Integrability of Double Cosine–Sine Series in the Sense of Improper Riemann Integral

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We consider the double cosine-sine series \((\ast) \sum \sum \lambda_i \alpha_{jk} \cos jx \sin ky\), where the coefficients \(\{\alpha_{jk} : j = 0, 1, \ldots; k = 1, 2, \ldots\}\) form a null sequence of bounded variation and \(\lambda_j = \frac{1}{2}\) or 1 according to \(j = 0\) or 1, 2, .... In this case, series \((\ast)\) converges pointwise in Pringsheim’s sense for all \(x, y\) such that \(x \neq 0 \text{ (mod } 2\pi)\) to a function \(h(x, y)\), say. We prove that \(h(x, y) \sin y\) is integrable in the sense of improper Riemann integral with respect to \(x\) and continuous with respect to \(y\), and series \((\ast)\) is the generalized Fourier sine series of its sum. This implies the important corollary that if \(h(x, y)\) is Lebesgue integrable, then series \((\ast)\) is the (Lebesgue) Fourier series of its sum. Furthermore, we give a sufficient condition for the integrability of \(h(x, y)\) in the sense of improper Riemann integral. This condition is also necessary in the special case when \(A_{jk} \neq 0\) for all \(j\) and \(k\). Finally, we give sufficient conditions, under which \(h(x, y)/y\) and \(h(x, y)/xy\) are integrable in the sense of improper Riemann integral.

1. INTRODUCTION

Let \(\{\alpha_{jk} : j = 0, 1, \ldots; k = 1, 2, \ldots\}\) be a double sequence of real numbers such that

\[
\lim_{j + k \to \infty} \alpha_{jk} = 0 \quad \quad \quad \quad \quad (1.1)
\]

and

\[
\sum_{j = 0}^{\infty} \sum_{k = 1}^{\infty} |A_{11} \alpha_{jk}| < \infty, \quad \quad \quad \quad \quad (1.2)
\]

where

\[
A_{11} \alpha_{jk} = A_{01} (A_{10} \alpha_{jk}) - A_{10} (A_{01} \alpha_{jk}),
\]

\[
A_{10} \alpha_{jk} = \alpha_{jk} - \alpha_{j+1,k}, \quad \text{and} \quad A_{01} \alpha_{jk} = \alpha_{jk} - \alpha_{j,k+1}.
\]

We say that \(\{\alpha_{jk}\}\) is a double null sequence of bounded variation.
We consider the double cosine–sine series

$$
\sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \lambda_j a_{jk} \cos jx \sin ky
$$

on the positive quadrant

$$
T_2^2 = [0, \pi] \times [0, \pi]
$$

of the two-dimensional torus, where \( \lambda_0 = \frac{1}{2} \) and \( \lambda_j = 1 \) for \( j = 1, 2, \ldots \). The pointwise convergence of series (1.3) is meant in Pringsheim’s sense. (See, e.g., [7, Vol. 2, Chap. 17, especially p. 303].) In other words, we form the rectangular partial sums

$$
s_{mn}(x, y) = \sum_{j=0}^{m} \sum_{k=1}^{n} \lambda_j a_{jk} \cos jx \sin ky \quad (m = 0, 1, \ldots; n = 1, 2, \ldots)
$$

of series (1.3), then let both \( m \) and \( n \) tend to \( \infty \) independently of one another, and assign the limit \( h(x, y) \) (if it exists) to series (1.3) as its sum.

As is known, under conditions (1.1) and (1.2), series (1.3) converges for all \( x, y \) such that \( x \neq 0 \) (mod \( 2\pi \)), and this convergence is uniform on each rectangle

$$
T_{\delta, \epsilon} = [\delta, \pi] \times [\epsilon, \pi] \quad (0 < \delta, \epsilon < \pi).
$$

(See also the first part of the proof of Theorem 2 in Section 4.)

We note that even more is true: under conditions (1.1) and (1.2), series (1.3) converges regularly for all \( x, y \) such that \( x \neq 0 \) (mod \( 2\pi \)). (Concerning the notion of regular convergence, we refer the reader to [2] and also [3].) As to the proof of the regular convergence of series (1.3) under conditions (1.1) and (1.2), see [4] where a proof is carried out in the case of double complex trigonometric series.

2. Main Results

Conditions (1.1) and (1.2) do not imply, in general, the Lebesgue integrability of the sum \( h(x, y) \) of series (1.3) (cf. [6] for single cosine and sine series) or the integrability of \( h(x, y) \) in the sense of improper Riemann integral (cf. [1] for single sine series). On the other hand, we will show that, under conditions (1.1) and (1.2), series (1.3) is the generalized Fourier sine series of its sum \( h(x, y) \). (Concerning this terminology, see [7, Vol. 1, p. 48].)
THEOREM 1. Under conditions (1.1) and (1.2),

(i) \( h(x, y) \sin y \) is integrable in the sense of improper Riemann integral with respect to \( x \) and continuous with respect to \( y \),

(ii) series (1.3) is the generalized Fourier sine series of \( h(x, y) \) in the sense that

\[
a_{mn} = \lim_{\delta \to 0} \frac{4}{\pi} \frac{1}{\delta} \int_{\delta}^{\pi} \int_{0}^{\pi} h(x, y) \cos mx \sin ny \, dx \, dy
\]

\((m = 0, 1, \ldots; n = 1, 2, \ldots). \quad (2.1)\)

We draw the following important corollary from Theorem 1.

COROLLARY. If conditions (1.1) and (1.2) are satisfied and the sum \( h(x, y) \) of series (1.3) is Lebesgue integrable, then (1.3) is the (Lebesgue) Fourier series of \( h(x, y) \).

The integrability of \( h(x, y) \) in the sense of improper Riemann integral is a delicate question. First, we treat this problem in the special case when

\[
A_{10}a_{jk} \geq 0 \quad (j = 0, 1, \ldots; k = 1, 2, \ldots). \quad (2.2)
\]

THEOREM 2. If conditions (1.1), (1.2), and (2.2) are satisfied, then the improper integral

\[
\lim_{\delta, \varepsilon \to 0} \int_{\delta}^{\pi} \int_{\varepsilon}^{\pi} h(x, y) \, dx \, dy
\]

exists if and only if

\[
\sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \frac{A_{10}a_{jk}}{k} < \infty. \quad (2.4)
\]

Since conditions (1.1) and (2.2) imply that \( \{a_{jk} : j = 0, 1, \ldots\} \) is a nonincreasing sequence of nonnegative numbers for each \( k = 1, 2, \ldots \), and condition (2.4) collapses into the condition

\[
\sum_{k=1}^{\infty} \frac{a_{0k}}{k} < \infty,
\]

Theorem 2 has a limited scope of applications. In the general case, we give a sufficient condition for the improper integrability of \( h(x, y) \) as follows.

THEOREM 3. If conditions (1.1) and (1.2) are satisfied, and the series

\[
\sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \frac{A_{10}a_{jk}}{k}
\]

converges regularly, \( (2.5) \) then relation (2.3) holds.
We raise two problems in connection with Theorem 3.

PROBLEM 1. The requirement of regular convergence in condition (2.5) seems to be essential. We conjecture that there exists a double sequence \( \{a_{jk}\} \) such that conditions (1.1) and (1.2) are satisfied, the double series in (2.5) converges in Pringsheim’s sense, and relation (2.3) does not hold.

PROBLEM 2. We guess that condition (2.5) is not necessary for the fulfillment of (2.3) in general (i.e., when (2.2) is not satisfied).

We note, however, that the conditions in Theorem 3 imply more than (2.3). To go into details, we consider the so-called “row” series (i.e., when \( k \) is fixed and \( \sin ky \) is deleted in series (1.3))

\[
\sum_{j=0}^{\infty} \lambda_j a_{jk} \cos jx = f_k(x) \quad (k = 1, 2, \ldots) \tag{2.6}
\]

and “column” series

\[
\sum_{k=1}^{\infty} a_{jk} \sin ky = g_j(y) \quad (j = 0, 1, \ldots). \tag{2.7}
\]

The pointwise convergence of these single series, except possibly at \( x = 0 \) (mod \( 2\pi \)), follows from the fact that the single sequences \( \{a_{jk} : j = 0, 1, \ldots\} \) and \( \{a_{jk} : k = 1, 2, \ldots\} \) are null sequences of bounded variation for each fixed \( k \) or \( j \), respectively, due to conditions (1.1) and (1.2). By virtue of the corresponding one-dimensional result of Boas [1], the improper integral

\[
\lim_{\varepsilon \downarrow 0} \int_{-\varepsilon}^{\varepsilon} g_j(y) \, dy \quad \text{exists} \tag{2.8}
\]

if and only if the series

\[
\sum_{k=1}^{\infty} \frac{a_{jk}}{k} \quad \text{converges} \quad (j = 0, 1, \ldots), \tag{2.9}
\]

while the improper integral

\[
\lim_{\varepsilon \downarrow 0} \int_{-\varepsilon}^{\varepsilon} f_k(x) \, dx \quad \text{exists} \quad (k = 1, 2, \ldots)
\]

without any further condition on the \( a_{jk} \). Since the regular convergence of the series in (2.5) implies the (ordinary) convergence of its columns, in particular, the fulfillment of (2.9), we can conclude (2.8) (together with (2.3)) from the conditions of Theorem 3.
In spite of this observation, we conjecture that there exists a double sequence \( \{a_{jk}\} \) such that conditions (1.1), (1.2), and (2.3) are satisfied, and the series in (2.5) does not converge regularly. By Theorem 2, if the differences \( A_{10}a_{jk} \) are of constant sign, then \( \{a_{jk}\} \) cannot be a counterexample.

Finally, we study the integrability of \( h(x, y)/y \) and \( h(x, y)/xy \) in the sense of improper Riemann integral, under stronger conditions than (1.2).

**Theorem 4.** If condition (1.1) is satisfied and

\[
\sum_{j=0}^{\infty} \sum_{k=1}^{\infty} |A_{10}a_{jk}| < \infty, \tag{2.10}
\]

then the improper integral

\[
\lim_{\delta \to 0} \int_{x=\delta}^{\pi} \int_{y=\delta}^{\pi} \frac{h(x,y)}{y} \, dx \, dy \quad \text{exists.} \tag{2.11}
\]

**Problem 3.** We note that if condition (2.2) is satisfied, then \( \{a_{jk} : j=0,1,\ldots\} \) is a nonincreasing sequence of nonnegative numbers for each \( k = 1, 2, \ldots \) and condition (2.10) collapses into the condition

\[
\sum_{k=1}^{\infty} a_{0k} < \infty.
\]

We conjecture that, under (1.1) and (2.2), conditions (2.10) and (2.11) are equivalent.

It is easy to obtain a natural preassumption in the problem of the integrability of \( h(x, y)/xy \). Namely, if \( h(x, y)/xy \) is integrable in the sense of improper Riemann integral, then we have necessarily \( h(0, y) = 0 \) for all \( y \), that is

\[
\sum_{j=0}^{\infty} \lambda_j a_{jk} = 0 \quad (k = 1, 2, \ldots). \tag{2.12}
\]

**Theorem 5.** If condition (2.12) its satisfied,

\[
\sum_{j=0}^{\infty} \sum_{k=1}^{\infty} |a_{jk}| < \infty, \tag{2.13}
\]

and the double series

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m} \sum_{j=0}^{m-1} \lambda_j a_{jn} \quad \text{converges regularly.} \tag{2.14}
\]
then the improper integral
\[
\lim_{\delta \to 0} \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} h(x, y) \frac{dx}{xy} \quad \text{exists.} \tag{2.15}
\]

We raise two problems in connection with Theorem 5.

**Problem 4.** We consider again the row series (2.6) and column series (2.7) of the double series (1.3). By virtue of the corresponding one-dimensional result of Boas [1] (observe that condition (2.12) is crucial), the improper integral
\[
\lim_{\delta \to 0} \int_{-\delta}^{\delta} f_k(x) \frac{dx}{x} \quad \text{exists} \tag{2.16}
\]
if and only if the series
\[
\sum_{m=1}^{\infty} \frac{1}{m} \sum_{j=0}^{m-1} \lambda_j a_{jk} \quad \text{converges} \quad (k = 1, 2, \ldots),
\]
while the improper integral
\[
\lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} g_j(y) \frac{dy}{y} \quad \text{exists} \quad (j = 0, 1, \ldots)
\]
without any further condition on the \(a_{jk}\). Since the regular convergence of series (2.14) implies the (ordinary) convergence of its rows, we can conclude (2.16) (together with (2.15)) from the conditions of Theorem 5.

In spite of this fact, we guess that there exists a double sequence \(\{a_{jk}\}\) such that conditions (2.12), (2.13), and (2.15) are satisfied, and the series in (2.14) does not converge regularly.

**Problem 5.** It would be of interest to know, under what further conditions imposed on \(\{a_{jk}\}\) (e.g., \(A_{11}a_{jk} \geq 0\) for all \(j\) and \(k\)), conditions (2.14) and (2.15) are equivalent.

On closing, we note that we dealt with the problem of the improper integrability of double cosine and double sine series in [5].

### 3. Auxiliary Results

We need four lemmas.

**Lemma 1 (Known).** Let \(\{b_k : k = 1, 2, \ldots\}\) be a sequence of real numbers, \(\{c_k\}\) a monotone sequence of positive numbers, and
\[
R_N = \sum_{k = n}^{N} b_k \quad (0 \leq n \leq N).
\]
Then
\[ \left| \sum_{k=n}^{N} b_k c_k \right| \leq \begin{cases} 2c_N \max |B_{n_1}| & \text{if } \{c_k\} \text{ is nondecreasing}, \\ c_n \max |B_{n_1}| & \text{if } \{c_k\} \text{ is nonincreasing}, \end{cases} \]
where the maximum is extended over \( n \leq n_1 \leq N \).

**Lemma 2.** If conditions (1.1) and (1.2) are satisfied, then there exist two sequences \( \{b_{jk}\} \) and \( \{c_{jk}\} \) of nonnegative numbers such that
\[ a_{jk} = b_{jk} - c_{jk} \quad (j = 0, 1, \ldots; k = 1, 2, \ldots), \]
\[ \lim_{j+k \to \infty} b_{jk} = \lim_{j+k \to \infty} c_{jk} = 0, \]
\[ \Delta_{11} b_{jk} \geq 0 \quad \text{and} \quad \Delta_{11} c_{jk} \geq 0. \]

We note that hence it follows that
\[ b_{jk} \geq 0, \quad \Delta_{10} b_{jk} \geq 0, \quad \Delta_{01} b_{jk} \geq 0, \]

analogous inequalities for \( c_{jk} \), and condition (1.2) is satisfied with \( b_{jk} \) and \( c_{jk} \), respectively, instead of \( a_{jk} \).

**Proof:** It is easy to check the fulfillment of the statements in Lemma 2 if we put
\[ b_{jk} = \frac{1}{2} \sum_{m=j}^{\infty} \sum_{n=k}^{\infty} (|\Delta_{11} a_{mn}| + \Delta_{11} a_{mn}) \]
and
\[ c_{jk} = \frac{1}{2} \sum_{m=j}^{\infty} \sum_{n=k}^{\infty} (|\Delta_{11} a_{mn}| - \Delta_{11} a_{mn}). \]

**Lemma 3.** If conditions (1.1) and (1.2) are satisfied, then the series
\[ \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \frac{\Delta_{10} a_{jk}}{k} \cos ky \]
converges regularly for all \( y \neq 0 \) (mod \( 2\pi \)).

**Proof:** We have to show that, for any \( 0 < y \leq \pi \), the sums
\[ S(m, p; n, q) = \sum_{j=m}^{p} \sum_{k=n}^{q} \frac{\Delta_{10} a_{jk}}{k} \cos ky \]
are arbitrarily small if \( \max(m, n) \) is large enough whenever \( p \geq m \) and \( q \geq n \). A summation by parts with respect to \( k \) yields

\[
S(m, p; n, q) = \sum_{j=m}^{p} \sum_{k=n}^{q-1} A_{01} \left( \frac{A_{10} a_{jk}}{k} \right) \times \sum_{l=n}^{k} \cos ly + \sum_{j=m}^{p} \frac{A_{10} a_{j\ell}}{q} \sum_{l=n}^{q} \cos ly.
\]

Since

\[
A_{01} \left( \frac{A_{10} a_{jk}}{k} \right) = \frac{A_{11} a_{jk}}{k} + \frac{A_{10} a_{j,k+1}}{k(k+1)}
\]

and, for \( 0 < y \leq \pi \),

\[
\left| \sum_{l=n}^{k} \cos ly \right| = \left| \frac{\sin(k + \frac{1}{2})y - \sin(n - \frac{1}{2})y}{2\sin(y/2)} \right| \leq \frac{\pi}{y},
\]

we can infer that

\[
\sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \frac{|A_{10} a_{j,k+1}|}{k(k+1)} \leq \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \sum_{l=k+1}^{\infty} |A_{11} a_{jl}| = \sum_{j=0}^{\infty} \sum_{l=2}^{\infty} |A_{11} a_{jl}| \sum_{k=1}^{l-1} \frac{1}{k(k+1)} < \infty
\]

and

\[
\sup_{q \geq 1} \sum_{j=0}^{\infty} |A_{10} a_{j\ell}| \leq \sum_{j=0}^{\infty} \sum_{l=1}^{\infty} |A_{11} a_{jl}| < \infty.
\]

These inequalities are enough to conclude the statement of Lemma 3.

In the sequel, let

\[
D_j(x) = \frac{1}{2} + \sum_{i=1}^{j} \cos ix = \frac{\sin(j + \frac{1}{2})x}{2\sin(x/2)} (j = 0, 1, \ldots)
\]

be the well-known Dirichlet kernel and let

\[
b_j(\delta) = \int_{-\delta}^{\delta} D_j(x) \, dx = \frac{\pi - \delta}{2} - \sum_{i=1}^{j} \frac{\sin \delta}{i} (0 < \delta < \pi).
\]

Since the partial sums of the series

\[
\sum_{i=1}^{\infty} \frac{\sin i\delta}{i}
\]
are uniformly bounded, we see that the $b_j^{(\delta)}$ are also uniformly bounded for all $j = 0, 1, \ldots$, and $0 < \delta < \pi$, and

$$\lim_{\delta \to 0} b_j^{(\delta)} = \frac{\pi}{2} \quad (j = 0, 1, \ldots). \quad (3.4)$$

**Lemma 4.** If conditions (1.1) and (1.2) are satisfied, then the series

$$S^{(\delta)}(y) = \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \frac{A_{10}a_{jk}}{k} b_j^{(\delta)} \cos ky$$

converges regularly for all $0 < \delta \leq \pi$ and

$$\lim_{\delta \to 0} S^{(\delta)}(y) = \frac{\pi}{2} \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \frac{A_{10}a_{jk}}{k} \cos ky.$$

**Proof:** It runs along the same lines as the proof of Lemma 3. Therefore we omit it.

4. THE PROOFS OF THE MAIN RESULTS

**Proof of Theorem 1.** We multiply series (1.3) by $2 \sin y$. As a result we obtain

$$2h(x, y) \sin y = \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \lambda_j a_{jk} \cos jx \{ \cos(k-1)y - \cos(k+1)y \}.$$

Hence a summation by parts with respect to $k$ yields

$$2h(x, y) \sin y = \sum_{j=0}^{\infty} \lambda_j \cos jx \left\{ a_{j1} + \sum_{k=1}^{\infty} (a_{j,k+1} - a_{j,k-1}) \cos ky \right\}$$

with the agreement that $a_{j0} = 0$ for all $j \geq 0$. Another summation by parts (this time with respect to $j$) gives

$$2h(x, y) \sin y = \sum_{j=0}^{\infty} A_{10}a_{j1} D_j(x)$$

$$+ \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} (A_{10}a_{j,k+1} - A_{10}a_{j,k-1}) D_j(x) \cos ky. \quad (4.1)$$

Since

$$A_{10}a_{j1} = \sum_{k=1}^{\infty} A_{11}a_{jk},$$

$$A_{10}a_{j,k+1} - A_{10}a_{j,k-1} = - (A_{11}a_{jk} + A_{11}a_{j,k-1}),$$
by (1.2), the series on the right-hand side of (4.1) converges uniformly on each rectangle \( \delta \leq x \leq \pi \) and \( 0 \leq y \leq \pi \), where \( 0 < \delta < \pi \). Thus, a term-by-term integration of (4.1) is allowed,

\[
I_{01}^{(\delta)} = \int_{-\delta}^{\delta} \int_{0}^{\pi} h(x, y) \sin y \, dx \, dy = \frac{\pi}{2} \sum_{j=0}^{\infty} b_{j}^{(\delta)} A_{10} a_{j},
\]

where \( b_{j}^{(\delta)} \) is defined by (3.3). By (3.4),

\[
\lim_{\delta \to 0} I_{01}^{(\delta)} = \frac{\pi^{2}}{4} \sum_{j=0}^{\infty} A_{10} a_{j} = \frac{\pi^{2}}{4} a_{01}, \tag{4.2}
\]

which proves (2.1) in the case of \( m = 0 \) and \( n = 1 \).

Now let \( m \geq 1 \) be fixed. Then, by (4.1),

\[
I_{m1}^{(\delta)} = \int_{-\delta}^{\delta} \int_{0}^{\pi} h(x, y) \cos mx \sin y \, dx \, dy = \frac{\pi}{2} \sum_{j=0}^{\infty} c_{jm}^{(\delta)} A_{10} a_{j},
\]

where

\[
c_{jm}^{(\delta)} = \int_{-\delta}^{\delta} D_{j}(x) \cos mx \, dx = \begin{cases} \frac{1}{2} \sum_{i=0}^{m-j} \frac{\sin i\delta}{i} & \text{if } j = 0, 1, \ldots, m-1; \\ \frac{\pi - \delta}{2} - \frac{1}{2} \sum_{i=0}^{j+m} \frac{\sin i\delta}{i} - \sum_{i=1}^{j-m} \frac{\sin i\delta}{i} & \text{if } j = m, m+1, \ldots. \end{cases}
\]

It follows again (cf. (3.3) and (3.4)) that the \( c_{jm}^{(\delta)} \) are uniformly bounded and

\[
\lim_{\delta \to 0} c_{jm}^{(\delta)} = \begin{cases} 0 & \text{if } j = 0, 1, \ldots, m-1; \\ \pi/2 & \text{if } j = m, m+1, \ldots. \end{cases} \tag{4.3}
\]

Consequently,

\[
\lim_{\delta \to 0} I_{m1}^{(\delta)} - \frac{\pi^{2}}{4} \sum_{j=m}^{\infty} A_{10} a_{j} = \frac{\pi^{2}}{4} a_{m1}, \tag{4.4}
\]

which proves (2.1) in the case of \( m \geq 1 \) and \( n = 1 \).

Finally, let \( n \geq 1 \) be fixed. On the one hand,

\[
2 \int_{-\delta}^{\delta} \int_{0}^{\pi} h(x, y) \cos mx \sin y \cos ny \, dx \, dy
= \int_{-\delta}^{\delta} \int_{0}^{\pi} h(x, y) \cos mx \{ \sin(n+1) y - \sin(n-1) y \} \, dx \, dy
= I_{m,n+1}^{(\delta)} - I_{m,n-1}^{(\delta)}.
\]
On the other hand, by (4.1),

\[ 2 \int_{\delta}^{\pi} \int_{0}^{\pi} h(x, y) \cos mx \sin y \sin ny \, dx \, dy \]

\[ = \pi \sum_{j=0}^{\infty} c(\beta, n)(A_{10}a_{j+n} - A_{10}a_{j-n}). \]

Hence, by (4.3),

\[ \lim_{\delta \to 0} \left( I_{m, n+1}^{(\beta)} - I_{m, n-1}^{(\beta)} \right) \]

\[ = \frac{\pi^2}{4} \sum_{j=m}^{\infty} (A_{10}a_{j+n} - A_{10}a_{j-n}) = \frac{\pi^2}{4} (a_{m+n} - a_{m-n}). \]

Applying an induction argument, while relying on the "initial values" occurring in (4.2) and (4.4), we can justify (2.1) in the general case, too.

**Proof of Theorem 2.** Assume that conditions (1.1) and (1.2) are satisfied. By a summation by parts with respect to \( j \) gives

\[ s_{mn}(x, y) = \sum_{j=0}^{m-1} \sum_{k=1}^{n} A_{10}a_{jk}D_j(x) \sin ky + \sum_{k=1}^{n} a_{mk}D_m(x) \sin ky. \quad (4.5) \]

We will prove that

\[ \lim_{m, n \to \infty} \sum_{k=1}^{n} a_{mk}D_m(x) \sin ky = 0 \quad \text{uniformly in } T_{\delta, \varepsilon} \quad (4.6) \]

for all \( 0 < \delta, \varepsilon < \pi \). To this effect, we perform a summation by parts as follows

\[ \sum_{k=1}^{n} a_{mk}D_m(x) \sin ky \]

\[ = \sum_{k=1}^{n-1} A_{01}a_{mk}D_m(x) \tilde{D}_k(y) + a_{mn}D_m(x) \tilde{D}_n(y), \quad (4.7) \]

where

\[ \tilde{D}_k(y) = \sum_{l=1}^{k} \sin ly = \frac{\cos(y/2) - \cos(k + \frac{1}{2})y}{2 \sin(y/2)} \quad (k = 1, 2, \ldots) \]

is the conjugate Dirichlet kernel. Since

\[ |D_m(x)| \leq \pi/2x \quad \text{and} \quad |\tilde{D}_n(y)| \leq \pi/y \quad (0 < x, y \leq \pi) \quad (4.8) \]
for all \( m = 0, 1, \ldots \), and \( n = 1, 2, \ldots \) (see, e.g., [7, Vol. 1, p. 51]), and

\[
\sum_{k=1}^{n} |\Delta a_{nk}| \leq \sum_{k=1}^{\infty} j=1 |\Delta a_{jk}|,
\]

from (4.7) we can conclude (4.6).

Now we consider the first sum on the right-hand side of (4.5). A summation by parts with respect to \( k \) yields

\[
\sum_{j=0}^{m-1} \sum_{k=1}^{n} \Delta a_{jk} D_{j}(x) \sin ky = \sum_{j=0}^{m-1} \sum_{k=1}^{n} \Delta a_{jk} D_{j}(x) \tilde{B}_{k}(y) + \sum_{j=0}^{m-1} \Delta a_{jn} D_{j}(x) \tilde{B}_{n}(y). \quad (4.9)
\]

Similarly to (4.6), we can prove again that

\[
\lim_{m,n \to \infty} \sum_{j=0}^{m-1} \sum_{k=1}^{n} \Delta a_{jk} D_{j}(x) \tilde{B}_{n}(y) = 0 \quad \text{uniformly in } T_{\delta, \varepsilon} \quad (4.10)
\]

for all \( 0 < \delta, \varepsilon < \pi \). By (1.2) and (4.8), the first sum on the right-hand side of (4.9) also converges uniformly in \( T_{\delta, \varepsilon} \) as \( m, n \to \infty \).

To sum up, by (4.5), (4.6), (4.9), and (4.10), we obtain that, under conditions (1.1) and (1.2),

\[
h(x, y) = \lim_{m,n \to \infty} s_{m,n}(x, y) = \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \Delta a_{jk} D_{j}(x) \sin ky \quad (4.11)
\]

and the convergence is uniform in \( T_{\delta, \varepsilon} \) for all \( 0 < \delta, \varepsilon < \pi \).

We note that in the above proof we did not use the full strength of (1.1); instead, we used only the following conditions:

\[
\lim_{j,k \to \infty} a_{jk} = 0,
\]

\[
\lim_{j \to \infty} \Delta_{01} a_{jk} = 0 \quad (k = 1, 2, \ldots)
\]

and

\[
\lim_{k \to \infty} \Delta a_{jk} = 0 \quad (j = 0, 1, \ldots).
\]

**Sufficiency.** We start with conditions (2.2) and (2.4). By (4.11), for all \( 0 < \delta, \varepsilon < \pi \) we have

\[
\int_{\delta}^{\pi} \int_{\varepsilon}^{\pi} h(x, y) \, dx \, dy = \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \frac{\Delta a_{jk}}{k} b_{j}^{(\delta)} (\cos k\varepsilon - \cos k\pi). \quad (4.12)
\]
By Lemma 4 and (2.4),
\[
\lim_{\delta, \varepsilon \downarrow 0} \int_0^\pi \int_0^\pi h(x, y) \, dx \, dy = \frac{\pi}{2} \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \frac{A_{10}a_{jk}}{k} (1 - \cos k\pi).
\]

**Necessity.** This time we start with conditions (2.2) and (2.3). By (4.12), we conclude that
\[
\sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \frac{A_{10}a_{jk}}{k} (1 - \cos k\pi) = 2 \sum_{j=0}^{\infty} \sum_{i=1}^{\infty} \frac{A_{10}a_{j,2i-1}}{2i-1} < \infty. \tag{4.13}
\]

Taking into account that
\[
A_{10}a_{j,2i} \leq A_{10}a_{j,2i-1} + |A_{11}a_{j,2i-1}|,
\]
relation (2.4) follows from (1.2) and (4.13).

**Proof of Theorem 3.** We begin with relation (4.12). By Lemma 4,
\[
\lim_{\delta, \varepsilon \downarrow 0} \int_0^\pi \int_0^\pi h(x, y) \, dx \, dy = \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \frac{A_{10}a_{jk}}{k} (\cos k\varepsilon - \cos k\pi).
\]

By Lemma 3, it is enough to prove that
\[
\lim_{\varepsilon \downarrow 0} \left\{ \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \frac{A_{10}a_{jk}}{k} \cos k\varepsilon - \sum_{j=0}^{\infty} \sum_{k=1}^{[1/\varepsilon]} \frac{A_{10}a_{jk}}{k} \right\} = 0, \tag{4.14}
\]
where \([u]\) denotes the greatest integral part of \(u\).

By (2.5), for every \(\eta > 0\) we can choose \(n\) so large that
\[
\left| \sum_{j=0}^{\infty} \sum_{k=k_n}^{k_{n+1}} \frac{A_{10}a_{jk}}{k} \right| \leq \eta \quad \text{if} \quad n \leq k_1 \leq k_2. \tag{4.15}
\]

In the sequel, we assume that \(0 < \varepsilon < 1/n\). Then
\[
k_0 = \lceil 1/\varepsilon \rceil > n. \tag{4.16}
\]

We decompose the double sums in (4.14) as follows
\[
\left\{ \sum_{j=0}^{\infty} \sum_{k=1}^{n} + \sum_{j=0}^{\infty} \sum_{k=n+1}^{k_0} + \sum_{j=0}^{k_0} \sum_{k=k_0+1}^{\infty} \right\} \frac{A_{10}a_{jk}}{k} \cos k\varepsilon = U_1 + U_2 + U_3,
\]
say, and
\[
\left\{ \sum_{j=0}^{\infty} \sum_{k=n}^{n-1} + \sum_{j=0}^{\infty} \sum_{k=n}^{k_0} \right\} \frac{A_{10}a_{jk}}{k} = V_1 + V_2,
\]
say.
Clearly,
\[ V_1 - U_1 - \sum_{k=1}^{n-1} (1 - \cos k\varepsilon) \sum_{j=0}^{\infty} \frac{A_{10}a_{jk}}{k} \to 0 \quad \text{as} \quad \varepsilon \downarrow 0, \quad (4.17) \]
provided \( n \) is fixed (cf. (3.2)).

Applying Lemma 1 with \( c_k = 1 - \cos k\varepsilon \) and using (4.15) and (4.16), we obtain that
\[ |V_2 - U_2| = \left| \sum_{k=n}^{k_0} (1 - \cos k\varepsilon) \sum_{j=0}^{\infty} \frac{A_{10}a_{jk}}{k} \right| \leq 2(1 - \cos 1) \max_{n \leq k_1 < k_0} \left| \sum_{k=n}^{k_1} \sum_{j=0}^{\infty} \frac{A_{10}a_{jk}}{k} \right| \leq 2(1 - \cos 1) \eta \leq \eta. \quad (4.18) \]

Finally, we estimate \( |U_3| \). By Lemma 2, we can write \( a_{jk} = b_{jk} - c_{jk} \), where the \( b_{jk} \) and \( c_{jk} \) possess the properties indicated there. Then \( U_3 \) is the difference of two similar sums, one with \( b_{jk} \) and one with \( c_{jk} \). By Lemma 1, the first of these sums
\[ U_3' = \sum_{k = k_0 + 1}^{\infty} \cos k\varepsilon \sum_{j=0}^{\infty} \frac{A_{10}b_{jk}}{k} \]
does not exceed in absolute value the quantity
\[ \sum_{j=0}^{\infty} \frac{A_{10}b_{j,k_0+1}}{k_0 + 1} \sup_{k_0 < k_1} \left| \sum_{k = k_0 + 1}^{k_1} \cos k\varepsilon \right|. \]
Hence, by (3.1) and (4.16),
\[ |U_3'| \leq \frac{\pi}{(k_0 + 1) \varepsilon} \sum_{j=0}^{\infty} A_{10}b_{j,k_0+1} \]
\[ = \frac{\pi b_{0,k_0+1}}{(k_0 + 1) \varepsilon} \leq \pi b_{0,k_0+1} \to 0 \quad \text{as} \quad k_0 \to \infty, \]
or equivalently, as \( \varepsilon \downarrow 0 \). The second part of \( U_3 \) is treated in the same way. Thus,
\[ U_3 \to 0 \quad \text{as} \quad \varepsilon \downarrow 0. \quad (4.19) \]
Combining (4.17)–(4.19) yields (4.14) to be proved.

**Proof of Theorem 4.** We introduce a new sequence \( \{b_{jk}\} \) defined by
\[ b_{jk} = \sum_{l=k}^{\infty} a_{jl} \quad (j = 0, 1, \ldots; k = 1, 2, \ldots). \]
This definition is justified by (1.1) and (2.10), according to which
\[ a_{ij} = \sum_{l=j}^{\infty} \Delta_{10} a_{il}, \]
whence
\[ |b_{jk}| \leq \sum_{l=k}^{\infty} |a_{il}| \leq \sum_{l=j}^{\infty} \sum_{l=k}^{\infty} |\Delta_{10} a_{il}|. \]

Clearly,
\[ \Delta_{01} b_{jk} = a_{jk} \quad \text{and} \quad \Delta_{11} b_{jk} = \Delta_{10} a_{jk}. \]

By (2.10),
\[ \lim_{j+k \to \infty} b_{jk} = 0 \quad \text{(4.20)} \]
and
\[ \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} |\Delta_{11} b_{jk}| = \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} |\Delta_{10} a_{jk}| < \infty. \quad \text{(4.21)} \]

A summation by parts with respect to \(k\) gives
\[ h(x, y) = \sum_{j=0}^{\infty} \lambda_j \cos jx \sum_{k=1}^{\infty} b_{jk} \{\sin ky - \sin(k-1)y\} \]
\[ = h_1(x, y) - h_2(x, y), \quad \text{(4.22)} \]
where
\[ h_1(x, y) = \sin y \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \lambda_j b_{jk} \cos jx \cos ky, \]
\[ h_2(x, y) = 2 \sin^2 \frac{y}{2} \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \lambda_j b_{jk} \cos jx \sin ky. \]

Since the function \((\sin y)/y\) has the limit 1 as \(y \downarrow 0\), the improper integral
\[ \lim_{\delta, \epsilon \to 0} \int_{\delta}^{\pi} \int_{\epsilon}^{\pi} \frac{h_1(x, y)}{y} \, dx \, dy \quad \text{exists} \quad \text{(4.23)} \]
if and only if the improper integral
\[ \lim_{\delta, \epsilon \to 0} \int_{\delta}^{\pi} \int_{\epsilon}^{\pi} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \lambda_j b_{jk} \cos jx \cos ky \, dx \, dy \quad \text{exists.} \]
Due to (4.20) and (4.21), [5, Theorem 1] is applicable and furnishes the existence of the latter improper integral. Consequently, (4.23) holds. Now we consider the function $2 \sin^2(y/2)/y \sin y$, which has the limit $\frac{1}{2}$ as $y \downarrow 0$, and conclude that the improper integral

$$
\lim_{\delta \to 0} \int_{-\delta}^{\delta} \int_{\epsilon}^{\pi} \frac{h_2(x, y)}{y} \, dx \, dy \text{ exists}
$$

if and only if

$$
\lim_{\delta \to 0} \int_{-\delta}^{\delta} \int_{\epsilon}^{\pi} \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \hat{a}_j \hat{b}_{jk} \cos jx \{\cos(k - 1) y - \cos(k + 1) y\} \, dx \, dy \text{ exists.}
$$

By (4.20) and (4.21), we can apply [5, Theorem 1] again and obtain that (4.24) holds. Putting (4.22)–(4.24) together provides (2.11) to be proved.

**Proof of Theorem 5.** We introduce a new sequence \( \{b_{mn}\} \) defined by

$$
b_{mn} = \sum_{j=0}^{m-1} \sum_{k=n}^{\infty} \hat{a}_j \hat{b}_{jk} \quad (m, n = 1, 2, \ldots). \tag{4.25}
$$

Then

$$
A_{10} b_{mn} = -\sum_{k=n}^{\infty} a_{mk}, \\
A_{01} b_{mn} = \sum_{j=0}^{m-1} \hat{a}_j a_{jn}, \\
A_{11} b_{mn} = -a_{mn},
$$

and (2.12)–(2.14) imply

$$
\lim_{m+n \to \infty} b_{mn} = 0, \tag{4.26}
$$

$$
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |A_{11} b_{mn}| < \infty, \tag{4.27}
$$

$$
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{A_{01} b_{mn}}{m} \text{ converges regularly.} \tag{4.28}
$$

We perform a double summation by parts to obtain

$$
h(x, y) = -\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn} \{\cos mx - \cos(m - 1) x\}
\times \{\sin ny - \sin(n - 1) y\}
= 4 \sin \frac{x}{2} \sin \frac{y}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn} \sin \left(\frac{m-1}{2}\right) x \cos \left(\frac{n-1}{2}\right) y.
$$
Hence
\[ h(x, y) = \sin x \sin y \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn} \sin mx \cos ny \]
\[ + 2 \sin x \sin^2 y \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn} \sin mx \sin ny \]
\[ - 2 \sin^2 x \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn} \cos mx \cos ny \]
\[ - 4 \sin^2 x \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn} \cos mx \sin ny \]
\[ = h_3(x, y) + h_4(x, y) + h_5(x, y) + h_6(x, y), \] (4.29)
say.

First, the improper integral
\[ \lim_{\delta, \epsilon \to 0} \int_{\delta}^{\epsilon} \int_{\delta}^{\epsilon} \frac{h_3(x, y)}{xy} \, dx \, dy \] exists (4.30)
if and only if
\[ \lim_{\delta, \epsilon \to 0} \int_{\delta}^{\epsilon} \int_{\delta}^{\epsilon} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn} \sin mx \cos ny \, dx \, dy \] exists.

By (4.26)–(4.28), the symmetric counterpart of Theorem 3 applies and results in (4.30).

Second, the improper integral
\[ \lim_{\delta, \epsilon \to 0} \int_{\delta}^{\epsilon} \int_{\delta}^{\epsilon} \frac{h_4(x, y)}{xy} \, dx \, dy \] exists (4.31)
if and only if
\[ \lim_{\delta, \epsilon \to 0} \int_{\delta}^{\epsilon} \int_{\delta}^{\epsilon} \sin y \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn} \sin mx \sin ny \, dx \, dy \] exists. (4.32)

A single summation by parts with respect to \( n \) yields
\[ \sin y \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn} \sin mx \sin ny \]
\[ = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{01} b_{mn} \sin mx \sum_{k=1}^{n} \sin ky \sin y \]
\[ = \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{01} b_{mn} \sin mx \right. \times \left\{ 1 + \cos y - \cos ny - \cos(n+1) y \right\}. \] (4.33)
Since

\[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{01} b_{mn} \sin mx = \sum_{m=1}^{\infty} b_{m1} \sin mx \]

and by (2.12)-(2.14)

\[ \lim_{m \to \infty} b_{m1} = \lim_{m \to \infty} \sum_{j=0}^{m-1} \sum_{k=1}^{\infty} \lambda_j a_{jk} = 0, \]

\[ \sum_{m=1}^{\infty} |b_{m1} - b_{m+1,1}| \leq \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} |a_{mk}| < \infty, \]

\[ \sum_{m=1}^{\infty} \frac{b_{m1}}{m} = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{j=0}^{m-1} \sum_{k=1}^{\infty} \lambda_j a_{jk} \]

converges,

we can apply the corresponding theorem of Boas [11] on single sine series to obtain that

\[ \lim_{\delta \to 0} \int_{-\delta}^{\delta} \sum_{m=1}^{\infty} b_{m1} \sin mx \, dx \]

exists. \hspace{1cm} (4.34)

Obviously, conditions (4.26)-(4.28) are also satisfied for \( \{A_{01} b_{mn}\} \) instead of \( \{b_{mn}\} \). Thus, we can apply the symmetric counterpart of Theorem 3 to obtain that

\[ \lim_{\delta \to 0} \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{01} b_{mn} \sin mx \cos ny \, dx \, dy \]

exists. \hspace{1cm} (4.35)

Combining (4.33)-(4.35) yields (4.32), and consequently (4.31).

Third, the case of \( h_3(x, y)/xy \) can be treated in an analogous way. That is, the improper integral

\[ \lim_{\delta, \epsilon \to 0} \int_{-\delta}^{\delta} \int_{-\epsilon}^{\epsilon} \int_{-\epsilon}^{\epsilon} h_3(x, y) \frac{dx \, dy}{xy} \]

exists \hspace{1cm} (4.36)

if and only if

\[ \lim_{\delta, \epsilon \to 0} \int_{-\delta}^{\delta} \int_{-\epsilon}^{\epsilon} \sin x \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} b_{mn} \cos mx \cos ny \, dx \, dy \]

exists.

A single summation by parts with respect to \( m \) gives

\[ \sin x \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn} \cos mx \cos ny \]

\[ = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{10} b_{mn} \cos ny \sum_{j=1}^{m} \cos jx \sin x \]

\[ = \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{10} b_{mn} \cos ny \{ -\sin x + \sin mx + \sin(m+1)x \}. \]
We can apply again Theorem 3, the corresponding theorem of Boas [1] on single cosine series in order to obtain (4.36).

Fourth, the improper integral

$$\lim_{\delta, \varepsilon \to 0} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} h_6(x, y) \, dx \, dy \quad \text{exists}$$

(4.37)

if and only if

$$\lim_{\delta, \varepsilon \to 0} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} 4 \sin \frac{x}{2} \sin \frac{y}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn} \cos mx \sin ny \, dx \, dy \quad \text{exists.}$$

(4.38)

A double summation by parts gives that the integrand in (4.38) equals

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{11} b_{mn} \sum_{j=1}^{m} \sum_{k=1}^{n} 2 \cos jx \sin \frac{x}{2} \sum_{k=1}^{n} 2 \sin ky \sin \frac{y}{2}$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{11} b_{mn} \left\{ -\sin \frac{x}{2} + \sin \left( m + \frac{1}{2} \right) x \right\}$$

$$\times \left\{ \cos \frac{y}{2} - \cos \left( n + \frac{1}{2} \right) y \right\}.$$ 

From here and (4.27) it follows that relation (4.38) holds (even in the sense of absolute Lebesgue integral). This proves (4.37).

Combining (4.29)-(4.31), (4.36), and (4.37) yields (2.15) to be proved.

REFERENCES