta$. By uniqueness, the solution of (73) coincides with the solution of (70) in the interval $(T - \delta, T]$. Define

$$z(t) = x(t), \quad t_0 \leqslant t < T,$$
$$z(T) = x_1,$$
$$z(t) = y(t), \quad T < t \leqslant T + \delta.$$

This gives a solution of (70) satisfying (71) in the interval $[t_0, T + \delta)$, contradicting the definition of T. Hence, $T = T_M$. □

5. The remaining proofs

We can now prove Theorem 1.

Proof of Theorem 1. If the theorem were not true, there would be a $\delta > 0$ such that

$$\big\|G'(u)\big\| \geqslant \psi\big(d(u, \tilde{A})\big) \tag{74}$$

would hold for all u in the set

$$Q = \big\{u \in E\colon\ b_0 - 3\delta \leqslant G(u) \leqslant a_0 + 3\delta\big\}. \tag{75}$$

We can find $\theta < 1$, $T > 0$ such that

$$a_0 - b_0 + 2\delta < \theta T, \quad T \leqslant \int_R^{\nu} \psi(t)\,dt. \tag{76}$$

Let

$$Q_0 = \{u \in Q\colon\ b_0 - 2\delta \leqslant G(u) \leqslant a_0 + 2\delta\}, \tag{77}$$

$$Q_1 = \{u \in Q\colon\ b_0 - \delta \leqslant G(u) \leqslant a + \delta\} \tag{78}$$

and

$$Q_2 = E \setminus Q_0, \qquad \eta(u) = d(u, Q_2)/\big[d(u, Q_1) + d(u, Q_2)\big]. \tag{79}$$

It is easily checked that $\eta(u)$ is locally Lipschitz continuous on E and satisfies

$$\eta(u) = 1, \quad u \in Q_1; \qquad \eta(u) = 0, \quad u \in \overline{Q}_2; \qquad 0 < \eta(u) < 1, \quad \text{otherwise}.$$

There is a locally Lipschitz continuous map $Y(u)$ of $\hat{E} = \{u \in E\colon\ G'(u) \neq 0\}$ into itself such that

$$\|Y(u)\| \leqslant 1, \qquad \theta\|G'(u)\| \leqslant \big(G'(u), Y(u)\big), \quad u \in \hat{E} \tag{80}$$

(cf., e.g., [6]). Let $\sigma(t)$ be the flow generated by

$$W(u) = -\eta(u)Y(u)\rho\big(d(u, \tilde{A})\big), \tag{81}$$

where $\rho(\tau) = 1/\psi(\tau)$. Since $\|W(u)\| \leqslant \rho(d(u, \tilde{A}))$ and $W(u)$ is locally Lipschitz continuous, $\sigma(t)$ exists for all $t \in [0, T]$ in view of Theorem 19. For $v \in A \cap \partial\mathbf{B}_\nu$, let $x(t) = \sigma(t)v - v$. Then

$$\|x'(t)\| \leqslant \rho\big(\|x(t)\|\big), \quad x(0) = 0.$$

Let $u(t)$ be the solution of (58) satisfying $u(0) = 0$. By Lemma 16,

$$\int_R^{R+u(T)} \psi(t)\,dt \leqslant \int_0^{u(T)} \psi(t)\,dt = T \leqslant \int_R^{\nu} \psi(t)\,dt.$$

Thus,

$$\|x(T)\| \leqslant u(T) \leqslant \nu - R.$$

Hence,

$$\|\sigma(t)v\| \geqslant R, \quad t \in [0, T],\ v \in A \cap \partial\mathbf{B}_\nu. \tag{82}$$

We also have

$$\begin{aligned} dG\big(\sigma(t)v\big)/dt &= \big(G'(\sigma),\sigma'\big) = -\eta(\sigma)\big(G'(\sigma),Y(\sigma)\big)\rho\big(d(\sigma,\tilde{A})\big) \\ &\leqslant -\theta\eta(\sigma)\big\|G'(\sigma)\big\|\rho\big(d(\sigma,\tilde{A})\big) \\ &\leqslant -\theta\eta(\sigma)\psi\big(d(\sigma,\tilde{A})\big)\rho\big(d(\sigma,\tilde{A})\big) \\ &= -\theta\eta(\sigma) \end{aligned} \tag{83}$$

in view of (74) and (80). Now suppose $v \in E_{a_0+\delta}$ is such that there is a $t_1 \in [0,T]$ for which $\sigma(t_1)v \notin Q_1$, where

$$E_\gamma = \big\{v \in E\colon\ G(v) \leqslant \gamma\big\}.$$

Then

$$G\big(\sigma(t_1)v\big) < b_0 - \delta,$$

since we cannot have $G(\sigma(t_1)v) > a_0 + \delta$ for $v \in E_{a_0+\delta}$ by (83). But this implies

$$G\big(\sigma(T)v\big) < b_0 - \delta. \tag{84}$$

On the other hand, if $\sigma(t)v \in Q_1$ for all $t \in [0,T]$, then

$$G\big(\sigma(T)v\big) \leqslant G(v) - \theta\int_0^T dt \leqslant a_0 + \delta - \theta T < b_0 - \delta$$

by (76). Thus, (84) holds for $v \in E_{a_0+\delta}$. In particular, this holds for all $v \in \tilde{A}$. But by (83),

$$G\big(\sigma(t)0\big) \leqslant G(0) - \int_0^t \eta\big(\sigma(s)0\big)\,ds, \quad 0 \leqslant t \leqslant T.$$

In order for $\sigma(\tau)0$ to intersect B for some $\tau \leqslant T$, we would need $G(0) = b_0$ and

$$\eta\big(\sigma(s)0\big) = 0, \quad 0 \leqslant s \leqslant \tau.$$

This would mean that

$$\sigma(s)0 \in \overline{Q}_2, \quad 0 \leqslant s \leqslant \tau.$$

This contradicts $G(\sigma(\tau)0) = b_0$. Hence, $\sigma(s)0$ remains in the interior of B for $0 \leqslant s \leqslant T$. In view of (82), this implies

$$\sigma(t)\tilde{A} \cap B \neq \emptyset, \quad t \in [0,T]. \tag{85}$$

But this is impossible by (84). Hence, there is a sequence satisfying (9). This completes the proof. □

Proof of Theorem 2. For some $\varepsilon > 0$, take

$$C = \gamma_\varepsilon = \frac{a_0 - b_0}{\ln(1+\varepsilon/3)}$$

and

$$\psi(t) = \frac{\gamma_\varepsilon}{2\nu + t}.$$

Then $\psi(t)$ satisfies (7) and (8). Consequently, there is a sequence satisfying

$$G(u_k) \to c, \quad b_0 \leqslant c \leqslant a_0, \quad \big(2\nu + d(u_k, \tilde{A})\big)\big\|G'(u_k)\big\| \leqslant \gamma_\varepsilon.$$

Since

$$\|u_k\| \leqslant \nu + d(u_k, \tilde{A}),$$

we have

$$\big(\nu + \|u_k\|\big)\big\|G'(u_k)\big\| \leqslant \gamma_\varepsilon.$$

The proof is complete. □

Proof of Theorem 3. Take γ_ε as above and

$$\psi_k(t) = \frac{\gamma_\varepsilon}{2\nu_k + t}.$$

Then $\psi_k(t)$ satisfies (7) and (8) for each k. Then there is a sequence satisfying

$$G(u_k) \to c, \quad b_0 \leqslant c \leqslant a_0, \quad \big(2\nu_k + d(u_k, \tilde{B}_k)\big)\big\|G'(u_k)\big\| \leqslant \gamma_\varepsilon.$$

Since

$$\|u_k\| \leqslant \nu_k + d(u_k, \tilde{B}_k),$$

we have

$$\big(\nu_k + \|u_k\|\big)\big\|G'(u_k)\big\| \leqslant \gamma_\varepsilon.$$

Apply Theorem 2. □

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