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State-Delayed Matrix Differential-Difference Equations

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Matrix differential-difference equations involving delayed state terms are solvable by the decomposition method without linearization. © 1986 Academic Press, Inc.

Differential-difference equations (or delay equations) arise naturally in complex physical systems involving time delay or lag in propagation of effects. Such systems can include a human physiological system, a national economy, control problems in large systems, and a host of other applications. The equations in general will be nonlinear and stochastic and, in special cases, may be linear or deterministic or both.

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Let us begin with the simple deterministic linear time-invariant state-delayed nth order matrix differential-difference equations with normalized delay of unity. In more general equations, of course, we can consider time-dependent or random delays as well [1]. We are considering the system

$$\dot{y}(t) = A_0\, y(t) + A_1\, y(t-1), \qquad t \geqslant 0$$

where y is an $n \times 1$ vector (the state vector), A_0, A_1 are constant[1] $n \times n$ matrices, and the initial state vector is specified. Thus we have

$$(d/dt)\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{n1} \\ b_{21} & & & \\ \vdots & & & \\ b_{n1} & \cdots & & b_{nn} \end{bmatrix} D \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

[1] Cases for time-varying or stochastic elements are considered elsewhere [2].

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where $Dy(t) = y(t-1)$. Thus

$$dy_1/dt = a_{11} y_1 + a_{12} y_2 + \cdots + a_{1n} y_n + b_{11} Dy_1 + \cdots + b_{n1} Dy_n$$
$$dy_2/dt = a_{21} y_1 + \cdots + a_{2n} y_n + b_{21} Dy_1 + \cdots + b_{21} Dy_n$$
$$\vdots$$
$$dy_n/dt = a_{n1} y_1 + \cdots + a_{nn} y_n + b_{n1} D_{y1} + \cdots + b_{n1} Dy_n$$

a system of n coupled equations. Let $L = d/dt$. Write each $y_k = \sum_{m=0}^{\infty} (y_k)_m$ and $y_k(0) = (y_k)_0$ as a component of the initial state vector. Now

$$y_1 = (y_1)_0 + L^{-1} \Bigg[a_{11} \sum_{m=0}^{\infty} (y_1)_m + \cdots + a_{1n} \sum_{m=0}^{\infty} (y_n)_m + b_{11} D \sum_{m=0}^{\infty} (y_1)_m + \cdots + b_{1n} D \sum_{m=0}^{\infty} (y_n)_m \Bigg]$$
$$\vdots$$
$$y_n = (y_n)_0 + L^{-1} [\cdot]$$

where the bracketed quantity is the same as in the expression above.

The second components $(y_1)_1, (y_2)_1, \ldots, (y_n)_1$ are given by

$$(y_1)_1 = L^{-1} [a_{11} (y_1)_0 + \cdots + a_{1n} (y_n)_0 + b_{11} D (y_1)_0 + \cdots + b_{1n} D (y_n)_0]$$
$$\vdots$$
$$(y_n)_1 = L^{-1} [a_{n1} (y_1)_0 + \cdots + a_{nn} (y_n)_0 + b_{n1} D (y_1)_0 + \cdots + b_{nn} D (y_n)_0]$$

which are calculable since they involve only components of the initial vector.

The third components $(y_1)_2, \ldots, (y_n)_2$ are

$$(y_1)_2 = L^{-1} [a_{11} (y_1)_1 + \cdots + a_{1n} (y_n)_1 + b_{11} D (y_1)_1 + \cdots + b_{1n} D (y_n)_1]$$
$$\vdots$$
$$(y_n)_2 = L^{-1} [a_{n1} (y_1)_1 + \cdots + a_{nn} (y_n)_1 + b_{n1} D (y_1)_1 + \cdots + b_{nn} D (y_n)_1]$$

again calculable since it depends only on the calculated components, etc., to determine $\sum_{m=0}^{\infty} (y_1)_m, \ldots, \sum_{m=0}^{\infty} (y_n)_m$ to some desired m which is a sufficient approximation. The operator D acts to delay by unity thus $D\phi(t) = \phi(t-1)$, etc.

2

Now we consider a nonlinear term Ny to be present. Its exact form does not matter so long as (Adomian's) polynomials [1, 2] can be generated for the nonlinearity which can be a product or composite nonlinearity as well as simple ones including polynomial, trigonometric, decimal power, etc. Ny is simply written as $\sum_{m=0}^{\infty} A_m$ and since each A_m involves only the first m components of the state vector, the calculation is as easy as for the linear case [1, 2]. For example, if we add to the previous two terms the vector whose components are $y_1^2, y_2^2, \ldots, y_n^2$ then

$$(y_1)_1 = L^{-1}[\cdot] + L^{-1}(y_1)_0^2$$
$$(y_1)_2 = L^{-1}[\cdot] + L^{-1}2(y_1)_0(y_1)_1$$
$$\vdots$$

since $A_0(y^2) = y_0^2$, $A_1(y^2) = 2y_0 y_1$, etc. [2].

3

Now suppose we have a stochastic matrix coefficient or coefficients. Then averaging is done after the components are determined to the desired m. Convergence is discussed in [2].

References

1. G. Adomian, "Stochastic Systems," Academic Press, New York, 1983.
2. G. Adomian, "Nonlinear Stochastic Operator Equations," Academic Press, New York, in press.