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Close-to-Normal Structure and Its Applications

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Let K be a bounded subset of a metric space (B, d). Let $W(K)$ be the supremum of the cardinals of all subsets H of K such that the distance between any two distinct points in H is equal to the diameter of K. This function W on the family of all bounded subsets of B is used to prove the following result. Let K be a weakly compact convex subset of a Banach space B. Then K has a close-to-normal structure if B satisfies any of the following conditions: (a) B is strictly convex; (b) B is separable; (c) B has the property A: For any sequence $\{x_n\}$ in B, $\{x_n\}$ converges to a point x in B if it converges weakly to x and $\{\| x_n \|\}$ converges to $\| x \|$. Applications of this result to the fixed point theory are given.

1. INTRODUCTION

Let K be a bounded subset of a metric space (B, d). Let H be a subset of K which has at least two points. H is a *diametral subset* of K if for any distinct points x, y in H, $d(x, y) = \delta(K)$, where $\delta(K)$ is the diameter of K. It is obvious and well known that every diametral subset of K is finite if K is compact. However, two maximal (with respect to the inclusion relation) diametral subsets of K may have different cardinals. For example, let A be the closed convex hull of the subset

$$\{(x, y) \in E^2 : x^2 + y^2 < 1, x, y > 0, \tan^{-1}(y/x) < \pi/3\}$$

of the two-dimensional Euclidean space E^2. Then both

$$\{(0, 0), (\cos(\pi/4), \sin(\pi/4))\}$$

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and

$$\{(0,0), (1,0), (\cos(\pi/3), \sin(\pi/3))\}$$

are maximal diametral subsets of A. Let $W(K)$ denote the supremum of $\{|H|: H$ is a (maximal) diametral subset of $K\}$, where $|H|$ is the cardinal of H. By the well ordering principle for cardinals, the cardinal $W(K)$ is well defined. By the usual inner product for the n-dimensional Euclidean space E^n, one can prove that $W(K) \leqslant n + 1$ if $B = E^n$. Also $W(K) \leqslant 2^n$ if $B = R^n$ with the supremum norm. Note that $W(K)$ is finite if K is compact. The same conclusion does not hold if we drop the condition compactness on K. In fact, for any cardinal α, there exists a bounded subset H (e.g. any orthonormal basis) of a Hilbert space B of dimension α such that $W(H) = \alpha$. By considering the closed convex hull K of H, we obtain $W(K) \geqslant \alpha$. So geometrically, the function W defined above can be used to investigate how badly a weakly compact convex set may behave. Also, it can be used to obtain significant generalizations and extensions of the fixed point theorems obtained by Kannan and Soardi in [5] and [8]. For this purpose, we need the following definition. Let K be a bounded convex subset of a normed linear space. K has a *close-to-normal structure* if for any closed convex subset H of K with $\delta(H) > 0$, there exists x in H such that $\| x - y \| < \delta(H)$ for all y in H. It is obvious that K has a close-to-normal structure if K has normal structure [2]. Let K be a weakly compact convex subset of a normed linear space B. There are two related open problems concerning the existence of a fixed point for a self map T on K: (i) T has a fixed point if it is nonexpansive ($\| T(x) - T(y)\| \leqslant \| x - y \|$, $x, y \in K$). (ii) T has a fixed point if T is a Kannan map on K, i.e.,

$$\| T(x) - T(y)\| \leqslant (\| x - T(x)\| + \| y - T(y)\|)/2$$

for all x, y in K. It was proved respectively by Kirk [7] and Soardi [8] that (i) and (ii) hold if K has normal structure. In [9], we obtain the following characterization of (ii).

THEOREM 1. *Let B be a normed linear space. Then the following propositions are equivalent*:

(i) *Every Kannan map of a nonempty weakly compact convex subset of B into itself has a fixed point*;

(ii) *Every weakly compact convex subset of B has a close-to-normal structure.*

The above theorem drops the open problem (ii) stated above from the fixed point theory and adds the following open problem to the geometry of Banach spaces: (iii) Every weakly compact convex subset of a Banach space B has a close-to-normal structure.
In Section 2, we shall use the function W defined above to prove that (iii) holds if any one of the following conditions is satisfied: (a) B is strictly convex; (b) B has the property A [3]: For any sequence $\{x_n\}$ in B, $\{x_n\}$ converges to a point x in B if it converges weakly to x and $\{\| x_n \|\}$ converges to $\| x \|$; (c) B is separable. From (c) and the example in [1], it follows that there exists a weakly compact convex subset of a Banach space which has a close-to-normal structure but has no normal structure. From (b), it follows that (iii) holds if B is locally uniformly convex [6].

2. Main Result and its Applications

Let B be a normed linear space. B is *strictly convex* if for any x, y, z in B, $\| x - z \| + \| z - y \| = \| x - y \|$ implies that $z \in [x, y]$ $(= \{(1 - t)x + ty: t \in [0, 1]\})$ [4, p. 111].

Theorem 2. *Let B be a Banach space which satisfies at least one of the following conditions:*

(a) *B is strictly convex;*

(b) *B has the property A:*

(c) *B is separable.*

Then every weakly compact convex subset of B has a close-to-normal structure.

Proof. Suppose to the contrary that there exists a weakly compact convex subset K of B which has no close-to-normal structure. Then K includes a closed convex set H such that $\delta(H) > 0$ and for any x in H, $\| x - T(x)\| = \delta(H)$ for some $T(x)$ in H. We shall first prove that $W(H)$ is uncountable. It is clear that $W(H) \geqslant 2$. Let $x_1, x_2, \ldots, x_n$ be n distinct elements in H such that $\| x_i - x_j \| = \delta(H)$ for all distinct i, j in $\{1, 2, \ldots, n\}$. Let

$$\bar{x} = \sum_{i=1}^{n} x_i/n, \qquad x_{n+1} = T(\bar{x}).$$

Then $\bar{x}$, $x_{n+1} \in H$ and

$$\begin{aligned}\delta(H) &= \| \bar{x} - T(\bar{x})\| \\ &= \left\| \sum_{i=1}^{n} (x_i - x_{n+1})/n \right\| \\ &\leqslant \sum_{i=1}^{n} \| x_i - x_{n+1} \|/n \\ &\leqslant \delta(H) \qquad (\| x_i - x_{n+1} \| \leqslant \delta(H)).\end{aligned}$$

So all of the above inequalities are equalities. Thus

$$\| x_i - x_{n+1} \| = \delta(H) \qquad \text{for all} \quad i = 1, 2, \ldots, n. \tag{1}$$

By induction, $W(H)$ is infinite. Let $\{x_n\}$ be a sequence in H such that

$$\| x_n - x_m \| = \delta(H) \text{ for all distinct positive integers } n, m.$$

It suffices to find an element x_∞ in H such that

$$\| x_\infty - x_n \| = \delta(H) \text{ for all positive integer } n.$$

Since $\sum_{i=1}^{\infty} (1/2^i)\, x_i$ is absolutely convergent and B is complete, $\{\sum_{i=1}^{n} (1/2^i)\, x_i\}$ converges to some element $\bar{x}$ in B. Since $\{\sum_{i=1}^{n} (1/2^i)\, x_i + (1/2^n)\, x_{n+1}\}$ is a sequence in H which converges to $\bar{x} = \sum_{i=1}^{\infty} (1/2^i)\, x_i$ and H is closed, $\bar{x}$ belongs to H. Let $x_\infty = T(\bar{x})$. Then

$$\begin{aligned}\delta(H) &= \| \bar{x} - T(\bar{x})\| \\ &= \left\| \sum_{i=1}^{\infty} (1/2^i)(x_i - x_\infty) \right\| \\ &\leqslant \sum_{i=1}^{\infty} (1/2^i)\, \| x_i - x_\infty \| \\ &\leqslant \sum_{i=1}^{\infty} (1/2^i)\, \delta(H) \\ &= \delta(H).\end{aligned}$$

So all of the above inequalities are equalities. Thus

$$\| x_\infty - x_n \| = \delta(H) \text{ for all positive integer } n.$$

Hence $W(H)$ is uncountable. This implies that B is not separable. Let

$$x_1 \in H, \qquad x_2 = T(x_1), \qquad \bar{x} = (1/2)(x_1 + x_2), \qquad x_3 = T(\bar{x}).$$

Then by (1),

$$\| x_i - x_j \| = \delta(H) \text{ for all distinct positive integers } i, j.$$

Let $x = x_1 + x_2 - x_3$. Then

$$\| x_1 - x_3 \| + \| x - x_1 \| = \| x_3 - x \| = 2\delta(H).$$

So B is not strictly convex (for otherwise, $x_1 = (1/2)(x_3 + x)$, i.e., $x_1 = x_2$, a contradiction to $\delta(H) > 0$). It follows then from hypothesis that B has the property A. Since $W(H)$ is infinite, there exists a sequence $\{x_n\}$ in H such that

$$\| x_n - x_m \| = \delta(H) \text{ for all distinct positive integers } n, m.$$

Since H is closed and convex, it is weakly closed. Since K is weakly compact, so is H. By Eberlein's theorem, H is sequentially compact. So by taking a subsequence, we may assume that $\{x_n\}$ converges weakly to some point x_∞ in H. Thus

$$\begin{aligned} \delta(H) &= \| x_\infty - T(x_\infty)\| \\ &\leqslant \liminf \| x_n - T(x_\infty)\| \\ &\leqslant \limsup \| x_n - T(x_\infty)\| \\ &\leqslant \delta(H). \end{aligned}$$

So $\lim \| x_n - T(x_\infty)\| = \delta(H)$. Since $\{x_n - T(x_\infty)\}$ converges weakly to $x_\infty - T(x_\infty)$ and

$$\lim \| x_n - T(x_\infty)\| = \delta(H) = \| x_\infty - T(x_\infty)\|,$$

by the property A of B, $\{x_n - T(x_\infty)\}$ converges to $x_\infty - T(x_\infty)$. So $\{x_n\}$ converges to x_∞. Hence $\{x_n\}$ is a Cauchy sequence, a contradiction to the choice of $\{x_n\}$. Q.E.D.

Note here that the unit ball of l_1 gives us an example of a bounded closed convex subset K of a Banach space B such that K has no normal structure and B has the property A and is not strictly convex (and therefore is not locally uniformly convex). It should be clear that through Theorem 1, one can apply the above result to obtain significant extensions and generalizations of the results of R. Kannan and P. Soardi in [5] and [8].

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