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GRAPHS WITH UNIQUE MAXIMUM INDEPENDENT SETS

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A graph is a unique independence graph if it has a unique maximum independent set. If, further, the complement of the maximum independent set is independent, the graph is a strong unique independence graph. We characterize strong unique independence graphs and unique independence trees.

An *independent set* in a graph G is a set of vertices no two of which are joined by an edge. A *maximum independent set* is an independent set of largest cardinality. A graph G will be called a *unique independence graph* if G has a unique maximum independent set I. If the complement of I is also independent, G will be called a *strong unique independence graph.* Clearly strong unique independence graphs are bipartite. We note that our strong unique independence graphs are precisely the semi-irreducible graphs of Harary and Plummer [3]. There, the interest was in characterizing graphs G such that $G = C(G)$, where $C(G)$ is the core of G.

In this article, our primary interest is in the relationship between strong unique independence graphs and their spanning trees. In addition, we characterize unique independence trees and we construct families of regular unique independence graphs.

To illustrate these concepts, we note that if $m \neq n$, the complete bipartite graphs $K_{m,n}$ are strong unique independence graphs, as are paths with an odd number of vertices. The tree in Fig. 1 is a unique independence graph, but not a strong one.

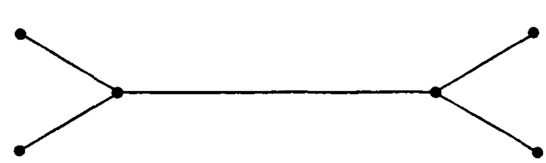

Fig. 1

Our first purpose here is to characterize strong unique independence trees in a concise fashion. We then show how to construct all unique independence trees from forests of unique independence trees. Finally, we obtain a characterization of strong unique independence graphs.

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In what follows, we will consistently use the following terminology. If G is a unique independence graph and I is the maximum independent set in G, then vertices in I will be called black vertices. Vertices in G not in I will be called white vertices.

Our first two lemmas provide constructions of larger unique independence graphs from smaller ones.

Lemma 1. *Suppose that G and H are unique independence graphs. Suppose that v is a white vertex of G and w is a vertex of H. Let J be the graph formed by adding the edge vw to the disjoint union of G and H. Then J is a unique independence graph. Further, if G and H are strong and w is black, then J is strong.*

Proof. We claim that $I(G) \cup I(H) = I$ is the unique maximum independent set in J. For, any independent set K in J with $|K| \geqslant |I|$ would have either at least as many vertices in G as $I(G)$ or at least as many vertices in H as $I(H)$. Now, if G and H are strong and w is black, it is easy to see that the complement of I in J is independent, so J is strong. □

Observe that the construction of Lemma 1 does not preserve strong unique independence. The graph of Fig. 1 may be gotten by applying this construction, where G and H are paths with three vertices.

Lemma 2. *Let $G_1, G_2, \ldots, G_n$ be (strong) unique independence graphs where $n \geqslant 2$. Let $v_1, v_2, \ldots, v_n$ be black vertices with v_i in G_i. Form a graph H as follows. Add a new vertex v to the disjoint union of the G_i's. Join v to each v_i by an edge. Then H is a (strong) unique independence graph.*

Proof. We claim that $I = \bigcup_{i=1}^{n} I(G_i)$ is the unique maximum independent set in H. Clearly any independent set as large as I must contain v, else it is independent in $\bigcup G_i$. But any independent set containing v can contain no more than $|I(G_i)| - 1$ vertices in each G_i. Since $n \geqslant 2$, this means that such an independent set must be smaller than I. Now, if each G_i is strong, then the white vertices in each G_i form an independent set. The new white vertex v is adjacent only to black vertices, so H is strong. □

Theorem 3. *A tree T is a strong unique independence tree if and only if the distance between any two end vertices is even.*

Proof. If T is strong, then the black and white vertices constitute a two-coloring. Hence any path in T passes through alternately black and white vertices, and certainly the distance between any two black vertices must be even. But an end vertex must certainly be black. For if v is an end vertex with neighbor w and if v were white, then $I - \{w\} \cup \{v\}$ would be another independent set as large as I.

Conversely, suppose that T has the property that the distance between any two end vertices is even. If T is a path, then it has an odd number of vertices and is strong. Otherwise, let v be an end vertex of T. Let w be the vertex of degree at least three which is closest to v in T. There is a unique path $v = v_0, v_1, v_2, \ldots, v_n = w$. Remove from T the path $v_0, \ldots, v_{n-1}$, leaving w. What remains is a tree T', and, in T', the only end vertices are in fact end vertices of T. Hence the distance between any two end vertices in T' is even. By induction T' is strong. Consider two cases. First, if w is white in T', we claim that n is odd. For, if u is an end vertex of T', then the distance from v to u in T is even. But, since w is white in T' and u is black in T', the distance from w to u in T' is odd. It follows that the distance from v to w must be odd, but this distance is n. We note that the path $v_0, v_1, v_2, \ldots, v_{n-1}$ is strong and that v_{n-1} is black in this path. We invoke Lemma 1 with $G = T'$ and $H =$ the path. Similarly, if w is black in T', it follows that n is even, so the path $v_0, v_1, \ldots, v_{n-2}$ is strong, and we invoke Lemma 2. □

The condition characterizing strong unique independence trees is the same condition which characterizes block-cutpoint trees of graphs, and appears in Harary and Plummer [3].

We now prove a lemma necessary for our characterization of strong unique independence graphs. Our lemma is a special case of a very nice theorem of Suškov [5]. We present a slightly different proof. In our proof, we denote bipartite graphs as triples (W, B, Γ) where (W, B) is the bipartition and Γ is the edge set. We refer to vertices in W and B as white and black vertices respectively. If $A \subseteq W$, we denote the set of vertices in B adjacent to vertices in A by $\Gamma(A)$.

Lemma 4. *Suppose that $G = (W, B, \Gamma)$ is a bipartite graph such that for each $\emptyset \neq A \subseteq W$ we have*

$$|\Gamma(A)| > |A|. \tag{*}$$

Then there is a spanning subgraph H of G such that H has property (*) and such that each $v \in W$ has degree two in H.

Proof. Suppose that v is a white vertex of degree $n \geq 3$. Let $e_1, e_2, \ldots, e_n$ be the edges incident with v, and let $v_1, v_2, \ldots, v_n$ be the respective end vertices of these edges. We will show that some e_i may be deleted from G, without losing property (*). Suppose by way of contradiction that the removal of any e_i results in the loss of property (*). Denote the graph obtained by deleting $e_i - (W, B, \Gamma_i)$. Then, for each i there is $A_i \subseteq W$ so that $|\Gamma_i(A_i)| \leq |A_i|$. Clearly, since only one edge was deleted, this means that $|\Gamma_i(A_i)| = |A_i|$. Note that each A_i contains v. Hence $|\Gamma(A_i)| = 1 + |A_i|$, and letting $A_i' = A_i - \{v\}$, we have $|\Gamma(A_i')| = 1 + |A_i'|$.

Note that $\Gamma_i(A_i')$ contains each v_j, $j \neq i$. Hence $\Gamma_i(A_i') \cap \Gamma_j(A_j') \neq \emptyset$, but this is the same as saying $\Gamma(A_i') \cap \Gamma(A_j') \neq \emptyset$.

We show by induction on k that for $1 \leq k \leq n$,

$$\left|\Gamma\left(\bigcup_{i=1}^{k} A_i'\right)\right| = 1 + \left|\bigcup_{i=1}^{k} A_i'\right|.$$

For $k = 1$, this has already been noted. Suppose now that

$$\left|\Gamma\left(\bigcup_{i=1}^{k-1} A_i'\right)\right| = 1 + \left|\bigcup_{i=1}^{k-1} A_i'\right|.$$

We now have

$$\begin{aligned}
\left|\Gamma\left(\bigcup_{i=1}^{k} A_i'\right)\right| &= \left|\Gamma\left(\bigcup_{i=1}^{k-1} A_i'\right) \cup \Gamma(A_k')\right| \\
&= \left|\Gamma\left(\bigcup_{i=1}^{k-1} A_i'\right)\right| + |\Gamma(A_k')| - \left|\Gamma\left(\bigcup_{i=1}^{k-1} A_i'\right) \cap \Gamma(A_k')\right| \\
&= \left(\left|\bigcup_{i=1}^{k-1} A_i'\right| + 1\right) + (|A_k'| + 1) - \left|\bigcup_{i=1}^{k-1} (\Gamma(A_i') \cap \Gamma(A_k'))\right| \\
&\leq \left|\bigcup_{i=1}^{k-1} A_i'\right| + |A_k'| + 2 - \left|\bigcup_{i=1}^{k-1} (\Gamma(A_i' \cap A_k'))\right| \\
&= \left|\bigcup_{i=1}^{k-1} A_i'\right| + |A_k'| + 2 - \left|\Gamma\left(\bigcup_{i=1}^{k-1} (A_i' \cap A_k')\right)\right|.
\end{aligned}$$

Now, if some $A_i' \cap A_k' \neq \emptyset$, this yields, using $(*)$,

$$\left|\Gamma\left(\bigcup_{i=1}^{k} A_i'\right)\right| \leq \left|\bigcup_{i=1}^{k-1} A_i'\right| + |A_k'| + 2 - \left|\left(\bigcup_{i=1}^{k-1} A_i' \cap A_k'\right)\right| - 1$$

or

$$\left|\Gamma\left(\bigcup_{i=1}^{k} A_i'\right)\right| \leq 1 + \left|\bigcup_{i=1}^{k} A_i'\right|.$$

In case each $A_i' \cap A_k' = \emptyset$, it is clear that

$$\left|\bigcup_{i=1}^{k-1} \Gamma(A_i') \cap \Gamma(A_k')\right| \geq 1 = 1 + 0 = 1 + \left|\bigcup_{i=1}^{k-1} A_i' \cap A_k'\right|$$

and the same result follows. Hence, when $k = n$ we have

$$\left|\Gamma\left(\bigcup_{i=1}^{n} A_i'\right)\right| = 1 + \left|\bigcup_{i=1}^{n} A_i'\right|.$$

Consider $\bigcup_{i=1}^{n} A_i = (\bigcup_{i=1}^{n} A_i') \cup \{v\}$. Since $\Gamma(v) \subseteq \bigcup_{i=1}^{n} \Gamma(A_i')$, we have

$$\left|\Gamma\left(\bigcup_{i=1}^{n} A_i\right)\right| = \left|\Gamma\left(\bigcup_{i=1}^{n} A_i'\right)\right| = \left|\bigcup_{i=1}^{n} A_i'\right| + 1 = \left|\bigcup_{i=1}^{n} A_i\right|,$$

which contradicts $(*)$.

By repeatedly deleting edges incident with white vertices of degree larger than two, we obtain a spanning subgraph H with the desired properties. □

Theorem 5. *A connected graph is a strong unique independence graph if and only if it is bipartite and has a spanning tree which is a strong unique independence tree.*

Proof. Suppose G is bipartite and has a spanning strong unique independence tree T. Let I be the maximum independent set in T. Then the complement of I is independent in T, since T is strong. We claim that each edge of G not in T joins a vertex in I to a vertex not in I. For, each such edge, added to T, determines a cycle. And if such an edge joined two vertices of I or two vertices not in I, then an odd cycle would be formed, contradicting the bipartiteness of G. It follows that I and its complement are independent in G. Hence I is certainly maximum independent in G, and G is a strong unique independence graph.

Conversely, suppose that G is a strong unique independence graph, with maximum independent set B. We may then use notation consistent with that in Lemma 4, $G = (W, B, \Gamma)$. We note first that G has property (*). For, if there were $\emptyset \neq A \subseteq W$ such that $|\Gamma(A)| \leq |A|$, then $B - \Gamma(A) \cup A$ would be another independent set at least as large as B, which is a contradiction. By Lemma 4, G has a spanning subgraph H which still enjoys property (*) and such that every $v \in W$ has degree two in H. We first claim that H is a forest. For, if H contained a cycle C, necessarily even, then the white vertices in C would dominate only the black vertices in C. Letting $A = C \cap W$, we would then have $|\Gamma_H(A)| = |A|$, contradicting (*) in H. Note that every end vertex of the forest H must be in B, since each vertex in W has degree two. In any component of H, then, the distance between any two end vertices is even, so each component of H is a strong unique independence tree. We now extend H to a spanning tree T of G by adding as many edges as necessary. Since all end vertices of T are in B, it is clear that B is a strong unique independence tree. □

We have characterized strong unique independence trees and strong unique independence graphs. Now we turn our attention to unique independence trees.

Theorem 6. *A tree T is a unique independence tree with independent set I if and only if T has a spanning forest F such that:*

(i) *Each component of F is a strong unique independence tree;*

(ii) *Each edge in $T - F$ joins two vertices not in I.*

Proof. One implication follows immediately from Lemma 1. For the other, assume T is a unique independence tree with n edges, and that the conditions hold for unique independence trees with fewer than n edges. If T has no edges which join two vertices not in I, then clearly the distance between any two end vertices of T is even, and T is itself a strong unique independence tree. Otherwise we remove an edge e which joins two vertices not in I, leaving two components T_1 and T_2. We claim that each of T_1 and T_2 is a unique independence tree. For, if in T_1 there is another independent set I_2 as large as $T_1 \cap I$, then $(T_2 \cap I) \cup I_2$ would be another independent set in T as large as I. It follows that T_1 is a unique independence tree. So likewise is T_2. By induction, each of T_1 and T_2 has a spanning forest with the desired properties. The union of these two forests is the desired forest. □

We now construct some families of regular unique independence graphs. As noted above, strong unique independence graphs are bipartite, but this restriction does not apply to unique independence graphs in general. First, we note some bounds on the size of independent sets in unique independence graphs.

Theorem 7. *Let G be a unique independence graph, with maximum degree r, and suppose that G has n vertices. Let I be the maximum independent set in G. Then:*

(i) $|I| \geqslant 2n/(r+2)$;
(ii) *if G is regular of degree r,* $|I| < n/2$.

Proof. In Lemma 1 of [4], it is proved that $|I| \geqslant 2n/(r+2) - \alpha_1/(r+2)$, where α_1 is the number of vertices in the complement of I with one neighbor in I. Clearly $\alpha_1 = 0$ and (i) follows. since $r\,|I|$ edges are incident with I, (ii) is clear. □

For each $r > 2$, we now construct a unique independence graph which is regular of degree r and whose maximum independent set has exactly the fraction $2/(r+2)$ of the vertices. Let e be an edge of a copy G of the complete graph K_{r+1}, and suppose e joins vertices v and w. Let e' joining v' and w' be an edge of another copy G' of K_{r+1}. Let a and b be the vertices of a copy H of K_2. To the graph $(G-e) \cup (G'-e') \cup H$, add the edges av, av', bw, and bw'. It is easy to check that this is a unique independence graph with 4 independent vertices out of $2r+4$. In case $r = 3$, the above graph is already regular. If $r \geqslant 4$, we regularize by the standard technique (see [1, p. 10]), and all the relevant properties are preserved.

Now we construct examples in which the unique maximum independent set has close to half the vertices. Let $r \geqslant 4$. Let H be a bipartite graph with bipartition (A, B), $A = \{a_1, a_2, \ldots, a_n\}$, $B = \{b_1, b_2, \ldots, b_n\}$, and suppose H is chosen so that every vertex has degree $r-1$, and so that for each i, a_i is not adjacent to b_i. Join a_{n-1} to a_n with a new edge, and for each i, $1 \leqslant i \leqslant n-2$, Join a_i to b_i with a new edge. The resulting graph G is a unique independence graph in which b_{n-1} and b_n have degree $r-1$ and all other vertices have degree r. Now, let G_1 and G_2 be copies of G, and let v be a new vertex. Join v to each of the four vertices of degree $r-1$ in G_1 and G_2. The resulting graph is a unique independence graph with $2n$ independent vertices out of $4n+1$. If $r = 4$, the example is regular. If $r > 4$, we regularize as above, and all relevant properties are preserved. In case $r = 3$, a similar construction works.

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