State-Delayed Matrix Differential-Difference Equations

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Matrix differential-difference equations involving delayed state terms are solvable by the decomposition method without linearization. © 1986 Academic Press, Inc.

Differential-difference equations (or delay equations) arise naturally in complex physical systems involving time delay or lag in propagation of effects. Such systems can include a human physiological system, a national economy, control problems in large systems, and a host of other applications. The equations in general will be nonlinear and stochastic and, in special cases, may be linear or deterministic or both.

Let us begin with the simple deterministic linear time-invariant state-delayed nth order matrix differential-difference equations with normalized delay of unity. In more general equations, of course, we can consider time-dependent or random delays as well [1]. We are considering the system

\[ Y(t) = A_0 Y(t) + A_1 Y(t-1), \quad t \geq 0 \]

where \( Y \) is an \( n \times 1 \) vector (the state vector), \( A_0, A_1 \) are constant \( n \times n \) matrices, and the initial state vector is specified. Thus we have

\[
\begin{bmatrix}
  y_1 \\
  y_2 \\
  \vdots \\
  y_n \\
\end{bmatrix}
= \begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & \ddots & \ddots & \vdots \\
  \vdots & \ddots & \ddots & \ddots \\
  a_{n1} & \cdots & a_{nn} \\
\end{bmatrix}
\begin{bmatrix}
  y_1 \\
  y_2 \\
  \vdots \\
  y_n \\
\end{bmatrix}

+ \begin{bmatrix}
  b_{11} & b_{12} & \cdots & b_{1n} \\
  b_{21} & \ddots & \ddots & \vdots \\
  \vdots & \ddots & \ddots & \ddots \\
  b_{n1} & \cdots & b_{nn} \\
\end{bmatrix}
\begin{bmatrix}
  y_1 \\
  y_2 \\
  \vdots \\
  y_n \\
\end{bmatrix}
\]

Cases for time-varying or stochastic elements are considered elsewhere [2].
where $Dy(t) = y(t - 1)$. Thus

$$\begin{align*}
dy_1/dt &= a_{11} y_1 + a_{12} y_2 + \cdots + a_{1n} y_n + b_{11} Dy_1 + \cdots + b_{n1} Dy_n \\
dy_2/dt &= a_{21} y_1 + \cdots + a_{2n} y_n + b_{21} Dy_1 + \cdots + b_{21} Dy_n \\
\vdots \\
dy_n/dt &= a_{n1} y_1 + \cdots + a_{nn} y_n + b_{n1} Dy_1 + \cdots + b_{n1} Dy_n
\end{align*}$$

a system of $n$ coupled equations. Let $L = d/dt$. Write each $y_k = \sum_{m=0}^{\infty} (y_k)_m$ and $y_k(0) = (y_k)_0$ as a component of the initial state vector. Now

$$\begin{align*}
y_1 &= (y_1)_0 + L^{-1} \left[ a_{11} \sum_{m=0}^{\infty} (y_1)_m + \cdots + a_{1n} \sum_{m=0}^{\infty} (y_n)_m + b_{11} D \sum_{m=0}^{\infty} (y_1)_m + \cdots + b_{n1} D \sum_{m=0}^{\infty} (y_n)_m \right] \\
\vdots \\
y_n &= (y_n)_0 + L^{-1} \left[ \cdots \right]
\end{align*}$$

where the bracketed quantity is the same as in the expression above.

The second components $(y_1)_1, (y_2)_1, \ldots, (y_n)_1$ are given by

$$\begin{align*}
(y_1)_1 &= L^{-1} \left[ a_{11} (y_1)_0 + \cdots + a_{1n} (y_n)_0 + b_{11} D (y_1)_0 + \cdots + b_{n1} D (y_n)_0 \right] \\
\vdots \\
(y_n)_1 &= L^{-1} \left[ a_{n1} (y_1)_0 + \cdots + a_{nn} (y_n)_0 + b_{n1} D (y_1)_0 + \cdots + b_{nn} D (y_n)_0 \right]
\end{align*}$$

which are calculable since they involve only components of the initial vector.

The third components $(y_1)_2, \ldots, (y_n)_2$ are

$$\begin{align*}
(y_1)_2 &= L^{-1} \left[ a_{11} (y_1)_1 + \cdots + a_{1n} (y_n)_1 + b_{11} D (y_1)_1 + \cdots + b_{n1} D (y_n)_1 \right] \\
\vdots \\
(y_n)_2 &= L^{-1} \left[ a_{n1} (y_1)_1 + \cdots + a_{nn} (y_n)_1 + b_{n1} D (y_1)_1 + \cdots + b_{nn} D (y_n)_1 \right]
\end{align*}$$

again calculable since it depends only on the calculated components, etc., to determine $\sum_{m=0}^{\infty} (y_1)_m, \ldots, \sum_{m=0}^{\infty} (y_n)_m$ to some desired $m$ which is a sufficient approximation. The operator $D$ acts to delay by unity thus $D\phi(t) = \phi(t - 1)$, etc.

Now we consider a nonlinear term $Ny$ to be present. Its exact form does not matter so long as (Adomian’s) polynomials [1, 2] can be generated for
the nonlinearity which can be a product or composite nonlinearity as well as simple ones including polynomial, trigonometric, decimal power, etc. $N_y$ is simply written as $\sum_{m=0}^{N} A_m$ and since each $A_m$ involves only the first $m$ components of the state vector, the calculation is as easy as for the linear case [1, 2]. For example, if we add to the previous two terms the vector whose components are $y_1^2, y_2^2, \ldots, y_n^2$, then

\[
\begin{align*}
(y_1)_1 &= L^{-1}[\cdot] + L^{-1}(y_1)_0^2 \\
(y_1)_2 &= L^{-1}[\cdot] + L^{-1}2(y_1)_0(y_1)_1 \\
&\quad \vdots \\
(y_1)_n &= L^{-1}[\cdot] + L^{-1}(y_1)_0(y_1)_n
\end{align*}
\]

since $A_0(y^2) = y_0^2$, $A_1(y^2) = 2y_0y_1$, etc. [2].

3

Now suppose we have a stochastic matrix coefficient or coefficients. Then averaging is done after the components are determined to the desired $m$. Convergence is discussed in [2].

REFERENCES