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Lie Coalgebras*

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DEDICATED TO SAUNDERS MAC LANE ON THE OCCASION OF HIS RECENT 70TH BIRTHDAY

A Lie coalgebra is a coalgebra whose comultiplication $\Delta : M \to M \otimes M$ satisfies the Lie conditions. Just as any algebra A whose multiplication $\varphi : A \otimes A \to A$ is associative gives rise to an associated Lie algebra $\mathfrak{L}(A)$, so any coalgebra C whose comultiplication $\Delta : C \to C \otimes C$ is associative gives rise to an associated Lie coalgebra $\mathfrak{L}^c(C)$. The assignment $C \mapsto \mathfrak{L}^c(C)$ is functorial. A universal coenveloping coalgebra $U^c(M)$ is defined for any Lie coalgebra M by asking for a right adjoint U^c to $\mathfrak{L}^c$. This is analogous to defining a universal enveloping algebra $U(L)$ for any Lie algebra L by asking for a left adjoint U to the functor $\mathfrak{L}$. In the case of Lie algebras, the unit (i.e., front adjunction) $1 \to \mathfrak{L} \circ U$ of the adjoint functor pair $U \dashv \mathfrak{L}$ is always injective. This follows from the Poincaré–Birkhoff–Witt theorem, and is equivalent to it in characteristic zero ($\chi = 0$). It is, therefore, natural to inquire about the counit (i.e., back adjunction) $\mathfrak{L}^c \circ U^c \to 1$ of the adjoint functor pair $\mathfrak{L}^c \dashv U^c$.

THEOREM. *For any Lie coalgebra M, the natural map $\mathfrak{L}^c(U^cM) \to M$ is surjective if and only if M is locally finite, (i.e., each element of M lies in a finite dimensional sub Lie coalgebra of M).*

An example is given of a non locally finite Lie coalgebra. The existence of such an example is surprising since any coalgebra C whose diagonal Δ is associative is necessarily locally finite by a result of that theory. The present paper concludes with a development of an analog of the Poincaré–Birkhoff–Witt theorem for Lie algebras which we choose to call the Dual Poincaré–Birkhoff–Witt Theorem and abbreviate by "The Dual PBWθ." The constraints of the present paper, however, allow only a sketch of this theorem. A complete proof will appear in a subsequent paper. The reader may, however, consult [12], in the meantime, for details. The Dual PBWθ shows for any locally finite Lie coalgebra M the existence (in $\chi = 0$) of a natural isomorphism of the graded Hopf algebras ${}_0E(U^cM)$ and ${}_0E(S^cM)$ associated to U^cM and to $S^cM = U^c(\text{Triv } M)$ when $U^c(M)$ and $S^c(M)$ are given the Lie filtrations. [Just as $U^c(M)$ is the analog of the enveloping algebra $U(L)$ of a Lie algebra L, so $S^c(V)$ is the analog of the symmetric algebra $S(V)$ on a vector space V. Triv(M) denotes the trivial Lie coalgebra structure on the underlying vector space of M obtained by taking the comultiplication to be the zero map.]

* The present paper is an account of some of my investigations into Lie coalgebras. These began in the spring of 1969 as an outgrowth of a seminar on Hopf algebras that I was giving at the time at the University of Washington.

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Contents.

1. Introductory Remarks, Definitions, Examples

Nonassociative algebras have been studied for some time. Among these, the Lie algebras in particular have shown themselves to be fruitful objects for research. In the case of coalgebras, however, attention has focused primarily on those with an associative diagonal. In the present work, we study coalgebras in which the diagonal satisfies the Lie conditions, the so-called Lie coalgebras. Lie coalgebras have also been considered by Michel André (cf. [1, 2]); however, the ones considered by him are graded and "reduced" (i.e., zero in degree zero) whereas those considered by us are ungraded. As such a distinction turns out to be significant, there is no overlap in our results.

As their name indicates, Lie coalgebras are defined dually to Lie algebras. To display this duality, one proceeds as in the classical case. (cf. Jonah [9]). Specifically, one considers Lie coalgebras (respectively, Lie algebras) to be Lie coalgebras (respectively, Lie algebras) over the monoidal category $(\mathscr{V}, \otimes)$ where $\mathscr{V}$ denotes the category of vector spaces over a field K and $\otimes: \mathscr{V} \times \mathscr{V} \to \mathscr{V}$ the functor assigning to any ordered pair (V, W) of vector spaces their tensor product $V \otimes W = V \otimes_K W$ over K. (For the definition of a monoidal category, the reader is referred to Mac Lane [11].) One then defines a Lie coalgebra over the monoidal category $(\mathscr{V}, \otimes)$ to be a Lie algebra over the monoidal category $(\mathscr{V}^{op}, \otimes^{op})$ where $\mathscr{V}^{op}$ denotes the opposite category of $\mathscr{V}$ and $\otimes^{op}: \mathscr{V}^{op} \times \mathscr{V}^{op} \to \mathscr{V}^{op}$ the functor induced in the obvious way by $\otimes$. To see what this means in down-to-earth language, we first recall that a Lie algebra consists of a vector space L together with a linear map $[\ ,\ ]: L \otimes L \to L$ (called the "bracket") such that

(1) $[x, x] = 0 \qquad \forall\, x \in L$

and

(2) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \qquad \forall\, x, y, z \in L.$

Condition (1) is a strong form of anti-commutativity whereas condition (2) is known as the *Jacobi identity*. In this form, the definition of a Lie algebra is not dualizable: What we must do is "get rid of the elements." To do so, we write $\varphi: L \otimes L \to L$ in place of $[\ ,\ ]$ (so that $\varphi(x \otimes y) = [x, y]$ for $x, y \in L$) and

$$\xi: L \otimes L \otimes L \to L \otimes L \otimes L$$

for the linear map induced by the map

$$L \times L \times L \to L \times L \times L$$

which cyclically permutes the co-ordinates [i.e., $(x, y, z) \mapsto (y, z, x)$]. In effect,

$$\xi = \xi_V \colon V \otimes V \otimes V \to V \otimes V \otimes V$$

may be defined for any vector space V as the composite

$$V \otimes V \otimes V \xrightarrow{\simeq} V \otimes (V \otimes V) \xrightarrow{\tau(V, V\otimes V)} (V \otimes V) \otimes V \xrightarrow{\simeq} V \otimes V \otimes V$$

in which the first and last maps are the natural isomorphisms sending $x \otimes y \otimes z$ to $x \otimes (y \otimes z)$ and $(x \otimes y) \otimes z$ to $x \otimes y \otimes z$, respectively, and in which

$$\tau_{(V,W)} \colon V \otimes W \to W \otimes V$$

is defined for any ordered pair (V, W) of vector spaces as the linear map which sends $v \otimes w$ to $w \otimes v$ (i.e., τ is the "twist" map). Thus, under ξ,

$$x \otimes y \otimes z \mapsto x \otimes (y \otimes z) \mapsto (y \otimes z) \otimes x \mapsto y \otimes z \otimes x$$

so under ξ^2,

$$x \otimes y \otimes z \mapsto z \otimes x \otimes y.$$

With these notational conventions, the Jacobi identity [condition (2)] evidently becomes

$$\varphi \circ (1_L \otimes \varphi) \circ (1_{L\otimes L\otimes L} + \xi_L + \xi_L^2) = 0 \colon L \otimes L \otimes L \to L;$$

and we have "gotten rid of the elements" in the second condition. To "get rid of the elements" in the first condition (*strong anti-commutativity*), observe that

$$[x, x] = 0 \quad \forall\, x \in L \Leftrightarrow \varphi(x \otimes x) = 0 \quad \forall\, x \in L$$
$$\Leftrightarrow \langle x \otimes x \mid x \in L\rangle \subset \ker \varphi$$

where $\langle x \otimes x \mid x \in L\rangle$ denotes the subspace of $L \otimes L$ generated by all $x \otimes x$ with $x \in L$. It is clear that one has the inclusion

$$\langle x \otimes x \mid x \in L\rangle \subset \ker[1_{L\otimes L} - \tau_{(L,L)}];$$

and we claim that the reverse inclusion also holds (and is straightforward to check). In passing, it is worthwhile to note that

$$\mathrm{Im}(1 + \tau) \subset \ker(1 - \tau)$$

[i.e., $(1-\tau)\circ(1+\tau)=0$] with equality holding precisely when the characteristic $\chi(K)$ of the ground field K is different from 2. Thus, condition (1) may be replaced by the condition

$$\ker(1-\tau)\subset\ker\varphi$$

[i.e., by $\operatorname{Im}(1+\tau)\subset\ker(1-\tau)\subset\ker\varphi$] and in case $\chi(K)\neq 2$ by $\varphi\circ(1+\tau)=0$, i.e., by $\varphi=-\varphi\circ\tau$.

A Lie algebra over $(\mathscr{V},\otimes)$ can thus be considered to be a vector space L together with a linear map $\varphi: L\otimes L\to L$ (called the "bracket") subject to the two conditions

(1) $\ker(1-\tau)\subset\ker\varphi$, and

(2) $\varphi\circ(1\otimes\varphi)\circ(1+\xi+\xi^2)=0: L\otimes L\otimes L\to L.$

Accordingly, we define a Lie coalgebra over $(\mathscr{V},\otimes)$ to consist of a vector space M together with a linear map $\Delta: M\to M\otimes M$ (called the "cobracket") subject to the two conditions

(1) $\operatorname{Im}\Delta\subset\operatorname{Im}(1-\tau)$, and

(2) $(1+\xi+\xi^2)\circ(1\otimes\Delta)\circ\Delta=0: M\to M\otimes M\otimes M.$

We note here that

$$\operatorname{Im}(1-\tau)\subset\ker(1+\tau)$$

[i.e., $(1+\tau)\circ(1-\tau)=0$] with equality holding in case $\chi(K)\neq 2$. Thus, in case $\chi(K)\neq 2$, we may replace condition (1) by condition

(1′) $\Delta=-\tau\circ\Delta.$

At this juncture, we shall switch to a parallel display format as an aid to the reader. We adopt this form of exposition the better to display the connection between Lie algebras and Lie coalgebras, and we begin by recapitulating the definitions of Lie algebra and of Lie coalgebra in this format.

Definition (*Algebra*)

A *Lie algebra* over $(\mathscr{V},\otimes)$ is a pair (L,φ) where L is an object of $\mathscr{V}$ and $\varphi: L\otimes L\to L$ is a morphism of $\mathscr{V}$ subject to

(1) $\ker(1-\tau)\subset\ker\varphi$,

and

(2) $\varphi\circ(1\otimes\varphi)\circ(1+\xi+\xi^2)=0.$

Note. $\operatorname{Im}(1+\tau)\subset\ker(1-\tau)\subset\ker\varphi$.

Definition (*Coalgebra*)

A *Lie coalgebra* over $(\mathscr{V},\otimes)$ is a pair (M,Δ) where M is an object of $\mathscr{V}$ and $\Delta: M\to M\otimes M$ is a morphism of $\mathscr{V}$ subject to

(1) $\operatorname{Im}\Delta\subset\operatorname{Im}(1-\tau)$,

and

(2) $(1+\xi+\xi^2)\circ(1\otimes\Delta)\circ\Delta=0.$

Note. $\operatorname{Im}\Delta\subset\operatorname{Im}(1-\tau)\subset\ker(1+\tau)$.

We remark in passing that, for any vector space V,

$$\operatorname{Im}(1 - \tau) = \bigcap_{f \in V^*} \ker(f \otimes f)$$

where V^* denotes the vector space dual of V.

We now turn to some examples of Lie coalgebras and, of course, of Lie algebras. First, we shall look at some familiar examples of Lie algebras, on the left side of the page. Then, on the right side of the page, we shall follow each Lie algebra example by its Lie coalgebra counterpart.

Examples of Lie Algebras

1. As our first example of a Lie algebra, we consider Euclidean 3-space $\mathbb{E}^3$ together with the vector cross product $\times$. Let $\mathbb{R}$ denote the field of real numbers. If we identify $\mathbb{E}^3$ with $\mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_3$ in which e_1, e_2, and e_3 denote, respectively, the standard basis elements (1, 0, 0), (0, 1, 0) and (0, 0, 1) of $\mathbb{E}^3$, and if we write $[v, w]$ in place of the vector cross product $v \times w$ (for $\{v, w\} \subset \mathbb{E}^3$), then one clearly has that $[e_i, e_i] = 0$ for each $i \in \{1, 2, 3\}$ and also that $[e_1, e_2] = e_3$, $[e_2, e_3] = e_1$, and $[e_3, e_1] = e_2$. Plainly, then, the Jacobi identity is satisfied for e_1, e_2, and e_3, i.e., $[e_1, [e_2, e_3]] + [e_2, [e_3, e_1]] + [e_3, [e_1, e_2]] = [e_1, e_1] + [e_2, e_2] + [e_3, e_3] = 0 + 0 + 0 = 0$. Since $[\ ,\ ]: \mathbb{E}^3 \otimes \mathbb{E}^3 \to \mathbb{E}^3$ is bilinear and $\{e_1, e_2, e_3\}$ is a basis for $\mathbb{E}^3$, it follows easily that $[x, x] = 0$ for all $x \in \mathbb{E}^3$ and that

$$[x,[y, z]] + [y,[z, x]] + [z,[x, y]] = 0$$

for all elements x, y, and z of $\mathbb{E}^3$. Alternatively, the Jacobi identity follows immediately from the fact that

$$[x, [y, z]] = (x \cdot z)y - (x \cdot y)z$$

where, for instance, $x \cdot z$ denotes the dot product of x with z:

$$\begin{aligned} x \cdot z &= (x_1, x_2, x_3) \cdot (z_1, z_2, z_3) \\ &= x_1 \cdot z_1 + x_2 \cdot z_2 + x_3 \cdot z_3 . \end{aligned}$$

Examples of Lie Coalgebras

1. As our first example of a Lie coalgebra, we consider $(\mathbb{E}^3)^*$, the vector space dual of $\mathbb{E}^3$, together with the diagonal $\Delta: (\mathbb{E}^3)^* \to (\mathbb{E}^3)^* \otimes (\mathbb{E}^3)^*$ defined as follows: Let $\{e^1, e^2, e^3\}$ denote the dual basis of $(\mathbb{E}^3)^*$ to $\{e_1, e_2, e_3\}$ of $\mathbb{E}^3$ [so that $e^i(e_j) = \delta_{ij} = \{{0, \text{ if } i \neq j \atop 1, \text{ if } i = j}\}$]. We then get a Lie coalgebra structure on $(\mathbb{E}^3)^*$ by putting

$$\Delta(e^1) = e^2 \otimes e^3 - e^3 \otimes e^2,$$
$$\Delta(e^2) = e^3 \otimes e^1 - e^1 \otimes e^3,$$

and

$$\Delta(e^3) = e^1 \otimes e^2 - e^2 \otimes e^1.$$

Indeed, we evidently have that

$$\operatorname{Im} \Delta \subset \operatorname{Im}(1 - \tau).$$

Moreover, upon applying $(1+\xi+\xi^2) \circ (1 \otimes \Delta) \circ \Delta$ to e^1, for instance, we find that

$$\begin{aligned} e^1 &\overset{\Delta}{\mapsto} e^2 \otimes e^3 - e^3 \otimes e^2 \\ &\qquad \downarrow 1 \otimes \Delta \\ &e^2 \otimes e^1 \otimes e^2 - e^2 \otimes e^2 \otimes e^1 \\ &- e^3 \otimes e^3 \otimes e^1 + e^3 \otimes e^1 \otimes e^3 \\ &\qquad \downarrow 1 + \xi + \xi^2 \\ &e^2 \otimes e^1 \otimes e^2 + e^1 \otimes e^2 \otimes e^2 \\ &+ e^2 \otimes e^2 \otimes e^1 - e^2 \otimes e^2 \otimes e^1 \\ &- e^2 \otimes e^1 \otimes e^2 - e^1 \otimes e^2 \otimes e^2 \\ &- e^3 \otimes e^3 \otimes e^1 - e^3 \otimes e^1 \otimes e^3 \\ &- e^1 \otimes e^3 \otimes e^3 + e^3 \otimes e^1 \otimes e^3 \\ &+ e^1 \otimes e^3 \otimes e^3 + e^3 \otimes e^3 \otimes e^1 \\ &= 0. \end{aligned}$$

Similarly, $(1 + \xi + \xi^2) \circ (1 \otimes \Delta) \circ \Delta(e^i) = 0$ for each $i \in \{2, 3\}$. Since Δ is defined as the unique linear map having the above values on e^1, e^2, and e^3, we get—in this way—the structure of a Lie coalgebra on $(\mathbb{E}^3)^*$.

The above example can readily be generalized: In a perfectly analogous way, which shall be examined in greater detail below, one can endow the vector space dual of any finite dimensional Lie algebra with the structure of a Lie coalgebra. [This type of assignment is functorial, and accounts for the fact that the categories of finite dimensional Lie algebras and of finite dimensional Lie coalgebras are dual to one another.] Even more generally, one can define a Lie coalgebra L^0 (read "L upper zero") for any Lie algebra L, in a functorial way. L^0 is a subspace of L^* and coincides with L^* in case L is finite dimensional. The details of this construction will appear below.

2. As our second example of a Lie algebra, we look not at a specific example of a Lie algebra, but rather at a standard procedure for manufacturing an entire class of Lie algebras.

To any algebra A with an associative multiplication

$$\varphi_A: A \otimes A \to A$$

we can associate a Lie algebra $\mathfrak{L}(A)$ in the following way: As vector spaces $\mathfrak{L}(A)$ and A are identical [i.e., the underlying vector spaces of $\mathfrak{L}(A)$ and of A coincide]; while for $x \in \mathfrak{L}(A)$ and $y \in \mathfrak{L}(A)$, $[x, y]_{\mathfrak{L}(A)}$ is defined by the equation

$$[x, y]_{\mathfrak{L}(A)} = x \cdot y - y \cdot x$$

2. Dually, we now show how to manufacture an entire class of Lie coalgebras. Just as one can define for any associative algebra its associated Lie algebra, so one can define for any coalgebra C having an associative diagonal

$$\Delta_C: C \to C \otimes C$$

its associated Lie coalgebra $\mathfrak{L}^c(C)$. This is done in the following way: As vector spaces $\mathfrak{L}^c(C)$ and C are identical [i.e., the underlying vector spaces of $\mathfrak{L}^c(C)$ and of C are the same]; and the diagonal $\Delta_{\mathfrak{L}^c(C)}$ of $\mathfrak{L}^c(C)$ is given by the identity

$$\Delta_{\mathfrak{L}^c(C)} = (1 - \tau) \circ \Delta_C$$

where the dot in $x \cdot y$ and in $y \cdot x$ denotes the multiplication in A. To "dualize" this example, one must, once again, "get rid of the elements," but this is easy. Simply note that the above identity simply says that $\varphi_{\mathfrak{L}(A)}$, the multiplication on $\mathfrak{L}(A)$, is given by the equation

$$\varphi_{\mathfrak{L}(A)} = \varphi_A \circ (1 - \tau),$$

where 1 is the identity on $A \otimes A$ and $\tau: A \otimes A \to A \otimes A$ is the twist map. Of course, one must check that $\varphi_{\mathfrak{L}(A)}$ satisfies the Lie conditions, but that is routine: Plainly,

$$[x, x] = x \cdot x - x \cdot x = 0$$

for each element x of $\mathfrak{L}(A)$ while the associativity of φ_A guarantees (what we choose to call) the *Jacobi associativity* of $\varphi_{\mathfrak{L}(A)}$. The verification that this is so is routine and requires only careful bookkeeping (and a cup of coffee); it is, therefore, omitted. [It has been suggested that this verification be done at most once in a lifetime!]

3. A third familiar example of a Lie algebra is the Lie algebra of "primitives" $P(H)$ of an (associative) Hopf algebra H. Specifically, if H is a Hopf algebra with an associative multiplication

$$\varphi_H: H \otimes H \to H,$$

then $P(H)$, the space of primitives of H, carries the structure of a Lie algebra, in a natural way, as a sub Lie algebra of $\mathfrak{L}(H)$, the Lie algebra associated to H:

$$P(H) \hookrightarrow \mathfrak{L}(H).$$

where 1 is the identity of $C \otimes C$ and $\tau: C \otimes C \to C \otimes C$ is the twist map. Thus, if

$$\Delta_C(c) = \sum_{i=1}^{n} c_{1i} \otimes c_{2i}$$

then

$$\Delta_{\mathfrak{L}^c(C)}(c) = \sum_{i=1}^{n} [c_{1i} \otimes c_{2i} - c_{2i} \otimes c_{1i}].$$

We shall use $\langle c \rangle$ to denote $\Delta_{\mathfrak{L}^c(C)}(c)$. Thus,

$$\langle c \rangle = \Delta_{\mathfrak{L}^c(C)}(c) = \sum_{i=1}^{n} [c_{1i} \otimes c_{2i} - c_{2i} \otimes c_{1i}].$$

To see that $\Delta_{\mathfrak{L}^c(C)}$ equips $\mathfrak{L}^c(C)$ with the structure of a Lie coalgebra, we must check that $\Delta_{\mathfrak{L}^c(C)}$ satisfies the Lie conditions. Plainly,

$$\operatorname{Im} \Delta_{\mathfrak{L}^c(C)} \subset \operatorname{Im}(1 - \tau);$$

and we claim that the associativity of Δ_C guarantees the Jacobi associativity of $\Delta_{\mathfrak{L}^c(C)}$. In this case, the "dual" of "at most once" is "at most once," and we omit the details.

3. When one studies Hopf algebras, it is useful to consider, for any Hopf algebra H, not only the "primitives" $P(H)$ of H, but also the "indecomposables" $Q(H)$ of H. Not surprisingly, one gets functors P and Q, and it turns out that for any biassociative Hopf algebra H, P and Q are defined dually to one another. Thus, one would expect, in the case of a Hopf algebra H with an associative diagonal

$$\Delta_H: H \to H \otimes H,$$

to be able to equip $Q(H)$—the space

To see why $P(H)$ inherits a Lie algebra structure in a natural way as a sub Lie algebra of $\mathfrak{L}(H)$, recall that $P(H)$ can be considered to be the space

$$\{x \in H \mid \Delta x = \mathbf{1} \otimes x + x \otimes \mathbf{1}\},$$

where $\mathbf{1} = \eta_H(1_K)$, $\eta_H: K \to H$ being the unit of H and 1_K being the identity of the ground field K. Thus $\mathbf{1}$ is the (multiplicative) unit of (the algebra) H. Since the diagonal Δ of a Hopf algebra is always (by definition) an algebra map, we find that if $x \in P(H)$ and $y \in P(H)$, then

$$\begin{aligned}
\Delta[x, y] &= \Delta(x \cdot y - y \cdot x) \\
&= \Delta x \cdot \Delta y - \Delta y \cdot \Delta x \\
&= (\mathbf{1} \otimes x + x \otimes \mathbf{1})(\mathbf{1} \otimes y + y \otimes \mathbf{1}) \\
&\quad - (\mathbf{1} \otimes y + y \otimes \mathbf{1})(\mathbf{1} \otimes x + x \otimes \mathbf{1}) \\
&= \mathbf{1} \otimes xy + y \otimes x + x \otimes y + xy \otimes \mathbf{1} \\
&\quad - \mathbf{1} \otimes yx - x \otimes y - y \otimes x - yx \otimes \mathbf{1} \\
&= \mathbf{1} \otimes xy - \mathbf{1} \otimes yx + xy \otimes \mathbf{1} - yx \otimes \mathbf{1} \\
&= \mathbf{1} \otimes (xy - yx) + (xy - yx) \otimes \mathbf{1} \\
&= \mathbf{1} \otimes [x, y] + [x, y] \otimes \mathbf{1}.
\end{aligned}$$

This shows that $[x, y] \in P(H)$ whenever $x \in P(H)$ and $y \in P(H)$; and it follows that $P(H)$ inherits the structure of a Lie algebra as a sub Lie algebra of $\mathfrak{L}(H)$, the Lie algebra associated to H. [The definition of a coalgebra appears below, while the definition of a Hopf algebra appears in Section 3 under *The Hopf Algebra Structure of* U^cM.]

of "indecomposables" of H—with the structure of a Lie coalgebra in a natural way as a quotient Lie coalgebra of $\mathfrak{L}^c(H)$, the Lie coalgebra associated to H:

$$\mathfrak{L}^c(H) \twoheadrightarrow Q(H).$$

This is indeed the case; to see why, recall that $Q(H)$ can be considered to be the space $\bar{H}/\bar{H}^2$, where $\bar{H}$ is the maximal ideal $\ker \epsilon_H$, $\epsilon_H: H \to K$ being the counit (augmentation) of H. The main observation necessary to showing that $Q(H)$ inherits a Lie coalgebra structure as a quotient of $\mathfrak{L}^c(H)$ is the observation that

$$\langle x \rangle = \Delta_{\mathfrak{L}^c(H)}(x) = (\Delta_H - \tau \circ \Delta_H)(x)$$

is an element of

$$\ker(1 \otimes \epsilon) \cap \ker(\epsilon \otimes 1),$$

i.e., of

$$(H \otimes \bar{H}) \cap (\bar{H} \otimes H) = \bar{H} \otimes \bar{H}.$$

From this, it follows that $\bar{H}$ is a sub Lie coalgebra of $\mathfrak{L}^c(H)$ under

$$\bar{H} \hookrightarrow \bar{H} \oplus \eta K = H$$

(where $\eta: K \to H$ is the unit of H), as well as a quotient Lie coalgebra of $\mathfrak{L}^c(H)$ under

$$H = \bar{H} \oplus \eta K \twoheadrightarrow \bar{H}.$$

[In any Hopf algebra, one has that $H = \ker \epsilon \oplus \operatorname{Im} \eta = \bar{H} \oplus \eta K$.] What is more, the square $\bar{H}^2$ of $\bar{H}$ is readily shown to be a *Lie coideal* of $\bar{H}$. [This means that for $x \in \bar{H}$ and $y \in \bar{H}$, $\langle x \cdot y \rangle \in \bar{H}^2 \otimes \bar{H} + \bar{H} \otimes \bar{H}^2$.] Accordingly, $Q(H)$ inherits the structure of a Lie coalgebra as a quotient Lie coalgebra of $\mathfrak{L}^c(H)$, the Lie coalgebra associated to H, under the map

$$\mathfrak{L}^c(H) \twoheadrightarrow \bar{H} \twoheadrightarrow \bar{H}/\bar{H}^2 = Q(H).$$

We now present a fourth example of a Lie coalgebra. The example that we are about to give is special in several respects. First of all, it comes from the air (or heaven) so to speak, as opposed to arising in some general, functorial way as did the examples considered so far. Secondly, this example turns out to be an example of a coalgebra that is not locally finite (a term defined below), and that fact is of interest both in the general context of coalgebra theory and in the particular context of Lie coalgebra theory. Both of these assertions, at this point necessarily vague, will be illuminated further on in this paper. We wish to emphasize here, however, that this example really is noteworthy; and was discovered not at the beginning (as it is presented here), but only later, when—for reasons that will be appreciated after Section 4—it became a question of paramount importance to know whether there were any Lie coalgebras that where not locally finite. [*Note.* It will shortly become clear that finite dimensional Lie coalgebras are trivially locally finite; that $\mathfrak{L}^c(C)$ is locally finite since C is; and that $Q(H)$ is locally finite as a quotient of $\mathfrak{L}^c(H)$.] Without further ado, let us then introduce this very special Lie coalgebra which we choose to denote by the pair (E, Δ). Here E denotes the vector space on the countably infinite set of basis elements $\{x_n\}_{n=0}^{\infty}$; and $\Delta\colon E \to E \otimes E$ denotes the linear map determined by the requirement that

$$\Delta(x_0) = 0$$

while

$$\Delta(x_n) = (1 - \tau)(x_0 \otimes x_{n+1}) \qquad \text{for} \quad n \geqslant 1.$$

Because of the importance we have attached to (E, Δ), it is appropriate to check that (E, Δ) is, in fact, a Lie coalgebra, and this we shall now do. Clearly $\operatorname{Im} \Delta \subset \operatorname{Im}(1 - \tau)$, so it remains to verify the Jacobi associativity of Δ. Let $n \geqslant 1$. Then

$$\begin{aligned}(1 \otimes \Delta) \circ \Delta(x_n) &= (1 \otimes \Delta)[x_0 \otimes x_{n+1} - x_{n+1} \otimes x_0] \\ &= x_0 \otimes \Delta(x_{n+1}) - x_{n+1} \otimes \Delta(x_0) \\ &= x_0 \otimes (x_0 \otimes x_{n+2} - x_{n+2} \otimes x_0) - x_{n+1} \otimes 0 \\ &= x_0 \otimes x_0 \otimes x_{n+2} - x_0 \otimes x_{n+2} \otimes x_0 .\end{aligned}$$

Therefore,

$$\begin{aligned}&(1 + \xi + \xi^2) \circ (1 \otimes \Delta) \circ \Delta(x_n) \\ &\quad = x_0 \otimes x_0 \otimes x_{n+2} + x_0 \otimes x_{n+2} \otimes x_0 + x_{n+2} \otimes x_0 \otimes x_0 \\ &\qquad - x_0 \otimes x_{n+2} \otimes x_0 - x_{n+2} \otimes x_0 \otimes x_0 - x_0 \otimes x_0 \otimes x_{n+2} \\ &\quad = 0,\end{aligned}$$

and we have shown that $(1 + \xi + \xi^2) \circ (1 \otimes \Delta) \circ \Delta(x_n) = 0$ for all $n \geqslant 1$. Since this identity holds trivially for $n = 0$, the validity of the Jacobi identity follows immediately. Thus, (E, Δ) is a Lie coalgebra.

We claim that x_1 does not lie in any finite dimensional sub Lie coalgebra of E. Why is this so? Well, our intuition tells us that if N denotes the smallest sub Lie coalgebra of E containing x_1, then N must contain x_0 and x_2 since $\Delta x_1 = x_0 \otimes x_2 - x_2 \otimes x_0$ and to say that N is a *sub Lie coalgebra* of E is to say that $\Delta(N) \subset N \otimes N$. But, the same reasoning leads to the conclusion that x_3 must lie in N whenever x_2 lies in N since $\Delta(x_2) = x_0 \otimes x_3 - x_3 \otimes x_0$. Similarly, if $x_3 \in N$, then $x_4 \in N$; and $x_4 \in N$ implies $x_5 \in N$, etc. Certainly this argument, as it stands, is not rigorous, but the above-mentioned suspicions can be confirmed. Their verifications use the annihilator transformations between subspaces of a Lie coalgebra and its dual Lie algebra, and also use the linear independence of the x_n; we shall omit the details.

DEFINITION. A coalgebra C is *locally finite* if and only if any $x \in C$ lies in some finite dimensional subcoalgebra $D \subset C$.

This definition furnishes us with the vocabulary necessary to assert that (E, Δ) is *not* locally finite.

Remark. The only element of $\{x_n\}_{n=0}^{\infty}$ which lies in a finite dimensional sub Lie coalgebra of E is x_0, as the reader is invited to check.

At this stage, as we suggested somewhat earlier, the significance of the above example lies in the fact that it is a basic result of the theory of associative coalgebras with counit that any such coalgebra is locally finite (cf. [16, p. 46, Theorem 2.2.1; 6, p. 65, Lemma III.1.8; or 10, p. 351, Proposition 2.5]). Later on, however, in Section 4, we shall observe a very concrete and significant consequence of this existence of non locally finite Lie coalgebras.

Remark. As an aid to a reader unfamiliar with coalgebra theory, we include a few definitions: The diagonal $\Delta: C \to C \otimes C$ of a coalgebra (C, Δ) is said to be associative in case the composites $(\Delta \otimes 1) \circ \Delta$ and $(1 \otimes \Delta) \circ \Delta$ coincide as maps from C to $C \otimes C \otimes C$. A coalgebra (C, Δ) is said to have a (two-sided) counit $\epsilon: C \to K$ [ϵ being a linear map] in case the diagram

$$\begin{array}{ccccc} C \otimes C & \xleftarrow{\Delta} & C & \xrightarrow{\Delta} & C \otimes C \\ \downarrow{\scriptstyle \epsilon \otimes 1_C} & & \downarrow{\scriptstyle 1_C} & & \downarrow{\scriptstyle 1_C \otimes \epsilon} \\ K \otimes C & \xrightarrow{\simeq} & C & \xleftarrow{\simeq} & C \otimes K \end{array}$$

is commutative. Here, the maps $K \otimes C \to^{\simeq} C$ and $C \otimes K \to^{\simeq} C$ are the obvious, natural isomorphisms $k \otimes c \mapsto k \cdot c$ and $c \otimes k \mapsto c \cdot k$, respectively, the dot denoting scalar multiplication. These definitions (of associativity of Δ, and of two-sided counit) result from "getting rid of the elements" in the usual algebra definitions (of associativity of φ, and of two-sided unit $\eta: K \to A$) via expressing those "pre-dual" conditions in terms of (diagrams with) arrows and then "turning the arrows around."

We conclude this section by observing that it is really the lack of associativity of (the diagonal of) a Lie coalgebra that accounts for the existence of non locally finite Lie coalgebras. [Of course, Lie coalgebras neither are associative nor do they have counits.] Indeed, although the classic result, from coalgebra theory, states that any coalgebra with an associative diagonal and counit is locally finite, one need not, in fact, assume the existence of a counit because one can prove that any coalgebra C with an associative diagonal is locally finite. The reason this is so is that one can show that any coalgebra C having an associative diagonal can always be obtained as a coalgebra quotient of a coalgebra C_ϵ having an associative diagonal *and* a counit, and this in a universal way. Now the image of a locally finite coalgebra under a coalgebra map is again locally finite because the image of a subcoalgebra under a coalgebra map is again a subcoalgebra.

Note. $f: (C, \Delta_C) \to (D, \Delta_D)$ is a coalgebra map in case $\Delta_D \circ f = (f \otimes f) \circ \Delta_C$; while $f: (C, \Delta_C, \epsilon_C) \to (D, \Delta_D, \epsilon_D)$ is a coalgebra map if, additionally, $\epsilon_D \circ f = \epsilon_C$.

Remark. The "universal way" mentioned above is an allusion to the fact that the assignment $C \mapsto C_\epsilon$ (referred to above) gives rise to a functor which is right adjoint to the forgetful functor. It is in this sense that the above-mentioned construction of obtaining an associative coalgebra C as a quotient of an associative, counitary coalgebra C_ϵ in a universal way is the analog of embedding an associative algebra A in an associative, unitary algebra A_η in a universal way. The details of this construction may be found in [12, p. 6].

Remark. The usefulness of local finiteness of each object of the category $\mathscr{C}$ of associative, counitary coalgebras results from the fact that one can establish many results about objects of $\mathscr{C}$ by dualizing results about finite dimensional objects of $\mathscr{A}$, the category of associative, unitary algebras. Similarly, one can expect to get results about locally finite Lie coalgebras by dualizing results that hold for finite dimensional Lie algebras.

2. Categories and Functors

(a) *Categories*

In what follows, it will be convenient for us to have some symbols to denote the categories frequently referred to. With morphisms defined in the obvious way, we shall let $\mathscr{V}$, $\mathscr{A}$, $\mathscr{C}$, $\mathscr{L}$, $\mathscr{L}^c$, and $\mathscr{L}^c_{l.f.}$ denote, respectively, the categories of vector spaces, associative unitary algebras, associative counitary coalgebras, Lie algebras, Lie coalgebras, and locally finite Lie coalgebras. In addition, $c\mathscr{A}$ and $c\mathscr{C}$ shall denote, respectively, the full subcategories of commutative (associative, unitary) algebras and commutative (associative, counitary) coalgebras. In other words, for $A \in \operatorname{Obj} \mathscr{A}$ and $C \in \operatorname{Obj} \mathscr{C}$, we have $A \in \operatorname{Obj} c\mathscr{A}$ and $C \in \operatorname{Obj} c\mathscr{C}$ if and only if $\varphi_A = \varphi_A \circ \tau$ and $\Delta_C = \tau \circ \Delta_C$, respectively.

(b) *Some Functors*

We now discuss a number of functors that arise in a natural way in connection with Lie coalgebras. The first one we shall discuss is the "upper zero."

1. $(\ )^0 \colon \mathscr{L}^{op} \to \mathscr{L}^c$

To any Lie algebra L, we can associate, in a functorial way, a Lie coalgebra L^0 (the "upper zero" of L) as described below. First, however, we recall that any time one has a coalgebra C, one can put an algebra structure on the vector space dual C^* of C. Lie coalgebras are no exception. If (M, Δ) is a Lie coalgebra, then the linear map

$$\Delta \colon M \to M \otimes M$$

gives rise to a linear map

$$\Delta^* \colon (M \otimes M)^* \to M^*.$$

Since there is always a (natural) linear injection

$$\rho \colon M^* \otimes M^* \to (M \otimes M)^*$$

from $M^* \otimes M^*$ to $(M \otimes M)^*$ defined by

$$\rho(f \otimes g)(x \otimes y) = f(x) \cdot g(y),$$

there is an obvious candidate, namely,

$$\Delta^* \circ \rho \colon M^* \otimes M^* \to (M \otimes M)^* \to M^*,$$

for a multiplication on M^*. It should come as no surprise that if $\Delta \colon M \to M \otimes M$ equips M with the structure of a Lie coalgebra, then $\Delta^* \circ \rho \colon M^* \otimes M^* \to M^*$ equips M^* with the structure of a Lie algebra. [In fact, if V is a vector space and $\Delta \colon V \to V \otimes V$ is a linear map, then (V, Δ) is a Lie coalgebra if and only if $(V^*, \Delta^* \circ \rho)$ is a Lie algebra.] Furthermore, the dual $f^* \colon M_2^* \to M_1^*$ of a map $f \colon M_1 \to M_2$ of $\mathscr{L}^c$ is a map of $\mathscr{L}$. In just this way, one gets a (contravariant) functor $*$ from $\mathscr{L}^c$ to $\mathscr{L}$, i.e., a (covariant) functor

$$(\ )^* \colon (\mathscr{L}^c)^{op} \to \mathscr{L}.$$

It is natural to inquire whether, conversely, L^* carries the structure of a Lie coalgebra in case L is a Lie algebra. Let's see. If $\varphi: L \otimes L \to L$ gives the multiplication on L, then $\varphi^*: L^* \to (L \otimes L)^*$ is a map from L^* to $(L \otimes L)^*$. We saw before that there was a natural injection $\rho: L^* \otimes L^* \to (L \otimes L)^*$ from $L^* \otimes L^*$ to $(L \otimes L)^*$. Unfortunately, ρ "goes the wrong way":

$$L^* \xrightarrow{\varphi^*} (L \otimes L)^* \xleftarrow{\rho} L^* \otimes L^*.$$

Of course in case L is finite dimensional, ρ is surjective as well (as injective) and hence invertible. In that case, L^* does carry the structure of a Lie coalgebra, as we saw earlier. [In fact, if V is a finite dimensional vector space and φ: $V \otimes V \to V$ is a linear map, then (V, φ) is a Lie algebra if and only if $(V^*, \rho^{-1} \circ \varphi^*)$ is a Lie coalgebra.] What can be done "in general"? Well, the map $\varphi: L \otimes L \to L$ induces a map $\varphi^*: L^* \to (L \otimes L)^*$ in any case. Consider the diagram

$$\begin{array}{ccc} L^* & \xrightarrow{\varphi^*} & (L \otimes L)^* \\ \big\uparrow & & \big\uparrow \rho_L \\ & & L^* \otimes L^* \\ & & \big\uparrow \\ V & \dashrightarrow & V \otimes V, \end{array}$$

in which V is a subspace of L^*. Basically, we would like to consider a subspace V of L^* to be a "good" subspace in case we can define a map from V to $V \otimes V$ filling in the above diagram.

Definition. A subspace $V \subset L^*$ is called "good" in case $\varphi^*(V) \subset \rho_L(V \otimes V)$.

It is easy to see that the sum of good subspaces of L^* is again a good subspace of L^* (cf. the proposition below).

Definition. For any Lie algebra L, put

$$L^0 = \sum_{V \in \mathscr{G}} V$$

where $\mathscr{G}$ denotes the set of all good subspaces of L^*.

Proposition. *L^0 is a good subspace of L^*, hence the maximal good subspace of L^*.*

Proof.

$$\varphi^*(L^0) = \varphi^*\left(\sum V\right) = \sum \varphi^*(V) \subset \sum \rho(V \otimes V) \subset \sum \rho\left[\left(\sum V\right) \otimes \left(\sum V\right)\right]$$

$$= \sum \rho(L^0 \otimes L^0) = \rho(L^0 \otimes L^0).$$

Hence L^0 is a good subspace of L^*.

Whenever V is a good subspace of L^*, we may define a map

$$\Delta_V \colon V \to V \otimes V$$

by requiring that

$$\rho_L[\Delta_V(f)] = \varphi^*(f) \qquad \forall f \in V \subset L^*.$$

This makes sense because $\operatorname{Im} \varphi^* \subset \operatorname{Im} \rho$ and ρ is injective. The map $\Delta_V\colon V \to V \otimes V$ so defined fills in the diagram

$$\begin{array}{ccc} L^* & \xrightarrow{\varphi^*} & (L \otimes L)^* \\ \uparrow & & \uparrow \rho_L \\ \Big| & & L^* \otimes L^* \\ \Big| & & \uparrow \\ V & \overset{\Delta_V}{\dashrightarrow} & V \otimes V. \end{array}$$

Notice that if we write $\Delta_V(f)$ as $\sum_{i=1}^n g_i \otimes h_i$, then

$$f[x, y] = \sum_{i=1}^{n} g_i(x) \cdot h_i(y) \qquad \forall x, y \in L.$$

PROPOSITION. *For any good subspace V of L^*, (V, Δ_V) is a Lie coalgebra. In particular, (L^0, Δ_{L^0}) is a Lie coalgebra.*

If (L_1, φ_1) and (L_2, φ_2) are Lie algebras and $f\colon L_1 \to L_2$ is a Lie algebra map, then $f^*\colon L_2^* \to L_1^*$ takes good subspaces (of L_2^*) to good subspaces (of L_1^*), because

$$\varphi_1^*(f^*V) = (f \otimes f)^*(\varphi_2^* V) \subset (f \otimes f)^* \rho_{L_2}(V \otimes V)$$

$$= \rho_{L_1}(f^* \otimes f^*)(V \otimes V) = \rho_{L_1}(f^*V \otimes f^*V).$$

Consequently, $f^*(L_2^0) \subset L_1^0$, so the restriction of f^* to L_2^0 induces a map

$$f^0\colon L_2^0 \to L_1^0,$$

the unique linear map making the following diagram commutative

$$\begin{array}{ccc} L_2^* & \xrightarrow{f^*} & L_1^* \\ \uparrow & & \uparrow \\ \cup & & \cup \\ L_2^0 & \dashrightarrow{f^0} & L_1^0 . \end{array}$$

It is easy to check that f^0 is a Lie coalgebra map. It follows that the assignment $L \mapsto L^0$ and $f \mapsto f^0$ defines a contravariant functor from $\mathscr{L}$ to $\mathscr{L}^c$.

THEOREM. *The contravariant functor* $^0: \mathscr{L} \to \mathscr{L}^c$ *is "adjoint on the right" to the contravariant functor* $*: \mathscr{L}^c \to \mathscr{L}$; *i.e., for every Lie algebra* L *and Lie coalgebra* M, *there is a natural set bijection*

$$\mathrm{Hom}_{\mathscr{L}}(L, M^*) \simeq \mathrm{Hom}_{\mathscr{L}^c}(M, L^0).$$

Remark. In the proof of the above, one must show that there exist natural transformations

$$\phi: 1_{\mathscr{L}^c} \to {}^0\circ{}^* \qquad \text{and} \qquad \psi: 1_{\mathscr{L}} \to {}^*\circ{}^0$$

such that the composites

$$M^* \xrightarrow{\psi_{M^*}} M^{*0*} \xrightarrow{(\phi_M)^*} M^* \qquad (\text{for } M \in \mathrm{Obj}\, \mathscr{L}^c)$$

and

$$L^0 \xrightarrow{\phi_{L^0}} L^{0*0} \xrightarrow{(\psi_L)^0} L^0 \qquad (\text{for } L \in \mathrm{Obj}\, \mathscr{L})$$

are the identities.

ϕ and ψ are defined by the commutative diagrams

$$\begin{array}{ccc} M & \xrightarrow{\chi_M} & M^{**} \\ & \searrow^{\phi_M} & \uparrow \iota_{M^{*0}} \\ & & M^{*0} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} L & \xrightarrow{\chi_L} & L^{**} \\ & \searrow^{\psi_L} & \downarrow (\iota_{L^0})^* \\ & & L^{0*} \end{array}$$

in which the maps $\iota_{(\,)}$ are inclusions, and $\chi_V: V \to V^{**}$ is defined for any vector space V by $\chi_V(v)(f) = f(v)$; in other words, χ_V is the natural injection of V into its double dual.

Note. The definition of ϕ_M makes sense because $\chi_M(M)$ is a good subspace of M^{**}, basically because we took the "upper zero" as "big as possible."

A Final Note. If L is finite dimensional, then $L^0 = L^*$. Clearly the categories $\mathscr{L}_{f.d.}$ and $\mathscr{L}^c_{f.d.}$ of finite dimensional Lie algebras and of finite dimensional Lie coalgebras are anti-equivalent.

This completes our discussion of the functor $(\ )^0$. We turn now to our next example.

2. $\mathfrak{L}^c\colon \mathscr{C} \to \mathscr{L}^c$

Earlier, when involved with the examples, we saw how to associate with any coalgebra C having an associative diagonal, a Lie coalgebra $\mathfrak{L}^c(C)$. If $f\colon C \to D$ is a morphism of $\mathscr{C}$, then $\mathfrak{L}^c(f)$ is a morphism of $\mathscr{L}^c$. [Here $\mathfrak{L}^c(f)$ and f coincide as vector space maps.] Notice that we distinguish between $\mathscr{L}^c$ (a category) and $\mathfrak{L}^c$ (a functor), and likewise between $\mathscr{L}$ and $\mathfrak{L}$.

It turns out that $\mathfrak{L}^c$ has a right adjoint U^c, the "universal coenvelop," described in Section 3, below.

Before moving on to our next example, we take note of a proposition to which we shall later refer. This result should, incidentally, come as no surprise.

PROPOSITION. *The following diagrams are commutative.*

(a)

$$\begin{array}{ccc} \mathscr{C}^{op} & \xrightarrow{\ *\ } & \mathscr{A} \\ {\scriptstyle (\mathfrak{L}^c)^{op}}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{L}} \\ (\mathscr{L}^c)^{op} & \xrightarrow{\ *\ } & \mathscr{L} \end{array}$$

and

(b)

$$\begin{array}{ccc} (\mathscr{A}_{f.d.})^{op} & \xrightarrow{\ *\ } & \mathscr{C}_{f.d.} \\ {\scriptstyle \mathfrak{L}^{op}}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{L}^c} \\ (\mathscr{L}_{f.d.})^{op} & \xrightarrow{\ *\ } & \mathscr{L}^c_{f.d.} \end{array}$$

Remark. In consequence of the above, we have that

$$[\mathfrak{L}^c(C)]^* = \mathfrak{L}(C^*) \qquad \text{for any object } C \text{ of } \mathscr{C},$$

and that

$$\mathfrak{L}^c(A^*) = [\mathfrak{L}(A)]^* \qquad \text{for any object } A \text{ of } \mathscr{A}_{f.d.}\,.$$

Here $\mathscr{A}_{f.d.}$ and $\mathscr{C}_{f.d.}$ denote the categories of finite dimensional objects of $\mathscr{A}$ and of $\mathscr{C}$, respectively.

3. Loc: $\mathscr{L}^c \to \mathscr{L}^c_{l.f.}$

The third functor we wish to consider is the functor Loc: $\mathscr{L}^c \to \mathscr{L}^c_{l.f.}$, which shall assign to each Lie coalgebra its locally finite part.

DEFINITION. For any Lie coalgebra M, set

$$\mathrm{Loc}(M) = \sum_{N,\ \text{a finite dimensional sub Lie coalgebra of } M} N.$$

$\mathrm{Loc}(M)$ is obviously also the sum of all locally finite sub Lie coalgebras of M, and hence the largest locally finite sub Lie coalgebra of M.

Denote by $\iota_{\mathrm{Loc}\,M}$: $\mathrm{Loc}\, M \hookrightarrow M$ the inclusion of Loc M into M. If f: $M_1 \to M_2$ is a Lie coalgebra map, then it is clear that $f(\mathrm{Loc}\, M_1) \subset \mathrm{Loc}\, M_2$. Thus, f: $M_1 \to M_2$ induces a map

$$\mathrm{Loc}(f)\colon \mathrm{Loc}(M_1) \to \mathrm{Loc}(M_2)$$

of $\mathscr{L}^c$ such that

$$\iota_{\mathrm{Loc}(M_2)} \circ \mathrm{Loc}(f) = f \circ \iota_{\mathrm{Loc}(M_1)} .$$

In this way, we get a functor from $\mathscr{L}^c$ to $\mathscr{L}^c_{l.f.}$ denoted *Loc*.

THEOREM. *The functor* Loc: $\mathscr{L}^c \to \mathscr{L}^c_{l.f.}$ *is right adjoint to the inclusion functor* I: $\mathscr{L}^c_{l.f.} \to \mathscr{L}^c$; *i.e., for every* $P \in \mathrm{Obj}\, \mathscr{L}^c_{l.f.}$ *and* $M \in \mathrm{Obj}\, \mathscr{L}^c$, *there is a natural set bijection*

$$\mathrm{Hom}_{\mathscr{L}^c}(IP, M) \simeq \mathrm{Hom}_{\mathscr{L}^c_{l.f.}}(P, \mathrm{Loc}\, M).$$

Hence $\mathscr{L}^c_{l.f.}$ *is a coreflective subcategory of* $\mathscr{L}^c$.

CONJECTURE. If $\mathrm{Loc}\, M = \{0\}$, then $M = \{0\}$.

If it is true that $M = \{0\}$ whenever $\mathrm{Loc}\, M = \{0\}$, it would follow that the example (E, Δ) of Section 1 is the best possible type of example of a non locally finite Lie coalgebra.

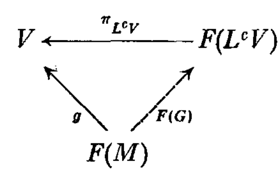
We now, momentarily, interrupt the flow of our presentation in order to present background material that will provide the context in which we can state our next result, a result which is an immediate consequence of the fact that Loc is a coreflector. Thereafter, we shall discuss a second application of the functor Loc, namely, the identification for any Lie algebra L, of the subspace

$$\{f \in L^* \mid \ker f \text{ contains a cofinite ideal of } L\}.$$

Recall that a category $\mathscr{K}$ is *complete* in case $\mathscr{K}$ has limits or equivalently products and equalizers. Dually, $\mathscr{K}$ is *cocomplete* in case $\mathscr{K}$ has colimits or equivalently coproducts and coequalizers. Now it is easy to see that $\mathscr{L}^c$ has coproducts, equalizers, and coequalizers (cf. [12, p. 9, Theorem I.1.12]). To show that $\mathscr{L}^c$ has products is, on the other hand, more involved: What one does is use the fact that the forgetful functor $F: \mathscr{L}^c \to \mathscr{V}$ from the category of Lie coalgebras to the category of vector spaces possesses a right adjoint

$$L^c: \mathscr{V} \to \mathscr{L}^c.$$

The fact that L^c is right adjoint to the forgetful functor expresses the fact that one has, for each vector space V, a Lie coalgebra $L^c(V)$ and a vector space map $\pi_{L^cV}: F(L^cV) \to V$ satisfying the following universal mapping property:

> If M is any object of $\mathscr{L}^c$ and $g: F(M) \to V$ is any morphism of $\mathscr{V}$, then there exists a unique morphism $G: M \to L^cV$ of $\mathscr{L}^c$ making the diagram
>
> 
>
>
> commutative.

This property clearly dualizes that satisfied by a free Lie algebra (LV, i_{LV}) on a vector space V. (For a definition of a free Lie algebra on a vector space V, see, for instance, [4, p. 285].) Accordingly, the pair (L^cV, π_{L^cV}) just described is called a *cofree Lie coalgebra on the vector space* V; it is determined up to canonical isomorphism by the universal mapping property it satisfies.

The details of the construction of L^c and of its subsequent use in the construction of products in $\mathscr{L}^c$ will appear in a subsequent paper. Those details provide a proof of the following theorem.

THEOREM. *$\mathscr{L}^c$ is complete and cocomplete.*

As an immediate consequence of the above, we find that $\mathscr{L}^c_{l.f.}$, as a full coreflective subcategory of $\mathscr{L}^c$, is likewise complete and cocomplete.

Remark. Since the forgetful functor $F: \mathscr{L}^c \to \mathscr{V}$ possesses a right adjoint L^c, the forgetful functor $\bar{F}: \mathscr{L}^c_{l.f.} \to \mathscr{V}$ likewise possesses a right adjoint, which shall be denoted by $L^c_{l.f.}: \mathscr{V} \to \mathscr{L}^c_{l.f.}$. $L^c_{l.f.}$ is given as the composite

$$\mathscr{V} \xrightarrow{L^c} \mathscr{L}^c \xrightarrow{\mathrm{Loc}} \mathscr{L}^c_{l.f.}$$

as the following diagram of categories and adjoint functor pairs reveals

$$\mathscr{L}^c_{l.f.} \underset{\mathrm{Loc}}{\overset{I}{\rightleftarrows}} \mathscr{L}^c \underset{L^c}{\overset{F}{\rightleftarrows}} \mathscr{V}.$$

[*Note*: $F \circ I$ is the forgetful functor.] As above there is, for each vector space V, a pair $(L^c_{l.f.}V, \pi_{L^c_{l.f.}V})$ where $L^c_{l.f.}V$ is an object of $\mathscr{L}^c_{l.f.}$ and $\pi_{L^c_{l.f.}V}: \bar{F}(L^c_{l.f.}V) \to V$ is a morphism of $\mathscr{V}$ satisfying the appropriate universal mapping property. The pair $(L^c_{l.f.}V, \pi_{L^c_{l.f.}V})$ is called a *locally finite cofree Lie coalgebra on the vector space* V.

The first functor that we looked at in this section was the functor "upper zero." Readers familiar with classical coalgebra theory and with the "upper zero" A^0 of an associative (unitary) algebra A as presented, for instance, in Sweedler's book [16] may wonder

(1) why we didn't define L^0, for a Lie algebra L, to consist of those elements of L^* whose kernel contains a cofinite ideal of L,

and

(2) what the connection is between the Lie coalgebra L^0 (that we have defined) and

$$\{f \in L^* \mid \ker f \text{ contains a cofinite ideal of } L\}.$$

Now that we have the functor Loc at our disposal, we can answer both of these questions at once. This is done in the following theorem.

THEOREM. *For any Lie algebra L,*

$$\mathrm{Loc}(L^0) = \{f \in L^* \mid \ker f \textit{ contains a cofinite ideal of } L\}.$$

In general, L^0 will not be locally finite, so in general $\mathrm{Loc}(L^0) \subsetneq L^0$. This may be seen by considering the example $L = M^*$ where M is a Lie coalgebra which is not locally finite. [Recall that $\phi_M: M \to M^{*0}$ is injective.] There is, however, a locally finite version of L^0, denoted L^{0f} and defined as the sum of all finite dimensional good subspaces of L^*:

$$L^{0f} = \sum_V \{V \mid V \in \mathscr{G} \text{ and } V \text{ is finite dimensional}\}.$$

Since $V \subset L^*$ is good if and only if V is a sub Lie coalgebra of L^0, it follows from the above that $L^{0f} = \mathrm{Loc}(L^0)$. Consequently, the functor ${}^{0f}: \mathscr{L}^{op} \to \mathscr{L}^c_{l.f.}$ is the composite $\mathscr{L}^{op} \to^{(\ )^0} \mathscr{L}^c \to^{\mathrm{Loc}} \mathscr{L}^c_{l.f.}$, and, as such, is the right adjoint of the functor $\mathscr{L}^c_{l.f.} \hookrightarrow \mathscr{L}^c \to^{(\ )^{*op}} \mathscr{L}^{op}$.

The fact that we may have $\mathrm{Loc}(L^0) \subsetneq L^0$ is in contrast to what may happen if we take an associative (unitary) algebra A instead of a Lie algebra L as the algebra to which we apply the "upper zero" construction. In case $A \in \mathrm{obj}\, \mathscr{A}$, $A^0 \in \mathrm{obj}\, \mathscr{C}$ and as such is locally finite. [Here we define A^0 to be the maximal good subspace of A^* where "good" again means that $\varphi_A^*(V) \subset \rho_A(V \otimes V)$,

$\varphi_A\colon A \otimes A \to A$ being the multiplication on A.] Since A^0 is locally finite, $A^0 \equiv \operatorname{Loc}(A^0)$ and consequently

$$A^0 = \{f \in A^* \mid \ker f \text{ contains a cofinite 2-sided ideal of } A\}.$$

This is the definition of A^0 given by Sweedler [16, p. 109]. In Sweedler's approach, the idea behind the construction of A^0 seems to be the following. Any linear map $f\colon A \to K$ whose kernel contains a cofinite two-sided ideal I of A gives rise to an element of $(A/I)^*$. Whenever I is a cofinite two-sided ideal of A, then A/I has the structure of a finite dimensional algebra, and hence $(A/I)^*$ has the structure of a finite dimensional coalgebra. Thus there is a linear map

$$(A/I)^* \to (A/I)^* \otimes (A/I)^*$$

giving the coalgebra structure of $(A/I)^*$. From a consideration of the exact sequence

$$0 \longrightarrow I \overset{i_I}{\hookrightarrow} A \overset{\pi_I}{\longrightarrow} A/I \longrightarrow 0$$

and then of the induced exact sequence

$$0 \longrightarrow (A/I)^* \xrightarrow{(\pi_I)^*} A^* \xrightarrow{(i_I)^*} I^* \longrightarrow 0$$

one sees that

$$\{f \in A^* \mid f(I) = 0\} = \ker[(i_I)^*] = \operatorname{Im}[(\pi_I)^*]$$

has the structure of a finite dimensional coalgebra. It is standard to denote $\ker(i_I^*)$ by the symbols $I^\perp$ (read, I-perp). Since the (cofinite) two-sided ideals I of A are directed [i.e., form a directed system], it follows that

$$A^0 = \bigcup_I I^\perp$$

has the structure of a coalgebra as a direct limit of finite dimensional coalgebras. In point of fact, Sweedler's construction of A^0 makes use of the algebra structure on $A \otimes A$. Since the tensor product of Lie algebras is not again a Lie algebra, a different approach was needed.

We conclude this section with a comparison of $\mathfrak{L}^c(A^0)$ with $(\mathfrak{L}A)^0$, where A is an associative algebra, $\mathfrak{L}A$ is the Lie algebra associated to A, and $\mathfrak{L}^c(A^0)$ is the Lie coalgebra associated to the associative coalgebra A^0. Such a comparison is of interest in view of the already established equalities

$$[\mathfrak{L}^c(C)]^* = \mathfrak{L}(C^*) \qquad \text{for} \quad C \in \operatorname{obj} \mathscr{C}$$

and

$$[\mathfrak{L}(A)]^* = \mathfrak{L}^c(A^*) \qquad \text{for} \quad A \in \operatorname{obj} \mathscr{A}_{f.d.}\,.$$

Here, by contrast, we do not have equality. Instead, we have the relation stated in the following proposition.

PROPOSITION. *For any associative (unitary) algebra A, $\mathfrak{L}^c(A^0)$ is a sub Lie coalgebra of* $\mathrm{Loc}[(\mathfrak{L}A)^0]$.

In general, $\mathfrak{L}^c(A^0) \subsetneq \mathrm{Loc}[(\mathfrak{L}A)^0]$, as may be seen by taking A to be an infinite dimensional, commutative, simple algebra (for example, an infinite dimensional, commutative field extension of the ground field K).

Remark. The elements of $\mathrm{Loc}(L^0)$ are in one-to-one correspondence with the finite dimensional representations of the Lie algebra L.

3. The Universal Coenveloping Coalgebra of a Lie Coalgebra

We now turn to the universal enveloping algebra of a Lie algebra and to the universal coenveloping coalgebra of a Lie coalgebra. We shall revert to our parallel display format in that we first list, on the left side of the page, results about Lie algebras. Then, on the right side of the page, we shall follow each Lie algebra result by its Lie coalgebra counterpart.

Since adjoint functors shall be appearing routinely in the exposition that follows, and since we wish to display the "duality" between the Lie algebra theory and the Lie coalgebra theory as succinctly as possible, we shall adopt the convention of writing

$$R \dashv S$$

to denote the fact that the functor $R\colon \mathscr{D} \to \mathscr{E}$ is left adjoint to the functor $S\colon \mathscr{E} \to \mathscr{D}$ and that the functor $S\colon \mathscr{E} \to \mathscr{D}$ is right adjoint to the functor $R\colon \mathscr{D} \to \mathscr{E}$, i.e., that there is a natural set bijection

$$\mathrm{Hom}_{\mathscr{E}}(RD, E) \xrightarrow{\cong} \mathrm{Hom}_{\mathscr{D}}(D, SE)$$

for each object (D, E) of $\mathscr{D}^{op} \times \mathscr{E}$.

The Lie Algebra Situation

In the case of Lie algebras, a universal enveloping algebra UL of a Lie algebra L is defined for every Lie algebra L by asking for a left adjoint

$$U \dashv \mathfrak{L}$$

The Lie Coalgebra Situation

"Dually," in the case of Lie coalgebras, one defines a universal coenveloping coalgebra U^cM for every Lie coalgebra M by asking for a right adjoint

$$\mathfrak{L}^c \dashv U^c$$

to the functor

$$\mathfrak{L}: \mathscr{A} \to \mathscr{L}.$$

In greater detail, *a universal enveloping algebra of a Lie algebra L* is defined to be an object UL of $\mathscr{A}$ together with a morphism

$$i_{UL}: L \to \mathfrak{L}(UL)$$

of $\mathscr{L}$ such that if A is any object of $\mathscr{A}$ and $f: L \to \mathfrak{L}(A)$ any morphism of $\mathscr{L}$, then there exists a unique morphism $F: UL \to A$ of $\mathscr{A}$ making the diagram

$$\begin{array}{ccc} L & \xrightarrow{i_{UL}} & \mathfrak{L}(UL) \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \mathfrak{L}(F)} & \\ & \mathfrak{L}(A) & \end{array}$$

commutative. The universal mapping property (hereafter abbreviated by U.M.P.) satisfied by (UL, i_{UL}) guarantees that there is a natural set bijection

$$\mathrm{Hom}_{\mathscr{A}}(UL, A) \xrightarrow{\simeq} \mathrm{Hom}_{\mathscr{L}}(L, \mathfrak{L}A),$$

i.e., that U is left adjoint to $\mathfrak{L}$ and $\mathfrak{L}$ is right adjoint to U:

$$U \dashv \mathfrak{L}.$$

Note. Here the map

$$\mathrm{Hom}_{\mathscr{A}}(UL, A) \to \mathrm{Hom}_{\mathscr{L}}(L, \mathfrak{L}A)$$

is given by

$$F \mapsto \mathfrak{L}(F) \circ i_{UL}.$$

to the functor

$$\mathfrak{L}^c: \mathscr{C} \to \mathscr{L}^c.$$

In greater detail, *a universal coenveloping coalgebra of a Lie coalgebra M* is defined to be an object U^cM of $\mathscr{C}$ together with a morphism

$$\pi_{U^cM}: \mathfrak{L}^c(U^cM) \to M$$

of $\mathscr{L}^c$ such that if C is any object of $\mathscr{C}$ and $f: \mathfrak{L}^c(C) \to M$ any morphism of $\mathscr{L}^c$, then there exists a unique morphism $F: C \to U^cM$ of $\mathscr{C}$ making the diagram

$$\begin{array}{ccc} M & \xleftarrow{\pi_{U^cM}} & \mathfrak{L}^c(U^cM) \\ & {\scriptstyle f}\nwarrow \quad \nearrow{\scriptstyle \mathfrak{L}^c(F)} & \\ & \mathfrak{L}^c(C) & \end{array}$$

commutative. The universal mapping property (hereafter abbreviated by U.M.P.) satisfied by (U^cM, π_{U^cM}) guarantees that there is a natural set bijection

$$\mathrm{Hom}_{\mathscr{L}^c}(\mathfrak{L}^cC, M) \xrightarrow{\simeq} \mathrm{Hom}_{\mathscr{C}}(C, U^cM),$$

i.e., that $\mathfrak{L}^c$ is left adjoint to U^c and U^c is right adjoint to $\mathfrak{L}^c$:

$$\mathfrak{L}^c \dashv U^c.$$

Note. Here the map

$$\mathrm{Hom}_{\mathscr{C}}(C, U^cM) \to \mathrm{Hom}_{\mathscr{L}^c}(\mathfrak{L}^cC, M)$$

is given by

$$F \mapsto \pi_{U^cM} \circ \mathfrak{L}^c(F).$$

Prior to the construction of (U^cM, π_{U^cM}), and of (UL, i_{UL}), it is worthwhile to look at a few special cases. We list these in a parallel display format where, for completeness, we restate the adjointness conditions that specify the desired properties of U and of U^c.

The Algebra Context

1. In the algebra context, we have that U is left adjoint to $\mathfrak{L}$:

$$U \dashv \mathfrak{L}.$$

The Coalgebra Context

1. "Dually," in the coalgebra context, we have that U^c is right adjoint to $\mathfrak{L}^c$:

$$\mathfrak{L}^c \dashv U^c.$$

We now look at some special cases.

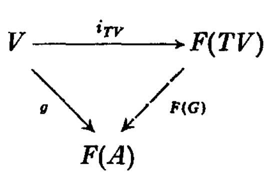
2. In the algebra context, one defines the tensor algebra functor $T: \mathscr{V} \to \mathscr{A}$ to be the left adjoint of the forgetful functor $F: \mathscr{A} \to \mathscr{V}$, i.e.,

$$T \dashv F.$$

2. "Dually," in the coalgebra context, we have a functor $T^c: \mathscr{V} \to \mathscr{C}$ which is right adjoint to the forgetful functor $F: \mathscr{C} \to \mathscr{V}$, i.e.,

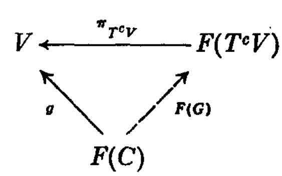
$$F \dashv T^c.$$

DEFINITION. *A tensor algebra on a vector space V* consists of an object TV of $\mathscr{A}$ together with a morphism $i_{TV}: V \to F(TV)$ of $\mathscr{V}$ such that if A is any object of $\mathscr{A}$ and $g: V \to F(A)$ any morphism of $\mathscr{V}$, then there is a unique morphism $G: TV \to A$ of $\mathscr{A}$ making the diagram



commutative. The pair (TV, i_{TV}) is called *a tensor algebra on the vector space V*, or sometimes, the *free associative unitary algebra on the vector space V.*

DEFINITION. *A tensor coalgebra on a vector space V* consists of an object T^cV of $\mathscr{C}$ together with a morphism $\pi_{T^cV}: F(T^cV) \to V$ of $\mathscr{V}$ such that if C is any object of $\mathscr{C}$ and $g: F(C) \to V$ any morphism of $\mathscr{V}$, then there is a unique morphism $G: C \to T^cV$ of $\mathscr{C}$ making the diagram



commutative. The pair (T^cV, π_{T^cV}) is called *a tensor coalgebra on the vector space V*, or sometimes, the *cofree associative counitary coalgebra on the vector space V.*

The construction of (TV, i_{TV}) is standard, but will be sketched in Section 5.

It turns out that T is a special case of U in that

$$T(V) = U[L(V)]$$

The construction of (T^cV, π_{T^cV}) is —by now—standard, but will be sketched below.

It turns out that T^c is a special case of U^c in that

$$T^c(V) = U^c[L^c(V)]$$

where $L(V)$ [more properly, $(L(V), i_{L(V)})$] denotes the free Lie algebra on the vector space V: Here the functor

$$L\colon \mathscr{V} \to \mathscr{L}$$

denotes the left adjoint of the forgetful functor

$$F\colon \mathscr{L} \to \mathscr{V},$$

i.e.,

$$L \dashv F.$$

The fact that $U[L(V)] = T(V)$, i.e., that

$$T = U \circ L,$$

is an immediate consequence of adjointness. To see this, one need merely consider the diagram

$$\underbrace{\mathscr{V} \underset{F}{\overset{L}{\rightleftarrows}} \mathscr{L}}_{L \dashv F} \underbrace{\underset{\mathfrak{L}}{\overset{U}{\rightleftarrows}} \mathscr{A}}_{U \dashv \mathfrak{L}}$$

of categories and adjoint functor pairs and observe that $F \circ \mathfrak{L}\colon \mathscr{A} \to \mathscr{L} \to \mathscr{V}$ is simply the forgetful functor from $\mathscr{A}$ to $\mathscr{V}$. It follows that

$$U \circ L\colon \mathscr{V} \to \mathscr{L} \to \mathscr{A}$$

is left adjoint to the forgetful functor, i.e.,

$$U \circ L \dashv F,$$

whence

$$U \circ L = T,$$

Remark 1. Prior to knowing of the existence of UL for an arbitrary Lie algebra L, one can show directly that TV is a universal envelop for LV. Specifically, one shows that TV to-

where $L^c(V)$ [more properly, $(L^c(V), \pi_{L^c(V)})$] denotes the cofree Lie coalgebra on the vector space V. Here $L^c\colon \mathscr{V} \to \mathscr{L}^c$ denotes the right adjoint of the forgetful functor

$$F\colon \mathscr{L}^c \to \mathscr{V},$$

i.e.,

$$F \dashv L^c,$$

as mentioned in Section 2. The fact that $U^c[L^c(V)] = T^c(V)$, i.e., that

$$T^c = U^c \circ L^c,$$

is an immediate consequence of adjointness. To see this, one need merely consider the diagram

$$\underbrace{\mathscr{C} \underset{U^c}{\overset{\mathfrak{L}^c}{\rightleftarrows}} \mathscr{L}^c}_{\mathfrak{L}^c \dashv U^c} \underbrace{\underset{L^c}{\overset{F}{\rightleftarrows}} \mathscr{V}}_{F \dashv L^c}$$

of categories and adjoint functor pairs and observe that $F \circ \mathfrak{L}^c\colon \mathscr{C} \to \mathscr{L}^c \to \mathscr{V}$ is simply the forgetful functor from $\mathscr{C}$ to $\mathscr{V}$. It follows that

$$U^c \circ L^c\colon \mathscr{V} \to \mathscr{L}^c \to \mathscr{C}$$

is right adjoint to the forgetful functor, i.e.,

$$F \dashv U^c \circ L^c,$$

whence

$$U^c \circ L^c = T^c.$$

Remark 1. Prior to knowing of the existence of U^cM for an arbitrary Lie coalgebra M, one can show directly that T^cV is a universal coenvelop for L^cV. Specifically, one shows that T^cV

gether with the map $i: LV \to \mathfrak{L}(TV)$ defined by the diagram

$$\begin{array}{ccc} V & \xrightarrow{i_{LV}} & F(LV) \\ & {\scriptstyle i_{TV}} \searrow \quad \swarrow {\scriptstyle F(i)} & \\ & \bar{F}(TV) = F[\mathfrak{L}(TV)] & \end{array}$$

(in which F and $\bar{F}$ are forgetful functors) is a universal enveloping algebra for LV. This is based on the natural equivalence

$$[TV, A]_{\mathscr{A}} \underset{T\dashv \bar{F}}{\simeq} [V, \bar{F}A]_{\mathscr{V}}$$

$$= [V, F(\mathfrak{L}A)]_{\mathscr{V}} \underset{L\dashv F}{\simeq} [LV, \mathfrak{L}A]_{\mathscr{L}}$$

where $[X, Y]_{\mathscr{E}}$ denotes the set of morphisms in the category $\mathscr{E}$ from X to Y.

Remark 2. In consequence of the Poincaré–Birkhoff–Witt theorem, one may take LV as the smallest sub Lie algebra of $\mathfrak{L}(TV)$ containing $\operatorname{Im} i_{TV}$, i.e., as the sub Lie algebra of $\mathfrak{L}(TV)$ generated by $\operatorname{Im} i_{TV}$. i_{LV} is then the map induced by i_{TV}, i.e., i_{LV} is the unique linear map making the following diagram commutative:

$$\begin{array}{ccc} V & \xrightarrow{i_{TV}} & F[\mathfrak{L}(TV)] \\ & {\scriptstyle i_{LV}} \searrow & \uparrow {\scriptstyle F[i]} \\ & & F[LV]. \end{array}$$

Here $i: LV \hookrightarrow \mathfrak{L}(TV)$ is the natural inclusion and $F: \mathscr{L} \to \mathscr{V}$ is the forgetful functor.

together with the map $\pi: \mathfrak{L}^c(T^cV) \to L^cV$ defined by the diagram

$$\begin{array}{ccc} V & \xleftarrow{\pi_{L^cV}} & F(L^cV) \\ & {\scriptstyle \pi_{T^cV}} \nwarrow \quad \nearrow {\scriptstyle F(\pi)} & \\ & \bar{F}(T^cV) = F[\mathfrak{L}^c(T^cV)] & \end{array}$$

(in which F and $\bar{F}$ are forgetful functors) is a universal coenveloping coalgebra for L^cV. This is based on the natural equivalence

$$[\mathfrak{L}^cC, L^cV]_{\mathscr{L}^c} \underset{F\dashv L^c}{\simeq} [F(\mathfrak{L}^cC), V]_{\mathscr{V}}$$

$$= [\bar{F}C, V]_{\mathscr{V}} \underset{\bar{F}\dashv T^c}{\simeq} [C, T^cV]_{\mathscr{C}}$$

where $[X, Y]_{\mathscr{E}}$ denotes the set of morphisms in the category $\mathscr{E}$ from X to Y.

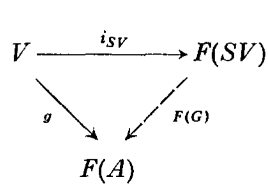
Remark 2. "Dual" to constructing the free Lie algebra on a vector space V as a sub Lie algebra of $\mathfrak{L}(TV)$, one may construct the *locally finite* cofree Lie coalgebra $L^c_{l.f.}(V)$ on a vector space V as a quotient of $\mathfrak{L}^c(T^cV)$. Specifically, one may take $L^c_{l.f.}(V)$ to be $\mathfrak{L}^c(T^cV)/I$ where $I \subset \mathfrak{L}^c(T^cV)$ is the largest Lie coideal of $\mathfrak{L}^c(T^cV)$ contained in $\ker \pi_{T^cV}$, i.e., where I is the coideal of $\mathfrak{L}^c(T^cV)$ "cogenerated" by $\ker \pi_{T^cV}$. $\pi_{L^c_{l.f.}(V)}$ is then the unique linear map making the following diagram commutative:

$$\begin{array}{ccc} V & \xleftarrow{\pi_{T^cV}} & F[\mathfrak{L}^c(T^cV)] \\ & {\scriptstyle \pi_{L^c_{l.f.}(V)}} \nwarrow & \downarrow {\scriptstyle F[p]} \\ & & F[\mathfrak{L}^c(T^cV)/I]. \end{array}$$

Here $p: \mathfrak{L}^c(T^cV) \to \mathfrak{L}^c(T^cV)/I$ is the natural projection and $F: \mathscr{L}^c_{l.f.} \to \mathscr{V}$ is the forgetful functor.

3. As our next (and second) example of a standard (type of) algebra which is a special instance of a universal enveloping algebra of a Lie algebra, we cite the symmetric algebra on a vector space.

DEFINITION. *A symmetric algebra on a vector space* V consists of an object SV of $c\mathscr{A}$ together with a morphism i_{SV}: $V \to F(SV)$ of $\mathscr{V}$ such that if A is any object of $c\mathscr{A}$ and g: $V \to F(A)$ any morphism of $\mathscr{V}$, then there is a unique morphism G: $SV \to A$ of $c\mathscr{A}$ making the diagram



commutative. The pair (SV, i_{SV}) is called a *symmetric algebra on the vector space* V, or sometimes, the *free commutative associative unitary algebra on the vector space* V.

Note. Ideals are what one factors algebras by to get quotient algebras. Dually, coideals are what one factors coalgebras by to get quotient coalgebras. If M is a Lie coalgebra and $I \subset M$ is a subspace of M, then I is a *coideal* of M in case

$$\Delta(I) \subset I \otimes M + M \otimes I.$$

Note. The above construction of $L^c_{l.f.}(V)$ as a quotient of $\mathfrak{L}^c(T^cV)$ utilizes the notion of "cogeneration" which shall be mentioned below and discussed more fully in Section 5.

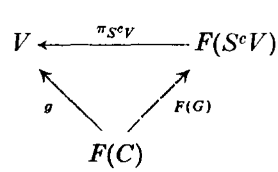
3. "Dually," in the coalgebra context, there is a functor S^c: $\mathscr{V} \to c\mathscr{C}$ from the category of vector spaces to the category of commutative, associative, counitary coalgebras which is right adjoint to the forgetful functor F: $c\mathscr{C} \to \mathscr{V}$, i.e.,

$$F \dashv S^c.$$

DEFINITION. *A symmetric coalgebra on a vector space* V consists of an object S^cV of $c\mathscr{C}$ together with a morphism π_{S^cV}: $F(S^cV) \to V$ of $\mathscr{V}$ such that if C is any object of $c\mathscr{C}$ and g: $F(C) \to V$ any morphism of $\mathscr{V}$, then there is a unique morphism

$$G\colon C \to S^cV$$

of $c\mathscr{C}$ making the diagram



commutative. The pair (S^cV, π_{S^cV}) is called a *symmetric coalgebra on the vector space* V, or sometimes, the

The construction of (SV, i_{SV}) is standard and hence is omitted (cf. the theorem which follows, below).

The U.M.P. satisfied by the pair (SV, i_{SV}) gives rise in a straightforward and standard way to a functor $S\colon \mathscr{V} \to c\mathscr{A}$ from the category of vector spaces to the category of commutative, associative, unitary algebras. S is called, for obvious reasons, the *symmetric algebra functor*. The U.M.P. satisfied by (SV, i_{SV}) guarantees that the symmetric algebra functor $S\colon \mathscr{V} \to c\mathscr{A}$ is left adjoint to the forgetful functor $F\colon c\mathscr{A} \to \mathscr{V}$, i.e., that

$$S \dashv F.$$

It turns out that S is a special case of U in that one can establish the following result.

THEOREM. *For any vector space V, let* Triv V *denote V considered as a Lie algebra in the trivial way: the map $V \otimes V \to V$ is the zero map. Then*

$$U(\text{Triv } V) = S(V).$$

cofree commutative associative counitary coalgebra on the vector space V.

Sweedler constructs S^cV as the largest commutative subcoalgebra of T^cV [i.e., as the sum of all commutative subcoalgebras of T^cV] and then takes π_{S^cV} to be the restriction of π_{T^cV} to S^cV. (See [16, p. 129, Theorem 6.4.3].)

It turns out that the *symmetric coalgebra functor* $S^c\colon \mathscr{V} \to c\mathscr{C}$ is a special instance of U^c in that one can establish the following result.

THEOREM. *For any vector space V, let* Triv V *denote V considered as a Lie coalgebra in the trivial way: the map $V \to V \otimes V$ is the zero map. Then*

$$U^c(\text{Triv } V) = S^c(V).$$

Remark. The proof of the theorem stated above uses the notion of a cogenerating subspace in that what one does is "dualize" the fact that if the elements of a generating subspace of an associative algebra pairwise commute then the algebra is commutative.

Recapitulation. Both TV and SV are special cases of the universal enveloping algebra of a Lie algebra. Likewise, their counterparts T^cV and S^cV are special cases of the universal coenveloping coalgebra of a Lie coalgebra. In consequence of the aforementioned facts, our U^cM generalizes the previously considered (known) T^cV and S^cV. Both T^cV and S^cV are discussed in Sweedler's book [16]; the notation used there, for them, is, however, different from the notation that we have adopted. We hope that the advantages of our notation are evident.

At this point, it will be convenient for us to review the construction of T^cV since that construction shall serve as a prototype for what follows. The construction of T^cV that we will be giving may be found in Sweedler (cf. [16, p. 125, Theorem 6.4.1]) though as mentioned above, the notation that we have adopted is our own and not his.

The construction of T^cV for an arbitrary vector space V, as given in [16], is a two-step process. One first shows that if (TV, i_{TV}) denotes the tensor algebra on V, then $(TV)^0$ together with the linear map

$$(TV)^0 \hookrightarrow (TV)^* \xrightarrow{(i_{TV})^*} V^*$$

satisfies the U.M.P. required of a cofree, associative, counitary coalgebra on the vector space V. Provided we keep in mind what is really being claimed, it should hopefully cause no confusion if we abbreviate this fact by simply writing

$$T^c(V^*) = (TV)^0.$$

Thus, as a first step, we see how to construct T^cW in case W is the vector space dual V^* of some vector space V. One then shows how to construct T^cW whenever W is a subspace of a vector space V for which T^cV is known. One simply observes that the diagram

$$\begin{array}{ccc} V & \xleftarrow{\pi_{T^cV}} & F[T^cV] \\ \uparrow & & \\ W & & \end{array}$$

may always be "filled in" to yield the commutative diagram

$$\begin{array}{ccc} V & \xleftarrow{\pi_{T^cV}} & F[T^cV] \\ \uparrow & & \uparrow \\ W & \xleftarrow[\pi_{T^cW}]{} & F[T^cW] \end{array}$$

giving a construction of T^cW for any subspace W of V. In other words, if T^cV is known and if W is a subspace of V, then T^cW may be constructed as a certain subspace [indeed subcoalgebra] of T^cV. Since any vector space V embeds in its double dual V^{**} via the map $\chi_V: V \to V^{**}$ where $\chi_V(v)(f) = f(v)$, the above two-step procedure yields a construction of T^cV for an arbitrary vector space V.

We now turn to the construction of UL and of U^cM. Once again, we shall resume our parallel display format in that we shall first describe, on the left-hand side of the page, the construction of UL. Following that description, we shall then describe, on the right-hand side of the page, the construction of U^cM.

The Construction of UL

Recall that the universal enveloping algebra UL of a Lie algebra L is constructed as a quotient algebra of the tensor algebra TL on L:

$$TL \twoheadrightarrow UL = TL/\langle\cdots\rangle.$$

In fact it is the universal mapping property (U.M.P.) that forces UL to be a quotient algebra of TL and indeed to be a specific quotient of TL. In order to remind ourselves of why and how this is the case, we consider the diagram

$$\begin{array}{ccc} L & \xrightarrow{i_{TL}} & F[TL] \\ & \searrow^{i_{UL}} & \\ & & F[UL] \end{array}$$

in which $F\colon \mathscr{A} \to \mathscr{V}$ is the forgetful functor. [If we were pedantic, we would instead consider the diagram

$$\begin{array}{ccc} \bar{F}(F) & \xrightarrow{i_{T(\bar{F}L)}} & \bar{F}[T(\bar{F}L)] \\ & \searrow^{\bar{F}(i_{UL})} & \\ & & \bar{F}[\mathfrak{L}(UL)] = F(UL) \end{array}$$

in which $\bar{F}\colon \mathscr{L} \to \mathscr{V}$ is the forgetful functor.] By the U.M.P. satisfied by (TL, i_{TL}), there exists a unique morphism $p\colon TL \to UL$ of $\mathscr{A}$ filling in the above diagram, i.e., making the diagram

$$\begin{array}{ccc} L & \xrightarrow{i_{TL}} & F(TL) \\ & \searrow^{i_{UL}} \quad \swarrow^{F(p)} & \\ & F(UL) & \end{array}$$

The Construction of U^cM

"Dually," we expect that U^cM should be a subcoalgebra of T^cM. As a matter of fact, returning for a moment to the construction of UL as a quotient of TL, we recall that TL has the structure of a Hopf algebra and that the ideal I of TL that one factors TL by to get UL is a Hopf ideal (i.e., an ideal and a coideal), so that UL inherits the structure of a Hopf algebra as a quotient Hopf algebra of TL. Thus, we should expect that T^cM has the structure of a Hopf algebra and that U^cM inherits a Hopf algebra structure from T^cM. This is indeed the case; but more of that later [cf. the second theorem under *The Hopf Algebra Structure of U^cM*].

CLAIM. *If (U^cM, π_{U^cM}) is a universal coenveloping coalgebra of a Lie coalgebra M, then U^cM must be a subcoalgebra of T^cM and π_{U^cM}: $\mathfrak{L}^c(U^cM) \to M$ must be the restriction to U^cM of $\pi_{T^cM}\colon F(T^cM) \to M$.*

Proof. "Dual" to the fact that $\operatorname{Im} i_{UL}$ is a generating subspace of UL [so that if B is a subalgebra of UL containing $\operatorname{Im} i_{UL}$, then $B = UL$], we have the fact that $\ker \pi_{U^cM}$ is a "cogenerating subspace" of U^cM [in the sense that if I is a coideal of U^cM contained in $\ker \pi_{U^cM}$, then $I = \{0\}$]. In each case, these facts are direct consequences of the universal mapping properties satisfied by (UL, i_{UL}) and by (U^cM, π_{U^cM}), respectively. Look at the diagram commutative. We claim that p is surjective. Why? Well p is an algebra map. Accordingly, $\operatorname{Im} p$ is a subalgebra of UL. But $i_{UL} = p \circ i_{TL}$ so that $\operatorname{Im} i_{UL} \subset \operatorname{Im} p$. Since $\operatorname{Im} i_{UL}$ generates UL as an algebra [by the U.M.P. satisfied by (UL, i_{UL})] it follows that $\operatorname{Im} p = UL$. Since the image of p is all of UL, it follows that p is surjective. But if E is any quotient of TL under an algebra map π: $TL \twoheadrightarrow E$, then a consideration of the (not necessarily commutative) diagram

$$\begin{array}{ccc} L \otimes L & \xrightarrow{[\ ,\ ]} & L \\ {\scriptstyle i_{TL}\otimes i_{TL}}\downarrow & & \downarrow{\scriptstyle i_{TL}} \\ TL \otimes TL & \xrightarrow[\varphi_{TL}\circ(1-\tau)]{} & TL \\ {\scriptstyle \pi\otimes\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ E \otimes E & \xrightarrow[\varphi_{E}\circ(1-\tau)]{} & E \end{array}$$

shows that $\pi \circ i_{TL}: L \to \mathfrak{L}(E)$ is a Lie algebra map *if and only if*

$$\begin{aligned} \pi \circ \varphi_{TL} \circ (1 - \tau) \circ (i_{TL} \otimes i_{TL}) \\ = \pi \circ i_{TL} \circ [\ ,\ ], \end{aligned}$$

i.e., *if and only if*

$$\operatorname{Im}\{\varphi_{TL} \circ (1 - \tau) \circ (i_{TL} \otimes i_{TL}) - i_{TL} \circ [\ ,\ ]\} \subset \ker \pi.$$

Accordingly, we let I be the smallest ideal of TL containing

$$\operatorname{Im}\{\varphi_{TL} \circ (1 - \tau) \circ (i_{TL} \otimes i_{TL}) - i_{TL} \circ [\ ,\ ]\}$$

and set $UL = TL/I$ and $i_{UL} = p \circ i_{TL}$.

$$\begin{array}{ccc} M & \xleftarrow{\pi_{T^cM}} & F[T^cM] \\ & {\scriptstyle \pi_{U^cM}}\nwarrow & \\ & & F[U^cM]. \end{array}$$

[The "dual" of pedantic is pedantic.] By the U.M.P. satisfied by (T^cM, π_{T^cM}), there exists a unique morphism i: $U^cM \to T^cM$ of $\mathscr{C}$ filling in this diagram, i.e., making the diagram

$$\begin{array}{ccc} M & \xleftarrow{\pi_{T^cM}} & F[T^cM] \\ {\scriptstyle \pi_{U^cM}}\nwarrow & & \nearrow{\scriptstyle F(i)} \\ & F[U^cM] & \end{array}$$

commutative. Since i: $U^cM \to T^cM$ is a map of $\mathscr{C}$, $\ker i$ is a coideal of U^cM. But the equality $\pi_{T^cM} \circ i = \pi_{U^cM}$ clearly implies that $\ker i \subset \ker \pi_{U^cM}$. Since $\ker i$ is a coideal of U^cM contained in $\ker \pi_{U^cM}$ and since $\ker \pi_{U^cM}$ is a cogenerating subspace, it follows that $\ker i = \{0\}$. In other words, (the map) i: $U^cM \to T^cM$ is injective and $\pi_{U^cM} = \pi_{T^cM} \circ i$. Thus, we have shown that if (U^cM, π_{U^cM}) is a universal coenveloping coalgebra of M, then U^cM must be a subcoalgebra of T^cM and π_{U^cM}: $\mathfrak{L}^c(U^cM) \to M$ must be the restriction to U^cM of $\pi_{T^cM}: F(T^cM) \to M$. But, if D is any subcoalgebra of T^cM and if j: $D \hookrightarrow T^cM$ denotes the inclusion, then a consideration of the (not necessarily commutative) diagram

$$\begin{array}{ccc} D & \xrightarrow{(1-\tau)\circ\Delta_D} & D \otimes D \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j\otimes j} \\ T^cM & \xrightarrow{(1-\tau)\circ\Delta_{T^cM}} & T^cM \otimes T^cM \\ {\scriptstyle \pi_{T^cM}=\pi}\downarrow & & \downarrow{\scriptstyle \pi\otimes\pi} \\ M & \xrightarrow[\Delta_M = \langle\ \rangle]{} & M \otimes M \end{array}$$

shows that $\pi_{T^cM} \circ j\colon \mathfrak{L}^c(D) \to M$ is a morphism of $\mathscr{L}^c$ *if and only if*

$$(\pi_{T^cM} \otimes \pi_{T^cM}) \circ (1 - \tau) \circ \Delta_{T^cM} \circ j = \langle\ \rangle \circ \pi_{T^cM} \circ j,$$

i.e., *if and only if*

$$D = \operatorname{Im} j \subset \ker\{(\pi_{T^cM} \otimes \pi_{T^cM}) \circ (1 - \tau) \circ \Delta_{T^cM} - \langle\ \rangle \circ \pi_{T^cM}\}$$

Accordingly, we *define* U^cM to be the largest subcoalgebra of T^cM contained in

$$\ker\{(\pi_{T^cM} \otimes \pi_{T^cM}) \circ (1 - \tau) \circ \Delta_{T^cM} - \langle\ \rangle \circ \pi_{T^cM}\}$$

(i.e., the sum of all such coalgebras), and we set $\pi_{U^cM} = \pi_{T^cM}|_{U^cM}$. It is then straightforward to check that (U^cM, π_{U^cM}) is a universal coenveloping coalgebra for the Lie coalgebra M.

Alternate Constructions of U^cM

1. We claim that for every Lie algebra L,

$$U^c(L^0) = (UL)^0.$$

What we have just written ought to be clarified a bit. What we are really saying is that if (UL, i_{UL}) is a universal enveloping algebra for a Lie algebra L and if we set

$$(i_{UL})^\natural\colon \mathfrak{L}^c[(UL)^0] \to L^0$$

equal to the composite

$$\mathfrak{L}^c[(UL)^0] \xhookrightarrow[j]{} [\mathfrak{L}(UL)]^0 \xhookrightarrow[(i_{UL})^0]{} L^0,$$

where j is the inclusion

$$\mathfrak{L}^c[(UL)^0] \hookrightarrow \operatorname{Loc}[\mathfrak{L}(UL)]^0 \hookrightarrow [\mathfrak{L}(UL)]^0,$$

then $((UL)^0, (i_{UL})^\natural)$ is a universal coenveloping coalgebra for the Lie coalgebra L^0. The bijective correspondence (between the appropriate Hom sets) necessary to establish adjointness is sketched below.

$$\begin{aligned}\mathrm{Hom}_{\mathscr{L}^c}[\mathfrak{L}^c(C), L^0] &\underset{*^{op}\dashv 0}{\simeq} \mathrm{Hom}_{\mathscr{L}}[L, [\mathfrak{L}^c(C)]^*] = \mathrm{Hom}_{\mathscr{L}}[L, \mathfrak{L}(C^*)] \\ &\underset{U\dashv \mathfrak{L}}{\simeq} \mathrm{Hom}_{\mathscr{A}}[UL, C^*] \underset{*^{op}\dashv 0}{\simeq} \mathrm{Hom}_{\mathscr{C}}[C, (UL)^0].\end{aligned}$$

Since any Lie coalgebra M can always be embedded in the "upper zero" of its dual Lie algebra (i.e., $M \rightarrowtail_{\phi_M} M^{*0}$), we can use the fact that $U^c(L^0) = (UL)^0$ to get an alternate construction of U^cM for an arbitrary Lie coalgebra M. [This two-step construction is clearly patterned on that given for T^cV.]

2. A second alternative construction of (U^cM, π_{U^cM}) depends on showing directly (i.e., prior to knowing of the existence of U^cM) that T^cV together with the Lie algebra map $\pi\colon \mathfrak{L}^c(T^cV) \to L^cV$ defined by the commutative diagram

$$\begin{array}{ccc} V & \xleftarrow{\;\pi_{L^cV}\;} & F[L^cV] \\ & {}_{\pi_{T^cV}}\nwarrow \quad \nearrow_{F[\pi]} & \\ & \bar{F}[T^cV] = F[\mathfrak{L}^c(T^cV)] & \end{array}$$

(in which F and $\bar{F}$ are forgetful functors) is a universal coenveloping coalgebra for L^cV. This verification is based on the natural equivalence

$$\begin{aligned}\mathrm{Hom}_{\mathscr{L}^c}[\mathfrak{L}^cC, L^cV] &\underset{F\dashv L^c}{\simeq} \mathrm{Hom}_{\mathscr{V}}[F(\mathfrak{L}^cC), V] = \mathrm{Hom}_{\mathscr{V}}[\bar{F}(C), V] \\ &\underset{\bar{F}\dashv T^c}{\simeq} \mathrm{Hom}_{\mathscr{C}}[C, T^cV]\end{aligned}$$

[cf. Remark 1 above]. Since any Lie coalgebra M can be embedded in the cofree Lie coalgebra on itself, i.e.,

$$\begin{array}{ccc} M & \xleftarrow{\;\pi_{L^cM}\;} & L^cM \\ & {}_{1_M}\nwarrow \quad \nearrow_{c} & \\ & M & \end{array},$$

we have (sketched) yet another proof for the existence of the universal coenveloping coalgebra of an arbitrary Lie coalgebra.

The Hopf Algebra Structure of U^cM

We begin by reviewing a few definitions and facts about Hopf algebras.

DEFINITION. A Hopf algebra consists of a vector space H together with linear maps

$$\varphi\colon H \otimes H \to H,$$
$$\eta\colon K \to H,$$
$$\Delta\colon H \to H \otimes H,$$
$$\epsilon\colon H \to K$$

such that

(1) (H, φ, η) is an algebra with multiplication φ and unit η;

(2) (H, Δ, ϵ) is a coalgebra with comultiplication Δ and counit ϵ; and

(3) the linear maps $\Delta\colon H \to H \otimes H$ and $\epsilon\colon H \to K$ are algebra homomorphisms.

Usually it will be assumed that (H, φ, η) is an object of $\mathscr{A}$ and that (H, Δ, ϵ) is an object of $\mathscr{C}$; but we may occasionally speak, for instance, of a biassociative Hopf algebra in order to emphasize this.

Note. If (A, φ_A, η_A) and (B, φ_B, η_B) are objects of $\mathscr{A}$, then so is $(A \otimes B, \varphi_{A\otimes B}, \eta_{A\otimes B})$, where $\varphi_{A\otimes B}$ and $\eta_{A\otimes B}$ are defined to be the (linear) composites

$$(A \otimes B) \otimes (A \otimes B) \xrightarrow{\simeq} A \otimes (B \otimes A) \otimes B \xrightarrow{1\otimes\tau\otimes 1} A \otimes (A \otimes B) \otimes B$$
$$\xrightarrow{\simeq} (A \otimes A) \otimes (B \otimes B) \xrightarrow{\varphi_A\otimes\varphi_B} A \otimes B$$

and

$$K \xrightarrow{\simeq} K \otimes K \xrightarrow{\eta_A\otimes\eta_B} A \otimes B,$$

respectively. Thus, $(a_1 \otimes b_1)\cdot(a_2 \otimes b_2) = a_1 \cdot a_2 \otimes b_1 \cdot b_2$ while, $\mathbf{1}_{A\otimes B} = \mathbf{1}_A \otimes \mathbf{1}_B$. Dually, if $(C, \Delta_C, \epsilon_C)$ and $(D, \Delta_D, \epsilon_D)$ are objects of $\mathscr{C}$, then so is $(C \otimes D, \Delta_{C\otimes D}, \epsilon_{C\otimes D})$, where $\Delta_{C\otimes D}$ and $\epsilon_{C\otimes D}$ are defined to be the (linear) composites

$$C \otimes D \xrightarrow{\Delta_C\otimes\Delta_D} (C \otimes C) \otimes (D \otimes D) \xrightarrow{\simeq} C \otimes (C \otimes D) \otimes D$$
$$\xrightarrow{1\otimes\tau\otimes 1} C \otimes (D \otimes C) \otimes D \xrightarrow{\simeq} (C \otimes D) \otimes (C \otimes D)$$

and

$$C \otimes D \xrightarrow{\epsilon_C\otimes\epsilon_D} K \otimes K \xrightarrow{\simeq} K,$$

respectively.

The requirement that Δ and ϵ be algebra maps translates to the requirement that four diagrams be commutative. It turns out that the four diagrams that arise in this way are precisely the same four that would arise were we to require that φ and η be coalgebra maps.

We may, therefore, view a Hopf algebra as a vector space H endowed simultaneously with an algebra structure and a coalgebra structure in such a way that these two structures are "compatible with one another." "Compatible" means that the two maps which give H its coalgebra structure should be algebra maps or equivalently that the two maps which give H its algebra structure should be coalgebra maps.

A map of Hopf algebras is a vector space map which is simultaneously an algebra and a coalgebra map.

Let $(C, \Delta_C, \epsilon_C)$ be an associative, counitary coalgebra with comultiplication $\Delta_C: C \to C \otimes C$ and counit $\epsilon_C: C \to K$; and let (A, φ_A, η_A) be an associative, unitary algebra with multiplication $\varphi_A: A \otimes A \to A$ and unit $\eta_A: K \to A$. Then for $f, g \in \mathrm{Hom}_K(C, A)$, we define the *convolution* $f * g \in \mathrm{Hom}_K(C, A)$ by

$$f * g = \varphi_A \circ (f \otimes g) \circ \Delta_C .$$

This makes $\mathrm{Hom}_K(C, A)$ into an associative algebra with unit

$$\eta_A \circ \epsilon_C: C \to K \to A.$$

Now let $(H, \varphi, \eta, \Delta, \epsilon)$ be a (biassociative) Hopf algebra and take $(C, \Delta_C, \epsilon_C)$ and (A, φ_A, η_A) to be the underlying coalgebra (H, Δ, ϵ) and algebra (H, φ, η) of $(H, \varphi, \eta, \Delta, \epsilon)$, respectively.

DEFINITION. An *involution* for (or on) H is a linear map

$$\omega: H \to H$$

such that ω and 1_H (the identity map on H) are inverse to each other in $\mathrm{Hom}_K(H, H)$, i.e., such that

$$\omega * 1_H = 1_H * \omega = \eta_H \circ \epsilon_H .$$

One can show that ω is an anti-algebra map and an anti-coalgebra map (i.e., that $\omega: H \to H^{op}$ is a map of Hopf algebras, where the multiplication and the comultiplication on H^{op} are defined, respectively, as $\varphi_H \circ \tau$ and $\tau \circ \Delta_H$, τ being the twisting map). If H has either a commutative multiplication φ_H (i.e., $\varphi_H = \varphi_H \circ \tau$) or a commutative comultiplication Δ_H (i.e., $\Delta_H = \tau \circ \Delta_H$), then it turns out that $\omega^2 = 1_H$; this justifies the name "involution."

Remark. Some authors (e.g., Sweedler [16]) use the words "Hopf algebra" to describe what for us would be a (biassociative) Hopf algebra with involution. Those same authors would consider our (biassociative) Hopf algebra to be merely a "bialgebra." Also, for these authors, a *Hopf ideal* is a *bi-ideal* (ideal and coideal) that is carried into itself by the involution. The bi-ideal I of TL discussed under *The Construction of* UL and under *The Construction of* U^cM is carried into itself by the involution that exists on TL and thus is a Hopf ideal.

If H has the structure of a Hopf algebra, then H carries, in particular, the structure of an algebra; so H^0 carries the structure of a coalgebra. On the other hand, H—as a Hopf algebra—also carries the structure of a coalgebra, so H^* is endowed with the structure of an algebra (as the dual of a coalgebra). As we know, H^0 is a vector subspace of H^*. It turns out that H^0 is closed under the multiplication of H^*. In just this way does H^0 acquire the structure of an algebra. Moreover, this algebra structure on H^0 turns out to be compatible with the coalgebra structure on H^0 in such a way that H^0 is in fact (endowed with the structure of) a Hopf algebra.

Since, as we have indicated above, the "upper zero" of a Hopf algebra is again a Hopf algebra; since $U^c(L^0) = (UL)^0$; and since UL carries—in addition to its algebra structure—a coalgebra structure making it into a (biassociative) Hopf algebra with involution, [the diagonal, for instance, being determined by the requirement that $\Delta(i_{UL}x) = i_{UL}x \otimes 1_{UL} + 1_{UL} \otimes i_{UL}x$ for $x \in L$], we should expect that U^cM carries—in addition to its coalgebra structure—an algebra structure making it into a (biassociative) Hopf algebra with involution. This is indeed the case. Moreover, even as UL inherits its (involutive) Hopf algebra structure as a quotient Hopf algebra of the (involutive) Hopf algebra TL, so U^cM inherits its (involutive) Hopf algebra structure as a sub Hopf algebra of the (involutive) Hopf algebra T^cM.

We shall now sketch some of the details of these assertions. Given a vector space V, let

$$\varphi_{T^cV}\colon T^cV \otimes T^cV \to T^cV,$$

$$\eta_{T^cV}\colon K \to T^cV,$$

and

$$\omega_{T^cV}\colon (T^cV)^{op} \to T^cV$$

denote the unique coalgebra maps which lift, respectively, the linear maps

$$T^cV \otimes T^cV \to V \quad \text{via} \quad x \otimes y \mapsto \epsilon_{T^cV}(y)\,\pi_{T^cV}(x) + \epsilon_{T^cV}(x)\,\pi_{T^cV}(y),$$

$$K \to V \quad \text{via} \quad k \mapsto 0,$$

and

$$(T^cV)^{op} \to V \quad \text{via} \quad x \mapsto -\pi_{T^cV}(x),$$

where $(T^cV)^{op}$ denotes the opposite coalgebra of T^cV: $\Delta_{(T^cV)^{op}} = \tau \circ \Delta_{T^cV}$. [These liftings exist by virtue of the U.M.P. satisfied by (T^cV, π_{T^cV}).] One then can establish the following results.

THEOREM. *$(T^cV, \varphi_{T^cV}, \eta_{T^cV}, \Delta_{T^cV}, \epsilon_{T^cV}, \omega_{T^cV})$ is a commutative, biassociative Hopf algebra with involution; moreover, for any linear map $f\colon V \to W$, the induced coalgebra map $T^c(f)\colon T^cV \to T^cW$ is a morphism of involutive Hopf algebras. Thus, there exists a functor $T^c_H : \mathscr{V} \to \mathscr{H}$, where $\mathscr{H}$ is the category of commutative, biassociative Hopf algebras with involution, such that $F \circ T^c_H = T^c$ where $F\colon \mathscr{H} \to \mathscr{C}$ is the forgetful functor.*

THEOREM. *For any Lie coalgebra M, U^cM can be given the structure of a commutative, biassociative Hopf algebra with involution as a sub Hopf algebra of $T^c_H(M)$. Moreover, if $f\colon M \to N$ is a map of $\mathscr{L}^c$, then the induced coalgebra map $U^c(f)\colon U^cM \to U^cN$ is compatible with the induced multiplications and units. Thus, there is a functor $U^c_H : \mathscr{L}^c \to \mathscr{H}$ (where $\mathscr{H}$ is the category of commutative, biassociative Hopf algebras with involution) such that $F \circ U^c_H = U^c$ where $F\colon \mathscr{H} \to \mathscr{C}$ is the forgetful functor.*

Sketch of Proof. Let φ, η, and ω denote, respectively, the multiplication, unit, and involution of $T^c_H(M)$. Since $\varphi(U^cM \otimes U^cM)$, $\eta(K)$, and $\omega(U^cM)$ are all subcoalgebras of T^cM [being the images under coalgebra maps of coalgebras], a verification that these are all contained in

$$\ker\{(\pi_{T^cM} \otimes \pi_{T^cM}) \circ (1 - \tau) \circ \Delta_{T^cM} - \langle\ \rangle \circ \pi_{T^cM}\}$$

will establish that $\varphi(U^cM \otimes U^cM) \subset U^cM$, $\eta(K) \subset U^cM$, and $\omega(U^cM) \subset U^cM$.

PROPOSITION. *For any Lie algebra L,*

$$U^c_H(L^0) = [U_H(L)]^0$$

where we write $U_H(L)$ to symbolize that we consider UL with the Hopf algebra structure it acquires as a quotient Hopf algebra of $T_H(L)$, where $T_H(L)$ symbolizes the (involutive) Hopf algebra structure one obtains from TL by defining a diagonal, counit, and involution appropriately.

PROPOSITION. *For every Lie coalgebra M, $U^c_H(M)$ is a* proper *algebra, i.e., the intersection of all cofinite two-sided ideals of $U^c_H(M)$ is zero.*[1]

THEOREM. *Given Lie coalgebras M and N, let*

$$\beta = \beta_{M,N}\colon U^cM \otimes U^cN \to U^c(M \oplus N)$$

denote the unique coalgebra map lifting the Lie coalgebra map

$$\xi = \xi_{M,N}\colon \mathfrak{L}^c(U^cM \otimes U^cN) \to M \oplus N$$

[1] *Note added in proof.* Classically, a result of Harish-Chandra guarantees that over a field of characteristic zero the universal enveloping algebra of a finite dimensional Lie algebra is proper [cf. Harish-Chandra, On representations of Lie algebras, *Ann. of Math.* **50** (1949), 900–915]. See also a forthcoming paper by the author in the *Proc. Amer. Math. Soc.*

determined by

$$\xi(x \otimes y) = (\epsilon_{U^cN}(y)\, \pi_{U^cM}(x), \epsilon_{U^cM}(x)\, \pi_{U^cN}(y)),$$

for $x \in U^cM$ and $y \in U^cN$. Then β is a natural isomorphism of involutive Hopf algebras.

COROLLARY. *The functor $U_H^c: \mathscr{L}^c \to \mathscr{H}$ preserves finite coproducts.*

Let $\mathscr{H}$ denote the category of biassociative Hopf algebras with involution.

THEOREM. *The functor $U_H^c: \mathscr{L}^c \to \mathscr{H}$ is right adjoint to the functor $Q: \mathscr{H} \to \mathscr{L}^c$, i.e.,*

$$Q \dashv U_H^c.$$

Remark. The result mentioned in the above theorem is "dual" to the fact that the functor $P: \mathscr{H} \to \mathscr{L}$ is right adjoint to the functor $U_H: \mathscr{L} \to \mathscr{H}$ where P is the functor assigning to each Hopf algebra H its Lie algebra of primitives (cf. Grünenfelder [6, p. 32, Theorem I.3.10]).

Note. For every Lie algebra L, $U_H(L)$ has the structure of a "cocommutative," biassociative Hopf algebra with involution. In the case of a Hopf algebra $(H, \varphi, \eta, \Delta, \epsilon)$ we say that $(H, \varphi, \eta, \Delta, \epsilon)$ is "commutative" in case $\varphi = \varphi \circ \tau$ and "cocommutative" in case $\Delta = \tau \circ \Delta$. Obviously $U_H(L)$ is *not* commutative unless $[x \cdot y]_L = 0$ for all $x, y \in L$.

We conclude this section by mentioning, in passing, that $(UL)^0$ is precisely *Hochschild's algebra of representative functions on UL* (cf. [7, p. 500]), a fact we shall make use of in Section 5. In view of the equality $(UL)^0 = U^c(L^0)$, we can thus view $(UL)^0$ in a new light.

4. A NATURAL QUESTION

In the case of Lie algebras, one knows that the natural map

$$i_{UL}: L \to \mathfrak{L}(UL)$$

is always injective. The injectivity of i_{UL} follows directly from the Poincaré–Birkhoff–Witt theorem, and is equivalent to it in case the characteristic $\chi(K)$ of the ground field K is zero.

It is, therefore, natural to inquire whether or not the natural map

$$\pi_{U^cM}: \mathfrak{L}^c(U^cM) \to M$$

is always (or ever) surjective. The answer to this inquiry is provided by the following theorem.

THEOREM. $\pi_{U^cM}\colon \mathfrak{L}^c(U^cM) \to M$ *is surjective* if and only if M *is locally finite.*

Since the proof of this result is a direct consequence of the theorem of Ado and Iwasawa for Lie algebras, it will be useful to remind ourselves of that result first.

THEOREM (Ado–Iwasawa). *If L is a finite dimensional Lie algebra, then UL contains a cofinite two-sided ideal I such that $i_{UL}(L) \cap I = \{0\}$.*

For a proof of this theorem, the reader is referred to Jacobson [8, p. 202].

As an immediate consequence of the above, we find that for each finite dimensional Lie algebra L there exists a finite dimensional associative, unitary algebra A and a Lie algebra injection

$$f\colon L \rightarrowtail \mathfrak{L}(A).$$

Indeed, one may take A to be UL/I, and

$$f\colon L \rightarrowtail \mathfrak{L}[UL/I]$$

to be the composite

$$L \xrightarrow{i_{UL}} \mathfrak{L}(UL) \xrightarrow{\mathfrak{L}(\pi)} \mathfrak{L}(UL/I)$$

where

$$\pi\colon UL \twoheadrightarrow UL/I$$

is the natural projection. [Here, of course, I is the cofinite two-sided ideal mentioned in the Ado–Iwasawa theorem.] The map $\mathfrak{L}(\pi) \circ i_{UL}$ is injective because

$$[\mathfrak{L}(\pi) \circ i_{UL}]^{-1}\{0\} = i_{UL}^{-1}[(\mathfrak{L}\pi)^{-1}\{0\}] = i_{UL}^{-1}(I)$$

$$= \{x \in L \mid i_{UL}(x) \in I\}$$

and

$$\{x \in L \mid i_{UL}(x) \in I\} = \{0\} \qquad \text{since} \quad i_{UL}(L) \cap I = \{0\}$$

and

$$i_{UL}\colon L \to \mathfrak{L}(UL)$$

is injective by the Poincaré–Birkhoff–Witt theorem.

Conversely, if—for a (finite dimensional) Lie algebra L—there exists a finite dimensional object A of $\mathscr{A}$ and a Lie algebra injection $f\colon L \rightarrowtail \mathfrak{L}(A)$, then there exists a cofinite two-sided ideal I of UL such that $i_{UL}(L) \cap I = \{0\}$. Indeed, by the U.M.P. satisfied by (UL, i_{UL}), the Lie algebra injection

$f\colon L \to \mathfrak{L}(A)$ extends to a map $F\colon UL \to A$ of $\mathscr{A}$ such that $f = \mathfrak{L}(F) \circ i_{UL}$. If we set I equal to the kernal of F, then I has the desired properties.

We may therefore view the Ado–Iwasawa theorem as saying the following: For every finite dimensional Lie algebra L, there exists a finite dimensional object A of $\mathscr{A}$ and an injective map $f\colon L \rightarrowtail \mathfrak{L}(A)$ of $\mathscr{L}$.

This result may be dualized to obtain the following result.

PROPOSITION. *For every finite dimensional Lie coalgebra M, there exists a finite dimensional object C of $\mathscr{C}$ and a surjective map $f\colon \mathfrak{L}^c(C) \twoheadrightarrow M$ of $\mathscr{L}^c$.*

Indeed, if M is a finite dimensional Lie coalgebra, then M^* is a finite dimensional Lie algebra, so there exists a finite dimensional object A of $\mathscr{A}$ and a Lie algebra injection

$$j\colon M^* = L \rightarrowtail \mathfrak{L}(A).$$

Upon applying the functor $* = \operatorname{Hom}_K(\ ;K)$ to this injection, we obtain a Lie coalgebra surjection

$$M \overset{\cong}{\twoheadleftarrow\!\!<} M^{**} = L^* \overset{j^*}{\twoheadleftarrow} [\mathfrak{L}(A)]^* = \mathfrak{L}^c(A^*)$$

in which $M^{**} \to^{\cong} M$ is the inverse of the natural Lie coalgebra isomorphism

$$\chi_M\colon M \to M^{**}$$

given by $\chi_M(x)(g) = g(x)$ for $x \in M$ and $g \in M^*$. We may therefore take C equal to A^* and $f\colon \mathfrak{L}^c(C) \twoheadrightarrow M$ equal to the composite

$$\mathfrak{L}^c(C) = \mathfrak{L}^c(A^*) = [\mathfrak{L}(A)]^* \overset{j^*}{\twoheadrightarrow} L^* = M^{**} \overset{\chi_M^{-1}}{\underset{\cong}{\twoheadrightarrow}} M.$$

We now prove the theorem announced at the beginning of this section.

THEOREM. *Let M be a Lie coalgebra, let U^cM be its coenvelop, and let $\pi_{U^cM}\colon \mathfrak{L}^c(U^cM) \to M$ be the canonical map. Then*

$$\pi_{U^cM} \textit{ is surjective} \Leftrightarrow M \textit{ is locally finite.}$$

Proof. $\Rightarrow$: Suppose that π_{U^cM} is surjective. Since U^cM is an object of $\mathscr{C}$, it follows that U^cM is locally finite. But if C is a subcoalgebra of U^cM, then $\mathfrak{L}^c(C)$ is a sub Lie coalgebra of $\mathfrak{L}^c(U^cM)$. Accordingly, the local finiteness of the object U^cM of $\mathscr{C}$ implies (entails) the local finiteness of $\mathfrak{L}^c(U^cM)$, which is of course an object of $\mathscr{L}^c$. But since the surjective image of a locally finite coalgebra under a coalgebra map is obviously locally finite (because the image of a subcoalgebra under a coalgebra map is again a subcoalgebra), it follows that M is locally finite whenever the Lie coalgebra map $\pi_{U^cM}\colon \mathfrak{L}^c(U^cM) \to M$ is surjective.

$\Leftarrow$: Conversely, suppose that M is locally finite. We must show that π_{U^cM}: $\mathfrak{L}^c(U^cM) \to M$ is surjective. Towards this end, let $x \in M$. Since M is locally finite, there is a finite dimensional sub Lie coalgebra $N \subset M$ such that $x \in N$. Since N is a finite dimensional Lie coalgebra, there exists a finite dimensional object C of $\mathscr{C}$ and a Lie coalgebra surjection

$$f\colon \mathfrak{L}^c(C) \twoheadrightarrow N.$$

Let

$$i_N\colon N \hookrightarrow M$$

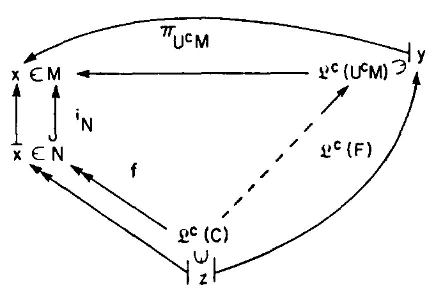
denote the natural inclusion. Then

$$i_N \circ f\colon \mathfrak{L}^c(C) \twoheadrightarrow N \hookrightarrow M$$

is a Lie coalgebra map and so, by the U.M.P. which (U^cM, π_{U^cM}) satisfies, there exists a unique morphism

$$F\colon C \to U^cM$$

of $\mathscr{C}$ filling in the following diagram.



Let us now take stock of the situation. We began with an element x of M, and found a sub Lie coalgebra N of M of finite dimension such that x already lay in N. Because N was finite dimensional, we were able to find a finite dimensional object C of $\mathscr{C}$ and a Lie coalgebra surjection $f\colon \mathfrak{L}^c(C) \twoheadrightarrow N$ from $\mathfrak{L}^c(C)$ onto N. Thus there is an element $z \in \mathfrak{L}^c(C)$ such that $f(z) = x$. [See the above diagram.] Set $y = \mathfrak{L}^c(F)(z)$. Then $y \in \mathfrak{L}^c(U^cM)$, and clearly

$$\begin{aligned}\pi_{U^cM}(y) &= \pi_{U^cM}[\mathfrak{L}^c(F)(z)] = [\pi_{U^cM} \circ \mathfrak{L}^c(F)](z)\\ &= (i_N \circ f)(z) = i_N(f(z)) = i_N(x)\\ &= x,\end{aligned}$$

so π_{U^cM} is surjective. This concludes the proof of the theorem.

Further insight into why π_{U^cM} may fail, in general, to be surjective may be obtained by observing that for any Lie coalgebra M,

$$U^c(M) \simeq U^c(\text{Loc}\, M)$$

since the functor Loc: $\mathscr{L}^c \to \mathscr{L}^c_{l.f.}$ is a coreflector (as we saw in Section 2). In fact, since $\text{Loc}\, M \subset M$, we may suppose that $U^c(\text{Loc}\, M) \subset U^c(M)$; then

$$U^c(M) = U^c(\text{Loc}\, M).$$

Accordingly, we have

$$\mathfrak{L}^c U^c(M) = \mathfrak{L}^c U^c(\text{Loc}\, M) \twoheadrightarrow \text{Loc}\, M \subset M;$$

that is, in general,

$$\text{Im}\, \pi_{U^cM} = \text{Loc}\, M.$$

We began our study of Lie coalgebras by defining them to be coalgebras whose diagonals satisfied the Lie conditions. In view of the above result, we may now state a proposition which permits a Milnor–Moore-like definition of locally finite Lie coalgebras.

PROPOSITION. *A vector space M, together with a linear map Δ_M: $M \to M \otimes M$, is a locally finite Lie coalgebra if and only if there exists an object C of $\mathscr{C}$ and a surjective linear map f: $C \to M$ making the diagram*

$$\begin{array}{ccc} C & \xrightarrow{(1-\tau)\circ\Delta_C} & C \otimes C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f\otimes f} \\ M & \xrightarrow[\Delta_M]{} & M \otimes M \end{array}$$

commutative.

Proof. If (M, Δ_M) is a locally finite Lie coalgebra, we may take C equal to U^cM and f equal to π_{U^cM} by virtue of what we have just established.

Conversely, if C is a coalgebra as in the proposition, then a routine verification shows that $(1 - \tau) \circ \Delta_C$ endows C with the structure of a Lie coalgebra; and the commutativity of the above diagram together with the fact that f is surjective insures that one can transfer the Lie coalgebra structure from C to M. $(C, (1 - \tau) \circ \Delta_C)$ and hence (M, Δ_M) are locally finite since any object of $\mathscr{C}$ is locally finite.

Remark. The definition of a graded (reduced) Lie coalgebra given by André in [2] is the analog of the above characterization of a locally finite Lie coalgebra.

5. Dualization of the Poincaré-Birkhoff-Witt Theorem for Lie Algebras

This final section concerns itself with an analog of the Poincaré–Birkhoff–Witt theorem for Lie algebras which we choose to call the Dual Poincaré–Birkhoff–Witt Theorem. In what follows, we shall abbreviate the former by "the PBWθ," and the latter by "the Dual PBWθ."

The constraints of the present paper allow only a sketch of the Dual PBWθ. A complete proof of this result may be found in [12], and will be presented in a subsequent paper.

We shall begin, shortly, by recalling the statement of the PBWθ and reminding ourselves of its proof. This will be done in order to suggest what the Dual PBWθ should be and how we might try to establish it. Following such a brief sketch in which we present—as it were—an aerial view of the terrain we must traverse, we shall return and fill in the broad strokes with somewhat finer detail.

One comment is in order prior to delving into the PBWθ. It is this. We can only expect to get a Dual PBWθ for locally finite Lie coalgebras. The reason for this stems from the fact that, on the one hand, the PBWθ is equivalent, in case the ground field K has characteristic zero, to the injectivity of the natural map $i_{UL}: L \to \mathfrak{L}(UL)$; whereas, on the other hand, the canonical map $\pi_{U^cM}: \mathfrak{L}^c(U^cM) \to M$ is surjective if and only if M is locally finite, in consequence of the deep result of Ado and Iwasawa.

In what follows, we shall restrict ourselves to the case in which the ground field K has characteristic zero.

Recall that the PBWθ gives a vector space basis for the universal enveloping algebra UL of a Lie algebra L in terms of a vector space basis for L. Specifically, if $\{x_\alpha\}_{\alpha \in A}$ is a well-ordered basis for L and if $z_\alpha = i_{UL}(x_\alpha)$, where $i_{UL}: L \to \mathfrak{L}(UL)$ is the composite

$$L \xrightarrow{i_{TL}} TL \xrightarrow{\pi} UL,$$

then the PBWθ asserts that

$$\{z_{\alpha_1} \cdot z_{\alpha_2} \cdots z_{\alpha_m}\}_{\alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_m}$$

is a basis for UL. This form of the PBWθ does not lend itself to dualization. There is, however, an alternate formulation which does, and to which we now turn.

Briefly put, the above-mentioned basis for UL gives rise to a filtration on UL, the so-called Lie filtration, which may be obtained (also) in the following manner. TL, the tensor algebra on (the underlying vector space of) L is a graded, hence a filtered, algebra. UL, as a quotient of TL, inherits a filtration from TL. What is critical is the fact that this (inherited) filtration on UL turns out to be a filtration by powers of a generating subspace of UL, specifically, by the generating subspace $\operatorname{Im} i_{UL}$ of UL. The associated graded algebra $E^0(UL)$ is in turn generated by the elements of $E^0(UL)$ of degree 1, i.e., by $E_1^0(UL)$. The fact that $i_{UL}: L \to \mathfrak{L}(UL)$ is a Lie map implies that the elements of $E_1^0(UL)$ pairwise commute so that $E^0(UL)$—which is generated by $E_1^0(UL)$—is a commutative algebra. The PBWθ asserts that $E^0(UL)$ is none other that SL, the symmetric algebra on L. In particular, if L is a Lie algebra of finite dimension n, then $E^0(UL)$ is the polynomial algebra (Hopf algebra) in n-variables $K[x_1, x_2, \ldots, x_n]$.

If we wish to dualize this situation, then presumably we will want to filter the co-envelop U^cM of a Lie co-algebra M by some sort of "co-powers" of a "co-generating" subspace. This strategy can—in fact—be carried out. The analog of the product of two subspaces of an algebra is the "wedge" of two subspaces of a coalgebra (to be defined below); while the analog of the fact that $\operatorname{Im} i_{UL}$ is a generating subspace of UL is the fact that $\ker \pi_{U^cM}$ is a "cogenerating" subspace of U^cM (also to be defined below). [Recall that $i_{UL}: L \to \mathfrak{L}(UL)$ and $\pi_{U^cM}: \mathfrak{L}^c(U^cM) \to M$ are the canonical maps.] In this type of "duality," $+$ and $\cap$ (sum and intersection) correspond under the annihilator transformations which send subspaces of a vector space V to subspaces of the dual space V^* and vice versa. [The definitions will be given below.] Moreover, UL is in fact filtered by powers of the generating subspace

$$\operatorname{Im} \eta_{UL} + \operatorname{Im} i_{UL}$$

where $\eta_{UL}: K \to UL$ is the unit of UL (i.e., $\eta_{UL}(1_K) = 1_{UL}$), so we can expect that U^cM will be filtered by "wedges" of the cogenerating subspace

$$\ker \epsilon_{U^cM} \cap \ker \pi_{U^cM}$$

where $\epsilon_{U^cM}: U^cM \to K$ is the counit of U^cM. This is indeed the case.

Here is how things are defined.

In the case of algebras, a subspace S of an algebra A is called a *generating subspace* if whenever B is a subalgebra of A containing S, then $B = A$.

Strictly speaking, one should speak of a generating map rather than a generating subspace.

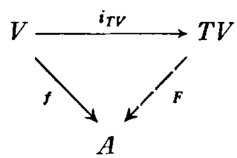
"Dually," in the case of coalgebras, a subspace S of a coalgebra C is called a *cogenerating subspace* if whenever I is a coideal of C contained in S, then $I = \{0\}$.

In point of fact, one should again really speak of a cogenerating map.

DEFINITION. A linear map $f: V \to A$ from a vector space V to an algebra A is called a *generating map* in case the only subalgebra of A containing $\operatorname{Im} f$ is A itself.

DEFINITION. A linear map $f: C \to V$ from a coalgebra C to a vector space V is called a *cogenerating map* in case any coideal of C which is contained in $\ker f$ is zero.

PROPOSITION. *Let A be an object of $\mathscr{A}$ and let f: $V \to A$ be a linear map from a vector space V to A. Let F: $TV \to A$ be the unique algebra map "extending" f, i.e., the unique morphism of $\mathscr{A}$ making the diagram*



commutative. Then f is generating $\Leftrightarrow F$ is *surjective.*

DEFINITION. Let (A, φ, η) be an object of $\mathscr{A}$, and let $S \subset A$ be a generating subspace. Then *the filtration of A by "powers of the generating subspace $\eta K + S$"* is defined via

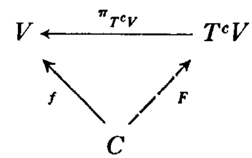
$$\begin{aligned} F_0 &= \eta K, \\ F_1 &= \eta K + S, \\ F_2 &= (\eta K + S)^2, \\ &\vdots \\ F_n &= (\eta K + S)^n, \qquad \text{for} \quad n \geqslant 1, \end{aligned}$$

where, for subspaces X and Y of an algebra A, the *product* $X \cdot Y$ of X and Y is defined by setting

$$X \cdot Y = \operatorname{Im}[X \otimes Y \hookrightarrow A \otimes A \xrightarrow{\varphi} A].$$

Note. This is an increasing (algebra) filtration.

PROPOSITION. *Let C be an object of $\mathscr{C}$ and let f: $C \to V$ be a linear map from C to a vector space V. Let F: $C \to T^cV$ be the unique coalgebra map "lifting" f, i.e., the unique morphism of $\mathscr{C}$ making the diagram*



commutative. Then f is cogenerating $\Leftrightarrow F$ *is* injective.

DEFINITION. Let (C, Δ, ϵ) be an object of $\mathscr{C}$, and let $S \subset C$ be a cogenerating subspace. For any subspace W of C, denote by W^+ the subspace of C defined by $W^+ = W \cap \ker \epsilon$. Then *the filtration of C by "wedges of the cogenerating subspace S^+"* is defined via

$$\begin{aligned} F_0 &= C, \\ F_1 &= C \cap \ker \epsilon = C^+ = \ker \epsilon, \\ F_2 &= S \cap \ker \epsilon = S^+, \\ F_3 &= S^+ \wedge S^+, \\ F_4 &= \wedge^3 S^+ = S^+ \wedge S^+ \wedge S^+, \\ &\vdots \\ F_n &= \wedge^{n-1} S^+, \qquad \text{for} \quad n \geqslant 1, \end{aligned}$$

where, by definition, $\wedge^0 S^+ = \ker \epsilon$, and where, for subspaces X and Y of a coalgebra C, the *wedge* $X \wedge Y$ of X and Y [not to be confused with the exterior product] is defined by setting

$$\begin{aligned} X \wedge Y &= \ker[C \xrightarrow{\Delta} C \otimes C \twoheadrightarrow C/X \otimes C/Y] \\ &= \Delta^{-1}[X \otimes C + C \otimes Y]. \end{aligned}$$

Note. This is a decreasing (coalgebra) filtration.

The definitions just given clearly display a duality between the product of two subspaces of an algebra and the wedge of two subspaces of a coalgebra. There is, in fact, a further, interesting relationship between "product" and "wedge." It is this: For subspaces X and Y of a coalgebra C,

$$X \wedge Y = [X^{\perp} \cdot Y^{\perp}]^{\perp\!\!\perp}$$

where $\perp$ and $\perp\!\!\perp$ are the annihilator transformations and where "$\cdot$" is the multiplication of the dual algebra C^*, i.e.,

$$X^{\perp} \cdot Y^{\perp} = \operatorname{Im}[X^{\perp} \otimes Y^{\perp} \hookrightarrow C^* \otimes C^* \xhookrightarrow{\rho_C} (C \otimes C)^* \xrightarrow{\Delta^*} C^*].$$

In other words, to find the wedge of subspaces X and Y of a coalgebra C, multiply their annihilators in the dual algebra C^* and then take the annihilator of the result to land back in C.

The annihilator transformations which appear above shall now be defined. Given a vector space V, let $s(V)$ denote the set of all subspaces of V. Then the annihilator transformations are functions

$$\perp: s(V) \to s(V^*)$$

and

$$\perp\!\!\perp: s(V^*) \to s(V),$$

defined as follows. If U is a subspace of V, let

$$i_U: U \hookrightarrow V$$

denote the inclusion, and set

$$U^{\perp} = \ker[(i_U)^*] = \{f \in V^* \mid f(U) = \{0\}\}.$$

If W is a subspace of V^* and

$$i_W: W \hookrightarrow V^*$$

denotes the inclusion, set

$$W^{\perp\!\!\perp} = \chi_V^{-1}(W^{\perp}) = \ker[(i_W)^* \circ \chi_V] = \{v \in V \mid W(v) = \{0\}\}$$

[where $\chi_V: V \to V^{**}$ is the natural injection]. $U^{\perp}$ is called the *annihilator of U in V^**, and $W^{\perp\!\!\perp}$ is called the *annihilator of W in V*. [We read $U^{\perp}$ as "U-perp," and $W^{\perp\!\!\perp}$ as "W-double-line-perp."] For further details, see [6, p. 61] or [12, p. 24].

We now return to the sketch given earlier of the PBWθ in order to fill in a few details. Thereafter, we shall give a brief sketch of the Dual PBWθ. We shall then conclude the section by mentioning two counterexamples.

We first recall the construction of the tensor algebra TV on a vector space V. By definition,

$$TV = \bigoplus_{n=0}^{\infty} T_n(V)$$

where

$$T_0(V) = K,\ T_1(V) = V, \quad \text{and} \quad T_n(V) = \bigotimes_1^n V = \underbrace{V \otimes \cdots \otimes V}_{n \text{ times}} \text{ for } n > 1.$$

If we denote the injection of $T_m(V)$ into $T(V)$ by

$$i_m\colon T_m(V) \to \bigoplus_{n=0}^{\infty} T_n(V),$$

then obviously

$$TV = \bigoplus_{m=0}^{\infty} \operatorname{Im} i_m \qquad \text{(internal direct sum).}$$

The multiplication on TV is defined by the obvious maps

$$T_p(V) \otimes T_q(V) \xrightarrow{\simeq} T_{p+q}(V) \xrightarrow{i_{p+q}} TV,$$

and the unit

$$\eta_{TV}\colon K \to TV$$

is defined by $\eta_{TV} = i_0$. These definitions make TV into an internally graded (associative, unitary) algebra. It follows that we can filter TV in an increasing way, as an algebra, by setting

$$F_n(TV) = \bigoplus_{m=0}^{n} \operatorname{Im} i_m \qquad \text{for } n \geqslant 0.$$

(For the basic facts concerning filtrations, the reader is referred to [3, 6, 13, 15, 16].) But

$$\operatorname{Im} i_n = (\operatorname{Im} i_1)^n \qquad \text{for} \quad n \geqslant 1,$$

from which it follows that

$$F_m(TV) = (\operatorname{Im} i_0 + \operatorname{Im} i_1)^m \qquad \text{if} \quad m \geqslant 1.$$

Since

$$i_{TV}\colon V \to TV$$

is defined by $i_{TV} = i_1$, this shows that the increasing filtration on TV given above (from the grading) can also be described by

$$F_0(TV) = \eta_{TV}K,$$

and

$$F_n(TV) = (\eta_{TV}K \oplus \operatorname{Im} i_{TV})^n = (\eta_{TV}K + \operatorname{Im} i_{TV})^n \qquad \text{for} \quad n \geqslant 1.$$

[Recall that the importance of i_{TV} (whence the special name for i_1) lies in the fact that (TV, i_{TV}) satisfies a universal mapping property.]

Since the enveloping algebra UL of a Lie algebra L is a quotient algebra of TL, it follows that we obtain an algebra filtration on UL from that on TL by setting

$$F_n(UL) = \pi[F_n(TL)] \qquad (\forall\, n)$$

where

$$\pi\colon TL \to UL = TL/\langle i_{TL}x \otimes i_{TL}y - i_{TL}y \otimes i_{TL}x - i_{TL}[x, y] \mid x, y \in L\rangle$$

is the projection. Since

$$\eta_{UL}\colon K \to UL$$

and

$$i_{UL}\colon L \to UL$$

are given, respectively, by

$$\eta_{UL} = \pi \circ \eta_{TL}$$

and

$$i_{UL} = \pi \circ i_{TL},$$

it follows that

$$F_0(UL) = \eta_{UL}K,$$

while

$$F_n(UL) = (\eta_{UL}K + \operatorname{Im} i_{UL})^n \qquad \text{for} \quad n \geqslant 1.$$

This filtration of UL, by powers of the generating subspace $\operatorname{Im} i_{UL}$, is called the *Lie filtration* on UL. Note that for $n \geqslant 1$, $F_n(UL)$ is the subspace of UL generated by all m-fold products of elements of the generating subspace $\operatorname{Im} i_{UL}$, where $m \leqslant n$, i.e., by the set of all

$$i_{UL}(x_{\alpha_1}) \cdots i_{UL}(x_{\alpha_m}),$$

where $x_{\alpha_j} \in L$ and $m \leqslant n$.

Let

$$E^0(UL) = \{E^0_n(UL) = F_n(UL)/F_{n-1}(UL)\}_{n=0}^{\infty}$$

denote the associated graded algebra. [By convention, $F_{-1}(UL) = 0$.] Since, for $n \geqslant 1$, the nth filtrand $F_n(UL)$ is the nth power of a generating subspace of UL, it follows easily that $E^0_1(UL)$ generates $E^0(UL)$. Because

$$i_{UL}: L \to \mathfrak{L}(UL)$$

is a Lie map, it is then easy to show that the elements of $E^0_1(UL)$ pairwise commute. It follows that $\bigoplus_{n=0}^{\infty} E^0_n(UL)$ is a commutative (associative, unitary) algebra.

Let SL denote the symmetric algebra on (the underlying vector space of) L; let

$$i_{SL}: L \to SL$$

denote the composite

$$L \xrightarrow{i_{TL}} TL \twoheadrightarrow SL = TL/\langle i_{TL}x \otimes i_{TL}y - i_{TL}y \otimes i_{TL}x \mid x, y \in L\rangle;$$

and let

$$\beta: E^0_1(UL) \to \bigoplus_{n=0}^{\infty} E^0_n(UL)$$

denote the injection of $E^0_1(UL)$ into $\bigoplus_{n=0}^{\infty} E^0_n(UL)$ as a direct summand. Since $\bigoplus_{n=0}^{\infty} E^0_n(UL)$ is commutative, the universal mapping property which (SL, i_{SL}) satisfies guarantees that there is a unique map

$$F: SL \to \bigoplus_{n=0}^{\infty} E^0_n(UL)$$

of associative unitary algebras making the following diagram commutative:

$$\begin{array}{ccccc} L & & \xrightarrow{\quad i_{SL} \quad} & & SL \\ {\scriptstyle i_{UL}}\downarrow & & & & \downarrow{\scriptstyle F} \\ i_{UL}(L) & \xrightarrow{\cong} & E^0_1(UL) & \xrightarrow{\beta} & \bigoplus_{n=0}^{\infty} E^0_n(UL). \end{array}$$

It is easy to see that F is a coalgebra map. [One must, of course, at some point, check that the Lie filtration (an algebra filtration) is also a coalgebra filtration; but this is easy since, for $x \in L$, $\Delta(i_{UL}x) = i_{UL}x \otimes 1_{UL} + 1_{UL} \otimes i_{UL}x$ and $\epsilon(i_{UL}x) = 0$.] It is also easy to see that F is surjective. [F maps a generating subspace of SL onto a generating subspace of $\bigoplus_{n=0}^{\infty} E^0_n(UL)$.] The PBW$\theta$ asserts that F is an isomorphism of Hopf algebras.

It follows immediately from the PBWθ that

$$i_{UL}: L \to \mathfrak{L}(UL)$$

is injective. Conversely, in case the characteristic of K is zero (a condition not used so far) it is not hard to show that the injectivity of i_{UL} implies that of F. Indeed, since SL is a pointed, irreducible coalgebra (Proposition 11.1.1 of [16]) it follows that

$$F \text{ is } 1\text{–}1 \Leftrightarrow F\,|_{P(SL)} \text{ is } 1\text{–}1,$$

where $P(SL)$ is the space of primitive elements of SL. [This follows, for example, from Lemma 11.0.1 of [16]. For the definitions, the reader is also referred to Sect. 8 of [16].] But $P(SV) \simeq V$ whenever $\chi(K) = 0$. [See Serre [14, pp. LA.3.10–LA.3.11, Theorem 3.5], for example, or Jacobson [8, p. 170, Theorem 9 (Friedrichs)].] Thus

$$F\,|_{P(SL)} = F \circ i_{SL},$$

from which the assertion follows. [For a more elementary proof of this equivalence, the reader is referred to Cartier [5, pp. 1-07–1-09, Lemma 2]. It is to be noted, however, that even though the proof in [5] does not use Hopf algebraic techniques so explicitly, it nonetheless employs the use of the diagonal on UL.]

We now transform the statement

$$F: SL \to \bigoplus_{n=0}^{\infty} E_n^0(UL) \text{ is an isomorphism of Hopf algebras,}$$

once more, to arrive at the form of the PBWθ that we wish to dualize.

Recall that we are denoting by $i_{SL}: L \to SL$ the injection of L into SL (as a direct summand), and by $\beta: E_1^0(UL) \to \bigoplus_{n=0}^{\infty} E_n^0(UL)$ the injection of $E_1^0(UL)$ into $\bigoplus_{n=0}^{\infty} E_n^0(UL)$ as a direct summand. It is easy to see that $\operatorname{Im} i_{SL}$ generates SL as an algebra and that $\operatorname{Im}\beta$ generates $\bigoplus_{n=0}^{\infty} E_n^0(UL)$ as an algebra. [These facts are used to show the surjectivity of F.] Filter SL by powers of the generating subspace $\operatorname{Im} i_{SL}$ and $\bigoplus_{n=0}^{\infty} E_n^0(UL)$ by powers of the generating subspace $\operatorname{Im}\beta$ by defining

$$F_0(SL) = \eta_{SL}K,$$
$$F_n(SL) = (\eta_{SL}K + \operatorname{Im} i_{SL})^n \qquad \text{for} \quad n \geqslant 1$$

and

$$F_0\left[\bigoplus_{n=0}^{\infty} E_n^0(UL)\right] = \eta_{\oplus_{n=0}^{\infty} E_n^0(UL)}K,$$

$$F_n\left[\bigoplus_{n=0}^{\infty} E_n^0(UL)\right] = [\eta_{\oplus_{n=0}^{\infty} E_n^0(UL)}K + \operatorname{Im}\beta]^n \qquad \text{for} \quad n \geqslant 1.$$

[Note that the filtration on SL (by powers of the generating subspace $\operatorname{Im} i_{SL}$) is simply the Lie filtration on $U(\operatorname{Triv} L) = SL$.] Since

$$F\colon SL \to \bigoplus_{n=0}^{\infty} E_n^0(UL)$$

is an algebra map, the commutativity of the diagram defining F guarantees that F is a map of filtered (Hopf) algebras and so induces a map

$$E^0(F)\colon E^0(SL) \to E^0\left(\bigoplus_{n=0}^{\infty} E_n^0(UL)\right)$$

of the associated graded (Hopf) algebras. But the filtration on $\bigoplus_{n=0}^{\infty} E_n^0(UL)$ is by the grading, so

$$E^0\left(\bigoplus_{n=0}^{\infty} E_n^0(UL)\right) \simeq E^0(UL).$$

Note. Hereafter, we shall write $\bigoplus E^0(UL)$ in place of $\bigoplus_{n=0}^{\infty} E_n^0(UL)$.

Moreover, SL is a graded algebra and the filtration on SL is by the grading; hence $E^0(SL) \simeq SL$. Since E^0 and $\bigoplus$ are functors, it follows that

$$F\colon SL \to \bigoplus E^0(UL)$$

is an isomorphism of Hopf algebras if and only if the natural map

$$E^0(F)\colon E^0(SL) \to E^0(UL)$$

is an isomorphism of (graded, connected, biassociative, bicommutative) Hopf algebras.

This is the form of the PBWθ which we shall dualize. To do so, we introduce a filtration on U^cM, analogous to the Lie filtration on UL and called the *Lie filtration on* U^cM. This filtration, which turns out also to be an algebra filtration, is given by "wedges" of the cogenerating subspace

$$(\ker \pi_{U^cM})^+ = \ker \pi_{U^cM} \cap \ker \epsilon_{U^cM}\,.$$

Let

$${}_0E(U^cM) = \{{}_0E_n(U^cM) = F_n(U^cM)/F_{n+1}(U^cM)\}_{n=0}^{\infty}$$

denote the associated graded coalgebra, and let

$$\alpha\colon \bigoplus_{n=0}^{\infty} {}_0E_n(U^cM) \to {}_0E_1(U^cM)$$

denote the projection of $\bigoplus_{n=0}^{\infty} {}_0E_n(U^cM)$ onto the component of degree one. Using the fact that ker π_{U^cM} is a cogenerating subspace of U^cM one is able to show that ker α is a cogenerating subspace of $\bigoplus_{n=0}^{\infty} {}_0E_n(U^cM)$. This, together with the fact that

$$\pi_{U^cM}\colon \mathfrak{L}^c(U^cM) \to M$$

is a Lie map, enables one to show that $\bigoplus_{n=0}^{\infty} {}_0E_n(U^cM)$ is cocommutative. Let (S^cV, π_{S^cV}) denote the symmetric coalgebra discussed in Section 3. Since $\bigoplus_{n=0}^{\infty} {}_0E_n(U^cM)$ is cocommutative, the universal mapping property which (S^cM, π_{S^cM}) satisfies guarantees that there exists a unique map

$$F\colon \bigoplus_{n=0}^{\infty} {}_0E_n(U^cM) \to S^cM$$

of counitary coalgebras making the following diagram commute:

$$\begin{array}{ccc} M & \xleftarrow{\pi_{S^cM}} & S^cM \\ \pi \uparrow & & \uparrow F \\ {}_0E_1(U^cM) & \xleftarrow{\alpha} & \bigoplus_{n=0}^{\infty} {}_0E_n(U^cM). \end{array}$$

Here $\pi\colon {}_0E_1(U^cM) \to M$ is induced by $\pi_{U^cM}\colon U^cM \to M$. [In case M is locally finite, π is a vector space isomorphism.] It is easy to see that F is injective and an algebra map. But, in contradistinction to what occurs in the case of Lie algebras, F is not (in general) an isomorphism since S^cM is not (in general) graded. [If $M = \{0\}$, then of course $S^c(0) = K$, and $\bigoplus_{n=0}^{\infty} {}_0E_n(U^c(0)) \simeq K$ as well.] We next filter S^cM by wedges of the cogenerating subspace (ker $\pi_{S^cM})^+$, and we filter $\bigoplus_{n=0}^{\infty} {}_0E_n(U^cM)$ by wedges of the cogenerating subspace (ker $\alpha)^+$. The filtration on S^cM is simply the Lie filtration on $U^c(\mathrm{Triv}\, M)$, whereas the filtration on $\bigoplus_{n=0}^{\infty} {}_0E_n(U^cM)$ turns out to be a (decreasing) Hopf algebra filtration arising from the grading. [From this latter fact, it follows that ${}_0E(\bigoplus {}_0E(U^cM)) = {}_0E(U^cM)$.] Since $F\colon \bigoplus_{n=0}^{\infty} {}_0E_n(U^cM) \to S^cM$ is a map of counitary coalgebras, the commutativity of the diagram defining F guarantees that F is a map of filtered Hopf algebras. It follows that

$$F\colon \bigoplus {}_0E(U^cM) \to S^cM$$

induces a map

$${}_0E(F)\colon {}_0E\left(\bigoplus {}_0E(U^cM)\right) \to {}_0E(S^cM),$$

i.e., a map

$${}_0E(F)\colon {}_0E(U^cM) \to {}_0E(S^cM)$$

of the associated graded Hopf algebras.

The Dual PBWθ asserts that in case the ground field K has characteristic 0, then the natural map

$$_0E(F)\colon {}_0E(U^cM) \to {}_0E(S^cM)$$

is an isomorphism of graded, connected, biassociative, bicommutative Hopf algebras whenever M is a locally finite Lie coalgebra (in fact, if and only if M is locally finite).

THEOREM [The Dual Poincaré–Birkhoff–Witt Theorem]. *Let M be a Lie coalgebra over a field of characteristic zero. Then the natural map*

$$_0E(F)\colon {}_0E(U^cM) \to {}_0E(S^cM)$$

is an isomorphism of graded, connected, bicommutative, biassociative Hopf algebras if and only if *M is locally finite.*

The ungraded version of this theorem asserts that, in characteristic zero, the bicommutative, biassociative, pointed, irreducible, involutive Hopf algebras $\bigoplus {}_0E(U^cM)$ and $\bigoplus {}_0E(S^cM)$ are isomorphic if and only if M is locally finite.

We conclude this section with a indication of two counterexamples. The first of these concerns the Hopf algebra map

$$F\colon \bigoplus {}_0E(U^cM) \to S^cM$$

where M is a locally finite Lie coalgebra. We have noted above that F is injective, so it is natural to ask whether F is not also surjective. [In the case of Lie algebras, the corresponding map is an isomorphism.] The following example shows that this is not the case. Let $M = L^*$ where L is a finite dimensional Lie algebra. Then

$$S^c(M) = S^c(L^*) = (SL)^0,$$

where SL denotes the symmetric algebra on (the underlying vector space of) L. We claim that $(SL)^0$ is not finitely generated as an algebra. The quickest way to see this is to apply a result of Sweedler [17, p. 266, Corollary 2.2.0] to conclude that if $(SL)^0$ were finitely generated, then the group

$$G(SL)^0$$

of group-like elements of $(SL)^0$ would be a finitely generated free abelian group, and hence countable. [By definition, an element x of a Hopf algebra H is *group-like* if $\Delta x = x \otimes x$.] But the group $G(SL)^0$ of group-like elements of $(SL)^0$ coincides with the group $\mathrm{Alg}(SL, K)$ of algebra homomorphisms from SL to K, where $\mathrm{Alg}(SL, K)$ has its group structure as a subgroup of the convolution algebra $\mathrm{Hom}_K(SL, K) = (SL)^*$, i.e., the dual algebra of the coalgebra SL. [This follows easily from the way in which the diagonal on the "upper zero" of a Hopf algebra is defined.] On the other hand, Alg(SL, K) is isomorphic as a group to the underlying additive group of L^*. Indeed, the universal mapping property satisfied by the symmetric algebra (SL, i_{SL}) on L guarantees that the set map

$$\mathrm{Hom}_{\mathscr{A}}(SL, K) \to \mathrm{Hom}_{\mathscr{V}}(L, K)$$

given by

$$f \mapsto f \circ i_{SL}$$

is bijective; in addition, this map is a group homomorphism since

$$\begin{aligned}(f * g) \circ i_{SL}(x) &= (f * g)(i_{SL}x) \\ &= f(i_{SL}x)\, g(1_{SL}) + f(1_{SL})\, g(i_{SL}x) \\ &= f(i_{SL}x) + g(i_{SL}x) \\ &= (f + g)(i_{SL}x) \\ &= (f + g) \circ i_{SL}(x).\end{aligned}$$

It follows that the underlying additive group of L^* is countable (being isomorphic to $G(SL)^0$), an absurdity. Thus, $(SL)^0$ cannot be finitely generated. On the other hand, it can be shown that $\bigoplus_0 E(U^cM)$ is finitely generated. Thus F cannot be an isomorphism and hence F cannot be surjective.

The second example concerns the question of whether or not U^cM and S^cM are isomorphic as augmented algebras. [Recall that in the case of a Lie algebra L, the Poincaré–Birkhoff–Witt theorem may be considered as saying that UL and SL are isomorphic as augmented coalgebras.] That the answer to this is negative is shown by the following example, called to our attention by M. Sweedler. Recall from the above that $(SL)^0$ is not finitely generated (cf. Sweedler [17, p. 266]). On the other hand, $(UL)^0$ is precisely Hochschild's algebra of representative functions on UL (cf. [7, p. 500]). But if L is a finite dimensional semi-simple Lie algebra, then a result of Harish–Chandra (cf. [7, p. 513]) says that $(UL)^0$ is finitely generated. Hence U^cM and S^cM cannot be isomorphic in general.

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