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Monogamous latin squares

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ABSTRACT

We show for all $n \notin \{1,2,4\}$ that there exists a latin square of order n that contains two entries γ_1 and γ_2 such that there are some transversals through γ_1 but they all include γ_2 as well. We use this result to show that if $n > 6$ and n is not of the form $2p$ for a prime $p \geqslant 11$ then there exists a latin square of order n that possesses an orthogonal mate but is not in any triple of MOLS. Such examples provide pairs of 2-maxMOLS.

1. Introduction

A *latin square of order* n is an $n \times n$ array in which each one of n symbols appears exactly once in each row and exactly once in each column. Two latin squares $A = [a_{ij}]$ and $B = [b_{ij}]$ are *orthogonal* if $(a_{ij}, b_{ij}) \neq (a_{i'j'}, b_{i'j'})$ whenever $i \neq i'$ or $j \neq j'$. A set of MOLS (*mutually orthogonal latin squares*) is a set of latin squares in which each pair is orthogonal. A *transversal* of a latin square of order n is a set of n entries containing no pair of entries that share a row, column or symbol. If two latin squares are orthogonal then the set of cells occupied by a fixed entry in one defines a transversal in the other. Further background and terminology of latin squares can be found in [2].

A set of k-maxMOLS(n) is a set of k MOLS of order n that is maximal in the sense that it is not contained in any set of $k+1$ MOLS. A *bachelor latin square* is a latin square which has no orthogonal mate; or equivalently, is a latin square with no decomposition into disjoint transversals. We define a *monogamous latin square* to be a latin square that has an orthogonal mate, but is in no triple of MOLS. Thus, a monogamous latin square and its orthogonal mate are a pair of 2-maxMOLS.

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The main purpose of this paper is to study the existence of monogamous latin squares and thereby prove the existence of a pair of 2-maxMOLS for many new orders. In Section 5 we prove the following result.

Theorem 1.1. *For each order $n > 6$ there exists a pair of 2-maxMOLS(n) except possibly when $n = 2p$ for some prime $p \geqslant 11$.*

It is well known (see e.g. [4]) that there does not exist a pair of 2-maxMOLS of order $n \in \{1, 2, 4, 5, 6\}$. We conjecture that these are the only orders for which a pair of 2-maxMOLS does not exist. The only orders less than 100 for which existence now remains in doubt are $n \in \{22, 26, 34, 38, 58, 62, 74, 86, 94\}$.

The study of “bachelor squares” dates back to Euler. However, the name [12] and the four distinct proofs [6,9,10,13] of their existence for all orders $\geqslant 4$, are comparatively recent.

Drake [4] showed that a pair of 2-maxMOLS exists for all orders $\neq 6$ that are 3 or 6 mod 9. Drake, van Rees and Wallis [5] showed that a pair of 2-maxMOLS exists for all orders that are one less than a power of two, as well as all orders that are 1 mod 9, 7 mod 9 or 11 mod 18.

2. Notation

Suppose that L is a latin square of order n. We will index the rows, columns and symbols of L with an abelian group G of order n, called the *index group* of L. We shall take $G = \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_t}$ where $p_1, p_2, \ldots, p_t$ are positive integers (typically, but not necessarily, primes). To save clutter, an element $(a_1, a_2, \ldots, a_t)$ of G will sometimes be written simply as $a_1 a_2 \cdots a_t$. We define an order $\prec$ on the elements of G by saying that $a_1 a_2 \cdots a_t \prec b_1 b_2 \cdots b_t$ iff $a_j < b_j$ where $j = \max\{i\colon a_i \neq b_i\}$ and a_j, b_j are taken to be the least non-negative integers representing their congruence class modulo p_j. Whenever we write a latin square the rows and columns will be listed in increasing order of their indices under $\prec$. When specifying an orthogonal mate M for a latin square L we will always assume the rows and columns of M are indexed by the index group of L. However, in the interests of readability we will use letters for the symbols in M rather than elements of the index group.

We let $L[x, y]$ denote the symbol in row x and column y of a latin square L. It will prove convenient to think of a latin square of order n as a set of n^2 triples, or *entries*, of the form $(x, y, L[x, y])$. Let $\vec{x} = (x_1, x_2, \ldots, x_t)$, $\vec{y} = (y_1, y_2, \ldots, y_t)$ and $\vec{z} = (z_1, z_2, \ldots, z_t)$ denote elements of G. For $i = 1, 2 \ldots, t$, we define a function $\Delta_i : L \mapsto \mathbb{Z}_{p_i}$ which on an entry $(\vec{x}, \vec{y}, \vec{z})$ of L evaluates to

$$\Delta_i(\vec{x}, \vec{y}, \vec{z}) = x_i + y_i - z_i \bmod p_i.$$

We define $\delta_i = \delta_i(G)$ by

$$\delta_i = \begin{cases} \frac{1}{2} p_i & \text{if } p_i \text{ is even and } n/p_i \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

The following simple lemma is crucial to our work, just as related results have been in [1,6–10,13].

Lemma 2.1. *The sum of the Δ_i values over the elements of a transversal T of a latin square indexed by G is*

$$\sum_{e \in T} \Delta_i(e) = \delta_i.$$

Proof. A transversal, by definition, comprises one entry from each row, one entry from each column, and one entry containing each symbol. Hence in $\mathbb{Z}_{p_i}$,

$$\sum_{e \in T} \Delta_i(e) = \frac{n}{p_i} \sum_{g \in \mathbb{Z}_{p_i}} g + \frac{n}{p_i} \sum_{g \in \mathbb{Z}_{p_i}} g - \frac{n}{p_i} \sum_{g \in \mathbb{Z}_{p_i}} g = \frac{n}{p_i} \sum_{j=0}^{p_i - 1} j = \frac{n(p_i - 1)}{2} = \delta_i. \qquad \square$$

For any latin square L, let $\Delta_i^* = \{e \in L : \Delta_i(e) \neq 0\}$. Let $\mathcal{S}_i$ be the set of subsets T of Δ_i^* satisfying:

(D1) If $(x_1, y_1, z_1) \in T$ and $(x_2, y_2, z_2) \in T$ then $x_1 \neq x_2$, $y_1 \neq y_2$ and $z_1 \neq z_2$.
(D2) $\sum_{e \in T} \Delta_i(e) = \delta_i$.

Sets of entries which satisfy (D1) will be called *independent*. By Lemma 2.1 each transversal of L intersects Δ_i^* in an element of $\mathcal{S}_i$.

We say a latin square L is Δ_i*-crimped* if there exist entries $\gamma_1, \gamma_2 \in L$ such that:

(C1) There is at least one transversal of L that includes γ_1.
(C2) There is no $T \in \mathcal{S}_i$ for which $\gamma_1 \in T$ but $\gamma_2 \notin T$.

Condition (C2) ensures that every transversal through γ_1 also includes γ_2. Condition (C1) ensures that (C2) is not vacuously true. Without condition (C1) the definition would be uninteresting. It is known [6,9,13] that for every order $n \geqslant 4$ there exists a latin square containing an entry γ_1 for which there is no $T \in \mathcal{S}_i$ with $\gamma_1 \in T$.

We say a latin square is Δ*-crimped* if there is some i for which it is Δ_i-crimped.

Example 2.1. The latin square L_{10} below, indexed by $\mathbb{Z}_5 \times \mathbb{Z}_2$, is Δ_1-crimped, with $\gamma_1 = (00, 00, 20)$ and $\gamma_2 = (20, 40, 40)$ shaded.

$$L_{10} = \begin{pmatrix}
\colorbox{lightgray}{$\mathbf{20}$} & \mathbf{00} & \mathbf{40} & 30 & \mathbf{10} & 01 & 11 & 21 & 31 & 41 \\
10 & 20 & 30 & 40 & 00 & 11 & 21 & 31 & 41 & 01 \\
\mathbf{00} & 30 & \mathbf{20} & \mathbf{10} & \colorbox{lightgray}{$\mathbf{40}$} & 21 & 31 & 41 & 01 & 11 \\
30 & 40 & \mathbf{10} & \mathbf{00} & \mathbf{21} & 31 & 41 & 01 & 11 & \mathbf{20} \\
40 & \mathbf{10} & \mathbf{00} & \mathbf{21} & \mathbf{31} & 41 & 01 & 11 & \mathbf{20} & \mathbf{30} \\
01 & 11 & 21 & 31 & 41 & 00 & 10 & 20 & 30 & 40 \\
11 & 21 & 31 & 41 & 01 & 10 & 20 & 30 & 40 & 00 \\
21 & 31 & 41 & 01 & 11 & 20 & 30 & 40 & 00 & 10 \\
31 & 41 & 01 & 11 & \mathbf{20} & 30 & 40 & 00 & 10 & \mathbf{21} \\
41 & 01 & 11 & \mathbf{20} & \mathbf{30} & 40 & 00 & 10 & \mathbf{21} & \mathbf{31}
\end{pmatrix},$$

$$L'_{10} = \begin{pmatrix}
a & b & c & d & e & f & g & h & i & j \\
g & d & a & h & i & e & b & j & f & c \\
c & h & e & f & a & i & d & g & j & b \\
f & g & b & j & d & a & h & e & c & i \\
i & j & f & c & h & b & a & d & g & e \\
h & a & j & e & c & g & i & f & b & d \\
j & f & g & a & b & d & c & i & e & h \\
e & c & i & g & f & h & j & b & d & a \\
b & e & d & i & j & c & f & a & h & g \\
d & i & h & b & g & j & e & c & a & f
\end{pmatrix}.$$

The square L'_{10} is orthogonal to L_{10}.

In L_{10} we have marked in **bold** all entries that are in Δ_i^* for some i. This practice will be adopted in all subsequent examples.

Theorem 2.2. *Every Δ-crimped latin square is either monogamous or a bachelor.*

Proof. Let L be a latin square that is Δ_i-crimped for some i, and let $\gamma_1 = (e_1, e_2, e_3)$ and $\gamma_2 = (f_1, f_2, f_3)$ be the two entries satisfying (C1) and (C2) in the definition of Δ_i-crimped. Suppose that L is orthogonal to latin squares A and B. Since A is orthogonal to L, we can locate a transversal of L by taking the cells in L that correspond to occurrences of the symbol $A[e_1, e_2]$ in A. This transversal includes γ_1 and hence also γ_2, by (C2). But now our method for choosing the transversal shows that $A[e_1, e_2] = A[f_1, f_2]$. Similarly $B[e_1, e_2] = B[f_1, f_2]$, which by definition then means that A and B are not orthogonal. Thus L cannot be in any triple of MOLS. □

Corollary 2.3. *If there exists a monogamous Δ-crimped latin square of order n then there exists a pair of* 2-maxMOLS(n).

For instance, in Example 2.1 we gave an orthogonal mate L'_{10} for a Δ-crimped latin square L_{10}. We now see that L_{10} must be monogamous and that $\{L_{10}, L'_{10}\}$ is a pair of 2-maxMOLS(10).

In this context we state one of the major results of this paper, which will be proved in Section 5.

Theorem 2.4. *There exists a Δ-crimped latin square of order n for every positive integer $n \notin \{1, 2, 4\}$.*

3. Monogamous crimped squares of odd order

Having demonstrated, in Corollary 2.3, the potential usefulness of monogamous Δ-crimped latin squares, we now pursue the question of their existence. In this section, we answer that question for all odd orders. We begin by defining a latin square $\mathcal{C}_n$ of odd order $n > 1$, indexed by $\mathbb{Z}_n$. We separate into three cases.

Case 0: $n \equiv 0 \pmod 3$. Let $n = 3k$ and define

$$\mathcal{C}_n[i, j] = \begin{cases} i + j + k & i = 0 \text{ and } j \in \{0, k, 2k\}, \\ i + j - k & i = k \text{ and } j \in \{0, k, 2k\}, \\ i + j & \text{otherwise.} \end{cases}$$

Case 1: $n \equiv 1 \pmod 3$. Let $n = 3k + 1$ and define

$$\mathcal{C}_n[i, j] = \begin{cases} i + j + k & i = 0 \text{ and } j \in \{0, k, 2k\}, \\ i + j - k & i = k \text{ and } j \in \{0, k, 2k\}, \\ i + j + 1 = 0 & i = n - j - 1 \text{ and } n - k \leqslant j < n, \\ i + j - 1 = -1 & i = n - j \text{ and } n - k \leqslant j < n, \\ i + j & \text{otherwise.} \end{cases}$$

Case 2: $n \equiv 2 \pmod 3$. Let $n = 3k - 1$ and define

$$\mathcal{C}_n[i, j] = \begin{cases} i + j + k & i = 0 \text{ and } j \in \{0, k, 2k\}, \\ i + j - k & i = k \text{ and } j \in \{0, k, 2k\}, \\ i + j + 1 = 1 & i = n - j \text{ and } 1 \leqslant j \leqslant n - k, \\ i + j - 1 = 0 & i = n - j + 1 \text{ and } 1 \leqslant j \leqslant n - k, \\ i + j & \text{otherwise.} \end{cases}$$

Lemma 3.1. *$\mathcal{C}_n$ is a Δ-crimped latin square for every odd $n > 1$.*

Proof. Let $k = \lfloor (n+1)/3 \rfloor$, $\gamma_1 = (0, 0, k)$ and $\gamma_2 = (k, 2k, 2k)$, so that $\Delta_1(\gamma_1) = -k$ and $\Delta_1(\gamma_2) = k$. It is easy to check that in $\mathcal{C}_3$ the only entry in Δ_1^* that is independent of γ_1 is γ_2, and hence the result holds for $n = 3$.

We now assume $n \geqslant 5$ and thus $2 \leqslant k \leqslant n - 2$. The entries in Δ_1^* all have Δ_1 values in $\{\pm 1, \pm k\}$. The only entry that is independent of γ_1 and has Δ_1 value in $\{\pm k\}$ is γ_2. All the entries which have Δ_1 value equal to 1 share the same symbol, and the same is true of those with Δ_1 value -1. Hence any independent set of entries can contain at most one of each. Thus, since the sum of Δ_i values in a transversal of $\mathcal{C}_n$ is 0 by Lemma 2.1, there is no $T \in \mathcal{S}_1$ for which $\gamma_1 \in T$ but $\gamma_2 \notin T$.

Thus we have shown that $\mathcal{C}_n$ satisfies condition (C2). It remains to construct a transversal through γ_1. Naturally, we include γ_1 and γ_2 in our transversal. Then in all rows $i \notin \{0, k\}$ we take the entry in cell $(i, 2k - 2i)$. Given that n is odd, it is routine to check that these choices produce a transversal. □

Table 1
Five special diagonals when $n = 3k + 1$.

0	$k-1$	k	$2k$	$3k$
$0, 0^*$	$0, k-1$	$0, k^*$	$0, 2k$	$0, 3k^*$
$1, k-1^*$	$j-1, k$	$1, 3k$	$1, 2k$	$j-1, 0$
j, k	$i, 0$	$j, 2k$	$j, k-1$	$i, 3k$
$i+1, 2k$	$\frac{1}{2}k-1, 0:3k$	$i+1, 3k$	$i+1, k-1$	$\frac{1}{2}k-1, k:0$
$\frac{1}{2}k, 0:2k$	$\frac{1}{2}k, k:0$	$\frac{1}{2}k, 2k:k-1$	$\frac{1}{2}k, 3k$	$\frac{1}{2}k, k-1:k$
$\frac{1}{2}k+1, 3k:0$	$k-i-1, k-1$	$\frac{1}{2}k+1, 2k:3k$	$\frac{1}{2}k+1, k$	$\frac{1}{2}k+1, 0:k-1$
$k-i, k-1$	$k-j, 2k$	$k-i, 3k$	$k-i, k$	$k-i, 0$
$k-j+1, 3k$	$k-1, k-1^*$	$k-j+1, 0$	$k-j+1, k$	$k-j+1, 2k$
$k, k-1$	$k, 0^*$	$k, 3k^*$	k, k^*	$k, 2k$
$k+1, 2k$	$k+1, 3k$	$k+1, k$	$k+1, 0$	$k+1, k-1$
$k+j, 0$	$k+j, k$	$k+j, 2k$	$k+j, k-1$	$k+j, 3k$
$k+i+1, k$	$k+i+1, 3k$	$k+i+1, 0$	$k+i+1, k-1$	$k+i+1, 2k$
$\frac{3}{2}k, 3k:0$	$\frac{3}{2}k, 0:k-1$	$\frac{3}{2}k, 2k:3k$	$\frac{3}{2}k, k$	$\frac{3}{2}k, k-1:2k$
$\frac{3}{2}k+1, 0:k$	$\frac{3}{2}k+1, 2k:0$	$\frac{3}{2}k+1, k:3k$	$\frac{3}{2}k+1, k-1$	$\frac{3}{2}k+1, 3k:2k$
$2k-i, 2k$	$2k-i, k-1$	$2k-i, 3k$	$2k-i, k$	$2k-i, 0$
$2k-j+1, k-1$	$2k-j+1, 0$	$2k-j+1, 2k$	$2k-j+1, k$	$2k-j+1, 3k$
$2k, 2k$	$2k, 3k$	$2k, k$	$2k, 0$	$2k, k-1$
$2k+1, 3k$	$2k+1, k$	$2k+1, k-1$	$2k+1, 0$	$2k+1, 2k$
$2k+2, k$	$2k+j, 2k$	$2k+j, 3k$	$2k+j, k-1$	$2k+j, 0$
$2k+j+1, 3k$	$2k+i+1, k$	$2k+i+1, 0$	$2k+i+1, k-1$	$2k+i+1, 2k$
$2k+i+2, k$	$\frac{5}{2}k, 2k:0$	$\frac{5}{2}k, 0:k$	$\frac{5}{2}k, k-1$	$\frac{5}{2}k, 3k:2k$
$\frac{5}{2}k+1, k-1:0$	$\frac{5}{2}k+1, 0:3k$	$\frac{5}{2}k+1, k:k-1$	$\frac{5}{2}k+1, 2k$	$\frac{5}{2}k+1, 3k:k$
$\frac{5}{2}k+2, 0:2k$	$3k-i, 3k$	$\frac{5}{2}k+2, k-1:0$	$3k-i, k$	$3k-i, 2k$
$3k-i+1, 0$	$3k-j+1, k-1$	$3k-i+1, 2k$	$3k-j+1, k$	$3k-j+1, 3k$
$3k-j+2, k-1$	$3k, k$	$3k-j+2, 0$	$3k, 3k$	$3k, 2k$

Lemma 3.2. *$\mathcal{C}_n$ is a monogamous latin square for odd $n \geqslant 3$ except $n \in \{5, 7\}$.*

Proof. By Theorem 2.2 and Lemma 3.1 it suffices to exhibit an orthogonal mate M_n for $\mathcal{C}_n$. All calculations of indices will be in $\mathbb{Z}_n$. We treat three cases. In each case we leave it to the reader to perform the (routine but laborious) check that M_n is orthogonal to $\mathcal{C}_n$.

Case: $n = 3k$. We let

$$M_n[i, j] \equiv \begin{cases} i + j - k & \text{if } i \in \{0, k\} \text{ and } j \in \{0, k, 2k\}, \\ 2i + j & \text{otherwise.} \end{cases}$$

Case: $n = 3k + 1$. We let $M_n[i, j] \equiv 2i + j \bmod n$ unless $2i + j \in \{0, k-1, k, 2k, 3k\}$. For the remaining entries we refer to Table 1. Each column of that table specifies the entries for the case when $2i + j$ is the value at the head of the column. An entry x, y in column z means that $M[x, z - 2x] = y$. An entry $x, y_1 : y_2$ in column z means that $M[x, z - 2x] = y_1$ if $k \equiv 0 \bmod 4$ and $M[x, z - 2x] = y_2$ if $k \equiv 2 \bmod 4$. An asterisk $*$ on an entry means that the corresponding entry in $\mathcal{C}_n$ has non-zero Δ_1 value. Finally, we note that the parameter i takes all values in $\{2, 4, 6, \ldots, 2\lfloor k/4 \rfloor - 2\}$ and parameter j takes all values in $\{2, 4, 6, \ldots, 2\lceil k/4 \rceil - 2\}$.

Case: $n = 3k - 1$. This case works very similarly to the previous case. We let $M_n[i, j] \equiv 2i + j \bmod n$ unless $2i + j \in \{0, 1, k, k + 1, 2k\}$. All other entries are provided by Table 2, using the same format as was used in Table 1. □

From the above results, it is a short step to:

Table 2
Five special diagonals when $n = 3k - 1$.

0	1	k	$k+1$	$2k$
$0, 0^*$	$0, 1^*$	$0, k^*$	$0, k+1$	$0, 2k^*$
$j-1, 0$	$1, 2k$	$j-1, 1$	$1, k+1$	$1, k$
$i, 2k$	$j, 1$	$i, 0$	j, k	$j, k+1$
$\frac{1}{2}k-1, 1:0$	$i+1, k$	$\frac{1}{2}k-1, 0:2k$	$i+1, 2k$	$i+1, k+1$
$\frac{1}{2}k, 0:2k$	$\frac{1}{2}k, k+1:k$	$\frac{1}{2}k, 2k:k+1$	$\frac{1}{2}k, k:0$	$\frac{1}{2}k, 1$
$k-i-1, 1$	$\frac{1}{2}k+1, k+1:0$	$k-i-1, k$	$\frac{1}{2}k+1, 0:k$	$k-i-1, 2k$
$k-j, k+1$	$k-i, k$	$k-j, 1$	$k-i, 0$	$k-j, 2k$
$k-1, 2k$	$k-j+1, 0$	$k-1, k$	$k-j+1, k+1$	$k-1, 1$
$k, 1$	$k, 2k^*$	$k, k+1^*$	$k, 0^*$	k, k^*
$k+j-1, 2k$	$k+j-1, 0$	$k+j-1, k$	$k+1, 1^*$	$k+j-1, k+1$
$k+i, k$	$k+i, 1$	$k+i, 0$	$k+j, 2k$	$k+i, k+1$
$\frac{3}{2}k-1, 0:k$	$\frac{3}{2}k-1, 2k:0$	$\frac{3}{2}k-1, k:1$	$k+i+1, 1$	$\frac{3}{2}k-1, k+1$
$\frac{3}{2}k, k:0$	$\frac{3}{2}k, k+1:2k$	$\frac{3}{2}k, 2k:k$	$\frac{3}{2}k, 0:k+1$	$\frac{3}{2}k, 1$
$2k-i-1, k+1$	$2k-i-1, 1$	$2k-i-1, 0$	$2k-i-1, k$	$2k-i-1, 2k$
$2k-j, 0$	$2k-j, k$	$2k-j, 1$	$2k-j, k+1$	$2k-j, 2k$
$2k-1, 1$	$2k-1, 2k$	$2k-1, k+1$	$2k-1, k$	$2k-1, 0^*$
$2k, 2k$	$2k, k$	$2k, k+1$	$2k, 1$	$2k, 0^*$
$2k+j-1, 1$	$2k+j-1, 0$	$2k+j-1, k$	$2k+j-1, 2k$	$2k+j-1, k+1$
$2k+i, 2k$	$2k+i, k$	$2k+i, 1$	$2k+i, 0$	$2k+i, k+1$
$\frac{5}{2}k-1, k+1:0$	$\frac{5}{2}k-1, 1:k$	$\frac{5}{2}k-1, k:k+1$	$\frac{5}{2}k-1, 0:1$	$\frac{5}{2}k-1, 2k$
$3k-i-2, k+1$	$\frac{5}{2}k, k:2k$	$\frac{5}{2}k, 0:k+1$	$\frac{5}{2}k, 2k:0$	$\frac{5}{2}k, 1$
$3k-j-1, k$	$3k-i-1, 1$	$3k-i-1, 0$	$3k-i-1, k+1$	$3k-i-1, 2k$
$3k, k+1^*$	$3k-j, 0$	$3k-j, k$	$3k-j, 1$	$3k-j, 2k$

Theorem 3.3. *There exists a monogamous Δ-crimped latin square of odd order n if and only if $n = 3$ or $n \geqslant 7$.*

Proof. For $n = 7$ the square L_7 below is Δ-crimped, with index group $\mathbb{Z}_7$ and $\gamma_1 = (5, 5, 1)$ and $\gamma_2 = (4, 3, 2)$ shaded. It has an orthogonal mate L'_7.

$$L_7 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & \mathbf{0} & 4 & 5 & 6 & \mathbf{3} \\ 2 & 3 & 4 & 5 & 6 & 0 & 1 \\ 3 & 4 & \mathbf{1} & 6 & 0 & \mathbf{2} & \mathbf{5} \\ 4 & 5 & 6 & \mathbf{2} & 1 & \mathbf{3} & \mathbf{0} \\ 5 & 6 & \mathbf{3} & \mathbf{0} & 2 & \mathbf{1} & 4 \\ 6 & 0 & \mathbf{5} & \mathbf{1} & 3 & 4 & \mathbf{2} \end{pmatrix}, \qquad L'_7 = \begin{pmatrix} a & b & c & d & e & f & g \\ g & d & e & f & b & c & a \\ e & f & g & c & a & b & d \\ c & a & f & b & d & g & e \\ b & g & d & a & c & e & f \\ d & e & b & g & f & a & c \\ f & c & a & e & g & d & b \end{pmatrix}.$$

For $n = 3$ and odd $n > 7$ we can use $\mathcal{C}_n$ by Lemma 3.1 and Lemma 3.2. It only remains to note that there are no monogamous latin squares of order $n \in \{1, 5\}$. All latin squares of order 5 are either bachelors or are contained in a set of 4 MOLS. □

4. A product construction

In this section we give a product construction that allows us to build larger crimped squares from smaller ones.

Let A and B be latin squares both indexed by a group $G_1 = \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_t}$ and let C be a latin square indexed by an arbitrary abelian group G_2 in which the identity is ε. We define a latin square indexed by $G_1 \times G_2$ and denoted $A \hookrightarrow B \times C$ by

$$A \hookrightarrow B \times C\big[(x_1, y_1), (x_2, y_2)\big] = \begin{cases} (A[x_1, x_2], C[y_1, y_2]) & \text{if } y_1 = y_2 = \varepsilon, \\ (B[x_1, x_2], C[y_1, y_2]) & \text{otherwise.} \end{cases}$$

Lemma 4.1. *Suppose that*

1. *A is Δ_i-crimped,*
2. *Δ_i is uniformly zero on B,*
3. *B has a transversal,*
4. *C has a transversal that includes the cell $[\varepsilon, \varepsilon]$,*
5. *p_i is odd or $|G_2|$ is odd or $|G_1|/p_i$ is even.*

Then $A \hookrightarrow B \times C$ is Δ_i-crimped.

Proof. Let $M = A \hookrightarrow B \times C$. Condition 5 ensures that $\delta_i(G_1 \times G_2) = \delta_i(G_1)$. We now argue that M is Δ_i-crimped. By assumption A is Δ_i-crimped, so there exist $\gamma_1 = (e_1, e_2, e_3)$ and $\gamma_2 = (f_1, f_2, f_3)$ satisfying (C1) and (C2). Let

$$\bar{\gamma}_1 = \big((e_1, \varepsilon), (e_2, \varepsilon), (e_3, C[\varepsilon, \varepsilon])\big) \quad \text{and} \quad \bar{\gamma}_2 = \big((f_1, \varepsilon), (f_2, \varepsilon), (f_3, C[\varepsilon, \varepsilon])\big).$$

Let Δ_i^*, $\bar{\Delta}_i^*$ be respectively the sets of entries of A and M with non-zero Δ_i values. Condition 2 means that there is a bijection from Δ_i^* to $\bar{\Delta}_i^*$ given by $(x, y, z) \mapsto ((x, \varepsilon), (y, \varepsilon), (z, C[\varepsilon, \varepsilon]))$. This map preserves Δ_i, so it is not hard to see that $\bar{\gamma}_1$ and $\bar{\gamma}_2$ satisfy (C2).

We next argue that (C1) is satisfied. By Condition 4, there exists a transversal, say $\{(i, y_i, z_i): i \in G_2\}$ in C, where $y_\varepsilon = \varepsilon$. We can then form a transversal of M by taking the union of transversals of each of the latin subsquares

$$S_i = \big\{\big((a, i), (b, y_i), (c, z_i)\big) \in M: a, b, c \in G_1\big\}$$

for $i \in G_2$. Note that S_ε is a copy of A and we may choose the transversal in it to include $\bar{\gamma}_1$ and $\bar{\gamma}_2$. If $i \neq \varepsilon$ then S_i is a copy of B, which has a transversal by Condition 3. Further, this transversal has all Δ_i values equal to 0 by Condition 2. Hence M satisfies (C1), and is Δ_i-crimped. □

Example 4.1. The latin square $\mathcal{C}_5 \hookrightarrow \mathbb{Z}_5 \times (\mathbb{Z}_2 \times \mathbb{Z}_2)$ is Δ-crimped by Lemma 3.1 and Lemma 4.1. To show that it is monogamous we provide the following orthogonal mate:

$$\begin{pmatrix}
p & m & q & l & s & a & b & c & d & e & k & n & r & t & o & f & g & h & i & j \\
o & r & n & t & k & e & a & b & c & d & p & l & s & m & q & j & f & g & h & i \\
q & t & s & r & p & d & e & a & b & c & m & o & k & n & l & i & j & f & g & h \\
s & k & m & n & q & c & d & e & a & b & l & r & t & o & p & h & i & j & f & g \\
l & n & r & k & o & b & c & d & e & a & t & p & q & s & m & g & h & i & j & f \\
f & g & h & i & j & l & p & t & r & m & a & b & c & d & e & o & s & k & n & q \\
j & f & g & h & i & k & l & q & o & t & e & a & b & c & d & m & n & r & p & s \\
i & j & f & g & h & o & m & n & p & q & d & e & a & b & c & t & k & s & l & r \\
h & i & j & f & g & p & r & s & t & n & c & d & e & a & b & q & l & m & k & o \\
g & h & i & j & f & s & o & l & m & k & b & c & d & e & a & n & r & t & q & p \\
a & b & c & d & e & r & s & p & k & o & f & g & h & i & j & l & m & q & t & n \\
e & a & b & c & d & n & k & o & q & s & j & f & g & h & i & r & p & l & m & t \\
d & e & a & b & c & m & t & r & n & p & i & j & f & g & h & k & q & o & s & l \\
c & d & e & a & b & q & n & k & l & r & h & i & j & f & g & s & t & p & o & m \\
b & c & d & e & a & t & q & m & s & l & g & h & i & j & f & p & o & n & r & k \\
t & q & o & m & l & f & g & h & i & j & r & s & n & p & k & a & b & c & d & e \\
k & l & p & s & m & j & f & g & h & i & q & t & o & r & n & e & a & b & c & d \\
n & s & k & p & r & i & j & f & g & h & o & m & l & q & t & d & e & a & b & c \\
r & p & l & o & t & h & i & j & f & g & n & q & m & k & s & c & d & e & a & b \\
m & o & t & q & n & g & h & i & j & f & s & k & p & l & r & b & c & d & e & a
\end{pmatrix}.$$

Our next construction will rely on 'turning intercalates'. That is, we replace some 2×2 subsquares by the other possible subsquare on the same symbols, like so:

$$\begin{array}{|c|c|}\hline a & b \\ \hline b & a \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|}\hline b & a \\ \hline a & b \\ \hline\end{array}.$$

Suppose $n = 2h$ for odd $h \geqslant 7$. We define a latin square $\mathcal{D}_n$ of order n indexed by $\mathbb{Z}_h \times \mathbb{Z}_2$, by turning some intercalates in $\mathcal{C}_h \hookrightarrow \mathbb{Z}_h \times \mathbb{Z}_2$. Specifically,

$$\mathcal{D}_n\big[(a_1, b_1), (a_2, b_2)\big] = \begin{cases} (a_1 + a_2, b_1 + b_2 + 1) & \text{if } a_1 = h - 2 \text{ and } a_2 = 2k + 2 \text{ or} \\ & \quad a_1 = h - 1 \text{ and } 2k - \frac{h-7}{2} \leqslant a_2 \leqslant 2k + 2, \\ (\mathcal{C}_h[a_1, a_2], 0) & \text{if } b_1 = b_2 = 0 \text{ and not defined above}, \\ (a_1 + a_2, b_1 + b_2) & \text{otherwise}, \end{cases}$$

where $k = \lfloor (h + 1)/3 \rfloor$.

Lemma 4.2. *For any odd $h \geqslant 3$ there exists a Δ-crimped latin square of order $2h$.*

Proof. Let $n = 2h$. For $n = 6$ the following square indexed by $\mathbb{Z}_6$ is Δ-crimped, with $\gamma_1 = (1, 5, 4)$ and $\gamma_2 = (0, 0, 5)$. A transversal through these two entries is shaded. This square is the member Q_6 of an infinite family of latin squares studied in [1] and [7].

$$\begin{pmatrix} \boxed{\mathbf{5}} & 1 & 2 & 3 & 4 & \mathbf{0} \\ 1 & \mathbf{0} & 3 & \mathbf{2} & 5 & \boxed{\mathbf{4}} \\ 2 & \boxed{3} & 4 & 5 & 0 & 1 \\ 3 & 4 & 5 & \boxed{0} & 1 & 2 \\ 4 & 5 & 0 & 1 & \boxed{2} & 3 \\ \mathbf{0} & \mathbf{2} & \boxed{1} & \mathbf{4} & 3 & \mathbf{5} \end{pmatrix}.$$

For $n = 10$ we have Example 2.1. So it suffices to show $\mathcal{D}_n$ is Δ-crimped for $n \geqslant 14$. Property (C2) is inherited from $\mathcal{C}_h$ (by an argument similar to that in the proof of Lemma 4.1) so it suffices to find a transversal to demonstrate (C1). Let $k = \lfloor (h + 1)/3 \rfloor$. The following set of cells defines a transversal that does the job:

$$\Big\{\big[(0,0),(0,0)\big], \big[(k,0),(2k,0)\big],$$
$$\big[(i,0),(2k-2i,0)\big] \quad \text{for } i = 1, \dots, k-1,$$
$$\big[(i,0),(h+2k+1-2i,1)\big] \quad \text{for } i = k+1, \dots, k+1+\frac{1}{2}(h-1),$$
$$\big[(i,0),(2h+2k+1-2i,0)\big] \quad \text{for } i = k+2+\frac{1}{2}(h-1), \dots, h-1,$$
$$\big[(i,1),(2k+1-2i,0)\big] \quad \text{for } i = 0, \dots, k,$$
$$\big[(i,1),(h+2k-2i,1)\big] \quad \text{for } i = k+1, \dots, k+\frac{1}{2}(h-1),$$
$$\big[(i,1),(2h+2k-2i,0)\big] \quad \text{for } i = k+1+\frac{1}{2}(h-1), \dots, h-1\Big\}. \qquad \square$$

Example 4.2. It is easy to show with Lemma 2.1 that $\mathcal{C}_7$ and $\mathcal{D}_{14}$ are bachelors. However, a monogamous Δ-crimped latin square of order 14 can be found by turning some intercalates in $L_7 \hookrightarrow \mathbb{Z}_7 \times \mathbb{Z}_2$, where L_7 is given in the proof of Theorem 3.3. Here is such an example, with an orthogonal mate specified by the subscripted letters.

$$
\begin{pmatrix}
\mathbf{01}_f & \mathbf{11}_c & 20_a & 30_i & 40_j & 50_k & 60_l & \mathbf{00}_g & \mathbf{10}_d & 21_h & 31_e & 41_b & 51_m & 61_n \\
\mathbf{11}_i & \mathbf{21}_m & \mathbf{00}_c & 40_a & 50_b & 60_g & \mathbf{30}_e & \mathbf{10}_j & \mathbf{20}_k & 31_n & 41_d & 51_l & 61_f & 01_h \\
20_d & 30_k & 40_h & 50_l & 60_e & 00_n & 10_a & 21_b & 31_m & 41_i & 51_j & 61_g & 01_c & 11_f \\
30_n & 40_l & \mathbf{10}_g & 60_j & 00_d & \mathbf{20}_m & \mathbf{50}_c & 31_f & 41_h & 51_b & 61_i & 01_k & 11_e & 21_a \\
40_k & 50_d & 60_n & \mathbf{20}_h & 10_i & \mathbf{30}_f & \mathbf{00}_b & 41_l & 51_e & 61_a & 01_g & 11_m & 21_j & 31_c \\
50_a & 60_i & \mathbf{30}_b & \mathbf{00}_m & 20_c & \mathbf{10}_h & 40_f & 51_k & 61_j & 01_e & 11_n & 21_d & 31_l & 41_g \\
60_b & 00_j & \mathbf{50}_e & \mathbf{10}_f & 30_m & 40_i & \mathbf{20}_g & 61_c & 01_n & 11_l & 21_k & 31_h & 41_a & 51_d \\
\mathbf{00}_h & \mathbf{10}_e & 21_f & 31_k & 41_n & 51_a & 61_d & \mathbf{01}_i & \mathbf{11}_b & 20_j & 30_l & 40_c & 50_g & 60_m \\
\mathbf{10}_l & \mathbf{20}_n & 31_d & 41_c & 51_f & 61_e & 01_m & \mathbf{11}_a & \mathbf{21}_i & 30_g & 40_b & 50_j & 60_h & 00_k \\
21_g & 31_a & 41_m & 51_n & 61_k & 01_l & 11_j & 20_e & 30_c & 40_d & 50_h & 60_f & 00_i & 10_b \\
31_j & 41_f & 51_i & 61_b & 01_a & 11_d & 21_n & 30_h & 40_g & 50_m & 60_c & 00_e & 10_k & 20_l \\
41_e & 51_g & 61_l & 01_d & 11_h & 21_c & 31_i & 40_m & 50_f & 60_k & 00_a & 10_n & 20_b & 30_j \\
51_c & 61_h & 01_j & 11_g & 21_l & 31_b & 41_k & 50_n & 60_a & 00_f & 10_m & 20_i & 30_d & 40_e \\
61_m & 01_b & 11_k & 21_e & 31_g & 41_j & 51_h & 60_d & 00_l & 10_c & 20_f & 30_a & 40_n & 50_i
\end{pmatrix}.
$$

We say that an abelian group G possesses an orthogonal mate if there is a latin square orthogonal to the Cayley table of G. A finite abelian group possesses an orthogonal mate unless it is isomorphic to the direct sum of cyclic groups of which exactly one has even order (see e.g. [2]).

Theorem 4.3. *Suppose L is a monogamous Δ-crimped latin square of order n, indexed by an abelian group G that possesses an orthogonal mate. Then for any integer $m \geqslant 3$ there exists a monogamous Δ-crimped latin square of order nm.*

Proof. We first treat the case when $m \neq 6$. In this case there exists a pair (A, A') of MOLS of order m. We use $\mathbb{Z}_m$ to index A and A'. By assumption, L has an orthogonal mate L' and there exists a latin square G' orthogonal to G. Define a pair (M, M') of latin squares of order mn, indexed by $G \times \mathbb{Z}_m$, by $M = L \hookrightarrow G \times A$ and $M' = L' \hookrightarrow G' \times A'$. It is routine to check that (M, M') are orthogonal, and Lemma 4.1 tells us that M is Δ-crimped.

Finally, we treat the case when $m = 6$. If $n > 3$ then $nm = 3k$ for some $k > 6$ so the existence of a monogamous Δ-crimped latin square of order nm is guaranteed by the $m \neq 6$ case treated above, given that there is a monogamous Δ-crimped latin square of order 3, by Theorem 3.3. If $n = 3$ then $nm = 18$ and we use $\mathcal{D}_{18}$, which we proved to be Δ-crimped in Lemma 4.2. To see that $\mathcal{D}_{18}$ is monogamous, we provide the following orthogonal mate:

$$
\begin{pmatrix}
r & b & i & q & j & c & p & d & a & k & f & l & e & h & g & m & n & o \\
c & n & j & k & r & a & l & b & g & p & h & m & d & e & f & o & i & q \\
d & i & r & e & k & o & m & h & j & n & p & f & l & b & a & q & g & c \\
q & m & f & p & h & i & r & o & l & a & b & k & n & d & e & c & j & g \\
g & p & d & i & f & m & a & j & c & q & r & o & h & k & l & e & b & n \\
i & o & q & g & c & k & e & n & f & h & l & r & b & j & m & p & d & a \\
f & e & h & r & m & p & b & q & n & j & g & a & o & l & k & i & c & d \\
a & d & l & n & o & q & h & k & e & f & j & b & g & i & c & r & m & p \\
e & j & g & b & l & n & f & c & p & d & m & i & q & r & o & a & h & k \\
n & h & c & l & e & b & k & m & r & o & i & q & f & a & d & g & p & j \\
b & r & k & m & n & h & c & p & d & l & a & g & i & q & j & f & o & e \\
m & q & o & a & i & e & n & r & k & g & d & p & j & c & b & l & f & h \\
j & k & p & d & q & l & i & a & o & e & c & h & m & g & n & b & r & f \\
l & c & a & j & g & f & q & e & h & b & o & n & r & p & i & d & k & m \\
o & g & b & c & p & r & d & f & i & m & e & j & k & n & q & h & a & l \\
p & l & m & h & d & j & o & g & q & i & n & c & a & f & r & k & e & b \\
k & f & e & o & a & g & j & i & b & c & q & d & p & m & h & n & l & r \\
h & a & n & f & b & d & g & l & m & r & k & e & c & o & p & j & q & i
\end{pmatrix}.
$$

If $n < 3$ there is no L that satisfies the hypothesis of the theorem. □

Corollary 4.4. *For any positive integer* $n \equiv 0 \bmod 8$ *there exists a monogamous* Δ*-crimped latin square of order* n*.*

Proof. The square L_8 below, indexed by $\mathbb{Z}_4 \times \mathbb{Z}_2$, is Δ_1-crimped on the shaded entries $\gamma_1 = (01, 00, 11)$ and $\gamma_2 = (31, 10, 31)$. It is orthogonal to L'_8. Thus, by Theorem 4.3 there are monogamous Δ-crimped latin squares for all orders which are divisible by 8, except possibly 16.

$$L_8 = \begin{pmatrix} 00 & 10 & 20 & 30 & 01 & 11 & 21 & 31 \\ 10 & 20 & 30 & 00 & 11 & 21 & 31 & 01 \\ 20 & 30 & 00 & 10 & 21 & 31 & 01 & 11 \\ 30 & 00 & 10 & 20 & 31 & 01 & 11 & 21 \\[6pt] \colorbox{lightgray}{$\mathbf{11}$} & \mathbf{01} & 21 & 31 & 00 & \mathbf{30} & 20 & \mathbf{10} \\ \mathbf{01} & 21 & 31 & \mathbf{11} & 10 & 20 & 30 & 00 \\ \mathbf{31} & \mathbf{11} & 01 & \mathbf{21} & 20 & \mathbf{10} & 00 & \mathbf{30} \\ \mathbf{21} & \colorbox{lightgray}{$\mathbf{31}$} & 11 & \mathbf{01} & 30 & 00 & 10 & 20 \end{pmatrix},$$

$$L'_8 = \begin{pmatrix} d & f & g & a & b & e & h & c \\ h & b & f & g & c & d & e & a \\ c & h & b & e & a & g & d & f \\ e & c & a & d & f & h & g & b \\[6pt] a & g & c & h & e & b & f & d \\ f & e & d & b & g & a & c & h \\ b & d & e & f & h & c & a & g \\ g & a & h & c & d & f & b & e \end{pmatrix}.$$

An example of order 16 is $L_8 \hookrightarrow (\mathbb{Z}_4 \times \mathbb{Z}_2) \times \mathbb{Z}_2$, which has the following orthogonal mate:

$$\begin{pmatrix} j & h & a & f & k & l & d & m & g & p & o & e & c & i & b & n \\ o & i & b & l & f & m & a & j & e & c & g & p & n & h & k & d \\ d & n & m & p & c & b & l & o & f & k & a & i & j & g & e & h \\ g & f & l & o & d & e & n & k & a & h & j & m & i & b & c & p \\[6pt] k & p & f & c & b & a & g & e & i & l & m & h & o & d & n & j \\ h & j & g & e & m & c & i & n & p & d & f & o & a & k & l & b \\ e & a & d & h & n & i & c & p & o & b & k & j & g & m & f & l \\ p & k & j & i & o & g & b & h & l & f & e & n & d & c & m & a \\[6pt] m & g & h & n & l & k & e & a & d & j & b & c & f & p & i & o \\ n & l & i & d & a & f & p & g & c & e & h & k & b & o & j & m \\ i & b & n & k & g & d & h & f & j & m & p & a & e & l & o & c \\ f & e & c & b & j & n & m & i & k & o & d & l & p & a & h & g \\[6pt] a & d & k & m & i & o & j & c & b & g & l & f & h & n & p & e \\ b & c & e & j & h & p & o & l & m & a & n & g & k & f & d & i \\ l & o & p & g & e & h & k & b & n & i & c & d & m & j & a & f \\ c & m & o & a & p & j & f & d & h & n & i & b & l & e & g & k \end{pmatrix}.$$

Moreover, $L_8 \hookrightarrow (\mathbb{Z}_4 \times \mathbb{Z}_2) \times \mathbb{Z}_2$ is Δ_1-crimped. Condition (C2) can be seen as per the proof of Lemma 4.1, and condition (C1) is immediate from the existence of an orthogonal mate. □

A turn-square, with underlying group G, is a latin square created from the Cayley table of G by turning some of the intercalates. For orders $n \equiv 2 \bmod 4$, where no group has a transversal (or an orthogonal mate), turn-squares are a convenient way to produce squares with many transversals [11] and many orthogonal mates [3]. Unsurprisingly then, turn-squares are useful for $n \equiv 2 \bmod 4$, where Theorem 4.3 does not apply.

Theorem 4.5. *Let $G = \mathbb{Z}_h \times \mathbb{Z}_2$ for some odd $h \geqslant 5$. Let T be a turn-square with underlying group G, such that T possesses an orthogonal mate. Let L be a monogamous Δ_1-crimped latin square that is indexed by G. Then for any integer $m \geqslant 3$ there exists a monogamous Δ-crimped latin square of order $2hm$.*

Proof. The proof is similar to that for Theorem 4.3. The $m = 6$ case is handled the same way, by writing $2hm = 3 \times 4h$. For $m \neq 6$ we again use a pair (A, A') of MOLS of order m. Define $M = L \hookrightarrow T \times A$ and $M' = L' \hookrightarrow T' \times A'$ where L' and T' are orthogonal mates for L and T respectively. Then M and M' are orthogonal. Moreover, M is Δ_1-crimped by Lemma 4.1, given that Δ_1 is uniformly zero on T. □

It seems likely that for all odd $h \geqslant 5$ there exists a turn-square with underlying group $\mathbb{Z}_h \times \mathbb{Z}_2$, and which has an orthogonal mate. This is certainly true [11] for $h \in \{5, 7\}$. The only case in which we need Theorem 4.5 in the present paper is the following example.

Example 4.3. A monogamous Δ-crimped latin square of order 50 can be found by applying Theorem 4.5 with $h = m = 5$ and $L = L_{10}$ as given in Example 2.1.

5. The main results

We are now in a position to draw the threads together and prove our main results.

Theorem 5.1. *There exists a monogamous Δ-crimped latin square for any order $n > 6$ not of the form $2p$ for a prime $p \geqslant 11$.*

Proof. Corollary 4.4 handles $n \equiv 0 \bmod 8$ and Theorem 3.3 takes care of all odd orders. The cases $n \in \{10, 14, 20, 50\}$ are solved by Examples 2.1, 4.1, 4.2 and 4.3. Every other order n under consideration can be written as $n = ab$ for $b > 2$ and odd $a \notin \{1, 5\}$. By Theorem 3.3 there exists a monogamous Δ-crimped latin square of order a. Furthermore, this square is indexed by $\mathbb{Z}_a$, which has an orthogonal mate, so the result follows by Theorem 4.3. □

Corollary 2.3 and Theorem 5.1 together imply Theorem 1.1. Our other main result, Theorem 2.4, is that Δ-crimped latin squares exist for all orders $n \notin \{1, 2, 4\}$. This follows for odd orders from Lemma 3.1 and for orders that are $2 \bmod 4$ from Lemma 4.2. All remaining cases are shown by Theorem 5.1.

To summarise: we have introduced the notion of a crimped latin square, which is one that contains an entry γ_1 that is in some transversal, but all transversals through γ_1 also include another specific entry γ_2. Using a product construction, we have shown (Theorem 2.4) that crimped latin squares exist for all orders $n \notin \{1, 2, 4\}$. A crimped latin square cannot be in any triple of MOLS. Thus if it has an orthogonal mate then together with that mate it forms a pair of 2-maxMOLS, a maximal set of 2 MOLS. By this means we have shown (Theorem 1.1) that a pair of 2-maxMOLS exist for all orders $n \notin \{1, 2, 4, 5, 6\}$ except possibly if $n = 2p$ for some prime $p \geqslant 11$.

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