Linear Algebra and its Applications 315 (2000) 145–154
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

Numerical ranges and matrix completions

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Received 28 August 1998; accepted 31 March 2000

Submitted by R.A. Brualdi

Abstract

There are two natural ways of defining the numerical range of a partial matrix. We show that for each partial matrix supported on a given pattern they give the same convex subset of the complex plane if and only if a graph associated with the pattern is chordal. This extends a previously known result (C.R. Johnson, M.E. Lundquist, Operator Theory: Adv. Appl. 50 (1991) 283–291) to patterns that are not necessarily reflexive and symmetric, and our proof overcomes an apparent gap in the proof given in the above-mentioned reference. We also define a stronger completion property that we show is equivalent to the pattern being an equivalence. © 2000 Elsevier Science Inc. All rights reserved.

Keywords: Matrix completion; Partial matrix; Numerical range; Inner and outer numerical ranges; Chordal graph

1. Introduction

A *pattern* $\mathscr{P}$ is a relation on $\{1, 2, \ldots, n\}$ for some positive integer n, i.e. a subset of $n \times n = \{(j, k) : 1 \leqslant j, k \leqslant n\}$, and a *partial matrix* on $\mathscr{P}$ (or a $\mathscr{P}$-matrix) is a complex-valued function defined on $\mathscr{P}$. Informally, a partial matrix is a square matrix whose entries may not all be specified.

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[1] The first author wishes to thank Murdoch University and the National Science Foundation for their support while this research was undertaken.

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If T_1 and T_2 are partial matrices on patterns $\mathscr{P}_1$ and $\mathscr{P}_2$, respectively, we write $T_1 \prec T_2$ if $\mathscr{P}_1 \subseteq \mathscr{P}_2$ and T_1 is the restriction of T_2 to $\mathscr{P}_1$. If $\mathscr{P}$ is a clique, i.e. a non-empty subset of $n \times n$ consisting of all pairs of the form (i, j), where i and j belong to some subset of $\{1, 2, \ldots, n\}$, then any $\mathscr{P}$-matrix can be regarded as a square matrix and an operator on $\mathbb{C}^m$ for some positive integer $m \leqslant n$. We say that T_2 is a *completion* of T_1 if T_2 is a square matrix and $T_1 \prec T_2$. Similarly, we say that T_1 is a *submatrix* of T_2 if T_1 is a square matrix and $T_1 \prec T_2$.

Matrix completion problems are typically concerned with finding completions of partial matrices T which preserve certain properties of T. In many cases their solutions can be easily described in terms of properties of an underlying graph $\mathscr{G}(\mathscr{P})$. This is the directed graph whose vertex set is $\{1, 2, \ldots, n\}$ and whose edge set is $\mathscr{P}$, i.e. there is an edge from vertex j to k if and only if $(j, k) \in \mathscr{P}$. Chordality often has an important role. We say that $\mathscr{P}$ is *chordal* if every cycle in $\mathscr{G}(\mathscr{P})$ of length 4 or more has a chord. Grone et al. [1] showed that if $\mathscr{P}$ is reflexive and symmetric, then every 'partially positive' $\mathscr{P}$-matrix has a positive completion if and only if $\mathscr{P}$ is chordal.

A related result concerning numerical ranges was proved by Johnson and Lundquist [4]. The *numerical range* of a bounded linear operator T on a complex Hilbert space $\mathscr{H}$ is the set $W(T) = \{(Te, e) : e \in \mathscr{H},\ \|e\| = 1\}$. The Toeplitz–Hausdorff theorem states that $W(T)$ is always convex. A useful computer algorithm for plotting the numerical range of matrices is given in [5].

There are two ways of defining the numerical range of a partial matrix T. The *inner numerical range* $W_1(T)$ is the convex hull of the numerical ranges of all submatrices of T, and the *outer numerical range* $W_2(T)$ is the intersection of the numerical ranges of all completions of T. If A and B are square matrices and $A \prec T \prec B$, then $W(A) \subseteq W(B)$, and it follows that $W_1(T) \subseteq W_2(T)$ and that $W_1(T) = W(T) = W_2(T)$ if T is also a square matrix. We say that a pattern $\mathscr{P}$ has the *numerical range completion property* if $W_1(T) = W_2(T)$ for every $\mathscr{P}$-matrix T. The main result in [4] states that a reflexive symmetric pattern has the *numerical range completion property* if and only if it is chordal.

We have discovered what appears to be a gap in the proof of the 'only if' part of this result. Their proof correctly shows that if $\mathscr{P}$ is not chordal and T is a partially positive $\mathscr{P}$-matrix with no positive completion, then there is a negative number λ that is contained in the intersection of the numerical ranges of all Hermitian completions of T. The problem is their assertion at this point of the proof that it follows that λ is in the numerical range of every completion of T. Since $\operatorname{Re} W(T) = W(\operatorname{Re} T)$, it follows that the numerical range of every completion of T contains a number whose real part is λ. However, it is conceivable that this may not imply their claim. We have not been able to prove or disprove their claim, but we have been able to construct a family of examples where their claim holds, and use this family to complete their proof. We also examine inner and outer numerical ranges of partial matrices defined on patterns which are not necessarily reflexive or symmetric.

In Section 4, we define a stronger numerical range completion property and prove it coincides with the pattern $\mathscr{P}$ being an equivalence.

2. Subpatterns

The *reflexive part*, $\mathscr{P}_{\mathrm{R}}$, of a pattern $\mathscr{P}$ is the largest reflexive subpattern of $\mathscr{P}$. Thus $\mathscr{P}_{\mathrm{R}}$ is given by $\mathscr{P}_{\mathrm{R}} = \mathscr{P} \cap \mathrm{Cl}(\mathscr{P})$, where $\mathrm{Cl}(\mathscr{P})$ is the clique $\{(j,k) : (j,j) \in \mathscr{P}$ and $(k,k) \in \mathscr{P}\}$. Similarly, the *reflexive symmetric part*, $\mathscr{P}_{\mathrm{RS}}$, of $\mathscr{P}$ is the largest reflexive and symmetric subpattern of $\mathscr{P}$, and is given by $\mathscr{P}_{\mathrm{RS}} = \mathscr{P}_{\mathrm{R}} \cap (\mathscr{P}_{\mathrm{R}})^*$, where for any pattern $\mathscr{Q}$, $\mathscr{Q}^* = \{(k,j) : (j,k) \in \mathscr{Q}\}$. Clearly $\mathscr{P}_{\mathrm{RS}} \subseteq \mathscr{P}_{\mathrm{R}} \subseteq \mathscr{P}$. We shall show that a pattern has the numerical range completion property if and only if its reflexive symmetric part has the same property. This claim is made in [4], but no proof is given there.

For any $\mathscr{P}$-matrix T, we let T_{R} and T_{RS} denote the restrictions of T to $\mathscr{P}_{\mathrm{R}}$ and $\mathscr{P}_{\mathrm{RS}}$, respectively. Clearly, if $T' \prec T$, then any completion of T is also a completion of T' and any submatrix of T' is also a submatrix of T. So it follows that $W_1(T') \subseteq W_1(T)$ and $W_2(T') \subseteq W_2(T)$. Furthermore, any submatrix of T is also a submatrix of T_{RS}. So we have

$$W_1(T_{\mathrm{RS}}) = W_1(T_{\mathrm{R}}) = W_1(T) \tag{1}$$

and

$$W_2(T_{\mathrm{RS}}) \subseteq W_2(T_{\mathrm{R}}) \subseteq W_2(T). \tag{2}$$

We shall show that equality holds in (2), and for this we need some preliminary results.

Lemma 1. *Suppose that $\mathscr{H}$ is a finite-dimensional Hilbert space, $A, E \in B(\mathscr{H})$ and that E is an orthogonal projection. Then*

$$\bigcap_{\lambda\in\mathbb{C}} W(\lambda E^{\perp} + A) = W(EA|_{E\mathscr{H}}), \quad \text{where } E^{\perp} = I - E.$$

Proof. The inclusion $\supseteq$ follows easily from the fact that $\lambda E^{\perp}|_{E\mathscr{H}} = 0$. So we need to show that any open half-plane containing $W(EA|_{E\mathscr{H}})$ also contains $\bigcap_{\lambda\in\mathbb{C}} W(\lambda E^{\perp} + A)$. By replacing A by $\mathrm{e}^{\mathrm{i}\theta}A + \mu I$ for suitably chosen positive real numbers θ and μ, we may assume that $W(EA|_{E\mathscr{H}})$ is contained in $\{z : \mathrm{Re}\, z > 0\}$. Since $W(EA|_{E\mathscr{H}})$ is compact, $W(EA|_{E\mathscr{H}}) \subset \{z : \mathrm{Re}\, z > r\}$ for some $r > 0$. Let $\lambda = r^{-1}\left\|A'\right\|^2 + \left\|A'\right\|$, where $A' = \mathrm{Re}\, A$. Then for any $e \in \mathscr{H}$,

$$\begin{aligned}
\mathrm{Re}((\lambda E^{\perp} + A)e, e) &= ((\lambda E^{\perp} + A')e, e) \\
&= \lambda \left\|E^{\perp}e\right\|^2 + (A'E^{\perp}e, E^{\perp}e) + (A'E^{\perp}e, Ee) + (A'Ee, E^{\perp}e) + (A'Ee, Ee) \\
&\geqslant \left(\lambda - \|A'\|\right)\|E^{\perp}e\|^2 - 2\|A'\|\|E^{\perp}e\|\|Ee\| + r\|Ee\|^2 \\
&= r^{-1}\left(\|A'\|\|E^{\perp}e\| - r\|Ee\|\right)^2 \geqslant 0,
\end{aligned}$$

and so $W(\lambda E^{\perp} + A)$ is contained in the right half plane, as required. □

Remark 1. Lemma 1 is a generalization of the equality $\bigcap_{\lambda\in\mathbb{C}} W(\lambda I + A) = \emptyset$, which holds for any operator A on a Hilbert space.

Remark 2. A simple modification of the proof of Lemma 1 shows that if $\mathscr{H}$ is infinite-dimensional, then

$$W(EA|_{E\mathscr{H}}) \subseteq \bigcap_{\lambda\in\mathbb{C}} W(\lambda E^{\perp} + A) \subseteq \overline{W}(EA|_{E\mathscr{H}})$$

for any operator A and any orthogonal projection E in $B(\mathscr{H})$.

To see that the closure is necessary, let $\mathscr{H} = \mathscr{K} \oplus \mathscr{K}$, where $\mathscr{K}$ is an infinite-dimensional Hilbert space, and choose $D \in B(\mathscr{K})$ such that $W(D) = (0, 1]$. Let

$$A = \begin{pmatrix} 0 & I \\ I & D \end{pmatrix} \quad \text{and} \quad E = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}.$$

Then $D = EA|_{E\mathscr{H}}$, but $\bigcap_{\lambda\in\mathbb{C}} W(\lambda E^{\perp} + A) = [0, 1]$.

Lemma 2. *Suppose that $\mathscr{H}$ is a Hilbert space, $A \in B(\mathscr{H})$ and that $\mathscr{M}$ is a linear subspace of $B(\mathscr{H})$. Then*

$$\bigcap_{X\in\mathscr{M}} W(A + X) = \bigcap_{X,Y\in\mathscr{M}} W(A + X + Y^*).$$

Proof. The inclusion $\supseteq$ is obvious. Now suppose that $W(A + X + Y^*) \subset \{z : \mathrm{Re}\, \mathrm{e}^{\mathrm{i}\theta} z \geqslant r\}$, where $X, Y \in \mathscr{M}$ and θ and r are real numbers. Then

$$\mathrm{Re}\left(\mathrm{e}^{\mathrm{i}\theta}(A + X + Y^*)\right) = \mathrm{Re}\left(\mathrm{e}^{\mathrm{i}\theta}(A + X + \mathrm{e}^{-2\mathrm{i}\theta} Y)\right).$$

So $W(A + Z) \subset \{z : \mathrm{Re}\, \mathrm{e}^{\mathrm{i}\theta} z \geqslant r\}$, where $Z = X + \mathrm{e}^{-2\mathrm{i}\theta} Y \in \mathscr{M}$, and the reverse inclusion follows. $\square$

We say that a matrix $A = (a_{ij})$ is *orthogonal* to a pattern $\mathscr{P}$ and we write $A \perp \mathscr{P}$ if $a_{ij} = 0$ for all $(i, j) \in \mathscr{P}$. If A_1 is any completion of a $\mathscr{P}$-matrix T, then A_2 is also a completion of T if and only if $(A_1 - A_2) \perp \mathscr{P}$.

Lemma 3. *For any partial matrix T we have*

$$W_2(T_{\mathrm{RS}}) = W_2(T_{\mathrm{R}}) = W_2(T).$$

Proof. Suppose that T is a $\mathscr{P}$-matrix and that A_0 is a completion of T_{R}. Then $A_1 = A_0 + B$ is a completion of T for some $B \perp \mathrm{Cl}(\mathscr{P})$. Let E denote the diagonal matrix $\mathrm{diag}(e_j)_{j=1}^{n}$, where

$$e_j = \begin{cases} 1 & \text{if } (j, j) \in \mathscr{P}, \\ 0 & \text{if } (j, j) \notin \mathscr{P}. \end{cases}$$

Then $E^{\perp} \perp \mathscr{P}$, and $T \prec \lambda E^{\perp} + A_1$ for any $\lambda \in \mathbb{C}$. So by Lemma 1,

$$W_2(T) \subseteq \bigcap_{\lambda \in \mathbb{C}} W(\lambda E^{\perp} + A_1) = W(EA_1|_{E\mathbb{C}^n}).$$

But $EA_1|_{E\mathbb{C}^n} = EA_0|_{E\mathbb{C}^n}$, and $W(EA_0|_{E\mathbb{C}^n}) \subseteq W(A_0)$. So

$$W_2(T) \subseteq \cap\{W(A_0)\colon A_0 \text{ is a completion of } T_{\mathrm{R}}\} = W_2(T_{\mathrm{R}}).$$

The reverse inclusion is in (2), and so $W_2(T) = W_2(T_{\mathrm{R}})$.

For the second equality, observe that B is a completion of T_{R} if and only if $B = A_0 + X$ for some $X \perp \mathscr{P}_{\mathrm{R}}$, and that C is a completion of T_{RS} if and only if $C = A_0 + X + Y^*$ for some $X, Y \perp \mathscr{P}_{\mathrm{R}}$. So by Lemma 2

$$W_2(T_{\mathrm{R}}) = \bigcap_{X \perp \mathscr{P}_{\mathrm{R}}} W(A_0 + X) = \bigcap_{X,Y \perp \mathscr{P}_{\mathrm{R}}} W(A_0 + X + Y^*) = W_2(T_{\mathrm{RS}}). \qquad \square$$

Theorem 4. *Suppose that $\mathscr{P}$ is a pattern which has the numerical range completion property. Then so does any pattern $\mathscr{Q}$ for which $\mathscr{P}_{\mathrm{RS}} = \mathscr{Q}_{\mathrm{RS}}$.*

Proof. This follows easily from (1), (2) and Lemma 3. $\square$

3. Chordality and numerical range

In this section, we prove that a pattern has the numerical range completion property if and only if its reflexive symmetric part is chordal. This extends the result of Johnson and Lundquist to patterns which are not reflexive and symmetric, and our proof overcomes the apparent gap in the proof given in [4]. We need to establish some preliminary results. The first shows that the numerical range completion property is hereditary.

Lemma 5. *Any induced subpattern of a pattern with the numerical range completion property also has the numerical range completion property.*

Proof. Suppose that $\mathscr{P}$ has the numerical range completion property and that $\mathscr{C}$ is the clique $\{(i, j) : i, j \in S\}$ for some subset S of $\{1, 2, \ldots, n\}$. Let T' be a $\mathscr{P}'$-matrix, where $\mathscr{P}'$ is the induced subpattern $\mathscr{P} \cap \mathscr{C}$. We need to show that $W_1(T') = W_2(T')$.

If $W_1(T') = \emptyset$, then $\mathscr{P}_{\mathrm{R}} = \emptyset$. Let A be a completion of T'. Then $\lambda I + A$ is also a completion of T' for any scalar λ, and so $W_2(T') \subseteq \bigcap_{\lambda \in \mathbb{C}} W(\lambda I + A) = \emptyset$.

If $W_1(T') \neq \emptyset$, then choose $\lambda \in W_1(T')$ and let T denote the $\mathscr{P}$-matrix $T = (t_{jk})$ which is an extension of T' and satisfies

$$t_{jk} = \begin{cases} \lambda & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases}$$

for each $(j, k) \in \mathscr{P} \setminus \mathscr{P}'$. Then $W_1(T) = W_1(T')$. Furthermore, since $T' \prec T$, and $W_2(T') \subseteq W_2(T)$, $W_2(T') \subseteq W_2(T) = W_1(T) = W_1(T')$. So $W_1(T') = W_2(T')$, and hence $\mathscr{P}'$ has the numerical range completion property. $\square$

For each $m \geqslant 4$ let $\mathscr{G}_m = \{(i, j) : |i - j| \leqslant 1\} \cup \{(1, m), (m, 1)\}$ (The $m \times m$ pattern $\mathscr{G}_m$ consists of the main 'tri-diagonal' and the 'corners'.). The next lemma is the key technical result we need to establish the main theorem.

Lemma 6. *Suppose that* V *is the m-square matrix* $(\cos((j-k)\theta))_{j,k=1}^{m}$, *where* $m \geqslant 4$ *and* $\theta = \pi/(2(m-1))$. *Then*
1. V *is positive definite*,
2. $\operatorname{rank} V = 2$, *and*
3. *for each* $D \perp \mathscr{G}_m$ *there is a non-zero vector* $e \in \mathbb{C}^m$ *such that* $Ve = 0$ *and* $(De, e) = 0$.

Proof. Properties 1 and 2 follow from the fact that $V = A^*A$, where

$$A = \begin{pmatrix} 1 & \cos\theta & \cos 2\theta & \dots & \cos(m-2)\theta & 0 \\ 0 & \sin\theta & \sin 2\theta & \dots & \sin(m-2)\theta & 1 \end{pmatrix}.$$

For property 3 let

$$c = (\cos\theta, \cos 2\theta, \dots, \cos(m-2)\theta)^*, \quad s = (\sin\theta, \sin 2\theta, \dots, \sin(m-2)\theta)^*,$$

and write

$$D = \begin{pmatrix} 0 & D_{12} & 0 \\ D_{21} & D_{22} & D_{23} \\ 0 & D_{32} & 0 \end{pmatrix},$$

where $D \perp \mathscr{G}_m$, $D_{21}, D_{23}, D_{12}^*, D_{32}^* \in \mathbb{C}^{m-2}$, and where D_{22} is an $(m-2)$-square matrix. Then $e \in \ker V$ if and only if $e = (-x^*c, x^*, -x^*s)^*$ for some $x \in \mathbb{C}^{m-2}$. For such an e, $(De, e) = -(Tx, x)$, where

$$T = cD_{12} + D_{21}c^* - D_{22} + sD_{32} + D_{23}s^*,$$

and so we need to show that $0 \in W(T)$.

Suppose that $0 \notin W(T)$. By multiplying by $e^{i\varphi}$ for a suitable real number φ, we may assume that $\operatorname{Re} T$ is strictly positive definite. Write $D_{12} + D_{21}^* = 2(\xi_1, \xi_2, \dots, \xi_{m-2})$, and $D_{32} + D_{23}^* = 2(\eta_1, \eta_2, \dots, \eta_{m-2})$, and let t_{jk} and δ_{jk} denote the (j, k) entries of $\operatorname{Re} T$ and $\operatorname{Re} D_{22}$, respectively. Then

$$t_{jk} = \xi_k \cos j\theta + \overline{\xi_k} \cos k\theta + \eta_k \sin j\theta + \overline{\eta_j} \sin k\theta - \delta_{jk}.$$

Let T_j denote the 2×2 principal submatrix

$$\begin{pmatrix} t_{j,j} & t_{j,j+1} \\ t_{j+1,j} & t_{j+1,j+1} \end{pmatrix}$$

of $\operatorname{Re} T$ for $1 \leqslant j \leqslant m-3$. Since $\delta_{jk} = 0$ if $|j-k| \leqslant 1$, we have

$$T_j = \begin{pmatrix} 2u_j & y_j + iz_j \\ y_j - iz_j & 2u_{j+1} \end{pmatrix},$$

where

$$u_j = \mathrm{Re}(\xi_j \cos j\theta + \eta_j \sin j\theta),$$
$$y_j = \mathrm{Re}(\xi_{j+1} \cos j\theta + \xi_j \cos(j+1)\theta + \eta_{j+1} \sin j\theta + \eta_j \sin(j+1)\theta),$$
$$z_j = \mathrm{Im}(\xi_{j+1} \cos j\theta - \xi_j \cos(j+1)\theta + \eta_{j+1} \sin j\theta - \eta_j \sin(j+1)\theta).$$

Since Re T is strictly positive definite, so too is each T_j. So $u_j > 0$ for $1 \leqslant j \leqslant m-2$ and $\det T_j = 4u_j u_{j+1} - y_j^2 - z_j^2 > 0$ for $1 \leqslant j \leqslant m-3$. Therefore $4u_j u_{j+1} > y_j^2$, and so $x_j x_{j+1} > y_j$ where $x_j^2 = 2u_j$. It follows that

$$\sum_{j=1}^{m-3} x_j x_{j+1} > \sum_{j=1}^{m-3} y_j.$$

Now

$$y_j = (u_j + u_{j+1}) \cos\theta - (v_j - v_{j+1}) \sin\theta,$$

where

$$v_j = \mathrm{Re}(\xi_j \sin j\theta - \eta_j \cos j\theta).$$

So

$$\sum_{j=1}^{m-3} y_j = \sum_{j=1}^{m-3} \left(u_j + u_{j+1}\right) \cos\theta + (v_{m-2} - v_1) \sin\theta.$$

Since $D \perp \mathscr{G}_m$ it follows that $\xi_1 = \eta_{m-2} = 0$. So

$$v_1 \sin\theta = -\mathrm{Re}\ \eta_1 \cos\theta \sin\theta = -u_1 \cos\theta,$$
$$v_{m-2} \sin\theta = \mathrm{Re}\ \xi_{m-2} \sin(m-2)\theta \sin\theta = u_{m-2} \cos\theta.$$

Therefore

$$\sum_{j=1}^{m-3} x_j x_{j+1} > \sum_{j=1}^{m-3} y_j = 2\sum_{j=1}^{m-2} u_j \cos\theta = \sum_{j=1}^{m-2} x_j^2 \cos\theta. \tag{3}$$

On the other hand, the tridiagonal matrix $G \in \mathscr{M}_{m-2}$ given by

$$G = \begin{pmatrix} 2\cos\theta & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2\cos\theta & -1 & \ddots & \ddots & 0 & 0 \\ 0 & -1 & 2\cos\theta & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 2\cos\theta & -1 & 0 \\ 0 & 0 & \ddots & \ddots & -1 & 2\cos\theta & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2\cos\theta \end{pmatrix},$$

is positive definite. (In fact $\det G = \sin(m-1)\theta/\sin\theta = \cot\theta > 0$, and the determinant of any principal submatrix of G is the product of positive factors of the form $\sin k\theta/\sin\theta$, where $2 \leqslant k \leqslant m-1$.) So

$$(Gx, x) = 2\left(\sum_{j=1}^{m-2} x_j^2 \cos\theta - \sum_{j=1}^{m-3} x_j x_{j+1}\right) \geqslant 0$$

for all $x = (x_1, x_2, \ldots, x_{m-2})^* \in \mathbb{R}^{m-2}$, and since this inequality contradicts (3), we conclude that no such T exists. So $0 \in W(T)$ and property 3 holds. □

Corollary 7. *No pattern of the form $\mathscr{G}_m$ for $m \geqslant 4$ has the numerical range completion property.*

Proof. Suppose that $m \geqslant 4$ and that T is the $\mathscr{G}_m$-matrix whose $(1, m)$ and $(m, 1)$ entries are both 0 and whose other entries are all 1. Then $W_1(T) \subset [0, \infty)$. Now let V and θ be as in Lemma 6, and let $B = (\sec\theta)V + (1 - \sec\theta)I$. Then B is a completion of T and, by Lemma 6, $\lambda = 1 - \sec\theta < 0$ is an eigenvalue of multiplicity $m-2$. Moreover, if A is any other completion of T and $D = B - A$, then by Lemma 6 there is a unit vector $e \in \operatorname{Ker}(B - \lambda I)$ such that $(De, e) = 0$. So $\lambda \in W_2(T)$, and therefore $W_1(T) \neq W_2(T)$. □

Theorem 8. *A pattern has the numerical range completion property if and only if its reflexive symmetric part is chordal.*

Proof. In view of Theorem 4 we can assume that $\mathscr{P}$ is reflexive and symmetric. The 'if' part is proved in [4]. So suppose that $\mathscr{P}$ is not chordal. Then $\mathscr{G}(P)$ contains a chordless cycle with at least four vertices, and there is an induced subpattern of $\mathscr{P}$ of the form $\mathscr{G}_m$ (after a possible reordering of the vertices $1, 2, \ldots, n$). It follows from Corollary 7 and Lemma 5 that neither $\mathscr{G}_m$ nor $\mathscr{P}$ has the matrix completion property. □

4. Strong numerical range completion property

We conclude with a related result. We say that a pattern $\mathscr{P}$ has the *strong numerical range completion property* if every $\mathscr{P}$-matrix T has a completion A for which $W_1(T) = W(A)$. This property is indeed stronger than the numerical range completion property because $W_1(T) \subseteq W_2(T) \subseteq W(A)$ for any completion A of a partial matrix T. We show that $\mathscr{P}$ has this property if and only if it is an equivalence relation, that is, up to permutation similarity, $\mathscr{P}$ is a block-diagonal pattern.

The following proof is a great simplification of our original proof and was kindly provided by the referee.

Theorem 9. *A pattern has the strong numerical range completion property if and only if it is an equivalence.*

Proof. The only if part is obvious. Conversely, suppose the pattern $\mathscr{P}$ has the strong numerical range completion property. Suppose that $\mathscr{P}$ is a pattern and that $\mathscr{P}_{\mathrm{RS}}$ is its reflexive and symmetric part. We define a partial matrix $T = (t_{jk})$ on $\mathscr{P}$ as follows:

$$t_{jk} = \begin{cases} 0 & \text{if } (j,k) \in \mathscr{P}_{\mathrm{RS}}, \\ 1 & \text{if } (j,k) \in \mathscr{P} \setminus \mathscr{P}_{\mathrm{RS}}. \end{cases}$$

Since $W_1(T) = W_1(T_{\mathrm{RS}})$, it follows that

$$W_1(T) = \begin{cases} \emptyset & \text{if } \mathscr{P}_{\mathrm{RS}} = \emptyset, \\ \{0\} & \text{if } \mathscr{P}_{\mathrm{RS}} \neq \emptyset. \end{cases}$$

Suppose that A is a completion of T. If $\mathscr{P} \neq \mathscr{P}_{\mathrm{RS}}$, then $A \neq 0$. But $W(A) = 0$ if and only if $A = 0$ [2], and hence $W(A) \neq W_1(T)$. Since this is a contradiction we conclude that $\mathscr{P} = \mathscr{P}_{\mathrm{RS}}$. This proves reflexivity and symmetry, and it remains to be shown that $\mathscr{P}$ is transitive.

If $\mathscr{P}$ is not transitive, it has an induced 3×3 subpattern $\mathscr{P}_1$ of the form

$$\begin{pmatrix} * & * & \\ * & * & * \\ & * & * \end{pmatrix}.$$

Let T be the $\mathscr{P}_1$-matrix

$$\begin{pmatrix} 0 & 1 & \\ 1 & 0 & \mathrm{i} \\ & \mathrm{i} & 0 \end{pmatrix}$$

and let A be the $\mathscr{P}$-matrix whose restriction to $\mathscr{P}_1$ is T and whose other entries are all 0. One readily checks that $W_1(A)$ is the convex hull of $\{1, -1, \mathrm{i}, -1\}$. Suppose that A' is a completion of A so that $W(A') = W_1(A)$, and that T' is the corresponding 3×3 completion of T. Then

$$W_1(A) \subseteq W(T') \subseteq W(A') = W_1(A),$$

and so $W(T') = W_1(A)$ and has four non-differentiable boundary points, namely $1, -1, \mathrm{i}, -1$. Each of them must be an eigenvalue of T', by Horn and Johnson [3, 1.6.3], which is impossible. Hence $\mathscr{P}$ must be transitive, and the proof is complete. □

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