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Zeta Functions of Finite Graphs and Coverings, Part II

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Galois theory for normal unramified coverings of finite irregular graphs (which may have multiedges and loops) is developed. Using Galois theory we provide a construction of intermediate coverings which generalizes the classical Cayley and Schreier graph constructions. Three different analogues of Artin L-functions are attached to these coverings. These three types are based on vertex variables, edge variables, and path variables. Analogues of all the standard Artin L-functions results for number fields are proved here for all three types of L-functions. In particular, we obtain factorization formulas for the zeta functions introduced in Part I as a product of L-functions. It is shown that the path L-functions, which depend only on the rank of the graph, can be specialized to give the edge L-functions, and these in turn can be specialized to give the vertex L-functions. The method of Bass is used to show that Ihara type quadratic formulas hold for vertex L-functions. Finally, we use the theory to give examples of two regular graphs (without multiple edges or loops) having the same vertex zeta functions. These graphs are also isospectral but not isomorphic. © 2000 Academic Press

1. INTRODUCTION

1.1. *Summary*

In Stark and Terras [14] and Stark [13], we investigated three sorts of zeta functions (vertex, edge, and path) that may be attached to irregular graphs X. These may be viewed as analogues of the Dedekind zeta function of a number field. Moreover, they have some analogous properties. When the graph is regular, the Riemann hypothesis for the vertex type, or Ihara, zeta function is equivalent to the graph being Ramanujan in the sense of Lubotzky *et al.* [9]. We gave in [14] a simple proof of the fact that for an unramified graph covering Y/X the Ihara zeta function of X divides the Ihara zeta function of Y. The analogue of this fact is an unproved conjecture in general for Dedekind zeta functions of extensions of number fields. This result and many examples in Part I suggested that there is a full graph

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analogue of Artin L-function theory. This will be provided in Sections 3, 4, and 5 for vertex, edge, and path L-functions, respectively.

To do this, we provide in Section 2 a complete Galois theory of normal unramified coverings of finite graphs. The analogue of the Frobenius automorphism of number theory plays an important role. We give a construction of covering graphs intermediate to normal coverings which generalizes the well known Cayley and Schreier graph constructions (see Terras [16]). We will use this construction in Section 6 to obtain an example of a non-isomorphic pair of isospectral regular graphs having neither multiple edges nor loops using the simple group of order 168, namely, $GL(3, \mathbb{F}_2)$. To see that the two graphs we construct are non-isomorphic, we make use of a lemma about counting triangles in graphs in terms of characters of permutation representations. This lemma is obtained as part of our proof of the induction property of L-functions.

1.2. *The Zeta and L-Functions*

Lang [8] gives the definitions and basic theory of zeta and L-functions in the number field setting. See also the survey article by Stark [12]. Artin L-functions are among the most important functions in algebraic number theory. Various conjectures are still open (e.g., the Artin conjecture that they are holomorphic for irreducible non-trivial characters of the Galois group). They appear prominently in much of the recent work on the subject. See, for example, Murty [10] and Stark [12].

Various authors such as Hashimoto [5, 6], and Sunada [15] have investigated the analogues for graphs. They obtain an analogue of the Chebotarev density theorem even when the covering is ramified though the point-of-view is rather different from ours.

Let us briefly summarize the definitions of the edge and path zeta functions. If X is any connected finite undirected graph with vertex set V and (undirected) edge set E, we orient its edges arbitrarily and obtain $2|E|$ oriented edges $e_1, e_2, ..., e_n, e_{n+1}=e_1^{-1}, ..., e_{2n}=e_n^{-1}$. "*Primes*" $[C]$ in X are equivalence classes of closed backtrackless tailless primitive paths C. Write $C=a_1a_2 \cdots a_s$, where a_j is an oriented edge of X. *Backtrackless* means that $a_{i+1} \neq a_i^{-1}$, for all i. *Tailless* means that $a_s \neq a_1^{-1}$. The *equivalence class* $[C]$ is the set

$$[C]=\{a_1a_2 \cdots a_s, a_2a_3 \cdots a_sa_1, ..., a_sa_1 \cdots a_{s-1}\}.$$

$[C]$ is *primitive* means $C \neq D^m$, for any integer $m \geqslant 2$ and path D in X.

For the multiedge zeta function of X, we need a $2|E| \times 2|E|$ matrix W with ij entry the complex variable w_{ij} if the associated edges e_i and e_j have the terminal vertex of e_i equal the starting vertex of e_j and $e_j \neq e_i^{-1}$.

Otherwise the ij entry $w_{ij}=0$. We will also set $w(e_i, e_j)=w_{ij}$, for edges e_i and e_j.

DEFINITION 1. For the prime $[C]$, $C=a_1 a \cdots a_s$, the *multiedge norm* of C is

$$\mathbb{N}_E(C)=w(a_s, a_1)\prod_{i=1}^{s-1} w(a_i, a_{i+1}).$$

DEFINITION 2. The *multiedge zeta function* of X is

$$\zeta_E(W, X)=\prod_{[C]} (1-\mathbb{N}_E(C))^{-1}.$$

If you specialize the non-zero w_{ij} to be the same variable u, then you obtain the *Ihara zeta function* of X,

$$\zeta_E(W, X)|_{non-0\ w_{ij}=u}=\zeta_X(u). \tag{1}$$

The multiedge zeta function is somewhat more general than that in Stark and Terras [14], where we had $2\,|E|$ variables $u_i=w_{ij}$. But the proof given in [14] for the special case extends to show that

$$\zeta_E(W, X)=\det(I-W)^{-1}. \tag{2}$$

This fact is much easier to prove than the corresponding result for the vertex Ihara zeta function

$$\zeta_X(u)^{-1}=(1-u^2)^{r-1}\det(I-Au+Qu^2). \tag{3}$$

Here r is the rank of the fundamental group of X, A is the adjacency matrix of X, and Q is the diagonal matrix with jth diagonal entry $Q_j=-1+\text{degree } j\text{th vertex of } X$. An elementary determinant identity due to Bass [1] allows one to derive (3) from (2). We will obtain the analogous result for Artin L-Functions in Section 4. See also Kotani and Sunada [7].

If you specialize the non-zero w_{ij} to be 1 and call the resulting matrix W_1, then

$$Tr((I-W_1 t)^{-1})=\sum_{n=0}^{\infty} c_n t^n,$$

where c_n is the number of closed backtrackless tailless paths of length n. Thus the magnitudes of the eigenvalues of W_1 control the backtrackless tailless random walks on X just as the magnitudes of the eigenvalues of the adjacency matrix A of X control the ordinary random walks on X (see

Terras [16]). Equivalently, it is the zeros of the reciprocal of the Ihara zeta function $\zeta_X(u)^{-1}$ which control the backtrackless tailless random walks on X rather than the eigenvalues of A.

The multipath zeta function of X is created in a similar manner to the multiedge zeta function replacing edges of X with generators of the fundamental group of X. We assume that X is a connected graph with vertex set V and (undirected) edge set E. Because X is connected, there is a subgraph T of X with $|V|$ vertices which is a tree. T is called a *spanning tree* for X and it has $|V|-1$ edges. We give each of the edges on the tree T a direction and label them $t_1, ..., t_{|V|-1}$. The inverse edges on T will be labeled $t_{|V|}, ..., t_{2|V|-2}$. We give each of the $r=|E|-|V|+1$ remaining edges on X a direction and label them $e_1, ..., e_r$. The inverse edges will be labeled $e_{r+1}, ..., e_{2r}$.

This allows us to identify the free group of rank r generated by the e_j's as the fundamental group of X. Any prime cycle C on X is uniquely (up to starting point on the tree between last and first e_k) determined by the ordered sequence of e_k's it passes through. See Section 5 for more details of this algorithm.

Consider a $2r\times 2r$ matrix $Z=(z_{ij})_{1\leqslant i,\, j\leqslant 2r}$ of complex variables where $z_{ij}=0$ if $e_j=e_i^{-1}$. Let $z(e_i, e_j)=z_{ij}$. Consider the prime $[C]$, $C=a_1a_2\cdots a_s$, where $a_j\in\{e_1, ..., e_r, e_{r+1}=e_1^{-1}, ..., e_{2r}=e_r^{-1}\}$ and C is a reduced product in the generators of the fundamental group. Here "reduced product" means that $a_{j+1}\neq a_j^{-1}$, for all j, and also $a_s\neq a_1^{-1}$.

DEFINITION 3. The *multipath norm* of the reduced prime $C=a_1a_2\cdots a_s$ is

$$\mathbb{N}_P(C)=z(a_s, a_1)\prod_{i=1}^{s-1} z(a_i, a_{i+1}).$$

DEFINITION 4. The *multipath zeta function* of X is

$$\zeta_P(Z, X)=\prod_{[C]}(1-\mathbb{N}_P(C))^{-1}.$$

Note that the multipath zeta function involves a matrix Z which is smaller than the matrix W in the multiedge zeta function. However, the matrix Z has far fewer zero entries than the matrix W.

We saw in Stark and Terras [14] that

$$\zeta_P(Z, X)=\det(I-Z)^{-1}.$$

Moreover it is possible to specialize the variables of the path zeta function to obtain the edge zeta function. This result is at first sight rather amazing, since ζ_P has fewer variables than ζ_E. The specialization algorithm of Stark and Terras [14] was improved by Stark [13] and that improved algorithm will be improved upon further and explained here. See Section 5.

The aim of this paper is to extend all the preceding results about zeta functions of X to Artin L-functions of unramified normal coverings Y of X with Galois group $G(Y/X)$. We need to consider primes $[D]$ of Y dividing primes $[C]$ of X and the Frobenius automorphism $[Y/X, D] \in G(Y/X)$. See the definitions in Subsections 2.3 and 2.4. Given a representation ρ of $G(Y/X)$, one defines the multipath Artin L-functions by replacing $\mathbb{N}_P(C)$ with $\rho([Y/X, D])\, \mathbb{N}_P(C)$. The multiedge Artin L-function is defined similarly.

These new Artin L-functions have all the properties of the old; e.g. the L-function corresponding to a representation ρ of a subgroup $H \subset G(Y/X)$ is the same as the L-function corresponding to the induced representation $\rho^{\#} = Ind_H^G\, \rho$, as defined in Terras [16], for example. Thus one obtains a factorization of the (specialized down to X) multiedge (respectively, multipath) zeta function of Y as a product over the inequivalent irreducible representations of $G(Y/X)$ of multiedge (respectively, multipath) Artin L functions. See Sections 4 and 5. We saw examples of such factorizations in Stark and Terras [14] and Stark [13]. More examples will be presented here.

2. THE BASICS: COVERINGS, GALOIS THEORY, AND FROBENIUS AUTOMORPHISMS

2.1. *Normal Unramified Coverings*

Throughout this paper "covering" means "unramified covering." We assume that we have a connected graph X with vertex set V and (undirected) edge set E. We will find that it simplifies the proofs of Section 5 significantly if we allow graphs to have loops and more than one edge between vertices. This is also useful in helping to explain the factorizations of zeta functions of graphs obtained in Stark and Terras [14].

For convenience in following paths, give each edge of X an arbitrary direction. Because X is connected, there is a subgraph T of X with $|V|$ vertices which is a tree. As such, T has $|V|-1$ edges. The remaining $r = |E| - |V| + 1$ edges of X give us a set of *r generators for the fundamental group* of X.

First we need some definitions. It is not enough to say that an abstract graph covers an abstract graph if you want to prove the fundamental theorem of Galois theory. For this we need to talk about neighborhoods and directed coverings.

DEFINITION 5. A *neighborhood* N of a vertex v in a directed graph X is obtained by taking one-third of each edge at v. The labels and directions are to be included. Think of it as two lists. The first list contains all edges with v as initial vertex. The second list contains all edges having v as terminal vertex. See Fig. 1.

DEFINITION 6. A *directed graph covering* Y of X means there is an assignment of directions on the edges of Y such that there is a *covering map* $\pi: Y \to X$ which is onto and maps neighborhoods of Y 1–1 and onto neighborhoods on X (i.e., preserving directions).

DEFINITION 7. A finite graph Y is a *covering* of a graph X if, after assigning directions arbitrarily to the edges of X, there exists an assignment of directions to the edges of Y such that Y is a directed graph covering of X.

Note that the fact that Y is a covering of X is independent of the choice of directions on X. See Fig. 2 for examples of invalid assignment of directions in Y over X. Note also that if you lift a loop you may get a graph with multiple edges. Thus once you allow loops, you cannot discuss the general covering without allowing multiple edges.

DEFINITION 8. A *d-sheeted normal covering* Y of the graph X means that there are d graph automorphisms $\sigma: Y \to Y$ such that $\pi \circ \sigma = \pi$. The Galois group $G(Y/X)$ is the set of these maps σ.

Many examples of coverings—normal and non-normal—can be found in Stark and Terras [14]. Here let us just consider a few basic examples.

EXAMPLE. *The cube is a quadratic covering of the tetrahedron.* See Fig. 3, where the edges in the tree of X are shown as dotted lines. The edges of the corresponding two sheets of Y are also shown as dotted lines.

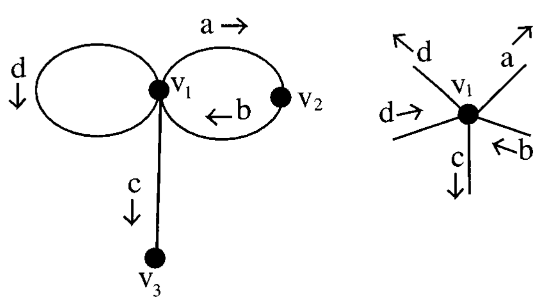

FIG. 1. A directed graph with a picture of a neighborhood of a vertex.

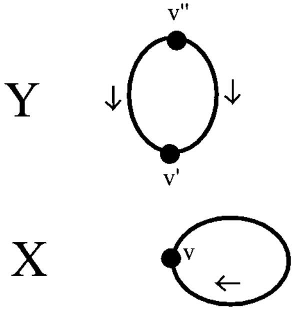

FIG. 2. Example of invalid covering maps.

In coverings of X involving multiedges or, worse still, loops, it is quite helpful to label the edges of X and give them directions as well, and then label the cover similarly to see that we have a cover.

PROPOSITION 1. *The Galois group $G = G(Y/X)$ acts transitively on the sheets of the covering. We view the d sheets of Y as d copies of a spanning tree of X. Points of Y can thus be labeled (x, g) for $x \in X$, $g \in G$.*

Proof. Every path downstairs in X has a unique lift once you specify the initial vertex in Y—even when there are loops and multiedges once you assign a direction to all edges. So now each spanning tree has a unique lift starting at any point in $\pi^{-1}(v_0)$, where v_0 is a fixed point in X. These d lifts are the sheets of the covering. An automorphism $\sigma \in G$ takes sheets to sheets. Any automorphism that fixes a sheet is the identity.

Equivalently any automorphism that fixes a point $\tilde{v}_0 \in \pi^{-1}(v_0)$ is the identity because a path from $\tilde{v}_0$ to $\tilde{v}$ in Y projects under π to a path from v_0 to $v = \pi(\tilde{v})$ in X and this path has a unique lift starting at $\tilde{v}_0$ which must be the path we started with. So if $\tilde{v}_0 = \sigma(\tilde{v}_0)$ then $\tilde{v} = \sigma(\tilde{v})$ and σ must be the identity.

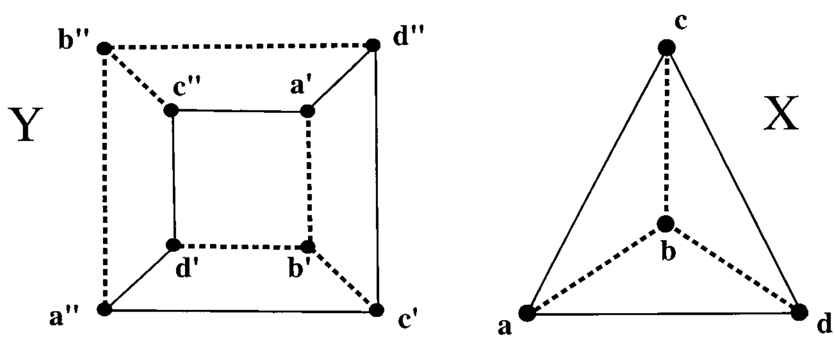

FIG. 3. The cube Y as a quadratic cover of the tetrahedron X.

So each distinct $\sigma \in G$ takes $\tilde{v}_0$ to a different point and there are only d different points in Y above v_0. It follows that the action of G is transitive. Otherwise two different automorphisms would take v_0 to the same point and we just showed that is impossible.

We choose one of the sheets of Y and call it sheet 1. The image of sheet 1 under an element g in G will be called *sheet g*. Any vertex $\tilde{x}$ on Y can then be uniquely denoted $\tilde{x} = (x, g)$, where $x = \pi(\tilde{x})$ and g is the sheet containing $\tilde{x}$. ▮

DEFINITION 9. The Galois group $G(Y/X)$ moves sheets of Y via $g \circ (\textit{sheet } h) = \textit{sheet}(gh)$,

$$g \circ (x, h) = (x, gh), \qquad \text{for} \quad x \in X, \quad g, h \in G.$$

It follows that g moves a path in Y as

$$g \circ (\text{path from } (a, h) \text{ to } (b, j)) = \text{path from } (a, gh) \text{ to } (b, gj). \tag{4}$$

EXAMPLE. *A directed n-cycle is a normal n-fold covering of a loop with cyclic Galois group*. See Fig. 4 for this example.

Consider the covering in Fig. 4. Note that the Ihara zeta function $\zeta_X(u) = (1-u)^{-2}$, and we have the factorization

$$\zeta_Y(u) = (1-u^n)^{-2} = \prod_{j=0}^{n-1} (1-w^j u)^{-2}, \qquad \text{where} \quad w = e^{2\pi i/n}.$$

The adjacency matrix of X is the 1×1 matrix $A = (2)$ and not (1). Here the degree of the graph is 2 and thus in Ihara's theorem (Theorem 1 of Stark and Terras [14]), we have $1 - Au + qu^2 = 1 - 2u + u^2 = (1-u)^2$. The

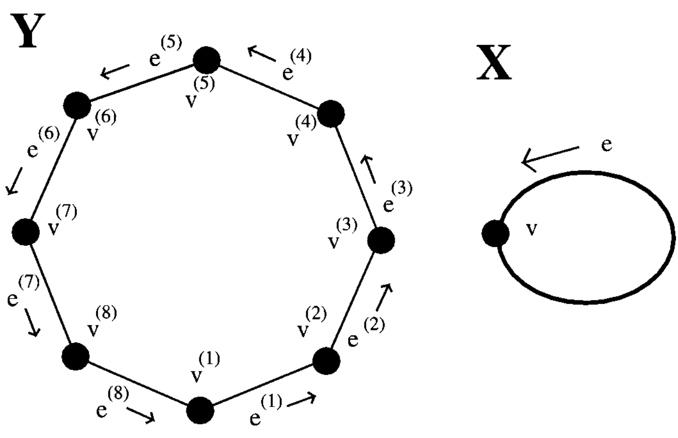

FIG. 4. An n-cycle covering a loop.

factorization of $\zeta_Y(u)$ will be seen here as a factorization of the Ihara zeta function into a product of Artin L-functions. See the Corollary to Proposition 3 in Section 3.

QUESTION. *Suppose a graph Y has a large symmetry group S and G is a subgroup of S. Is there a graph X such that Y is a normal cover of X with group G?*

Answer. Not always. For example, the cube has S_4 symmetry group and $G = S_4$ cannot be the Galois group $G(Y/X)$ since for X to exist $|G|$ divides the number $|V|$ of vertices of Y as well as the number $|E|$ of edges of Y and thus it also divides $r-1 = |E| - |V|$.

EXAMPLES. *X with the cube as a normal cover.* See Figs. 5 and 6 for these examples.

Let Y be the cube. Then $|V| = 8$, $|E| = 12$ and $|G|$ divides $g.c.d(8, 12) = 4$. We present a normal covering Y/X such that $G = G(Y/X)$ is a cyclic group of order 4 in Fig. 5 and another such covering Y/X in Fig. 6 where $G = G(Y/X)$ is the Klein 4-group.

EXAMPLE. *The octahedron as a cyclic 6-fold cover of 2 loops.* The octahedron has $|V| = 6$, $|E| = 12$ which implies that $|G| = 6$ may be possible. An example where X is a double loop is given in Fig. 7.

2.2. *Intermediate Coverings*

What is the meaning of $\tilde{X}$ is intermediate to Y/X? It is more than just Y covers $\tilde{X}$ and $\tilde{X}$ covers X. Consider Fig. 6. There are three intermediate

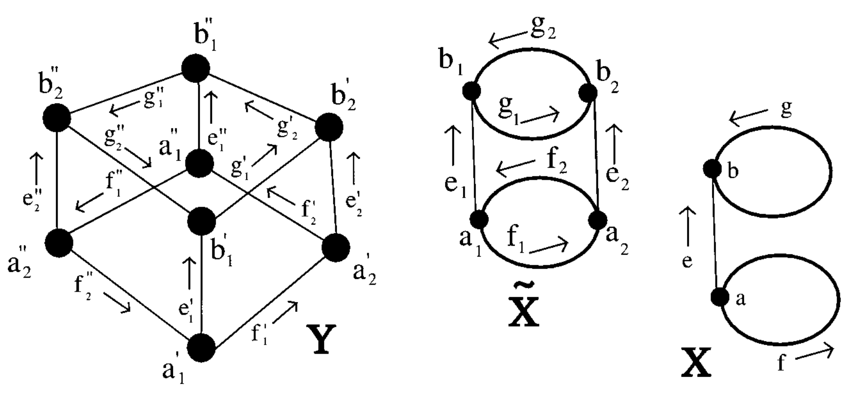

FIG. 5. An order 4 cyclic cover Y/X, where Y is the cube. Included is the intermediate quadratic cover $\tilde{X}$. The notation makes clear the covering projections π: $Y \to X$, π_2: $Y \to \tilde{X}$, $\pi_1 : \tilde{X} \to X$,

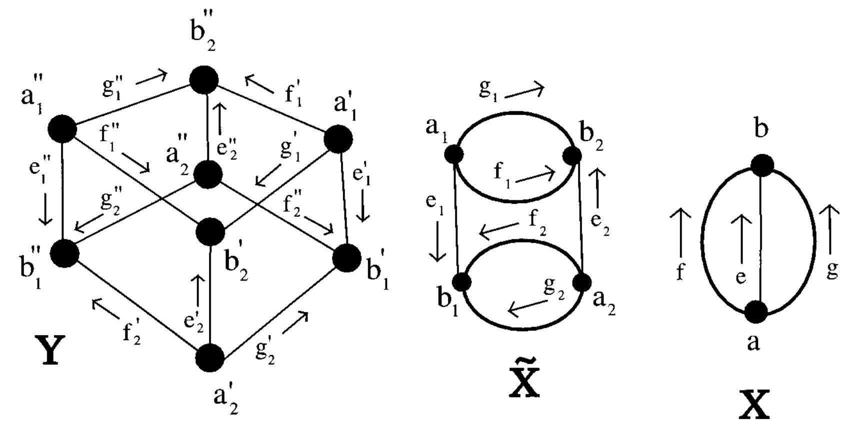

FIG. 6. A Klein 4-group cover Y/X, where $Y=$ the cube. Included is one of the 3 intermediate quadratic covers.

quadratic covers of which one is drawn. Abstractly Y covers all three intermediate graphs. Since each of the intermediate graphs $\tilde{X}_1$, $\tilde{X}_2$, $\tilde{X}_3$ is isomorphic to the other two, each covers the other two and thus each $\tilde{X}_i$ is intermediate between Y and $\tilde{X}_j$. That would negate the fundamental theorem of Galois theory.

The cure for this lies in the following definition.

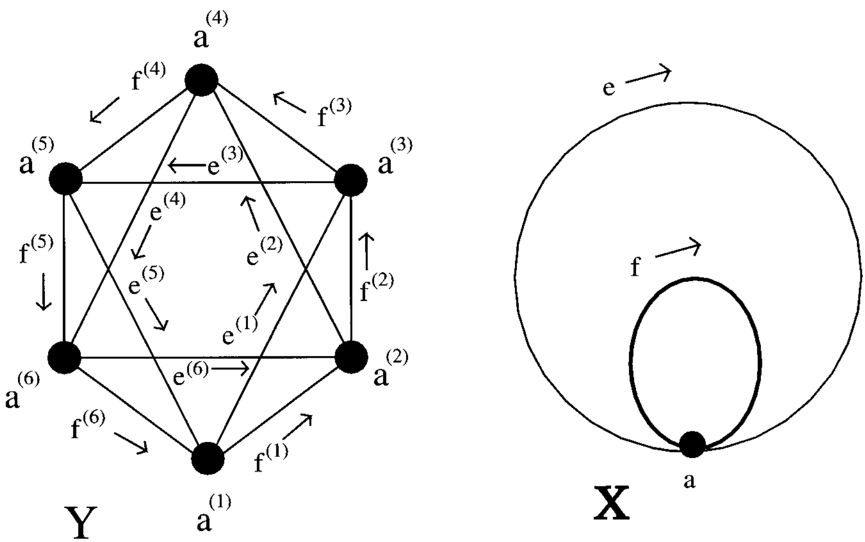

FIG. 7. A cyclic 6-fold cover Y/X, where Y is the octahedron.

Definition 10. If we have three graphs X, $\tilde{X}$, Y with Y covering both X and $\tilde{X}$ and with $\tilde{X}$ covering X, having projection maps $\pi\colon Y \to X$, $\pi_2\colon Y \to \tilde{X}$, $\pi_1\colon \tilde{X} \to X$, we will say that $\tilde{X}$ is *intermediate* to Y/X iff $\pi = \pi_1 \circ \pi_2$.

A projection map means more than just a pointwise map on vertices but also a map of neighborhoods and thus the preceding definition requires that the neighborhoods on $\tilde{X}$ are consistent with those on X and Y.

In Galois theory, if Y/X is normal with Galois group G, there will be a $1-1$ correspondence between intermediate graphs and subgroups. Thus we will later speak of **THE** intermediate graph corresponding to a subgroup H of G. In order to be able to do this, we need to define what it means for two abstract intermediate graphs to be "the same" or "equal".

Definition 11. Suppose Y is an (unramified) covering of X with the projection map π and that edges in Y and X have been assigned directions consistent with π. Suppose $\tilde{X}$ and $\tilde{X}'$ are intermediate graphs with projections π_2 and π_2' from Y to $\tilde{X}$ and $\tilde{X}'$, respectively, and projections π_1 and π_1' from $\tilde{X}$ and $\tilde{X}'$, respectively, to X, where $\pi = \pi_1 \circ \pi_2 = \pi_1' \circ \pi_2'$. Thus the edges of $\tilde{X}$ and $\tilde{X}'$ inherit directions consistent with these projections. We say $\tilde{X}$ and $\tilde{X}'$ are *the same or equal* if there is a (directed) graph isomorphism $i\colon \tilde{X} \to \tilde{X}'$ such that $\pi_2' = i \circ \pi_2$ and $\pi_1 = \pi_1' \circ i$.

Clearly Definition 11 yields an equivalence relation between intermediate graphs $\tilde{X}$ to a covering Y/X.

Theorem 1 (Fundamental Theorem of Galois Theory). *Suppose Y/X is an unramified normal covering with Galois group $G = G(Y/X)$.*

- (1) *Given a subgroup H of G, there exists a graph $\tilde{X}$ intermediate to Y/X such that $H = G(Y/\tilde{X})$. Write $\tilde{X} = \tilde{X}(H)$.*
- (2) *Suppose $\tilde{X}$ is intermediate to Y/X. Then there is a subgroup $H = H(\tilde{X})$ of G which is $G(Y/\tilde{X})$.*
- (3) *Two intermediate graphs $\tilde{X}$ and $\tilde{X}'$ are equal (as in Definition 11) if and only if $H(\tilde{X}) = H(\tilde{X}')$.*
- (4) *We have $H(\tilde{X}(H)) = H$ and $\tilde{X}(H(\tilde{X})) = \tilde{X}$. So we write $\tilde{X} \leftrightarrow H$ for the correspondence between intermediate graphs $\tilde{X}$ to Y/X and subgroups H of the Galois group $G = G(Y/X)$.*
- (5) *If $\tilde{X}_1 \leftrightarrow H_1$ and $\tilde{X}_2 \leftrightarrow H_2$ then $\tilde{X}_1$ is intermediate to $Y/\tilde{X}_2$ iff $H_1 \subset H_2$.*

Proof. (1) Let H be a subgroup of G. Recall that points of Y have the form (x, σ), with $x \in X$ and $\sigma \in G$. Now define the vertices of $\tilde{X}$ by $\tilde{X} = \{(x, H\sigma) \mid x \in X,\ H\sigma \in H\backslash G\}$. Then put an edge between $(a, H\sigma)$ and

$(b, H\tau)$, for $a, b \in X$ and $\sigma, \tau \in G$ iff there are $h, h' \in H$ such that $(a, h\sigma)$ and $(b, h'\tau)$ have an edge joining them in Y. The edge between $(a, H\sigma)$ and $(b, H\tau)$ in $\tilde{X}$ is given the label and direction of the projected edge between a and b in X. It is easy to see that $\tilde{X}$ is well-defined, intermediate to Y/X, and connected.

(2) Let $\tilde{X}$ be intermediate, with projections π: $Y \to X$, π_2: $Y \to \tilde{X}$, π_1: $\tilde{X} \to X$. Fix a point $v_0 \in X$ and let $\tilde{v}_0 = \pi_2(\tilde{\tilde{v}}_0) \in \tilde{X}$ be such that $\tilde{\tilde{v}}_0 \in \pi^{-1}(v_0)$ is on sheet 1 of Y. That is, $\tilde{\tilde{v}} = (v_0, 1)$ as in Proposition 1. Define $H = \{h \in G \mid h(\tilde{\tilde{v}}_0) \in \pi_2^{-1}(\tilde{v}_0)\}$.

We need only show that H is closed under multiplication to see that H is a subgroup of G. Let h_1 and h_2 be elements of H. Then the vertices (v_0, h_1) and (v_0, h_2) project under π_2 to $\tilde{v}_0$. Let $\tilde{\tilde{p}}_1$ and $\tilde{\tilde{p}}_2$ be paths on Y from $(v_0, 1)$ to the vertices (v_0, h_1) and (v_0, h_2), respectively. Then $\tilde{\tilde{p}}_1$ and $\tilde{\tilde{p}}_2$ project under π_2 to closed paths $\tilde{p}_1$ and $\tilde{p}_2$ in $\tilde{X}$ beginning and ending at $\tilde{v}_0$; $\tilde{\tilde{p}}_1$ and $\tilde{\tilde{p}}_2$ also project under $\pi = \pi_1 \circ \pi_2$ to closed paths p_1 and p_2 in X beginning and ending at v_0. By Eq. (4), $h_1 \circ \tilde{\tilde{p}}_2$ starts at (v_0, h_1) and terminates at $(v_0, h_1 h_2)$. Therefore the lift of $\tilde{p}_1 \tilde{p}_2$ from $\tilde{X}$ to Y beginning at $(v_0, 1)$, which is the same as the lift of $p_1 p_2$ from X to Y beginning at $(v_0, 1)$, terminates at $(v_0, h_1 h_2)$. Therefore $h_1 h_2$ is in H and H is a subgroup of G.

Taking this further, let $\tilde{v}$ be another vertex of $\tilde{X}$ and assume $\pi(\tilde{v}) = v \in X$. Let $\tilde{\tilde{v}} = (v, g_0)$ be a vertex in Y with $\tilde{v} = \pi_2(\tilde{\tilde{v}}) \in \tilde{X}$. Let $\tilde{\tilde{q}}$ be a path in Y from $(v_0, 1)$ to (v, g_0) and $\tilde{\tilde{p}}$ a path in Y from $(v_0, 1)$ to (v_0, h), where $h \in H$. These two paths project to paths $\tilde{q}$ from $\tilde{v}_0$ to $\tilde{v}$ in $\tilde{X}$, and $\tilde{p}$ from $\tilde{v}_0$ to $\tilde{v}_0$ in $\tilde{X}$. Projected all the way to X, we get paths q from v_0 to v, and p from v_0 to v_0, respectively. Again, by Eq. (4), we see that the path $\tilde{p}\tilde{q}$ from $\tilde{v}_0$ to $\tilde{v}$ in $\tilde{X}$ lifts to a path from $(v_0, 1)$ to (v_0, hg_0) in Y. Thus, for all g in the coset Hg_0, the vertex (v, g) projects under π_2 to $\tilde{v}$, and this provides the complete inverse image $\pi_2^{-1}(\tilde{v})$ in Y since $|\pi_2^{-1}(\tilde{v})| = |H|$. See Fig. 8 below.

It is clear now that the coset graph $\tilde{X}(H)$ is equal to $\tilde{X}$ in the sense defined above. We may think of the projection π_2 corresponding to the abstract intermediate graph $\tilde{X}$ as providing the analogy of an embedding of intermediate number fields. The graph $\tilde{X}(H)$ would then be the corresponding embedded version of $\tilde{X}$.

Parts (3), (4), and (5) are left to the reader. ∎

2.3. *Conjugate and Normal Intermediate Graphs*

Suppose $\tilde{X}$ is a graph intermediate to Y/X, where Y/X is normal (unramified) with Galois group G. We presume that all edges of Y, $\tilde{X}$, and X have been given directions consistent with the projection maps $Y \xrightarrow{\pi_2} \tilde{X} \xrightarrow{\pi_1} X$. If we think of π_2 as providing the *embedding* of $\tilde{X}$ into Y,

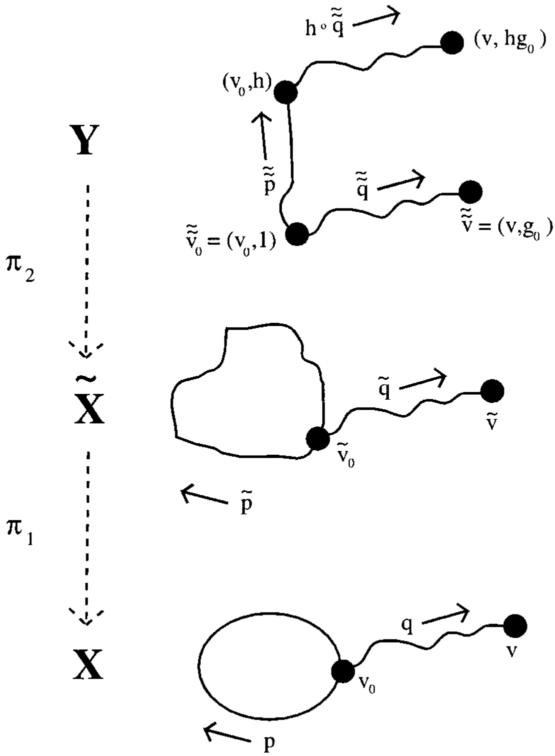

FIG. 8. Part of the proof of Part (2) of Theorem 1, showing $Hg_0 = \pi_2^{-1}(\tilde{v})$. The dashed lines are the projection maps π_1 and π_2.

then we should let $\pi_2 \colon Y \to \tilde{X}$ vary (as a projection of directed graphs) subject to $\pi = \pi_1 \circ \pi_2$ and think of these different π_2's as "conjugate embeddings" of $\tilde{X}$ in Y. The following definition accomplishes the equivalent.

Definition 12. Suppose Y/X is normal with Galois group G and that $\tilde{X}$ and $\tilde{X}'$ are intermediate graphs with projection maps π_2 and π_2' from Y to $\tilde{X}$, $\tilde{X}'$, respectively, and π_1, π_1' from $\tilde{X}$, $\tilde{X}'$ to X. We assume $\pi = \pi_1 \circ \pi_2 = \pi_1' \circ \pi_2'$ is the projection from Y to X and that directions have been assigned to all edges in all four graphs consistent with these projections. We say that $\tilde{X}$ and $\tilde{X}'$ are *conjugate intermediate graphs* if there is an isomorphism (of directed graphs) $i \colon \tilde{X} \to \tilde{X}'$ such that $\pi_1 = \pi_1' \circ i$.

Note that there is no condition involving i, π_2 and π_2'; the natural extra condition $\pi_2' = i \circ \pi_2$ would make $\tilde{X}$ and $\tilde{X}'$ not only conjugate but equal (as in Definition 11). With this definition, we can lift vertices or paths from $\tilde{X}$ to Y via π_2^{-1} and also via i followed by $\pi_2'^{-1}$. These 2 lifts need not be the same.

THEOREM 2. *With the notation of Definition* 12, *let H and H' be the subgroups of G corresponding to $\tilde{X}$ and $\tilde{X}'$ via the fundamental theorem of Galois Theory. Then $\tilde{X}$ and $\tilde{X}'$ are conjugate intermediate graphs iff H and H' are conjugate subgroups of G.*

Proof. Suppose that H and $H' = g_0 H g_0^{-1}$ are conjugate subgroups of G, where $g_0 \in G$. We want to show that the corresponding intermediate graphs $\tilde{X} = \tilde{X}(H)$ and $\tilde{X}' = \tilde{X}(H')$ (using the notation of the fundamental theorem of Galois theory) are conjugate. We have the disjoint coset decompositions

$$G = \bigcup_{j=1}^{n} Hg_j \qquad \text{and} \qquad G = \bigcup_{j=1}^{n} H' g_0 g_j.$$

Thus the graphs $\tilde{X}$ and $\tilde{X}'$ have vertices $\{(v, Hg_j) \mid v \in X,\ 1 \leqslant j \leqslant n\}$ and $\{(v, H'g_0 g_j) \mid v \in X,\ 1 \leqslant j \leqslant n\}$, respectively. The isomorphism $i\colon \tilde{X} \to \tilde{X}'$ is defined by $i(v, Hg) = (v, H'g_0 g)$.

To prove the converse, we suppose that $\tilde{X}$ and $\tilde{X}'$ are conjugate intermediate graphs. We must show that the corresponding subgroups $H = H(\tilde{X})$ and $H' = H(\tilde{X}')$ (using the notation of the fundamental theorem of Galois theory) are conjugate. So we have an isomorphism $i\colon \tilde{X} \to \tilde{X}'$ such that $\pi_1 = \pi'_1 \circ i$. Choose a vertex v_0 in X and let $\tilde{v}_0 = \pi_2(\tilde{\tilde{v}}_0)$ in $\tilde{X}$, where $\tilde{\tilde{v}} = (v_0, 1)$ is on sheet 1 of Y. Let $\tilde{v}$ be an arbitrary vertex of $\tilde{X}$ projecting to a vertex v in X, and let $\tilde{\tilde{v}} = (v, g)$ be a vertex in Y projecting to $\tilde{v}$ under π_2. See Fig. 9. The collection of all $g \in G$ such that $(v, g) \in Y$ projects to $\tilde{v}$

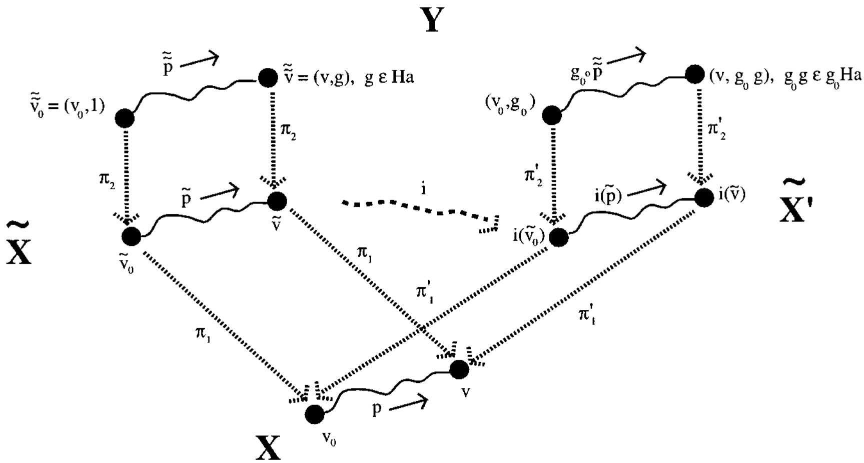

FIG. 9. Proof that conjugate graphs correspond to conjugate subgroups.

is a right coset Ha of G. Let $\tilde{\tilde{p}}$ be a path on Y from $\tilde{\tilde{v}}_0$ to $\tilde{\tilde{v}}$. It projects to a path $\tilde{p}$ in $\tilde{X}$ from $\tilde{v}_0$ to $\tilde{v}$ and to a path p in X from v_0 to v.

Now $i(\tilde{p})$ is a path in $\tilde{X}'$ from $i(\tilde{v}_0)$ to $i(\tilde{v})$ which also projects under π_1' to p by hypothesis. Since $i(\tilde{v}_0)$ projects under π_1' to v_0, we may assume there is an element $g_0 \in G$ such that the vertex (v_0, g_0) in Y projects via π_2' to $i(\tilde{v}_0)$. Now $\pi(g_0 \circ \tilde{\tilde{p}}) = \pi(\tilde{\tilde{p}}) = p$ and since $\pi = \pi_1' \circ \pi_2'$, we see that the path $\pi_2'(g_0 \circ \tilde{\tilde{p}})$ in $\tilde{X}'$ is a path with initial vertex $i(\tilde{v}_0)$ projecting to p in X. By the uniqueness of lifts, we find that $i(\tilde{p}) = \pi_2'(g_0 \circ \tilde{\tilde{p}})$. However $g_0 \circ \tilde{\tilde{p}}$ terminates at $(v, g_0 g)$. Therefore π_2' takes $(v, g_0 g)$ to $i(\tilde{v})$. In particular, the set of all such $g_0 g$ is $g_0 Ha = (g_0 H g_0^{-1})\, g_0 a$. Therefore the subgroup of G corresponding to $\tilde{X}'$ is $g_0 H g_0^{-1}$.

Note that we have actually proved much more: the effect of the isomorphism i can be accomplished by the element $g_0 \in G$. Further, g_0 itself may be replaced by any element of the right coset $(g_0 H g_0^{-1})\, g_0 = g_0 H$, a left coset of H. In this way, there is a 1-1 correspondence between left cosets of H and all possible "embeddings" of $\tilde{X}$ in Y. ∎

Last we come to the question of classifying normal intermediate coverings.

THEOREM 3. *Suppose Y/X is a normal covering with Galois group G and $\tilde{X}$ is an intermediate covering corresponding to the subgroup H of G. We may consider $\tilde{X}$ as a covering of X in its own right with the same projection map used as an intermediate covering. Then $\tilde{X}$ is itself a normal covering of X if and only if H is a normal subgroup of G and when this happens $G(\tilde{X}/X) \cong G/H$.*

Proof. We continue the development begun in the proof of Theorem 2. We may think of $\tilde{X}$ as being given by $\tilde{X}(H)$ (using the notation of the fundamental theorem of Galois theory), whose vertex set is $\{(v, Hg_j) \mid v \in X,\ 1 \leqslant j \leqslant n\}$, where the g_j are right coset representatives for $H \backslash G$.

Now suppose H is a normal subgroup of G. A coset Hg acts on $\tilde{X}(H)$ by taking (v, Hg_j) to (v, Hgg_j). This action preserves edges. It is also transitive on the cosets Hg_j since the identity coset $H \cdot 1$ can be taken to any other coset Hg_j by setting $g = g_j$. Thus we have $n = |G/H|$ automorphisms of $\tilde{X}(H)$. Hence $\tilde{X}(H)$ is normal over X with Galois group G/H.

Conversely, suppose $\tilde{X}/X$ is normal and that i is an automorphism of $\tilde{X}$ in $G(\tilde{X}/X)$. We will think of $\tilde{X}'$ as $\tilde{X}$ with $\pi_1 = \pi_1'$ and $\pi_2 = \pi_2'$. Although i is not the map that makes $\tilde{X}' = \tilde{X}$ (that map is the identity map), nevertheless, i is an isomorphism between $\tilde{X}$ and $\tilde{X}'$ and it is a conjugation map since $\pi_1' \circ i = \pi_1 \circ i = \pi_1$. Thus Theorem 2 applies and there is an element $g_0 \in G$ such that $\tilde{X}'$ corresponds to $g_0 H g_0^{-1}$. Since $\tilde{X}' = \tilde{X}$, we have $g_0 H g_0^{-1} = H$. Further, if $\tilde{v}_0$ in $\tilde{X}$ is chosen as in the proof of Theorem 2, then $\pi_2((v_0, g_0)) = \pi_2'((v_0, g_0)) = i(\tilde{v}_0)$. As i runs through the n elements of

$G(\tilde{X}/X)$, $i(\tilde{v}_0)$ runs through the n lifts of v_0 to $\tilde{X}$. Thus the corresponding n different g_0's run through all n left cosets of H in G, and we have $g_0 H g_0^{-1} = H$ for all of these. Therefore H is normal in G. ▮

2.4. *Primes in Coverings*

In order to discuss Artin L-functions of a normal graph covering Y/X, we need to recall the analogue of primes in graphs and the analogue of a prime in Y dividing a prime in X. See Stark and Terras [14] and Stark [13] for more details on the graph theory and Lang [8] for the number theoretic version.

As we said in Section 1 a *path* C on X follows a finite succession of directed edges. When there are no loops or multiedges, it may also be designated by listing the vertices through which C passes in order $C = \langle v_1, ..., v_k \rangle$, $v_i \in X$. The path C is closed if $v_1 = v_k$. A closed path is called a *cycle*. The *length* of C denoted $v(C)$ is the number of edges in the path. A closed path such that 2 consecutive edges in the path are inverses of each other is said to have *backtracking*. A path such that the first and last edges are inverses of each other is said to have a *tail*. The closed path which runs through the same vertices (and edges) as C but starts at v_j rather than v_1 is an *equivalent* path to C.

DEFINITION 13. A "*prime*" $[C]$ in X is an equivalence class of primitive closed backtrackless, tailless cycles C such that $C \neq D^m$, $m \geqslant 2$ and D a path in X. For brevity, we will sometimes refer to primitive closed backtrackless tailless cycles C as *prime cycles*.

We do not have unique factorization into primes here. Nevertheless we do have primes D in covers Y/X above primes C in X. This allows one to find analogues of the corresponding concepts regarding prime ideals in extensions of algebraic number fields

Suppose D is a prime cycle in a covering Y of X with projection map π. Then $\pi(D)$ is backtrackless and tailless in X but may not be primitive. However, there is a prime cycle C in X and an integer f such that $\pi(D) = C^f$. The integer f is independent of the choice of prime D in $[D]$.

DEFINITION 14. If D is a prime cycle in a covering Y/X with projection map π and $\pi(D) = C^f$, where C is a prime cycle of X, we will say that $[D]$ is a *prime of* Y *above* $[C]$, or more loosely, that D is a *prime above* C and we write $D \mid C$ and $f = f(D, Y/X)$. We say that f is the *residual degree* of D with respect to Y/X.

When Y/X is normal, then for a prime C of X and a given integer j, either every lift of C^j is closed in Y or no lift is closed. Thus the residual degree of $[D]$ above C is the same for all $[D]$ above C. Let $g = g(D, Y/X)$

be the number of primes $[D]$ above $[C]$. Since we are dealing with unramified covers, the analogue of ramification is $e = e(D, Y/X) = 1$ and we have the familiar formula from algebraic number theory for normal covers,

$$efg = d = \text{number of sheets of the cover.} \tag{5}$$

DEFINITION 15. If Y/X is normal and $[D]$ is a prime of Y over $[C]$ in X and σ is in $G(Y/X)$, we refer to $[\sigma \circ D]$ as a *conjugate prime* of Y over C.

We then have $f(\sigma \circ D, Y/X) = f(D, Y/X)$. If $f = f(D, Y/X)$, then as g runs through $G(Y/X)$, $g \circ D$ runs through all possible lifts of C^f from X to Y and thus the conjugates of $[D]$ account for all the primes of Y above $[C]$.

Next we consider the analogues of the familiar transitivity property for f. Given Y/X a finite unramified normal graph covering with Galois group G and suppose $\tilde{X}$ is a legal intermediate cover with $H = Gal(Y/\tilde{X})$. Suppose $\pi_1 : \tilde{X} \to X$ and $\pi_2 : Y \to \tilde{X}$ are the covering maps. Then $\pi : Y \to X$ is the covering map $\pi = \pi_1 \circ \pi_2$. Let E be a prime of Y over the prime C of X and let $\pi_2(E) = D^{f_2}$, where $f_2 = f(E, Y/\tilde{X})$. Then we have the *transitivity property*,

$$f(E, Y/X) = f(E, Y/\tilde{X})\, f(D, \tilde{X}/X). \tag{6}$$

Note that we do not actually need to assume Y/X normal.

EXAMPLE. *Primes in the cube Y over primes in the tetrahedron X.* In Fig. 10 we show a prime $[C]$ of length 3 in X defined by $C = \langle a, d, c, a \rangle$. The prime $[D]$ of Y, with $D = \langle a', d'', c', a'', d', c'', a' \rangle$, has length 6 and is over $[C]$ in X. Note that we can write $D = C_1(\sigma \circ C_1)$, where $C_1 = \langle a', d'', c', a'' \rangle$ and the Galois group is $G = G(Y/X) = \{1, \sigma\}$. We are using the notation $x' = (x, 1)$ and $x'' = (x, \sigma)$ in Y, for $x \in X$. Here $\nu(D) = 2\nu(C) = 6$. In this example $f = 2$ and $g = 1$.

A second example in Y/X is also shown in Fig. 10. It has the prime $[D']$ of Y represented by $D' = \langle a'', c', d'', b'', a'' \rangle$ over the prime $[C']$ represented by $C' = \langle a, c, d, b, a \rangle$ in X. We have $\nu(D') = \nu(C') = 4$. Here $f = 1$ and $g = 2$ since there is another prime D'' in Y over C' also pictured in Fig. 10.

2.5. *Frobenius Automorphisms*

As usual, Y is a normal cover of the graph X with Galois group G. Next we introduce the Frobenius automorphism $[Y/X, [D]]$ for a prime $[D]$ in Y over the prime $[C]$ in X. It has properties analogous to the Frobenius automorphism associated to Galois extensions of algebraic number fields.

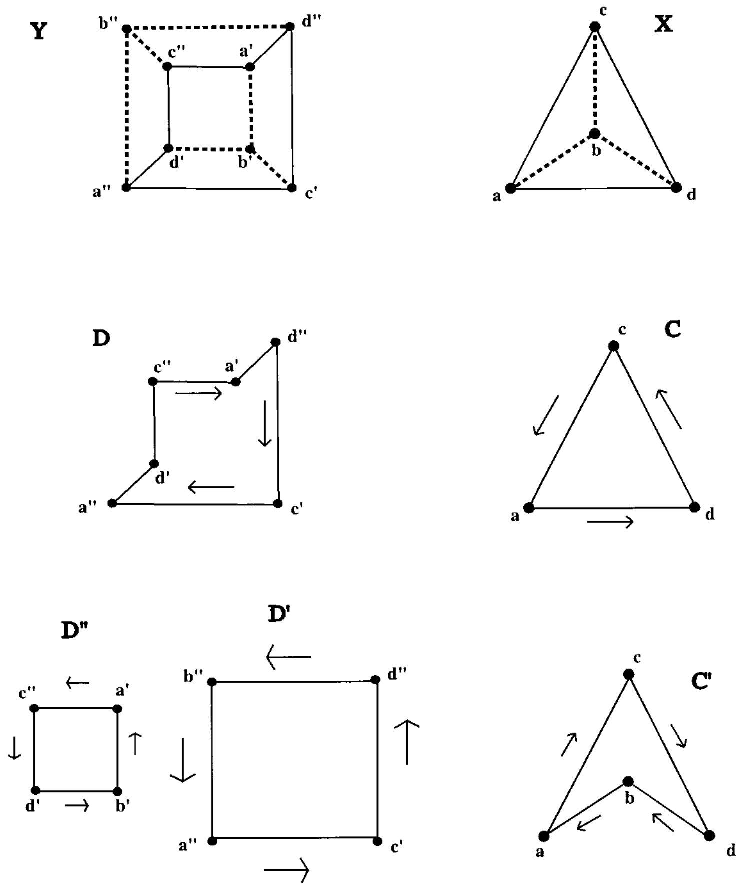

FIG. 10. Primes in the cube dividing primes in the tetrahedron. Here $\nu(D) = 6$, $\nu(C) = 3$, $f = 2$, $g = 1$ and $D'' = \sigma \circ D'$, $\nu(D') = \nu(D'') = \nu(C) = 4$, $f = 1$, $g = 2$.

See Lang [8]. We first define the automorphism $\sigma(p) \in G = G(Y/X)$ associated to a directed path p of X.

DEFINITION 16. Given any path p of X, there is a unique lifting of p to a path $\tilde{p}$ of Y starting on sheet 1. If $\tilde{p}$ has its terminal vertex on sheet γ, we define the *normalized Frobenius automorphism* $\sigma(p)$ in G by $\sigma(p) = \gamma$.

The basic calculational rule for normalized Frobenius automorphisms is given by the following lemma.

LEMMA 1. • (1) *Suppose that p_1 and p_2 are two paths on X with the terminal vertex of p_1 being the initial vertex of p_2. Then $\sigma(p_1 p_2) = \sigma(p_1)\,\sigma(p_2)$.*

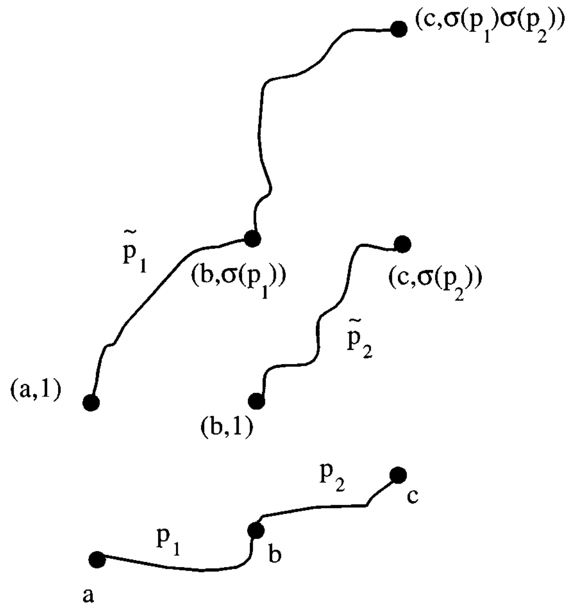

FIG. 11. The map σ preserves composition of paths.

• (2) *If a path p is composed of edges* $e_1, ..., e_n$; *that is*, $p = e_1 \cdots e_n$, *then* $\sigma(p) = \sigma(e_1) \cdots \sigma(e_n)$

Proof. Suppose p_1 goes from a to b in X and p_2 goes from b to c in X. Then the lift $\tilde{p}_1$ of p_1 starting on sheet 1 of Y goes from $(a, 1)$ to $(b, \sigma(p_1))$. The lift $\tilde{p}_2$ of p_2 starting on sheet 1 of Y goes from $(b, 1)$ to $(c, \sigma(p_2))$. See Fig. 11. Therefore the lift of p_2 starting on sheet $\sigma(p_1)$ goes from $(b, \sigma(p_1))$ to $(c, \sigma(p_1)\,\sigma(p_2))$. Thus the lift of $p_1 p_2$ beginning on sheet 1 of Y will end on sheet $\sigma(p_1)\,\sigma(p_2)$. ∎

Now we can define the Frobenius automorphisms and decomposition groups.

DEFINITION 17. Suppose C is a prime cycle on X starting and ending at vertex a. Let D be a prime cycle of Y over C starting and ending at vertex (a, μ) on sheet μ. If the residual degree of D/C is f, then D is the lifting of C^f which begins on sheet μ. Suppose C itself lifts to a path $\tilde{C}$ on Y starting on sheet μ at (a, μ) and ending on sheet v at (a, v). We define the *Frobenius automorphism*

$$[\,Y/X, [D]\,] = v\mu^{-1}.$$

DEFINITION 18. The *decomposition group* of D with respect to Y/X is

$$Z(D) = Z(D,\ Y/X) = \{\tau \in G \mid [\tau \circ D] = [D]\}.$$

Next we need to find analogues of the usual properties of the Frobenius automorphism (as in Lang [8]).

PROPOSITION 2 (Properties of the Frobenius Automorphism).

- (1) *For a prime cycle D in Y over C in X, the Frobenius automorphism is independent of the choice of D in its equivalence class $[D]$. Thus we can define $[Y/X, [D]] = [Y/X, D]$, without ambiguity.*
- (2) *The order of $[Y/X, D]$ in G is the residual degree $f = f(D,\ Y/X)$.*
- (3) *If $\tau \in G$, then $[Y/X, \tau \circ D] = \tau [Y/X, D] \tau^{-1}$.*
- (4) *If D begins on sheet 1, then $[Y/X, D] = \sigma(C)$.*
- (5) *The decomposition group $Z(D)$ is the cyclic subgroup of G of order f generated by $[Y/X, D]$. In particular, $Z(D)$ does not depend on the choice of D in its equivalence class $[D]$.*

Proof. Part (4) is proved by noting that the respective definitions are the same.

Now we prove Part (1). Suppose C has initial (and terminal) point vertex a in X and D is the lifting of C^f beginning at vertex (a, μ_0) on sheet μ_0. In lifting C^f, we lift C a total of f times consecutively, beginning at (a, μ_0) and ending respectively at $(a, \mu_1), (a, \mu_2), ..., (a, \mu_{f-1}), (a, \mu_f)$, where $\mu_f = \mu_0$, and $\mu_j \neq \mu_0$, for $j = 1, 2, ..., f-1$.

Suppose that (b, κ) is another vertex on D, where b is on C. Thus (b, κ) lies on one of the f consecutive lifts of C referred to above, say the rth. See Fig. 12.

The vertex b splits C into two paths $C = p_1 p_2$, where b is the terminal vertex of p_1 and the initial vertex of p_2. The vertex (b, κ) on Y is the terminal vertex of the lift of p_1 to D beginning at (a, μ_{r-1}). The lift of the version of C in $[C]$ beginning at b, namely $p_2 p_1$ to a path on Y which starts at (b, κ) then ends at a vertex (b, λ) on D which lies on the $(r+1)$st consecutive lift of C.

Suppose $\tilde{C}'$ is a path on Y from $(a, 1)$ to (a, μ_0) and that C' is the projection of $\tilde{C}'$ to X. The vertices (a, μ_0), (a, μ_1), (b, κ), and (b, λ) of Y are then the terminal points of the lifts of the paths

$$C',\ C'C,\ C'C^{r-1}p_1,\ C'C^r p_1,$$

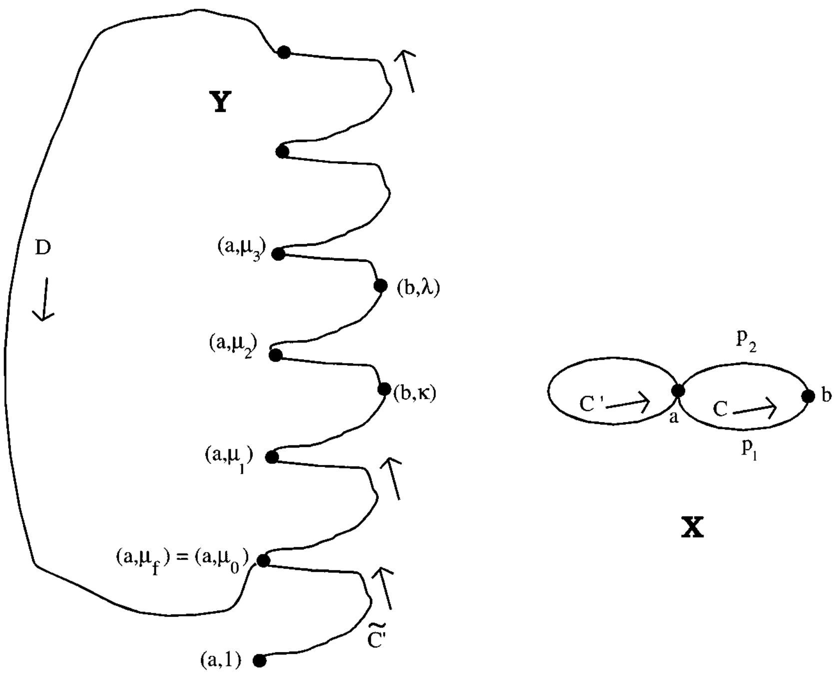

FIG. 12. The vertex (b, κ) lies on the rth consecutive lift of C (shown with $r=2$). The lift to a path in Y starting at (b, κ) of the version of C in $[C]$ starting at b ends at a vertex (b, λ) which arises on the $(r+1)$ st consecutive lift of C.

respectively, to paths on D starting at (a,1). Therefore, by Lemma 1 we have

$$\mu_0 = \sigma(C'), \qquad \mu_1 = \sigma(C'C) = \sigma(C')\,\sigma(C);$$

$$\kappa = \sigma(C'C^{r-1}p_1) = \sigma(C')\,\sigma(C)^{r-1}\,\sigma(p_1);$$

$$\lambda = \sigma(C'C^r p_1) = \sigma(C')\,\sigma(C)^r\,\sigma(p_1).$$

It follows that $[Y/X, D]$ is the common value of

$$\lambda\kappa^{-1} = \mu_1\mu_0^{-1} = \sigma(C')\,\sigma(C)\,\sigma(C')^{-1}.$$

This proves (1). It also proves (3) in the case $\tau = \mu_0^{-1} = \sigma(C')^{-1}$ and this suffices to prove (3) in general.

In the same manner as above, we also find that each

$$\mu_j = \sigma(C'C^j) = \sigma(C')\,\sigma(C)^j$$

and thus

$$\mu_j \mu_0^{-1} = \sigma(C')\, \sigma(C)^j\, \sigma(C')^{-1} = [\, Y/X, D]^j. \tag{7}$$

This proves (2).

Last, we prove (5). If $\tau \circ D$ is equivalent to D, then since $\tau \circ D$ also starts at a vertex projecting to a, we must have $\tau\mu_0 = \mu_j$, for one of the μ_j above. Thus, for some j, $\tau = \mu_j \mu_0^{-1} = [\, Y/X, D]^j$ by (7). Conversely, any such τ has $[\tau \circ D] = [D]$. Since the Frobenius automorphism of D is independent of the choice of starting point, the decomposition group of D depends only on $[D]$. ∎

It remains only to discuss the Frobenius automorphism with respect to intermediate coverings.

THEOREM 4 (More Properties of the Frobenius Automorphism).

- (1) *Suppose $\tilde{X}$ is an intermediate covering to Y/X and corresponds to the subgroup H of $G = G(\, Y/X)$. Let $[D]$ be an equivalence class of prime cycles in Y such that D lies above $\tilde{C}$ in $\tilde{X}$. Let $f = f(D, Y/X) = f_1 f_2$, where $f_2 = f(D, Y/\tilde{X})$ and $f_1 = f(\tilde{C}, \tilde{X}/X)$. Then f_1 is the minimal power of $[\, Y/X, D]$ which lies in H and we have*

$$[\, Y/X, D]^{f_1} = [\, Y/\tilde{X}, D]. \tag{8}$$

- (2) *If further $\tilde{X}$ is normal over X, then as an element of $H \backslash G$, we have*

$$[\tilde{X}/X, \tilde{C}] = H[\, Y/X, D].$$

Proof. (1) Let C be the prime of X below $\tilde{C}$. The Frobenius automorphism $[\, Y/\tilde{X}, D]$ is found by lifting $\tilde{C}$ from $\tilde{X}$ to Y. This is the same as lifting C^{f_1} from X to Y and the analysis in the proof of Proposition 2 (Eq. (7) in particular) gives Eq. (8) of (1). The fact that f_1 is the minimal power of $[\, Y/X, D]$ which lies in H follows from the fact that

$$Z(\, Y/\tilde{X}, D) = Z(\, Y/X, D) \cap H.$$

which we know to be cyclic of order f_2. Therefore since $[\, Y/X, D]$ is of order $f_1 f_2$, we see that $[\, Y/X, D]^j$ cannot be in H if $j < f_1$.

(2) Now suppose that $\tilde{X}$ is normal over X. We think of $\tilde{X}$ as given by our earlier construction where vertices (v, τ) of Y project to vertices $(v, H\tau)$ of $\tilde{X}$. We suppose D starts and ends at (a, μ_0) in Y and that $\tilde{C}$ starts and ends at $(a, H\mu_0)$ in $\tilde{X}$. If C lifts to a path in Y starting at (a, μ_0) and terminating at (a, μ_1), then C lifts to a path in $\tilde{X}$ starting at $(a, H\mu_0)$ and

terminating at $(a, H\mu_1)$. Then (2) follows from the definition of the Frobenius automorphism. ∎

2.6. *The construction of Intermediate Coverings and Minimal Normal Coverings via Frobenius Automorphisms*

LEMMA 2. *Suppose Y/X is normal with Galois group G. Let $e_1, ..., e_r$ be the cut edges of X with directions assigned. The r normalized Frobenius automorphisms $\sigma(e_j)$, $j = 1, ..., r$, generate G.*

Proof. Since $\sigma(t) = 1$ for all edges t on the tree of X, for any path p on X, $\sigma(p)$ is a product of the $\sigma(e_j)$ and their inverses, by Lemma 1. But, by lifting all paths of X to paths starting on sheet 1 of Y, we can get to every sheet of Y and thus we have obtained the whole of G. ∎

LEMMA 3. *Suppose Y/X is normal with Galois group G and $\tilde{X}$ is an intermediate graph corresponding to the subgroup H of G. Let $H_0 = \bigcap_{g \in G} gHg^{-1}$. Then $H_0 = \{1\}$ if and only if there are no intermediate graphs, other than Y, which are normal over X and intermediate between Y and $\tilde{X}$.*

Proof. This is a standard fact in Galois theory. A normal intermediate graph covering $\tilde{X}$ would correspond to a normal subgroup of G contained in H and conversely. Any normal subgroup of G contained in H is also contained in every conjugate of H and hence is contained in H_0. Since H_0 is a normal subgroup of G, we are done. ∎

LEMMA 4. *Suppose $\tilde{X}$ is a covering of X and that Y/X is a normal covering of X of minimal degree such that $\tilde{X}$ is intermediate to Y/X. Let $G = G(Y/X)$ and $H = G(Y/\tilde{X})$. Let $Hg_1, ..., Hg_n$ be the right cosets of H. Any element $g \in G$ gives rise to a permutation of the Hg_j by multiplication on the right. Let $\mu(g)$ denote the corresponding permutation of* 1, ..., n. *That is $\mu(g)(i) = j$ if $Hg_i g = Hg_j$. Then μ is a faithful (i.e.,* $1-1$) *permutation representation of G.*

Proof. By definition $\mu(g)\,\mu(g') = \mu(gg')$, and thus μ is a homomorphism. The homomorphism μ will be 1-1 if and only if its kernel is the identity. The kernel consists of all $g \in G$ such that $Hg'g = Hg'$, $\forall g' \in G$. This happens iff $Hg'gg'^{-1} = H$, $\forall g' \in G$. This is equivalent to $g \in g'^{-1}Hg'$, $\forall g' \in G$. By Lemma 3, $g = 1$ and μ is 1-1. ∎

Now we put these three lemmas together.

THEOREM 5. *Let the graphs Y, $\tilde{X}$, X, the groups G, H, and the representation μ be as in Lemma* 4. *Suppose that e is one of the (directed) cut edges* *of X (i.e., edges not in a given spanning tree of X). Let $\sigma(e)$ be the corresponding normalized Frobenius automorphism of G. Suppose that v is the initial vertex of e and v' is the terminal vertex of e. If $\mu=\mu(\sigma(e))$ is the permutation of $1, \ldots, n$ such that $\mu(i)=\mu(\sigma(e))(i)=j$, then the directed edge e lifts to an edge in $\tilde{X}$ starting at (v, Hg_i) and terminating at (v', Hg_j).*

Proof. By the definition of μ, $Hg_i\sigma(e)=Hg_j$. This means that $g_i\sigma(e)=hg_j$ for some element $h\in H$. By definition of $\sigma(e)$, the edge e lifts to an edge on Y from $(v, 1)$ to $(v', \sigma(e))$. If we apply g_i to this edge, we get an edge on Y commencing at (v, g_i) and terminating at $(v', g_i\sigma(e))=(v', hg_j)$. Hence e lifts to a directed edge on $\tilde{X}$ from (v, Hg_i) to (v', Hg_j). ∎

This theorem allows us to construct intermediate graphs given a normal cover and it also allows us to construct the minimal normal cover Y of X having a given intermediate covering graph $\tilde{X}$ of X as well as the Galois group $G(Y/X)$. We illustrate this with the S_3 example from Fig. 11 of Stark and Terras [14]. We reproduce this here as our Fig. 13. We will present another series of examples based on the simple group of order 168 in Section 6.

EXAMPLE. *An S_3 Cover*. Consider the covering Y_6/X given in Fig. 13. Two intermediate graphs Y_2 and Y_3 are shown. Note that the covering Y_3/X is not normal.

We can view our Galois group $G(Y_6/X)$ as the dihedral group of motions of a regular triangle and write $S_3=D_3=\{I, R, R^2, F, FR, FR^2\}$, where F denotes a flip of the triangle and R a rotation by $120°$, $F^2=I$, $R^3=I$, $FR=R^2F$. We can identify the sheets of the graph Y via: $a'=(a, I)$, $a''=(a, FR^2)$, $a'''=(a, R^2)$, $a^{(4)}=(a, FR)$, $a^{(5)}=(a, R)$, $a^{(6)}=(a, F)$, for $a\in X$.

We define c to be the directed cut edge of X from vertex 2 to vertex 4 and d is the directed cut edge of X from vertex 4 to vertex 3. With the sheets assigned group elements as above, we see that c lifts to an edge of Y_6 from sheet I to sheet FR and hence $\sigma(c)=FR$. Similarly $\sigma(d)=FR^2$. Let $H=\{I, FR\}$, a subgroup of S_3. There are three cosets Hg_i, $i=1, 2, 3$, where we may take $g_1=I$, $g_2=FR^2$, $g_3=F$.

Then $Hg_1\sigma(c)=Hg_1$, $Hg_2\sigma(c)=Hg_3$, $Hg_3\sigma(c)=Hg_2$, and the cycle decomposition of the permutation corresponding to $\sigma(c)$ is (1)(23). Likewise the cycle decomposition of the permutation corresponding to $\sigma(d)$ is (12)(3).

We can now construct the intermediate graph corresponding to H. We take three copies of the tree of X with the vertices in each copy labeled $'$, $''$, $'''$. The permutation (1)(23) corresponding to $\sigma(c)$ tells us to lift c to three edges in the cover; from $2'$ to $4'$, from $2''$ to $4'''$, and from $2'''$ to $4''$.

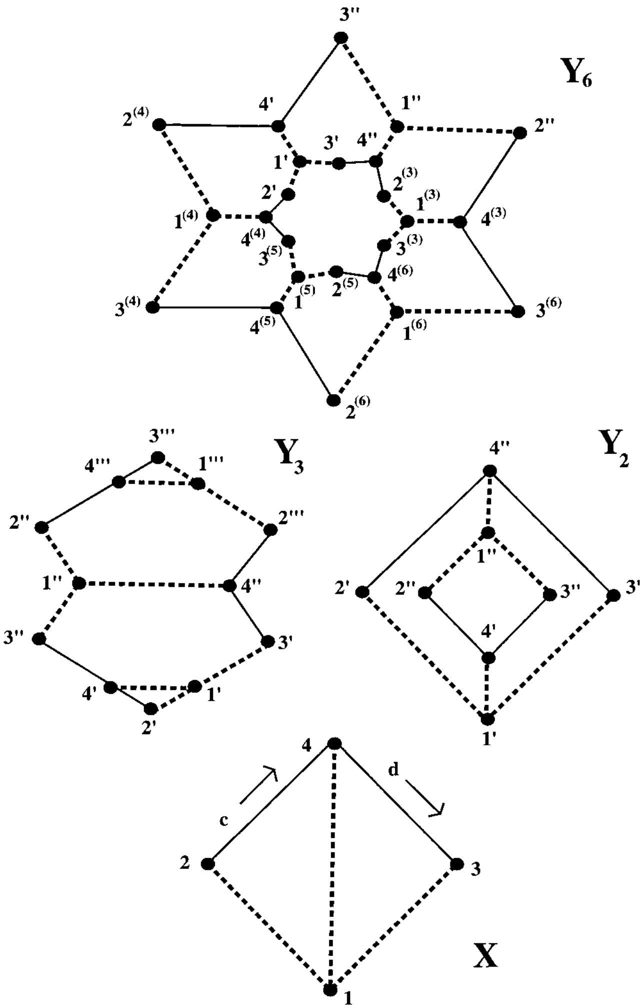

FIG. 13. A sextic S_3 covering Y_6 of a subgraph X of K_4 with intermediate quadratic Y_2 and cubic Y_3 covers. A spanning tree in X is given by dotted lines. The sheets of the coverings are copies of this tree.

Likewise the permutation $(12)(3)$ for $\sigma(d)$ says we lift d to the three edges from $4'$ to $3''$, from $4''$ to $3'$, from $4'''$ to $3'''$. The resulting graph is precisely Y_3 in Fig. 13. Actually the three two element subgroups of S_3 are the conjugates of H. Each has three cosets and in each case with the correct labeling of the cosets (6 choices), we will get Y_3. This is because we have not provided in Fig. 13 the projections from Y_6 to Y_3. Without knowing this projection, the three conjugate intermediate cubic covers of X are all isomorphic with the isomorphism preserving the projections to X by

Theorem 2. In other words, as abstract covers of X, with no reference to Y_6, they are all exactly alike.

Now we discuss the reverse question. Suppose we are given the cover Y_3 of X and wish to construct the minimal normal cover of X having Y_3 as an intermediate graph. We suppose Y is this unknown cover, G the Galois group $G(Y/X)$, and H the subgroup corresponding to Y_3. Thus H has three cosets in G; we label the cosets so that Hg_1, Hg_2, Hg_3 correspond to the sheets $'$, $''$, $'''$, respectively of Y_3. According to Lemma 4, the permutation corresponding to $\sigma(c)$ is $(1)(23)$ and the permutation corresponding to $\sigma(d)$ is $(12)(3)$. Further, by Lemmas 2 and 4, these permutations generate an isomorphic copy of G inside S_3 and under this isomorphism, $\sigma(c)$ becomes $(1)(23)$ and $\sigma(d)$ becomes $(12)(3)$. These two permutations generate all of S_3 and thus $G = S_3$. The corresponding normal cover is then a sextic cover Y_6 of X. The six sheets of Y_6 are copies of the tree of X labeled by the six group elements. If g is one of these elements, we connect vertex $(2, g)$ on sheet g to vertex $(4, g\sigma(c))$ on sheet $g\sigma(c)$. The resulting directed edge with initial vertex $(2, g)$ projects to c. Likewise, we connect vertex $(4, g)$ on sheet g to vertex $(3, g\sigma(d))$ on sheet $g\sigma(d)$ and project the resulting directed edge to d. The resulting normal cover is Y_6 as shown in Fig. 13.

Can we say which subgroup of S_3 corresponds to Y_3? No; we can identify H only up to conjugation. There are several ways of thinking about this. One way is that we do not know which of the three cosets Hg_1, Hg_2, Hg_3 contains the identity element of the group. The three choices give the three embeddings. Equivalently, we can relabel the sheets of Y_3 and the three cosets. On S_n, a relabeling is equivalent to conjugation.

3. VERTEX ARTIN–IHARA L-FUNCTIONS

Suppose that Y is a normal unramified covering of X with Galois group $G = G(Y/X)$.

Definition 19. If ρ is a representation of G with degree $d = d_\rho$, and u is a complex variable with $|u|$ sufficiently small, define the *vertex Artin–Ihara L-function* by

$$L_V(u, \rho, Y/X) = L_V(u, \rho) = \prod_{[C]} \det(I - \rho([Y/X, D])\, u^{\nu(C)})^{-1},$$

where the product runs over primes $[C]$ of X and $[D]$ is arbitrarily chosen from the primes in Y above C. Here $[Y/X, D]$ is the Frobenius automorphism of Definition 17. The subscript V stands for vertex L-function and $\nu(C)$ is the length of a curve C representing the prime $[C]$.

When the representation ρ is trivial ($=1$), this is the Ihara zeta function (for irregular graphs) of formula (1) in the introduction. It was considered in Stark and Terras [14], formula (1.2)

$$L_V(u, 1, Y/X) = \zeta_X(u). \tag{9}$$

For the representation theory of finite groups; e.g., the definition of induced representation needed in the next proposition, see, for example, Terras [16].

Proposition 3 (Formal Properties of the Vertex Artin-L-Function).

- (1) $L_V(u, \rho_1 \oplus \rho_2) = L_V(u, \rho_1)\, L_V(u, \rho_2)$.
- (2) *Let Y/X be an unramified normal covering. Suppose $\tilde{X}$ is intermediate to Y/X and assume $\tilde{X}/X$ is normal, $G = Gal(Y/X)$, $H = Gal(Y/\tilde{X})$. Let ρ be a representation of $G/H \cong Gal(\tilde{X}/X)$. Thus ρ can be viewed as a representation of G, often called the lift of ρ. Then*

$$L_V(u, \rho, Y/X) = L_V(u, \rho, \tilde{X}/X).$$

- (3) *If $\tilde{X}$ is an intermediate cover of the unramified normal graph cover Y/X and ρ is a representation of $H = Gal(Y/\tilde{X})$, then let $\rho^{\#} = Ind_H^G\, \rho$, that is, the representation induced by ρ from H up to G. Then*

$$L_V(u, \rho^{\#}, Y/X) = L_V(u, \rho, Y/\tilde{X}).$$

Proof. Only property (3) requires some effort. We will postpone the proof of (3) until the next section when we do the more general case of edge L-functions. ∎

Corollary (Factorization of the Ihara Zeta Function of an Unramified Normal Extension of Graphs). *Suppose that Y is a normal unramified covering of X with Galois group $G = G(Y/X)$. Let $\hat{G}$ be a complete set of inequivalent irreducible unitary representations of G. Then*

$$\zeta_Y(u) = L_V(u, 1, Y/Y) = \prod_{\rho \in \hat{G}} L_V(u, \rho, Y/X)^{d_\rho}.$$

Proof. Use the fact that

$$Ind_{\{e\}}^G\, 1 = \sum_{\rho \in \hat{G}}{}^{\oplus}\, d_\rho\, \rho. \quad ∎$$

One also has the useful fact that the representations of $G(Y/X)$ can be used to block diagonalize the adjacency matrix of Y. See Dedeo *et al.* [3].

We define some matrices associated to a representation ρ of $G(Y/X)$, where Y/X is a finite unramified normal covering of graphs.

DEFINITION 20. For σ, $\tau \in G$ and vertices a, $b \in X$, define the $|X| \times |X|$ matrix $A(\sigma, \tau)$ by setting the entry $A(\sigma, \tau)_{a,b} =$ the number of directed edges in Y from (a, σ) to (b, τ). Here every undirected edge of Y has been given both directions.

Except when (a, σ) and (b, τ) are the same vertex on Y (i.e., $a = b$ and $\sigma = \tau$), and even then if there is no loop at $(a, \sigma) = (b, \tau)$, $A(\sigma, \tau)_{a,b}$ is simply the number of undirected edges on Y connecting (a, σ) to (b, τ). However if there is a loop at $(a, \sigma) = (b, \tau)$, then it is counted in both directions and thus the undirected loop is counted twice.

Since we can easily show that

$$A(\sigma, \tau) = A(1, \sigma^{-1}\tau),$$

we can write

$$A(\sigma, \tau) = A(\sigma^{-1}\tau).$$

DEFINITION 21. If ρ is a representation of $G(Y/X)$, define

$$A_\rho = \sum_{\sigma \in G} A(\sigma) \otimes \rho(\sigma).$$

Also set

$$Q_\rho = Q \otimes I_d,$$

where $Q =$ the $|X| \times |X|$ diagonal matrix with diagonal entry corresponding to $a \in X$ given by $q_a = (\textit{degree } a) - 1$ and d is the degree of ρ.

Now we can generalize Ihara's theorem (3).

THEOREM 6. *With the hypotheses and definitions above, we have*

$$L_V(u, \rho, Y/X)^{-1} = (1 - u^2)^{(r-1)d} \det(I - A_\rho u + Q_\rho u^2).$$

Here r is the rank of the fundamental group of X.

Proof. See the next section for a proof. ∎

Note. One could simply generalize the proof for the case that the representation is trivial given in Stark and Terras [14, proof of Theorem 2]. Instead we generalize Bass's proof in [1] or at least our version of his proof.

EXAMPLE. *The Cube over the Tetrahedron.* See Fig. 3, where the action of the group $G = G(Y/X) = \{1, \sigma\}$ on Y is denoted with primes; i.e., $x' = (x, 1)$ and $x'' = (x, \sigma)$, for $x \in X$. In this case the representations of G are the trivial representation $\rho_0 = 1$ and the representation ρ defined by $\rho(1) = 1$, $\rho(\sigma) = -1$. So $Q_\rho = 2I_4$. There are two cases.

Case 1. *The representation* $\rho_0 = 1$. Here $A_1 = A(1) + A(\sigma) = A$, where

$$A(1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad A(\sigma) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix},$$

and A is the adjacency matrix of X.

Case 2. *The representation* ρ. Here we find

$$A_\rho = A(1) - A(\sigma) = \begin{pmatrix} 0 & 1 & -1 & -1 \\ 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & 1 & -1 & 0 \end{pmatrix}.$$

Now we proceed to check our formulas for this case. We know by the corollary to Proposition 3 that

$$\begin{aligned} \zeta_Y(u) &= L_V(u, 1, Y/Y) = L_V(u, 1, Y/X)\, L_V(u, \rho, Y/X) \\ &= \zeta_X(u)\, L_V(u, \rho, Y/X). \end{aligned} \tag{10}$$

In Stark and Terras [14], we found that

$$\zeta_X(u)^{-1} = (1-u^2)^2\,(1-u)(1-2u)(1+u+2u^2)^3$$

and

$$\zeta_Y(u)^{-1} = (1-u^2)^2\,(1+u)(1+2u)(1-u+2u^2)^3\,\zeta_X(u)^{-1}.$$

Indeed, we easily check that (since $r = 3$)

$$\begin{aligned} L_V(u, \rho, Y/X)^{-1} &= (1-u^2)^2 \det(I_4 - A'_\rho u + 2u^2 I_4) \\ &= (1-u^2)^2\,(1+u)(1+2u)(1-u+2u^2)^3. \end{aligned}$$

It is noteworthy that here $L_V(u, \rho, Y/X) = \zeta_X(-u)$, although this is not instantly apparent in the determinant formula where $-A \neq A_\rho$.

Note. Thanks to Theorem 6, Eq. (10) can be viewed as a factorization of an 8×8 determinant,

$$\det(I_8 - A_Y u + 2I_8 u^2) = \det(I_4 - A_X u + 2I_4 u^2) \cdot \det(I_4 - A'_\rho u + 2I_4 u^2).$$

EXAMPLE. *The Cube over a Dumbbell.* The covering we consider in this example is Y/X in Fig. 5. The covering group $G(Y/X)$ is the integers mod 4 denoted $\mathbb{Z}_4 = \{0, 1, 2, 3 \pmod 4\}$. We label the sheets as

$$x'_1 = (x, 0 \pmod 4), \qquad x'_2 = (x, 1 \pmod 4),$$
$$x''_1 = (x, 2 \pmod 4), \qquad x''_2 = (x, 3 \pmod 4).$$

The irreducible representations are all one-dimensional and may be written $\chi_\nu(j) = \exp(\frac{2\pi i \nu j}{4}) = i^{\nu j}$, for $j, \nu \in \mathbb{Z}_4$. Note that although X has loops, Y does not. It follows that

$$A(0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad A(1) = A(3) = I_2, \qquad A(2) = 0.$$

Thus

$$A_{\chi_0} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \qquad A_{\chi_1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = A_{\chi_3}, \qquad A_{\chi_2} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}.$$

The corresponding L-functions are

$$L(u, \chi_0, Y/X)^{-1} = (1-u^2) \det \begin{pmatrix} 1-2u+2u^2 & -u \\ -u & 1-2u+2u^2 \end{pmatrix}$$
$$= (1-u^2)(1-u)(1-2u)(1-u+2u^2);$$

$$L(u, \chi_1, Y/X)^{-1} = L(u, \chi_3, Y/X)^{-1} = (1-u^2) \det \begin{pmatrix} 1+2u^2 & -u \\ -u & 1+2u^2 \end{pmatrix}$$
$$= (1-u^2)(1+u+2u^2)(1-u+2u^2)$$

$$L(u, \chi_2, Y/X)^{-1} = (1-u^2) \det \begin{pmatrix} 1+2u+2u^2 & -u \\ -u & 1+2u+2u^2 \end{pmatrix}$$
$$= (1-u^2)(1+u)(1+2u)(1+u+2u^2).$$

One sees again that as in the corollary to Proposition 3

$$\zeta_Y(u) = L(u, \chi_0, Y/X)\, L(u, \chi_1, Y/X)\, L(u, \chi_2, Y/X)\, L(u, \chi_3, Y/X).$$

Note. Again you can view the preceding equality as a factorization of the determinant of an 8×8 matrix as a product of 4 determinants of 2×2 matrices.

EXAMPLE. *An S_3 Cover*. We consider the example from Fig. 13. Then we find our matrices

$$A(I)=\begin{pmatrix}0&1&1&1\\1&0&0&0\\1&0&0&0\\1&0&0&0\end{pmatrix},\qquad A(FR^2)=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix},$$

$$A(FR)=\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&1&0&0\end{pmatrix},\qquad A(R^2)=0,\quad A(R)=0,\quad A(F)=0.$$

Next we need to know the representations of S_3. See Terras [16, Chaps. 16 and 17]. The non-trivial 1-dimensional representation of S_3 has the values $\chi_1(FR)=-1$ and $\chi_1(FR^2)=-1$. The 2-dimensional representation ρ has the values

$$\rho(FR)=\begin{pmatrix}0&\omega^2\\\omega&0\end{pmatrix},\qquad\text{and}\qquad \rho(FR^2)=\begin{pmatrix}0&\omega\\\omega^2&0\end{pmatrix},\quad\text{where}\quad \omega=e^{2\pi i/3}.$$

Now we can compute the matrices in our L-functions,

$$A_{\chi_0}=A,\qquad A_{\chi_1}=\begin{pmatrix}0&1&1&1\\1&0&0&-1\\1&0&0&-1\\1&-1&-1&0\end{pmatrix},$$

$$A_\rho=A_1(I)\otimes\rho(I)+A_1(FR)\otimes\rho(FR)+A_1(FR^2)\otimes\rho(FR^2)$$

$$=\begin{pmatrix}0&1&1&1&0&0&0&0\\1&0&0&0&0&0&0&\omega^2\\1&0&0&0&0&0&0&\omega\\1&0&0&0&0&\omega^2&\omega&0\\0&0&0&0&0&1&1&1\\0&0&0&\omega&1&0&0&0\\0&0&0&\omega^2&1&0&0&0\\0&\omega&\omega^2&0&1&0&0&0\end{pmatrix}.$$

It follows that

$$\begin{aligned}
&L_V(u, \chi_0, Y_6/X)^{-1}\\
&\quad = (1-u^2)\det\begin{pmatrix} 1+2u^2 & -u & -u & -u \\ -u & 1+u^2 & 0 & -u \\ -u & 0 & 1+u^2 & -u \\ -u & -u & -u & 1+2u^2 \end{pmatrix}\\
&\quad = (1-u^2)(1-u)(1+u^2)(1+u+2u^2)(1-u^2-2u^3);\\
&L_V(u, \chi_1, Y_6/X)^{-1}\\
&\quad = (1-u^2)\det\begin{pmatrix} 1+2u^2 & -u & -u & -u \\ -u & 1+u^2 & 0 & u \\ -u & 0 & 1+u^2 & u \\ -u & u & u & 1+2u^2 \end{pmatrix}\\
&\quad = (1-u^2)(1+u)(1+u^2)(1-u+2u^2)(1-u^2+2u^3);\\
&L_V(u, \rho, Y_6/X)^{-1}\\
&\quad = (1-u^2)^2 \det(I_8 - A_\rho u + u^2 Q_\rho).\\
&\quad = (1-u^2)^2\,(1+u+2u^2+u^3+2u^4)(1+u+u^3+2u^4)\\
&\qquad \times (1-u+2u^2-u^3+2u^4)(1-u-u^3+2u^4).
\end{aligned}$$

Putting all our result together, using Theorem 6, we have

$$\begin{aligned}
\zeta_X(u)^{-1} &= L_V(u, \chi_0, Y_6/X)^{-1}\\
&= (1-u^2)(1-u)(1+u^2)(1+u+2u^2)(1-u^2-2u^3);\\
\zeta_{Y_2}(u)^{-1}\,\zeta_X(u) &= L_V(u, \chi_1, Y_2/X)^{-1} = L_V(u, \chi_1, Y_6/X)^{-1}\\
&= (1-u^2)(1+u)(1+u^2)(1-u+2u^2)(1-u^2+2u^3);\\
\zeta_{Y_3}(u)^{-1}\,\zeta_X(u) &= L_V(u, \rho, Y_6/X)^{-1}\\
&= (1-u^2)^2\,(1+u+2u^2+u^3+2u^4)(1+u+u^3+2u^4)\\
&\quad \times (1-u+2u^2-u^3+2u^4)(1-u-u^3+2u^4);
\end{aligned}$$

and

$$\begin{aligned}
\zeta_{Y_6}(u) &= L_V(u, \chi_0, Y_6/X)\, L_V(u, \chi_1, Y_6/X)\, L_V(u, \rho, Y_6/X)^2\\
&= \zeta_X(u)\,\frac{\zeta_{Y_2}(u)}{\zeta_X(u)}\left[\frac{\zeta_{Y_3}(u)}{\zeta_X(u)}\right]^2.
\end{aligned}$$

Thus it follows from the theory that

$$\zeta_X(u)^2\,\zeta_{Y_6}(u)=\zeta_{Y_2}(u)\,\zeta_{Y_3}(u)^2$$

as we noted in Stark and Terras [14, p. 149]. This is analogous to an example of zeta functions of pure cubic extensions of number fields that goes back to Dedekind.

Note. Again the last equality above says that a certain 24×24 determinant involving polynomials in u can be factored as a product of 2 determinants of 4×4 matrices times the square of an 8×8 determinant.

EXAMPLE. *A Klein 4-Group Cover Y/X from Fig.* 6. Here we can identify the Galois group $G=G(Y/X)$ with $\mathbb{Z}_2^2$. The identification is given by $x_1'=(x,(1,0))$, $x_1''=(x,(1,1))$, $x_2'=(x,(0,0))$, $x_2''=(x,(0,1))$.

The characters of G are $\chi_{r,s}(u,v)=(-1)^{ru+sv}$, for $r, s, u, v\in\mathbb{Z}_2$. We find that

$$L_V(u,\chi_{0,0},Y/X)^{-1}=(1-u^2)\det\begin{pmatrix}1+2u^2 & -3u\\ -3u & 1+2u^2\end{pmatrix}$$
$$=Z_X(u)^{-1}=(1-u^2)(1-u)(1+u)(1-2u)(1+2u).$$

Similarly

$$L_V(u,\chi_{0,1},Y/X)^{-1}=(1-u^2)\det\begin{pmatrix}1+2u^2 & -u\\ -u & 1+2u^2\end{pmatrix}$$
$$=L_V(u,\chi_{1,1},Y/X)^{-1}$$
$$=Z_X(u)^{-1}=(1-u^2)(1-u+2u^2)(1+u+2u^2).$$

Also

$$L_V(u,\chi_{1,0},Y/X)^{-1}=(1-u^2)\det\begin{pmatrix}1+2u^2 & u\\ u & 1+2u^2\end{pmatrix}$$
$$=(1-u^2)(1-u+2u^2)(1+u+2u^2).$$

Thus all 3 L-functions with non-trivial characters are equal. This happens here because all 3 intermediate quadratic covers of X are isomorphic as abstract graphs (although not conjugate) and so they have equal zeta functions. Each intermediate zeta function is of the form $\zeta_{\tilde{X}}(u)=\zeta_X(u)\,L_V(u,\chi,Y/X)$, where χ runs through the 3 non-trivial characters of G as $\tilde{X}$ runs through the 3 intermediate quadratic covers of X. For $\zeta_Y(u)$ we have

$$\begin{aligned}\zeta_Y(u)^{-1} &= \prod_{\chi\in\hat{G}} L_V(u,\chi,\,Y/X)^{-1}\\ &= (1-u^2)^4\,(1-u)(1+u)(1-2u)(1+2u)(1-u+2u^2)^3\\ &\quad\times(1+u+2u^2)^3.\end{aligned}$$

We also obtain the relation

$$\zeta_X^2(u)\,\zeta_Y(u)=\zeta_{\tilde{X}}(u)^3$$

valid for all 3 intermediate quadratic covers $\tilde{X}$ of X.

EXAMPLE. *a Cyclic 6-Fold Cover Y/X from Fig.* 7. The covering group $G=G(Y/X)\cong\mathbb{Z}_6=\{1,\ 2,\ 3,\ 4,\ 5,\ 6\pmod 6\}$, with identity element 6 (mod 6). Let $\omega=e^{2\pi i/6}$. The characters are $\chi_a(b)=\omega^{ab}$, for $a,\ b\in\mathbb{Z}_6$. Here the matrices $A(\tau)$ are 1×1. We obtain

$$A(6)=A(3)=0,\qquad A(1)=A(2)=A(4)=A(5)=1.$$

We find that

$$\begin{aligned}&A_{\chi_0}=4=A=\text{adjacency matrix of } X;\\ &A_{\chi_j}=0, \qquad \text{for}\quad j=1,3,5;\\ &A_{\chi_j}=-2, \qquad \text{for}\quad j=2,4.\end{aligned}$$

Then

$$\begin{aligned}&L_V(u,\chi_0,\,Y/X)^{-1}=\zeta_X(u)^{-1}=(1-u^2)(1-u)(1-3u);\\ &L_V(u,\chi_j,\,Y/X)^{-1}=(1-u^2)(1+3u^2), \qquad \text{for}\quad j=1,3,5;\\ &L_V(u,\chi_j,\,Y/X)^{-1}=Z_X(u)^{-1}=(1-u^2)(1+2u+3u^2), \qquad \text{for}\quad j=2,4.\end{aligned}$$

Set

$$m=\begin{pmatrix}1+3u^2 & -u & -u & 0 & -u & -u\\ -u & 1+3u^2 & -u & -u & 0 & -u\\ -u & -u & 1+3u^2 & -u & -u & 0\\ 0 & -u & -u & 1+3u^2 & -u & -u\\ -u & 0 & -u & -u & 1+3u^2 & -u\\ -u & -u & 0 & -u & -u & 1+3u^2\end{pmatrix}.$$

By Ihara's formula

$$\begin{aligned}\zeta_Y(u)^{-1} &= (1-u^2)^6 \det(m)\\ &= (1-u^2)^6\,(3u-1)(u-1)(3u^2+2u+1)^2\,(1+3u^2)^3,\end{aligned}$$

which agrees with the product

$$\zeta_Y(u) = \prod_{\chi \in \hat{G}} L_V(u, \chi, Y/X).$$

4. MULTIEDGE ARTIN L-FUNCTIONS

A variable attached to a directed edge can be thought of as attached to an ordered pair of vertices—the initial and terminal vertices of that edge. It will be very fruitful to attach variables to pairs of edges, and later to pairs of paths. Suppose that the directed edges of X are $e_1, ..., e_{2|E|}$. We attach a variable w_{ij} to each ordered pair of edges e_i and e_j such that e_i "feeds into" e_j; that is, whenever the terminal vertex of e_i is the initial vertex of e_j, PROVIDED THAT e_i is not the inverse of e_j. To all other pairs i, j we set $w_{ij}=0$. In this manner, we create a $2|E| \times 2|E|$ matrix $W=(w_{ij})$ whose entries are the multiedge variables together with zeros elsewhere. We call W the *multiedge matrix*.

We recall the edge norm of a cycle from Definition 1. For a backtrackless, tail-less cycle C made up of the n directed edges $a_1, ..., a_n$ (that is, a path of minimum length in its free homotopy class), we define the *edge norm* of C to be

$$\mathbb{N}(C) = \mathbb{N}_E(C) = w(a_1, a_2)\, w(a_2, a_3) \cdots w(a_{n-1}, a_n)\, w(a_n, a_1).$$

Here if $a' = e_i$, $a'' = e_j$, then $w(a', a'') = w_{ij}$. Let ρ be a representation of $G = G(Y/X)$.

DEFINITION 22. The *multiedge Artin L-function* is defined by

$$L(W, \rho) = L_E(W, \rho, Y/X) = \prod_{[C]} \det(I - \rho([Y/X, D])\, \mathbb{N}_E(C))^{-1}.$$

Here $W=(w_{ij})$ is the multiedge matrix above. The product runs over all primes $[C]$ of X, D is arbitrarily chosen from the primes in Y over C, and $[Y/X, D]$ is the Frobenius automorphism of Definition 17.

As for the vertex L-function, the determinant does not depend on the choice of D above C since the various $[Y/X, D]$ are conjugate to each other.

The multiedge Artin L-function clearly specializes to the vertex Artin–Ihara L-function $L_V(u, \rho, Y/X)$ of Definition 19 when all non-zero variables w_{ij} are set equal to the same complex variable u.

LEMMA 5. *Suppose that ρ is a representation of $G = G(Y/X)$ and χ is the corresponding character. Then*

$$\log(L_E(W, \rho, Y/X)) = \sum_{[C]} \sum_{j=1}^{\infty} \frac{1}{j} \chi(\sigma(C)^j) \, \mathbb{N}_E(C)^j.$$

Thus $L_E(W, \rho, Y/X)$ depends only on the character χ of the representation ρ and not on the particular representation ρ in it equivalence class.

Proof. This is a standard formula in the theory of Artin L-functions. ∎

One can, as usual, relate the multiedge Artin L-function of the trivial representation to what we called the edge zeta function $\zeta_E(W, X)$ in Definition 2,

$$\zeta_E(W, X) = \prod_{[C]} \det(I - \mathbb{N}_E(C))^{-1}.$$

The following proposition lists some of the formal properties of the multi-edge L-function. However, we postpone discussion of the induction property (which is the analogue of part (3) of Proposition 3) until later in this section.

PROPOSITION 4 (Some Formal Properties of the Multiedge Artin L-Function).

- (1) $L_E(W, 1, Y/X) = \zeta_E(W, X)$.
- (2) $L_E(W, \rho_1 \oplus \rho_2, Y/X) = L_E(W, \rho_1, Y/X) \, L_E(W, \rho_2, Y/X)$.
- (3) *Let Y/X be an unramified normal covering. Suppose $\tilde{X}$ is intermediate to Y/X and assume $\tilde{X}/X$ is normal, $G = Gal(Y/X)$, $H = Gal(Y/\tilde{X})$. Let ρ be a representation of $G/H \cong Gal(\tilde{X}/X)$. Thus ρ can be viewed as a representation of G, often called the lift of ρ. Then*

$$L_E(W, \rho, Y/X) = L_E(W, \rho, \tilde{X}/X).$$

Proof. These are the exact analogues of formula (9) and parts (1) and (2) of Proposition 3 on the vertex Artin L-function. ∎

We need the following lemma as in [14, Part I].

LEMMA 6. *Suppose we have two power series*

$$f(x_1, ..., x_n) = \sum_{i_1, ..., i_n} a(i_1, ..., i_n)\, x_1^{i_1} \cdots x_n^{i_n},$$

and

$$g(x_1, ..., x_n) = \sum_{i_1, ..., i_n} b(i_1, ..., i_n)\, x_1^{i_1} \cdots x_n^{i_n}$$

(*i.e., the sums are over all n-tuples of non-negative integers*) *with* $a(0, ..., 0) = b(0, ..., 0) = 0$. *Then* $f(x_1, ..., x_n) = g(x_1, ..., x_n)$ *iff*

$$\sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} f(x_1, ..., x_n) = \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} g(x_1, ..., x_n).$$

Proof. This is clear since the differential operator takes each monomial in the x_j to itself times the total degree of the monomial. ∎

THEOREM 7 (The Multiedge Artin L-Function Is the Inverse of a Polynomial). *Suppose the representation ρ has degree d. If, in block form, we set $W_\rho = (w_{ij}\rho(\sigma(e_i)))$, with the excess w_{ij} all zero, and I is the $2\,|E|\,d \times 2\,|E|\,d$ identity matrix, then*

$$L_E(W, \rho, Y/X) = \det(I - W_\rho)^{-1}.$$

Proof. By Lemma 5, we have, with $\chi = \mathrm{Tr}(\rho)$,

$$\begin{aligned}\sum_{w_{ij}} w_{ij} \frac{\partial}{\partial w_{ij}} \log(L_E(W, \rho, Y/X)) &= \sum_{[C]} \sum_{j=1}^{\infty} \nu(C)\, \chi(\sigma(C)^j)\, \mathbb{N}_E(C^j) \\ &= \sum_{C} \sum_{j=1}^{\infty} \chi(\sigma(C)^j)\, \mathbb{N}_E(C^j).\end{aligned}$$

Now the sum is over paths C rather than classes $[C]$.

On the other hand, the block i_1, i_{n+1} entry of W_ρ^n is

$$\begin{aligned}&\sum_{i_2, ..., i_n} w(i_1, i_2) \cdots w(i_n, i_{n+1})\, \rho(\sigma(i_1)) \cdots \rho(\sigma(i_n)) \\ &\quad = \sum_{\substack{C = i_1 \cdots i_n \\ \nu(C) = n}} w(i_1, i_2) \cdots w(i_n, i_{n+1})\, \rho(\sigma(C)).\end{aligned}$$

Here the sum is over all paths C on X of length n with leading edge i_1. The sum may be restricted to those paths C whose initial edge is i_1 and whose terminal edge i_n feeds into i_{n+1} with the additional stipulation that i_{n+1} is not the inverse edge to i_n, since all remaining paths contribute 0 to the sum. Thus when $i_{n+1}=i_1$, we are talking about closed backtrackless, tailless cycles. Hence

$$\sum_{w_{ij}} w_{ij} \frac{\partial}{\partial w_{ij}} \log(L_E(W, \rho, Y/X)) = \mathrm{Tr}((I - W_\rho)^{-1})$$

$$= \sum_{w_{ij}} w_{ij} \frac{\partial}{\partial w_{ij}} \log(\det((I - W_\rho)^{-1})).$$

By Lemma 6, the proof is complete. ∎

For Artin's fundamental induction theorem to hold, we must relate, or specialize, the edge variables of a covering graph of X to the edge variables of X. We prove the induction theorem after defining more precisely what we mean by specializing.

Definition 23. Suppose that $\tilde{X}$ is an unramified covering of X and that $\tilde{W}$ and W are the corresponding multiedge matrices. Suppose $\tilde{e}_i$ and $\tilde{e}_j$ are two edges of $\tilde{X}$ and e_i and e_j are the projections of these edges in X. If $\tilde{e}_i$ feeds into $\tilde{e}_j$ and $\tilde{e}_i \neq \tilde{e}_j^{-1}$, then e_i feeds into e_j and $e_i \neq e_j^{-1}$. Thus we can set the variable $\tilde{w}(\tilde{e}_i, \tilde{e}_j) = w(e_i, e_j)$. When we do this for all the variables of $\tilde{W}$, we say we have *specialized the multiedge variables of* $\tilde{X}$ *to the projected multiedge variables of* X. We denote the resulting specialized matrix by $\tilde{W}_{spec}$. For a prime $\tilde{C}$ of $\tilde{X}$ we write $\mathbb{N}_E(\tilde{C})_{spec}$ to mean that we take the edge norm $\mathbb{N}_E(\tilde{C})$ and specialize all the $\tilde{w}(\tilde{e}_i, \tilde{e}_j)$ as above.

Now we come to the induction theorem for L-functions. The heart of the classical number theory proof is contained in the following lemma, whose corollary we will need in Section 6.

Lemma 7. *Suppose Y/X is a finite unramified normal covering with Galois group G and that H is the subgroup of G corresponding to an intermediate covering $\tilde{X}$. Suppose χ is a character of H and $\chi^{\#}$ is the corresponding induced character of G. For any prime $[C]$ of X, we have*

$$\sum_{j=1}^{\infty} \frac{1}{j} \chi^{\#}(\sigma(C)^j)\, \mathbb{N}_E(C)^j = \sum_{[\tilde{C}] \mid [C]} \sum_{j=1}^{\infty} \frac{1}{j} \chi(\tilde{\sigma}(\tilde{C})^j)\, \mathbb{N}_E(\tilde{C})^j_{spec}. \quad (11)$$

Here $\sigma(C)$ is the normalized Frobenius automorphism in G corresponding to C in X and $\tilde{\sigma}(\tilde{C})$ is the normalized Frobenius automorphism in H corresponding to $\tilde{C}$ in $\tilde{X}$.

Proof. Let D_1 be the prime of Y above C starting on sheet 1 so that $\sigma(C) = [Y/X, D_1]$. Using the Frobenius formula for the induced character, we have

$$\sum_{j=1}^{\infty} \frac{1}{j} \chi^{\#}(\sigma(C)^j)\, \mathbb{N}_E(C)^j$$

$$= \sum_{j=1}^{\infty} \sum_{\substack{g \in G \\ (g\sigma(C)\, g^{-1})^j \in H}} \frac{1}{j\,|H|} \chi((g\sigma(C)\, g^{-1})^j)\, \mathbb{N}_E(C)^j.$$

Each distinct $[D]$ of Y above C is of the form $D = g \circ D_1$ and occurs for $f = f(D, Y/X)$ elements of G, where f is the residual degree of Definition 14. From Proposition 2 we obtain

$$\sum_{j=1}^{\infty} \frac{1}{j\,|H|} \sum_{\substack{g \in G \\ (g\sigma(C)\, g^{-1})^j \in H}} \chi((g\sigma g^{-1})^j)\, \mathbb{N}(C)^j$$

$$= \sum_{[D] \mid [C]} \sum_{\substack{j \geqslant 1 \\ [Y/X, D]^j \in H}} \frac{f}{j\,|H|} \chi([Y/X, D]^j)\, \mathbb{N}(C)^j.$$

We group the various D over C into those over a fixed $\tilde{C}$ and then sum over the $\tilde{C}$. Once we do this, for a fixed $\tilde{C}$, all D dividing $\tilde{C}$ have the same minimal power $j = f_1 = f(\tilde{C}, \tilde{X}/X)$ such that $[Y/X, D]^j \in H$. This power gives the Frobenius automorphism of D with respect to $Y/\tilde{X}$ by Theorem 4. Thus our last double sum is

$$\sum_{[\tilde{C}] \mid [C]} \sum_{[D] \mid [\tilde{C}]} \sum_{j \geqslant 1} \frac{f}{f_1 j\,|H|} \chi([Y/\tilde{X}, D]^j)\, \mathbb{N}(C)^{f_1 j}.$$

For all $[D] \mid [\tilde{C}]$, the $[Y/\tilde{X}, D]$ are conjugate to each other in H and there are g_2 such D where $g_2 f_2 = |H|$. Here $f_2 = f(D, Y/\tilde{X})$ and $g_2 = g(D, Y/\tilde{X})$. If we pick one fixed D above $\tilde{C}$, we therefore get

$$\sum_{[D] \mid [\tilde{C}]} \sum_{j \geqslant 1} \frac{f}{f_1 j\,|H|} \chi([Y/\tilde{X}, D]^j)\, \mathbb{N}_E(C)^{f_1 j}$$

$$= \sum_{j \geqslant 1} \frac{f g_2}{f_1 j\,|H|} \chi([Y/\tilde{X}, D]^j)\, \mathbb{N}(C)^{f_1 j}$$

$$= \sum_{j \geqslant 1} \frac{1}{j} \chi([Y/\tilde{X}, D]^j)\, \mathbb{N}(C)^{f_1 j}.$$

Putting the chain of equalities together proves the lemma, since

$$\mathbb{N}(C)^{f_1} = \mathbb{N}(\tilde{C})_{spec}. \quad \blacksquare$$

COROLLARY. *Suppose Y/X is a finite unramified normal covering with Galois group G and that H is the subgroup of G corresponding to an intermediate cover $\tilde{X}$. Let $\chi_1^{\#}$ be the character of G induced from the trivial character χ_1 of H. Then the number of primes $[\tilde{C}]$ of $\tilde{X}$ above a prime $[C]$ of X with lengths $\nu(\tilde{C}) = \nu(C)$ is $\chi_1^{\#}(\sigma(C))$, where $\sigma(C)$ denotes the normalized Frobenius automorphism of Definition* 16. *In other words, $\chi_1^{\#}(\sigma(C))$ is the number of primes of $\tilde{X}$ above $[C]$ with residual degree* 1.

Proof. We take $\chi = \chi_1$ in Lemma 7 and set each non-zero edge variable $w_{ij} = u$. After this substitution, $\mathbb{N}_E(C) = u^{\nu(C)}$ and $\mathbb{N}_E(\tilde{C})_{spec} = u^{\nu(\tilde{C})}$. We look at the $u^{\nu(C)}$ term on both sides of Eq. (11). On the left side we are looking at the $j=1$ term and get $\chi^{\#}(\sigma(C))$ as the coefficient of $u^{\nu(C)}$. On the right side, only $j=1$ occurs and only when $\nu(\tilde{C}) = \nu(C)$. Thus the coefficient of the $u^{\nu(C)}$ term on the right is the number of $[\tilde{C}]$ above $[C]$ with $\nu(\tilde{C}) = \nu(C)$. $\blacksquare$

THEOREM 8 (Induction Property of Multiedge L-Functions). *Suppose Y/X is a finite unramified normal graph covering. If H is a subgroup of G corresponding to the intermediate covering $\tilde{X}$ and $\rho^{\#}$ is the representation of G induced by a representation ρ of H, then*

$$L_E(\tilde{W}_{spec}, \rho, Y/\tilde{X}) = L_E(W, \rho^{\#}, Y/X).$$

Proof. Note that the two determinants arising from Theorem 7 are the same size. For the reader familiar with induced representations, looked at properly, the two determinants are the same. However, here we will complete the classical number theory proof based on Lemma 7.

By Lemma 5 we have

$$\log(L_E(W, \rho^{\#}, Y/X)) = \sum_{[C]} \sum_{j=1}^{\infty} \frac{1}{j} \chi^{\#}(\sigma(C)^j) \, \mathbb{N}(C)^j.$$

By Lemma 7 the right side is

$$\sum_{[\tilde{C}]} \sum_{j=1}^{\infty} \frac{1}{j} \chi(\tilde{\sigma}(\tilde{C})^j) \, \mathbb{N}_E(\tilde{C})^j_{spec},$$

where the sum is over all primes $\tilde{C}$ of $\tilde{X}$ and $\tilde{\sigma}(\tilde{C})$ is the corresponding normalized Frobenius automorphism in H. By Lemma 5 again, we are done. $\blacksquare$

COROLLARY. *Suppose Y/X is an unramified normal graph covering with Galois group $G(Y/X)$. Then*

$$\zeta_E(\tilde{W}_{spec}, Y) = \prod_{\rho \in \hat{G}} L_E(W, \rho, Y/X)^{d_\rho}.$$

Next we prove Theorem 6 in the preceding section.

Proof of Theorem 6. Bass [1] gave the first proof of the quadratic formula for zeta functions of irregular graphs (Theorem 6 when ρ is the trivial representation). His ingenious proof, which we generalize here for any representation ρ, simply transforms the multiedge formula in Theorem 7 when all the non-zero w_{ij} are specialized to equal u into the quadratic formula of Theorem 6. To accomplish this, in our language, Bass introduces two fundamental $|V| \times 2|E|$ zero-one matrices relating vertices to edges. The first of these is the *starting matrix* (we would call it the initial matrix, but the letter "I" is already taken), $S = (s_{ve})$ indexed by vertices v and directed edges e with $s_{ve} = 1$ if v is the starting (initial) vertex of e and $s_{ve} = 0$ otherwise. We assume that the e are ordered as usual so that $e_{j+|E|}$ is the inverse edge to e_j. The second matrix is the *terminal matrix* $T = (t_{ve})$ with the same indices and $t_{ve} = 1$ if v is the terminal vertex of e and $t_{ve} = 0$ otherwise. Note that each column of S and T has precisely one non-zero entry. These matrices are also related by a simple formula, upon setting

$$J = \begin{pmatrix} 0 & I_{|E|} \\ I_{|E|} & 0 \end{pmatrix}.$$

We then have

$$SJ = T, \qquad TJ = S,$$

since the starting (terminal) vertex of an edge e_j is the terminal (respectively starting) vertex of $e_{j+|E|}$.

From these matrices, we obtain all the matrices of our theorems when the representation ρ is trivial. Indeed we easily find that the $|V| \times |V|$ matrices A and Q are given by

$$A = S\,{}^tT, \qquad Q + I_{|V|} = S\,{}^tS = T\,{}^tT,$$

where ${}^tS$ denotes the transpose of the matrix S. Notice that the product for A counts exactly the number of undirected edges connecting two distinct vertices $v_i \neq v_j$, and counts two times the number of loops at each vertex.

Also define the $2\,|E| \times 2\,|E|$ matrix W_1 to be the matrix obtained from W by setting all the non-zero w_{ij} equal to 1. We will multiply W_1 by u to match our zeta function. Then we have

$$W_1 + J = {}^tTS.$$

The J compensates for the fact that we do not allow edges e_j to feed into edges $e_{j \pm |E|}$, whereas the product on the right counts such pairs.

We are ready to generalize all these matrices to include representations. We suppose we have a normal unramified cover Y/X with group $G = G(Y/X)$ and that ρ is a d-dimensional representation of G. If $B = (b_{ij})$ is any matrix, by $B \otimes I_d$ we mean explicitly the block matrix $(b_{ij} I_d)$. With this agreement define

$$S_\rho = S \otimes I_d, \qquad T_\rho = T \otimes I_d, \qquad J_\rho = J \otimes I_d, \qquad Q_\rho = Q \otimes I_d.$$

Again we have

$$S_\rho J_\rho = T_\rho, \qquad T_\rho J_\rho = S_\rho, \qquad Q_\rho + I_{|V|\,d} = S_\rho\, {}^tS_\rho = T_\rho\, {}^tT_\rho. \tag{12}$$

Our most important two matrices, the analogues of A and W_1, have the representation ρ itself built in and not just the dimension of ρ. For them, we need an auxiliary block diagonal matrix R_ρ incorporating the normalized Frobenius automorphism of each edge. Define

$$R_\rho = \begin{pmatrix} \rho(\sigma(e_1)) & 0 & \cdots & 0 \\ 0 & \rho(\sigma(e_2)) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \rho(\sigma(e_{2\,|E|})) \end{pmatrix}.$$

The matrix A_ρ introduced in Definition 13 of the preceding section is then given by

$$A_\rho = S_\rho R_\rho\, {}^tT_\rho. \tag{13}$$

The definition for W_ρ (with all w_{ij} set equal to 1 and which will be multiplied by u in proving our theorem) becomes

$$W_\rho = R_\rho W_1 \otimes I_d.$$

Thus we have

$$W_\rho + R_\rho J_\rho = R_\rho\, {}^tT_\rho S_\rho. \tag{14}$$

The matrix R_ρ has one important calculational property of its own. Since any edge e has the property that ee^{-1} lifts from X to Y to a two edge path starting and ending on sheet 1, we see that

$$\sigma(e)\,\sigma(e^{-1})=\sigma(ee^{-1})=1.$$

Hence $\rho(\sigma(e^{-1}))=\rho(\sigma(e))^{-1}$ as well. This means that we can write the matrix R_ρ in block form

$$R_\rho=\begin{pmatrix} U & 0 \\ 0 & U^{-1} \end{pmatrix}. \tag{15}$$

Then

$$(R_\rho J_\rho)^2=I_{2\,|E|\,d}. \tag{16}$$

We now come to the heart of our generalization of Bass's proof. In the following Lemma all the matrices are square of size $(|V|+2\,|E|)\,d$ written in block form. The first block of each matrix row has $|V|\,d$ rows of the matrix and the first block column has $|V|\,d$ columns. The second block row has $2\,|E|\,d$ rows and the same is true of the second block column.

LEMMA 8. *We have the block matrix identity*

$$\begin{pmatrix} I_{|V|\,d} & 0 \\ R_\rho\,{}^tT_\rho & I_{2\,|E|\,d} \end{pmatrix}\begin{pmatrix} I_{|V|\,d}(1-u^2) & S_\rho u \\ 0 & I_{2\,|E|\,d}-W_\rho u \end{pmatrix}$$
$$=\begin{pmatrix} I_{|V|\,d}-A_\rho u+Q_\rho u^2 & S_\rho u \\ 0 & I_{2\,|E|\,d}+R_\rho J_\rho u \end{pmatrix}\begin{pmatrix} I_{|V|\,d} & 0 \\ R_\rho\,{}^tT_\rho-{}^tS_\rho u & I_{2\,|E|\,d} \end{pmatrix}.$$

Proof. Using the relation (14), the matrix product on the left in the lemma is

$$\begin{pmatrix} I_{|V|\,d}(1-u^2) & S_\rho u \\ R_\rho\,{}^tT_\rho(1-u^2) & I_{2\,|E|\,d}+R_\rho J_\rho u \end{pmatrix}. \tag{17}$$

Using (12), (13), and (16) we find that the right hand side of our formula is given by (17) as well. This proves the lemma. ∎

Now we complete the proof of Theorem 6. Taking the determinant of the matrix equality in Lemma 8, we get

$$(1-u^2)^{|V|\,d}\det(I_{2\,|E|\,d}-W_\rho u)$$
$$=\det(I_{|V|\,d}-A_\rho u+Q_\rho u^2)\det(I_{2\,|E|\,d}+R_\rho J_\rho u).$$

We need to calculate the last determinant. By (15), we can split up $I_{2|E|d} + R_\rho J_\rho u$ into $|E|\, d \times |E|\, d$ blocks,

$$I_{2|E|d} + R_\rho J_\rho u = \begin{pmatrix} I_{|E|d} & Uu \\ U^{-1}u & I_{|E|d} \end{pmatrix}.$$

Hence

$$\begin{pmatrix} I_{|E|d} & 0 \\ -U^{-1}u & I_{|E|d} \end{pmatrix} (I_{2|E|d} + R_\rho J_\rho u) = \begin{pmatrix} I_{|E|d} & Uu \\ 0 & I_{|E|d}(1-u^2) \end{pmatrix}$$

and

$$\det(I_{2|E|d} + R_\rho J_\rho u) = (1-u^2)^{|E|d}.$$

Theorem 6 follows. ∎

Remarks. Note that the multiedge L-functions of a graph become those of other graphs by specializing variables in certain ways. For example, you can contract an edge to a point by setting the corresponding variable equal to 1. You can cut an edge by setting a variable equal to 0. If you set $w_{ij} = w_{i+|E|,\, j} = 0$, for fixed i, you have eliminated the edge e_i from the graph. If you just set $w_{ij} = 0$, you have made e_i a one-way edge.

Example. *The Multiedge L-Function of a Cube Covering a Dumbbell.* See Fig. 5 for the covering of the cube over the dumbbell. The multiedge L-functions for the representations of the Galois group of Y/X, which is $\mathbb{Z}_4$, require the matrix W which has entries w_{ij}, when edge e_i feeds into edge e_j. For the labeling of the edges of the dumbbell, see Fig. 14. We find that the matrix W is

$$W = \begin{pmatrix} w_{11} & w_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & w_{23} & 0 & 0 & w_{26} \\ 0 & 0 & w_{33} & 0 & w_{35} & 0 \\ 0 & w_{42} & 0 & w_{44} & 0 & 0 \\ w_{51} & 0 & 0 & w_{54} & 0 & 0 \\ 0 & 0 & 0 & 0 & w_{65} & w_{66} \end{pmatrix}.$$

Next we need to compute $\sigma(e_i)$ for each edge e_i and each $\sigma \in G(Y/X)$. We will write $G(Y/X) = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$, where $(x, \sigma_j) = x^{(j)}$, for $x \in X$. The identification of $G(Y/X)$ with $\mathbb{Z}_4$ sends σ_j to $(j-1 \pmod 4)$). Then compute the Galois group elements associated to the edges: $\sigma(e_1) = \sigma_2$, $\sigma(e_2) = \sigma_1$, $\sigma(e_3) = \sigma_2$, $\sigma(e_4) = \sigma_4$, $\sigma(e_5) = \sigma_1$, $\sigma(e_6) = \sigma_4$. The representations of our group are 1-dimensional, given by $\chi_a(\sigma_b) = i^{a(b-1)}$, for $a, b \in \mathbb{Z}_4$.

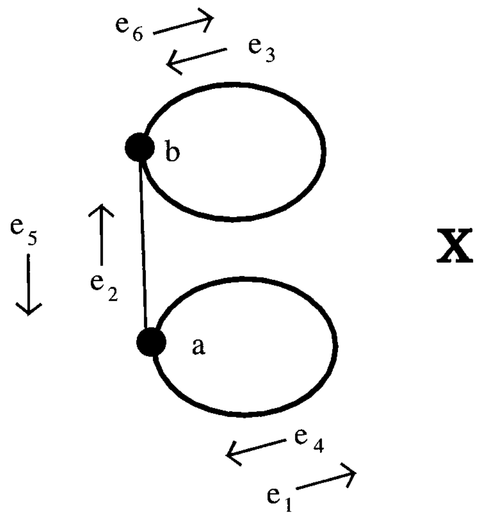

FIG. 14. Edge labelings for the dumbbell.

So we obtain

$$L_E(W, \chi_0, Y/X)^{-1}$$

$$= \zeta_E(W, X)^{-1}$$

$$= \det \begin{pmatrix} w_{11}-1 & w_{12} & 0 & 0 & 0 & 0 \\ 0 & -1 & w_{23} & 0 & 0 & w_{26} \\ 0 & 0 & w_{33}-1 & 0 & w_{35} & 0 \\ 0 & w_{42} & 0 & w_{44}-1 & 0 & 0 \\ w_{51} & 0 & 0 & w_{54} & -1 & 0 \\ 0 & 0 & 0 & 0 & w_{65} & w_{66}-1 \end{pmatrix};$$

$$L_E(W, \chi_1, Y/X)^{-1}$$

$$= \det(I - W_{\chi_1})$$

$$= \det \begin{pmatrix} iw_{11}-1 & iw_{12} & 0 & 0 & 0 & 0 \\ 0 & -1 & w_{23} & 0 & 0 & w_{26} \\ 0 & 0 & iw_{33}-1 & 0 & iw_{35} & 0 \\ 0 & -iw_{42} & 0 & -iw_{44}-1 & 0 & 0 \\ w_{51} & 0 & 0 & w_{54} & -1 & 0 \\ 0 & 0 & 0 & 0 & -iw_{65} & -iw_{66}-1 \end{pmatrix};$$

$$L_E(W, \chi_2, Y/X)^{-1}$$
$$= \det(I - W_{\chi_2})$$
$$= \det \begin{pmatrix} -w_{11}-1 & -w_{12} & 0 & 0 & 0 & 0 \\ 0 & -1 & w_{23} & 0 & 0 & w_{26} \\ 0 & 0 & -w_{33}-1 & 0 & -w_{35} & 0 \\ 0 & -w_{42} & 0 & -w_{44}-1 & 0 & 0 \\ w_{51} & 0 & 0 & w_{54} & -1 & 0 \\ 0 & 0 & 0 & 0 & -w_{65} & -w_{66}-1 \end{pmatrix};$$

$$L_E(W, \chi_3, Y/X)^{-1}$$
$$= \det(I - W_{\chi_3})$$
$$= \det \begin{pmatrix} -iw_{11}-1 & -iw_{12} & 0 & 0 & 0 & 0 \\ 0 & -1 & w_{23} & 0 & 0 & w_{26} \\ 0 & 0 & -iw_{33}-1 & 0 & -iw_{35} & 0 \\ 0 & iw_{42} & 0 & iw_{44}-1 & 0 & 0 \\ w_{51} & 0 & 0 & w_{54} & -1 & 0 \\ 0 & 0 & 0 & 0 & iw_{65} & iw_{66}-1 \end{pmatrix}.$$

Note that, by the corollary to Theorem 8, the product of these four 6×6 determinants has to be a 24×24 determinant,

$$\det(I - \widetilde{W}_{spec}) = \prod_{i=0}^{3} \det(I - W_{\chi_i}),$$

where $\widetilde{W}_{spec}$ is available from the authors upon request.

You can use Maple to check the identity.

5. THE MULTIPATH ARTIN L-FUNCTIONS

It is well known that the fundamental group Γ of X is a free group on $r = |E| - |V| + 1$ generators. We have previously defined the multipath zeta function in terms of variables attached to pairs of generating paths in Γ (see Stark and Terras [14] and Stark [13]). The algorithms of Stark and Terras [14] were improved in Stark [13]. As we need these new algorithms here, we review them briefly. A fuller discussion appears in [13]. We present an equivalent version here, but in a manner which makes the specialization algorithms more transparent.

As above, we assume that X is a connected graph with vertex set V, (undirected) edge set E, and a base vertex denoted v_0. Because X is connected, there is a subgraph T of X with $|V|$ vertices which is a tree. As such T has $|V|-1$ edges. We give each of the $r=|E|-|V|+1$ remaining edges a direction and label them $e_1, \ldots, e_r$. The inverse edges will be labeled $e_{r+1}, \ldots, e_{2r}$. We will give each of the $|V|-1$ edges on the tree T a direction and label them $t_1, \ldots, t_{|V|-1}$. The inverse edges on T will be labeled $t_{|V|}, \ldots, t_{2|V|-2}$. Any backtrackless, tailless cycle on X is uniquely (up to starting point on the tree between last and first e_k) determined by the ordered sequence of e_k's it passes through. In particular, if e_i and e_j are two consecutive e_k's in this sequence, then the part of the cycle between e_i and e_j is the unique backtrackless path on T joining the terminal vertex of e_i to the initial vertex of e_j. Further for such e_i and e_j, e_j is not the inverse of e_i, since the cycle is backtrackless. Nor is the last edge the inverse of the first. Conversely, given any ordered sequence of edges from the e_k's, with no two consecutive edges being inverses of each other and with the last edge not being the inverse of the first, there is a unique (up to starting point on the tree between the last and first e_k) backtrackless tailless cycle on X whose sequence of e_k's is the given sequence.

The free group of rank r generated by the e_k's puts a group structure on backtrackless tailless cycles which is completely equivalent to the fundamental group of X. When dealing with the fundamental group of X, any closed path starting at a fixed vertex v_0 on X is completely determined up to homotopy by the ordered sequence of e_k's through which it passes. When backtracking is eliminated, such a path consists of a tail on the tree followed by a backtrackless, tailless cycle corresponding to the same sequence of e_k's, followed by the original tail in the reverse direction, ending at v_0 again. Any sequence of e_k's determines such a path on X with initial and terminal vertex at v_0. In this manner, the free group of rank r generated by the e_k's becomes identified with the fundamental group of X, and for this reason, we will refer to the free group generated by the e_k's as the *fundamental group* of X. Compare this version with the presentation in Stark [13] based on paths beginning and ending at v_0.

Just as there are two elementary reduction operations for paths expressed in terms of directed edges, so there are the corresponding elementary reduction operations for words in the fundamental group of X. As we are presenting it here, this means that if $a_1, \ldots, a_n$ and e are taken from the e_i or their inverses, the *two elementary reduction operations* for words in the fundamental group of X are:

(1) $a_1 \cdots a_{i-1} e e^{-1} a_{i+2} \cdots a_n \cong a_1 \cdots a_{i-1} a_{i+2} \cdots a_n$;

(2) $a_1 \cdots a_n \cong a_2 \cdots a_n a_1$.

Using just the first elementary reduction, each equivalence class of words corresponds to a group element and a word of minimum length in an equivalence class is a *reduced* word in the language of group theory. Since the second operation is equivalent to conjugating by a_1, an equivalence class using both elementary reductions corresponds to a conjugacy class in the fundamental group and a word of minimum length using both equivalence operations corresponds to finding words of minimum length in a conjugacy class of the fundamental group. If $a_1, ..., a_n$ are taken from $e_1, ..., e_{2r}$, a word $C = a_1 \cdots a_n$ is of minimum length in its conjugacy class (denoted by $[C]$) if and only if $a_{i+1} \neq a_i^{-1}$, for $1 \leqslant i \leqslant n-1$ and $a_1 \neq a_n^{-1}$. This is equivalent to saying that C corresponds to a *backtrackless, tailless* cycle under the correspondence at the start of this section. Equivalent cycles correspond to conjugate elements of the fundamental group. A conjugacy class $[C]$ is *primitive* if a word of minimal length in $[C]$ is not a power of another word. Equivalently, a conjugacy class is primitive if and only if the corresponding backtrackless tailless cycles (all equivalent to each other) are prime. We will say that a word of minimal length in its conjugacy class is *reduced in its conjugacy class*. From this point on, when we write a conjugacy class $[C]$, it will be assumed that a representative element C has been chosen which is reduced in $[C]$.

For i and j in the range from 1 to $2r$, we attach a variable z_{ij} to each ordered pair of paths e_i and e_j, PROVIDED that e_i is not the inverse of e_j. To the remaining $2r$ pairs i, j such that $i = j \pm r$, we set $z_{ij} = 0$. In this manner, we create a $2r \times 2r$ matrix $Z = (z_{ij})$ whose entries are the multipath variables together with zeros elsewhere. We call Z the *multipath matrix*.

For a reduced path $C = a_1 \cdots a_n$ in a conjugacy class $[C]$, we define the *multipath norm* of C to be (as in Definition 3)

$$\mathbb{N}(C) = \mathbb{N}_P(C) = z(a_1, a_2)\, z(a_2, a_3) \cdots z(a_{n-1}, a_n)\, z(a_n, a_1),$$

where $z(e_i, e_j) = z_{ij}$. This is non-zero by the reduction assumption and is well defined since the only reduced elements of a conjugacy class are cyclic reorderings of each other.

Definition 24. The *multipath Artin L-function* of the normal unramified graph covering Y/X associated to a representation ρ of the Galois group $G(Y/X)$ is

$$L_P(Z, \rho) = L_P(Z, \rho, Y/X) = \prod_{[C]} \det(1 - \mathbb{N}_P(C)\, \rho([Y/X, D]))^{-1},$$

where the product runs through equivalence classes of primitive reduced conjugacy classes other than the identity class. Here D is arbitrarily chosen from the conjugate primes in Y above the cycle corresponding to C and $[Y/X, D]$ is the Frobenius automorphism of Definition 17.

Since Frobenius automorphisms of conjugate primes are conjugate, the determinant in the definition of $L_P(Z, \rho)$ is independent of the choice of D above C.

The multipath zeta function (see Definition 4) introduced in Stark and Terras [14] is given by $L_P(Z, 1, X)$, where 1 is the trivial representation.

LEMMA 9. *Suppose that ρ is a representation of $G = G(Y/X)$ and χ is the corresponding character. Then*

$$\log(L_P(Z, \rho, Y/X)) = \sum_{[C]} \sum_{j=1}^{\infty} \frac{1}{j} \chi([Y/X, D]^j) \, \mathbb{N}_P(C)^j.$$

In particular, $L_P(Z, \rho, Y/X)$ depends only on the character χ of the representation ρ and not on the particular representation ρ in its equivalence class.

Proof. This is the exact analogue of Lemma 5. ∎

PROPOSITION 5 (Some Formal Properties of the Multipath Artin L-Function).

- (1) $L_P(Z, 1, Y/X) = \zeta_P(Z, X)$.
- (2) $L_P(Z, \rho_1 \oplus \rho_2, Y/X) = L_P(Z, \rho_1, Y/X) \, L_P(Z, \rho_2, Y/X)$.
- (3) *Let Y/X be a unramified normal covering. Suppose $\tilde{X}$ is intermediate to Y/X and $\tilde{X}/X$ is normal with $G = G(Y/X)$ and $H = G(Y/\tilde{X})$. Let ρ be a representation of $G/H \cong G(\tilde{X}/X)$. Thus ρ can be viewed as a representation of G, often called the lift of ρ. Then*

$$L_P(Z, \rho, Y/X) = L_P(Z, \rho, \tilde{X}/X).$$

Proof. The proofs are exact analogues of the proofs of Proposition 4 above. ∎

THEOREM 9 ([14], The Multipath Artin L-Function Is the Inverse of a Polynomial).

The multipath L-function satisfies

$$L_P(Z, \rho, Y/X) = \det(I - Z_\rho)^{-1}.$$

Here $Z_\rho = (z_{ij} \rho(\sigma(e_i)))$ and I is the $2dr \times 2dr$ identity matrix, where d is the degree of ρ.

Proof. The proof is the same as that in Theorem 7 for the multiedge L-functions. We choose D so that $[Y/X, D] = \sigma(C)$, the normalized Frobenius automorphism corresponding to C as in Definition 16. Note that if a path C on X corresponds to a sequence of cut edges $a_1, ..., a_n$, then

$\sigma(C)=\sigma(a_1)\cdots\sigma(a_n)$, since all the remaining edges of the path are on the tree and for any edge b on the tree $\sigma(b)=1$. ∎

We now turn to our *improved specialization algorithm.* We wish to specialize the multipath variables z_{ij} so that each $\mathbb{N}_P(C)$ becomes the multiedge $\mathbb{N}_E(C)$, where in the first instance C is a product of generators of the fundamental group of X and in the second C is a product of edges of X. We use the generators of the fundamental group of X which were identified above with (directed) edges e_j of X left out of a spanning tree T. Thus we use the elementary reduction operations above to write C as a reduced word in the generators e_j of the fundamental group.

As above, let the t_j denote directed edges on the spanning tree T. Then for $i \neq j \pm r$ (that is, $e_i \neq e_j^{-1}$), we write the part of the corresponding path between e_i and e_j as the (unique) product $t_{v_1}\cdots t_{v_n}$, where the t_{v_k}'s are directed edges on T. Thus our cycle C is first a product of generators e_i of the fundamental group and then a product of actual edges e_i and t_k of X. We now set $Z=Z(W)=(z_{ij})_{1\leqslant i,\, j\leqslant 2r}$ where

$$z_{ij}=z_{ij}(W)=w(e_i,t_{v_1})\left[\prod_{k=1}^{n-1} w(t_{v_k},t_{v_{k+1}})\right] w(t_{v_n},e_j). \tag{18}$$

This accomplishes the desired specialization.

Theorem 10. *With the specialization* (18), *we have the equality of the path and edge L-functions,*

$$L_P(Z(W),\rho,Y/X)=L_E(W,\rho,Y/X).$$

Proof. This is clear: with the specialization, the defining infinite product for $L_P(Z(W),\rho,Y/X)$ becomes the defining infinite product for $L_E(W,\rho,Y/X)$. ∎

Our improved specialization algorithm allows us to conclude directly from Theorem 9 that the multiedge and Ihara zeta functions are inverses of polynomials. However, we do not instantly get our earlier explicit expressions for these zeta functions.

Example. We show the specialization for the graph in Fig. 13 which we redraw in Fig. 15 with the edges labeled. To avoid conflicting numbering, we will use letters as subscripts. We introduce the five edges inverse to a, b, c, d, e and label them A, B, C, D, E. We take the cut edges to be the

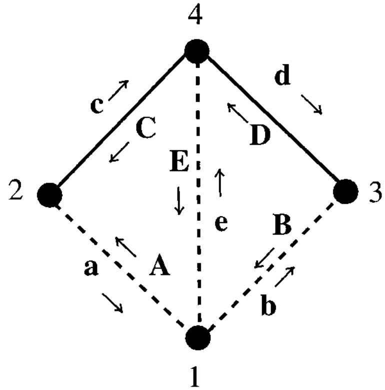

FIG. 15. The rank 2 graph of Fig. 13. The edges are denoted by lower-case letters with a direction arbitrarily assigned as indicated by the arrows. The directed edges in the inverse directions are labeled with the corresponding capital letters. The three edges of a tree connecting every vertex of the graph are shown in dotted lines; the remaining two edges correspond to generators of the fundamental group and are shown with solid lines.

cC and dD pairs. Removing the cut edges gives a tree T indicated by the dotted lines in Fig. 13 and 15. The matrix Z is then

$$Z = \begin{pmatrix} z_{cc} & z_{cd} & 0 & z_{cD} \\ z_{dc} & z_{dd} & z_{dC} & 0 \\ 0 & z_{Cd} & z_{CC} & z_{CD} \\ z_{Dc} & 0 & z_{DC} & z_{DD} \end{pmatrix}.$$

To specialize a variable as in (18), for example z_{dc}, we write the path starting with d, following the tree and ending with c as $dBAc$. We then specialize z_{dc} to $w_{dB} w_{BA} w_{Ac}$. In this way the matrix Z specializes to the matrix

$$Z(W) = \begin{pmatrix} w_{cE} w_{EA} w_{Ac} & w_{cd} & 0 & w_{cE} w_{Eb} w_{bD} \\ w_{dB} w_{BA} w_{Ac} & w_{dB} w_{Be} w_{ed} & w_{dB} w_{Be} w_{eC} & 0 \\ 0 & w_{Ca} w_{ae} w_{ed} & w_{Ca} w_{ae} w_{eC} & w_{Ca} w_{ab} w_{bD} \\ w_{DE} w_{EA} w_{Ac} & 0 & w_{DC} & w_{DE} w_{Eb} w_{bD} \end{pmatrix}.$$

It is particularly easy to specialize Z so as to get the Ihara zeta function without even writing down $Z(W)$. Since all the edge variables are set equal to u, this means we have only to count the number of edges on the tree connecting each e_i and e_j (from the terminal vertex of e_i to the initial vertex of e_j) and add one for the initial e_i; the result is the power of u replacing z_{ij} in Z. Again with z_{dc}, d together with the two edges B and A constitute the three edges on the backtrackless path starting with d and following the tree up to the initial vertex of c. Thus we specialize z_{dc} to u^3. We then get

$$\zeta_X(u)^{-1} = \det(I - Z(u)),$$

where

$$Z(u) = \begin{pmatrix} u^3 & u & 0 & u^3 \\ u^3 & u^3 & u^3 & 0 \\ 0 & u^3 & u^3 & u^3 \\ u^3 & 0 & u & u^3 \end{pmatrix}.$$

It is interesting to note that a different choice of cut edges and resulting tree can yield a completely different $Z(u)$. To illustrate, if in our example graph, we use the tree of Fig. 16, we are led to the specialized matrix (rows and columns indexed in order by c, e, C, E).

$$Z(u) = \begin{pmatrix} u^4 & u^3 & 0 & u \\ u^4 & u^3 & u & 0 \\ 0 & u^2 & u^4 & u^4 \\ u^2 & 0 & u^3 & u^3 \end{pmatrix}.$$

which still results in the same vertex zeta function. The 2 different $Z(u)$ matrices here are not even similar since their traces are not equal.

Example. *The Multipath L-Functions of an S_3 Cover*. See Fig. 13 for a picture of the covering. We compute the path L-functions for the case that all variables are specialized down to u using the tree of Fig. 15. This requires us to find the 3 matrices corresponding to the representations

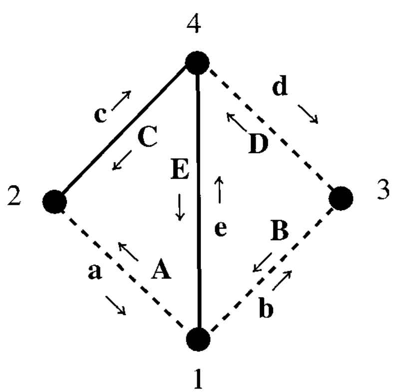

FIG. 16. The graph of Fig. 15 with a different spanning tree.

of $G(Y/X)$ described in our earlier discussion of the S_3 example. Here $\omega = \exp(\frac{2\pi i}{3})$. Since $\sigma(c) = FR = \sigma(c)^{-1}$ and $\sigma(d) = FR^2 = \sigma(d)^{-1}$, we obtain

$$Z(u) = Z_{\chi_0} = \begin{pmatrix} u^3 & u & 0 & u^3 \\ u^3 & u^3 & u^3 & 0 \\ 0 & u^3 & u^3 & u^3 \\ u^3 & 0 & u & u^3 \end{pmatrix}; \qquad Z = Z_{\chi_1} = -Z;$$

$$Z_\rho(u) = \begin{pmatrix} 0 & u^3\omega^2 & 0 & u\omega^2 & 0 & 0 & 0 & u^3\omega^2 \\ u^3\omega & 0 & u\omega & 0 & 0 & 0 & u^3\omega & 0 \\ 0 & u^3\omega & 0 & u^3\omega & 0 & u^3\omega & 0 & 0 \\ u^3\omega^2 & 0 & u^3\omega^2 & 0 & u^3\omega^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & u^3\omega^2 & 0 & u^3\omega^2 & 0 & u^3\omega^2 \\ 0 & 0 & u^3\omega & 0 & u^3\omega & 0 & u^3\omega & 0 \\ 0 & u^3\omega & 0 & 0 & 0 & u\omega & 0 & u^3\omega \\ u^3\omega^2 & 0 & 0 & 0 & u\omega^2 & 0 & u^3\omega^2 & 0 \end{pmatrix}.$$

So we have the L-functions,

$$L_P(Z(u), \chi_0, Y/X)^{-1}$$

$$= \zeta_X(u)^{-1} = \det \begin{pmatrix} u^3-1 & u & 0 & u^3 \\ u^3 & u^3-1 & u^3 & 0 \\ 0 & u^3 & u^3-1 & u^3 \\ u^3 & 0 & u & u^3-1 \end{pmatrix}$$

$$= 1 - 4u^3 - 2u^4 + 4u^6 + 4u^7 + u^8 - 4u^{10}$$

$$= -(u+1)(2u^2+u+1)(u^2+1)(2u^3+u^2-1)(u-1)^2;$$

$$L_P(Z(u), \chi_1, Y/X)^{-1}$$

$$= L_V(u, \chi_1, Y/X)^{-1} = \zeta_X(-u)^{-1}$$

$$= 1 + 4u^3 - 2u^4 + 4u^6 - 4u^7 + u^8 - 4u^{10}$$

$$= -(u-1)(2u^2-u+1)(u^2+1)(2u^3-u^2+1)(u+1)^2;$$

$$
\begin{aligned}
&L_P(Z(u), \rho, Y/X)^{-1}\\
&\quad = L_V(u, \rho, Y/X)^{-1}\\
&\quad = \det \begin{pmatrix}
-1 & u^3\omega^2 & 0 & u\omega^2 & 0 & 0 & 0 & u^3\omega^2 \\
u^3\omega & -1 & u\omega & 0 & 0 & 0 & u^3\omega & 0 \\
0 & u^3\omega & -1 & u^3\omega & 0 & u^3\omega & 0 & 0 \\
u^3\omega^2 & 0 & u^3\omega^2 & -1 & u^3\omega^2 & 0 & 0 & 0 \\
0 & 0 & 0 & u^3\omega^2 & -1 & u^3\omega^2 & 0 & u^3\omega^2 \\
0 & 0 & u^3\omega & 0 & u^3\omega & -1 & u^3\omega & 0 \\
0 & u^3\omega & 0 & 0 & 0 & u\omega & -1 & u^3\omega \\
u^3\omega^2 & 0 & 0 & 0 & u\omega^2 & 0 & u^3\omega^2 & -1
\end{pmatrix}\\
&\quad = 1 + 2u^4 - 2u^6 + 3u^8 + 2u^{10} + 3u^{12} - 10u^{14} - 7u^{16} - 8u^{18} + 16u^{20}\\
&\quad = (2u^4 + u^3 + 2u^2 + u + 1)(2u^4 - u^3 - u + 1)\\
&\qquad \times (2u^4 - u^3 + 2u^2 - u + 1)(2u^4 + u^3 + u + 1)(u-1)^2\,(u+1)^2.
\end{aligned}
$$

This agrees with our example in the preceding section.

Now we come to the induction property of the multipath L-functions. For this, if $\tilde{X}$ is a covering of X, we need to specialize the path matrix $\tilde{Z}$ of $\tilde{X}$ to the variables in the path matrix Z of X. This must be done in such a way that if $\tilde{C}$ is a reduced cycle in its conjugacy class of the fundamental group of $\tilde{X}$, then under the specialization, $\mathbb{N}_E(\tilde{C})$ becomes $\mathbb{N}_E(C)$ where C is the projected cycle of $\tilde{C}$ in X. This is easy to achieve by building upon an analogy between edge and path zeta functions developed in Stark [13] to treat the case of quadratic coverings.

The desired analogy rests upon the fact that in X, we may contract the base tree T to a point without losing any information about the fundamental group of X. In the contracted graph, the path and edge zeta functions are the same. In the process, in the cover $\tilde{X}$, we also contract each sheet (the connected inverse images of T) to a point and here the lifts of the r generating paths of X become the edges of the contracted $\tilde{X}$. What makes the process interesting is that if $\tilde{X}$ is an n-fold covering of X, then $n-1$ of the lifted edges from the contracted X must be set aside to complete the tree of $\tilde{X}$. The remaining $nr-(n-1)=n(r-1)+1$ edges of the contracted $\tilde{X}$ give rise to the generating paths of the fundamental group of $\tilde{X}$. The specialization algorithm needs to take account of the tree edges.

Everything is in place to do this. First we specialize the $\tilde{Z}$ variables to the edge variables on the contracted $\tilde{X}$. Each $\mathbb{N}_P(\tilde{C})$ becomes $\mathbb{N}_E(\tilde{C})$ on the contracted $\tilde{X}$. We follow this by specializing the edge variables of the contracted $\tilde{X}$ to the edge variables of the contracted X. This turns $\mathbb{N}_E(\tilde{C})$

into $\mathbb{N}_E(C)$ which, on the contracted X, is the same as $\mathbb{N}_P(C)$. This accomplishes the desired specialization.

For example, the contracted versions of X and Y_3 from Fig. 13 are shown in Fig. 17.

The tree $\tilde{T}$ of Y_3 is completed with one of the lifts of the cC pair between the top two sheets of Y_3 and one of the lifts of the dD pair between the bottom two sheets. The remaining four undirected edges of the contracted Y_3 give rise to the fundamental group of Y_3 and the resulting 8×8 path matrix $\tilde{Z}$. We give these edges directions projecting to either c or d, rather than C or D, and labels I, II, III, IV, as shown. The inverse edges, projecting to C and D, are given labels V, VI, VII, $VIII$, as shown. The rows and columns of $\tilde{Z}$ are then labeled by the roman numerals I–$VIII$.

Following our algorithm, the resulting specialized matrix $\tilde{Z}_{spec}$ is then

$$\begin{pmatrix}
z_{dd} & z_{dc}z_{cc} & z_{dc}z_{cd}z_{dd} & z_{dc}z_{cd}z_{dc} & 0 & z_{dC} & z_{dc}z_{cD} & z_{dc}z_{cd}z_{dC} \\
z_{cd} & z_{cc}z_{cc} & z_{cc}z_{cd}z_{dd} & z_{cc}z_{cd}z_{dc} & z_{cD} & 0 & z_{cc}z_{cD} & z_{cc}z_{cd}z_{dC} \\
z_{dC}z_{Cd} & z_{dc} & z_{dd}z_{dd} & z_{dd}z_{dc} & z_{dC}z_{CD} & z_{dC}z_{CC} & 0 & z_{dd}z_{dC} \\
z_{cD}z_{DC}z_{Cd} & z_{cD}z_{Dc} & z_{cd} & z_{cc} & z_{cD}z_{DC}z_{CD} & z_{cD}z_{DC}z_{CC} & z_{cD}z_{DD} & 0 \\
0 & z_{Dc}z_{cc} & z_{Dc}z_{cd}z_{dd} & z_{Dc}z_{cd}z_{dc} & z_{DD} & z_{DC} & z_{Dc}z_{cD} & z_{Dc}z_{cd}z_{dC} \\
z_{CC}z_{Cd} & 0 & z_{Cd}z_{dd} & z_{Cd}z_{dc} & z_{CC}z_{CD} & z_{CC}z_{CC} & z_{CD} & z_{Cd}z_{dC} \\
z_{DD}z_{DC}z_{Cd} & z_{DD}z_{Dc} & 0 & z_{Dc} & z_{DD}z_{DC}z_{CD} & z_{DD}z_{DC}z_{CC} & z_{DD}z_{DD} & z_{DC} \\
z_{CD}z_{DC}z_{Cd} & z_{CD}z_{Dc} & z_{Cd} & 0 & z_{CD}z_{DC}z_{CD} & z_{CD}z_{DC}z_{CC} & z_{CD}z_{DD} & z_{CC}
\end{pmatrix}.$$

As an illustration, the IV, I entry $\tilde{Z}_{IV,\,I}$ follows directed edge IV (projecting to c), through two edges of $\tilde{T}$ (projecting to D and C consecutively) to edge I (projecting to d) resulting in the specialized value $z_{cD}z_{DC}z_{Cd}$. This agrees with the fact that any path on Y_3 going through consecutive cut edges IV and I must project to a path on X going consecutively through c, D, C, d.

THEOREM 11 (Induction Property for Multipath L-Functions). *If H is a subgroup of G corresponding to the intermediate covering $\tilde{X}$ and ρ is a representation of H and $\rho^{\#}$ is the representation of G induced by ρ, then*

$$L(\tilde{Z}_{spec}, \rho, Y/\tilde{X}) = L(Z, \rho^{\#}, Y/X).$$

Proof. We follow the idea of contracting each copy of the tree T to a point, both in X and in $\tilde{X}$. In this manner, both sides of the equality of this theorem become multiedge L-functions attached to a graph with one vertex and r loops and the corresponding covering of it. Since the induction theorem has been proved in Theorem 8 for multiedge L-functions, we are finished. ∎

Remark. From Theorem 9, the equality of Theorem 11 becomes

$$\det(I - \tilde{Z}_{spec,\,\rho}) = \det(I - Z_{\rho^{\#}}).$$

Unlike the analogous equality that we get for the multiedge L-functions by combining Theorem 7 and Theorem 8, here these determinants are of different sizes!

The following corollary is obtained just as for the corollary to Theorem 8.

COROLLARY. *Suppose Y/X is an unramified normal graph covering with Galois group $G(Y/X)$. Then*

$$\zeta_P(\tilde{Z}_{spec}, Y) = \prod_{\rho \in \hat{G}} L_P(Z, \rho, Y/X)^{d_\rho}.$$

EXAMPLE. *Factorization of the path zeta function of a non-normal cubic cover Y_3 over X from Fig.* 13. This is analogous to the example from zeta functions of number fields which goes back to Dedekind (see [12, Sect. 3.3]).

Here we re-consider the last example. Set $\omega = e^{2\pi i/3}$ and

$$u_1 = z_{cc}, \quad u_2 = z_{cd}, \quad u_3 = z_{cD}, \quad u_4 = z_{dc}, \quad u_5 = z_{dd}, \quad u_6 = z_{dC},$$
$$u_7 = z_{Cd}, \quad u_8 = z_{CC}, \quad u_9 = z_{CD}, \quad u_{10} = z_{Dc}, \quad u_{11} = z_{DC}, \quad u_{12} = z_{DD}.$$

Then by the corollary to Theorem 11, the product of

$$\det \begin{pmatrix} u_1 - 1 & u_2 & 0 & u_3 \\ u_4 & u_5 - 1 & u_6 & 0 \\ 0 & u_7 & u_8 - 1 & u_9 \\ u_{10} & 0 & u_{11} & u_{12} - 1 \end{pmatrix}$$

and

$$\det\begin{pmatrix} -1 & \omega^2 u_1 & 0 & \omega^2 u_2 & 0 & 0 & 0 & \omega^2 u_3 \\ \omega u_1 & -1 & \omega u_2 & 0 & 0 & 0 & \omega u_3 & 0 \\ 0 & \omega u_4 & -1 & \omega u_5 & 0 & \omega u_6 & 0 & 0 \\ \omega^2 u_4 & 0 & \omega^2 u_5 & -1 & \omega^2 u_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega^2 u_7 & -1 & \omega^2 u_8 & 0 & \omega^2 u_9 \\ 0 & 0 & \omega u_7 & 0 & \omega u_8 & -1 & \omega u_9 & 0 \\ 0 & \omega u_{10} & 0 & 0 & 0 & \omega u_{11} & -1 & \omega u_{12} \\ \omega^2 u_{10} & 0 & 0 & 0 & \omega^2 u_{11} & 0 & \omega^2 u_{12} & -1 \end{pmatrix}$$

must equal the determinant of the matrix

$$\tilde{Z}_{spec} - I = \begin{pmatrix} u_5 - 1 & u_4 u_1 & u_4 u_2 u_5 & u_4 u_2 u_4 & 0 & u_6 & u_4 u_3 & u_4 u_2 u_6 \\ u_2 & u_1 u_1 - 1 & u_1 u_2 u_5 & u_1 u_2 u_4 & u_3 & 0 & u_1 u_3 & u_1 u_2 u_6 \\ u_6 u_7 & u_4 & u_5 u_5 - 1 & u_5 u_4 & u_6 u_9 & u_6 u_8 & 0 & u_5 u_6 \\ u_3 u_{11} u_7 & u_3 u_{10} & u_2 & u_1 - 1 & u_3 u_{11} u_9 & u_3 u_{11} u_8 & u_3 u_{12} & 0 \\ 0 & u_{10} u_1 & u_{10} u_2 u_5 & u_{10} u_2 u_4 & u_{12} - 1 & u_{11} & u_{10} u_3 & u_{10} u_2 u_6 \\ u_8 u_7 & 0 & u_7 u_5 & u_7 u_4 & u_8 u_9 & u_8 u_8 - 1 & u_9 & u_7 u_6 \\ u_{12} u_{11} u_7 & u_{12} u_{10} & 0 & u_{10} & u_{12} u_{11} u_9 & u_{12} u_{11} u_8 & u_{12} u_{12} - 1 & u_{11} \\ u_9 u_{11} u_7 & u_9 u_{10} & u_7 & 0 & u_9 u_{11} u_9 & u_9 u_{11} u_8 & u_9 u_{12} & u_8 - 1 \end{pmatrix}.$$

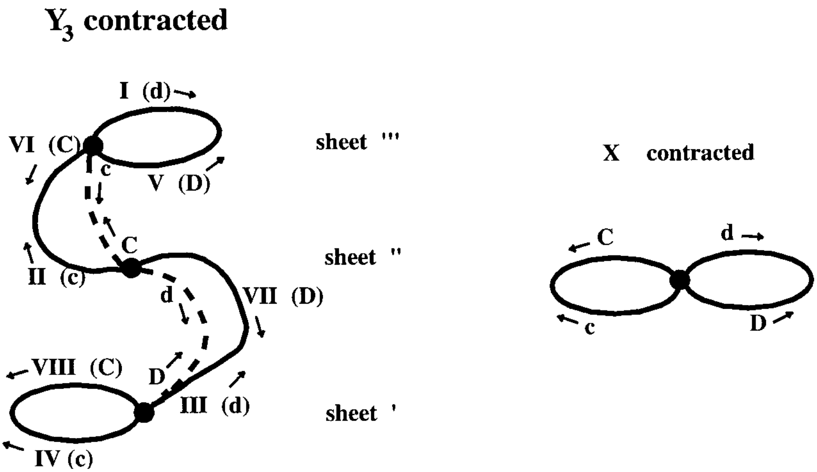

FIG. 17. The contracted versions of X and Y_3 from Fig. 13. Solid edges are the cut edges generating the fundamental group.

6. NON-ISOMORPHIC REGULAR GRAPHS WITHOUT LOOPS OR MULTIEDGES HAVING THE SAME IHARA ZETA FUNCTION

Algebraic number fields K_1, K_2 can have the same Dedekind zeta functions without being isomorphic. See Perlis [11]. The smallest examples have degree 7 over $\mathbb{Q}$ and come from Artin L-functions of induced representations from subgroups of $G = GL(3, \mathbb{F}_2)$, the simple group of order 168. An analogous example of 2 graphs (each having 7 vertices) which are isospectral but not isomorphic was given by P. Buser. These graphs are found in Fig. 18. See Buser [2] or Terras [16, Chap. 22]. Buser's graphs ultimately lead to 2 planar isospectral drums which are not obtained from each other by rotation and translation. See Gordon *et al.* [4]. But Buser's graphs are not simple. That is, they have multiple edges as well as loops. We can use our theory to obtain examples of simple regular graphs with 28 vertices which are isospectral but not isomorphic. See Fig. 19. The graphs in Fig. 19 are constructed using the same group G and subgroups H_j as in Buser's examples.

Define $G = GL(3, \mathbb{F}_2)$, which is a simple group of order 168. Two subgroups H_j of index 7 in G are

$$H_1 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix} \right\} \quad \text{and} \quad H_2 = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\}.$$

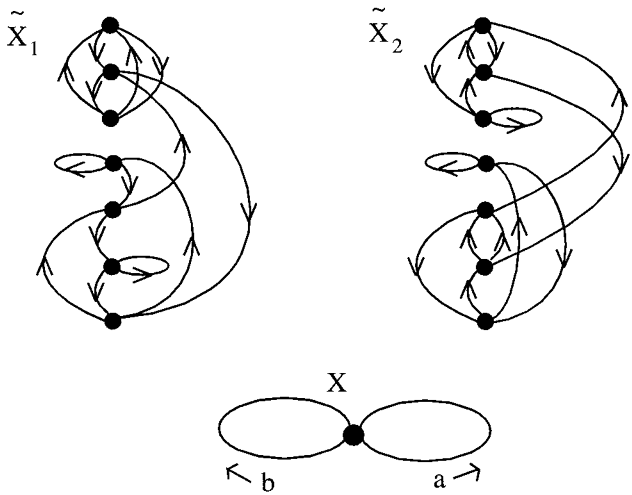

FIG. 18. Buser's isospectral non-isomorphic Schreier graphs. See Buser [2]. The sheets of $\tilde{X}_1$ and $\tilde{X}_2$ are numbered 1 to 7 from bottom to top. Lifts of a are on the right in each graph; lifts of b are on the left.

It is well known that these two subgroups are not conjugate in G. This fact is also a trivial consequence of our upcoming construction of two non-isomorphic intermediate graphs corresponding to these subgroups H_j of G. More importantly, these two groups give rise to equivalent permutation representations of G and this means we will get graphs with the same zeta function.

Given $g \in G$, all 24 elements of $H_1 g$ have the same first row. The 7 possible non-zero first rows correspond naturally to the numbers 1–7 in binary. Thus we have a natural way of tabulating the 7 right cosets of H_1 as $H_1 g_j$, $j = 1, ..., 7$, where the first row of each g_j represents j in binary. For example, the first row of g_6 is (110); as another example, $H_1 g_4$ is the identity coset. Given g, it is also very easy to calculate what coset $H_1 g_j g$ is, since the first row of the product $g_j g$ depends only on the first row of g_j. Thus, for a given $g \in G$, it is easy to calculate the permutation $\mu(g)$ (introduced in Subsection 2.6) corresponding to multiplying the 7 cosets $H_1 g_j$ by g on the right.

We will need the permutations $\mu(A)$ and $\mu(B)$ for the two matrices used by Buser,

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

We find that

$$\mu(A)=(1436)(2)(57) \qquad \text{and} \qquad \mu(B)=(132)(4)(576).$$

For example, to find $H_1 g_3 A$, we want the first row of

$$\begin{pmatrix} 0 & 1 & 1 \\ * & * & * \\ * & * & * \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}=\begin{pmatrix} 1 & 1 & 0 \\ * & * & * \\ * & * & * \end{pmatrix}\in H_1 g_6$$

and so $\mu(A)$ takes 3 to 6. Indeed, we need only to add appropriate rows of A and B to find both permutations $\mu(A)$ and $\mu(B)$.

We will want to do the same permutation calculation with the matrices A and B acting on the right cosets of H_2. At first glance, the right cosets of H_2 seem more difficult to deal with, but fortunately a very useful automorphism of G helps out. The map $\varphi(g)={}^t g^{-1}$, where ${}^t g$ denotes the transpose of $g\in G$, provides an automorphism of G in which $\varphi(H_1)=H_2$. If we apply φ to the right cosets $H_1 g_j$, we get G as a union of the 7 right cosets $H_2\, {}^t g_j^{-1}$. Further, to find how a given element $g\in G$ permutes these cosets, it is sufficient to look at the action of ${}^t g^{-1}$ on the $H_1 g_j$. We have

$${}^t A^{-1}=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad {}^t B^{-1}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Therefore the action of ${}^t A^{-1}$ and ${}^t B^{-1}$ on the right cosets $H_1 g_j$ is given by the permutations

$$\mu({}^t A^{-1})=(14)(2376)(5) \qquad \text{and} \qquad \mu({}^t B^{-1})=(123)(4)(567).$$

These same permutations give the actions of A and B on the right cosets $H_2\, {}^t g_j^{-1}$.

The matrices A and B generate the group G. Buser [2] used these matrices to construct 2 Schreier graphs corresponding to the 2 subgroups H_1 and H_2. In our terminology, this means that one constructs coverings $\tilde{X}_1$ and $\tilde{X}_2$ of X, where X is the graph consisting of a single vertex and a double loop. We give each loop a direction resulting in two directed edges a and b, say. We assign the normalized Frobenius elements $\sigma(a)=A$ and $\sigma(b)=B$. The resulting normal cover of X is the Cayley graph of G corresponding to the generators A and B. We are interested in the two intermediate graphs $\tilde{X}_1$ and $\tilde{X}_2$ corresponding to the subgroups H_1 and H_2 by Theorem 1; which are Schreier graphs. The previously calculated permutations $\mu(A)$ and $\mu(B)$ instantly give the graphs $\tilde{X}_1$ and $\tilde{X}_2$. See Fig. 18

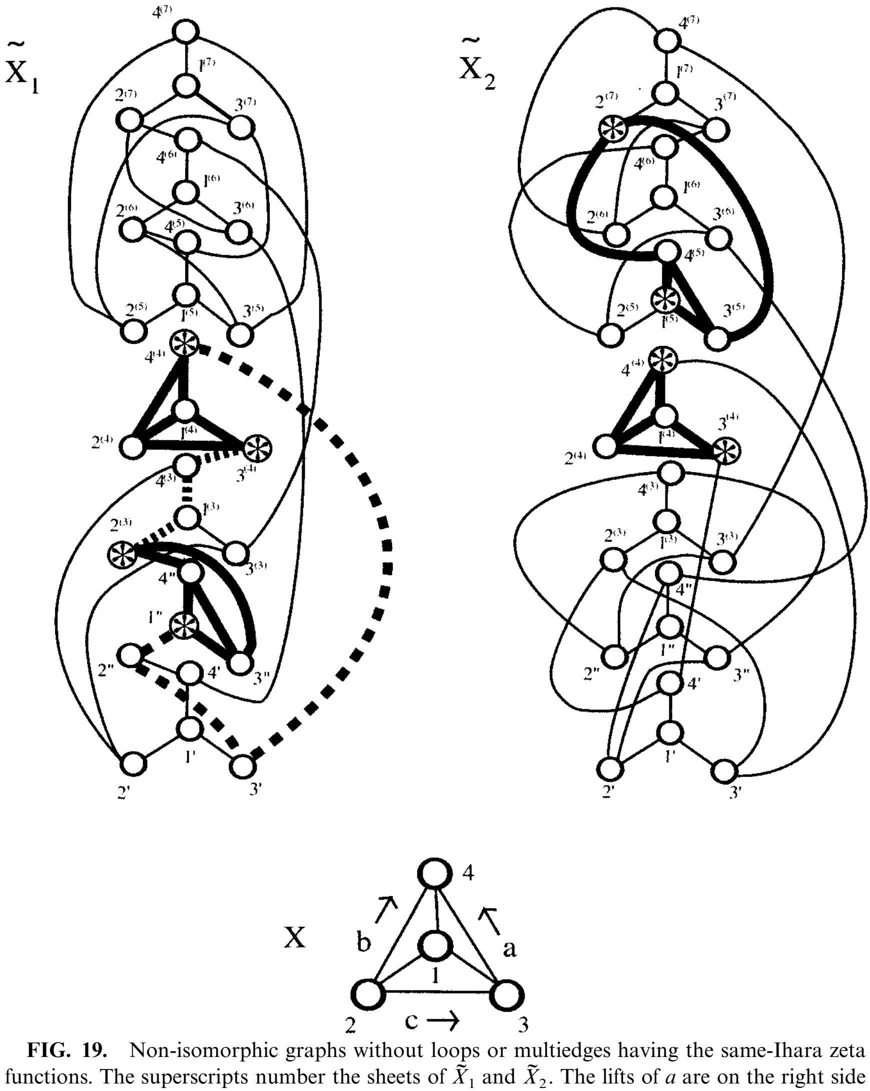

FIG. 19. Non-isomorphic graphs without loops or multiedges having the same-Ihara zeta functions. The superscripts number the sheets of $\tilde{X}_1$ and $\tilde{X}_2$. The lifts of a are on the right side of each graph, lifts of b are on the left, and lifts of c cross from the left to the right.

For many different reasons, the 2 graphs in Fig. 18 are not isomorphic—even as undirected graphs. Look at triple edges; look at double edges; look at distances between loops, etc. Therefore H_1 and H_2 are not conjugate in G. Both graphs are 4-regular; they have the same zeta function and their adjacency matrices have the same spectrum (i.e., they are isospectral graphs).

While regular, the graphs of Fig. 18 have loops and multiedges. We now construct an example of two regular non-isomorphic graphs with the same zeta function such that the graphs have no loops or multiedges. We use the same G, H_1 and H_2, but now take X to be a tetrahedron. Thus X has rank 3. We take the cut edges (directed as in Fig. 19) to be a, b, c. As normalized Frobenius automorphisms, we choose

$$\sigma(a) = A, \qquad \sigma(b) = \sigma(c) = B.$$

We then take 7 copies of the tree of X for the sheets in $\tilde{X}_1$ and again for $\tilde{X}_2$. On $\tilde{X}_1$, we lift a, b, c using the permutations $\mu(A)$ and $\mu(B)$ above to connect appropriate vertices. On $\tilde{X}_2$ we lift a, b, c using the permutations $\mu({}^tA^{-1})$ and $\mu({}^tB^{-1})$ to connect appropriate vertices. This results in the 3-regular graphs $\tilde{X}_1$ and $\tilde{X}_2$ shown in Fig. 19.

Let us give a few more details of this construction. The edge c goes from vertex 2 to vertex 3 in X and has the normalized Frobenius automorphism $\sigma(c) = B$. The lifts of c to $\tilde{X}_1$ are determined by the permutation $\mu(B) =$ (132)(4)(576). This means that c in X lifts to an edge in $\tilde{X}_1$ from $2'$ to $3^{(3)}$, an edge from $2^{(3)}$ to $3^{(2)}$, and edge from $2''$ to $3'$ and then (beginning a new cycle) to an edge from $2^{(4)}$ to $3^{(4)}$, etc. The edge b lifts in exactly the same manner as c. Similarly, for $\tilde{X}_1$, the edge a in X corresponds to the permutations (1436)(2)(57). This means that edge a in X lifts to an edge in $\tilde{X}_1$ from $3'$ to $4^{(4)}$, an edge from $3^{(4)}$ to $4^{(3)}$, an edge from $3^{(3)}$ to $4^{(6)}$, etc.

To see that graphs $\tilde{X}_1$ and $\tilde{X}_2$ in Fig. 18 are not isomorphic, proceed as follows. There are exactly 4 triangles in each graph (indicated by very thick solid lines in Fig. 19) and they are connected in pairs in both graphs. This distinguishes in each pair the 2 vertices not on common edges (starred vertices). In $\tilde{X}_1$ we can go in 3 steps (via dotted lines) from a starred vertex in one pair to a starred vertex in the other pair and, in fact, in 2 different ways. This cannot be done at all in $\tilde{X}_2$.

We have said that each of $\tilde{X}_1$ and $\tilde{X}_2$ has 4 triangles and it is not too hard to verify this by systematically checking both graphs. However, the corollary to Lemma 7 gives us a usable formula. We will count the triangles on $\tilde{X}_1$ and $\tilde{X}_2$ up to equivalence and choice of direction on the triangle (this is what we mean when we say there are 4 triangles on each graph). Since X has no loops or multiedges, any triangle on $\tilde{X}_1$ or $\tilde{X}_2$ projects to a triangle on X. Up to equivalence and choice of direction, there are four triangles on X.

Let χ_1 be the trivial character on H_1 or H_2. The induced character $\chi_1^{\#}$ on G is the same in both cases. According to the Corollary to Lemma 7, for any directed triangle C on X, there are $\chi_1^{\#}(\sigma(C))$ directed triangles above C on $\tilde{X}_1$ and also above C on $\tilde{X}_2$. Reversing the direction of C reverses the direction of the covering triangles. We choose the most convenient direction for each triangle.

Three of the triangles on X have 2 edges on the tree of X with normalized Frobenius elements $= 1$ automatically. Thus, with appropriate choice of direction in each case, $\sigma(C) = A, B, B$, for each triangle. The fourth triangle may be taken to be the path $ab^{-1}c$ whose normalized Frobenius is $\sigma(a)\,\sigma(b)^{-1}\,\sigma(c) = AB^{-1}B = A$. For $g \in G$, $\chi_1^{\#}(g)$ is simply the number of 1 cycles in the permutation $\mu(g)$. In particular, $\chi_1^{\#}(A) = \chi_1^{\#}(B) = 1$ (the same for both H_1 and H_2). Thus each of the 4 triangles of X has precisely 1 triangle of $\tilde{X}_j$ above it for $j = 1$ or 2. It follows that the triangles shown in Fig. 19 account for all triangles on $\tilde{X}_1$ and $\tilde{X}_2$ as claimed.

To see that the graphs are isospectral, one just notices that as above

$$\zeta_{\tilde{X}_j}(u) = L_V(u, \rho_j).$$

The same argument as we used in Terras [16, *loc. cit.*] for Buser's graphs says that the representations $\rho_j = Ind_{H_j}^{G}\,1$ are equivalent because the subgroups H_j are almost conjugate (i.e. $|H_1 \cap \{g\}| = |H_2 \cap \{g\}|$, for every conjugacy class $\{g\}$ in G).

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