PHANTOM MAPS AND HOMOLOGY THEORIES

J. Daniel Christensen and Neil P. Strickland

We study phantom maps and homology theories in a stable homotopy category $\mathscr{S}$ via a certain Abelian category $\mathscr{A}$. We express the group $\mathscr{P}(X, Y)$ of phantom maps $X \to Y$ as an Ext group in $\mathscr{A}$, and give conditions on X or Y which guarantee that it vanishes. We also determine $\mathscr{P}(X, HB)$. We show that any composite of two phantom maps is zero, and use this to reduce Margolis's axiomatisation conjecture to an extension problem. We show that a certain functor $\mathscr{S} \to \mathscr{A}$ is the universal example of a homology theory with values in an AB 5 category and compare this with some results of Freyd. © 1997 Elsevier Science Ltd. All rights reserved

1. INTRODUCTION

In this paper we collect together a number of results about the homotopy category of spectra. A central theme is the problem of reconstructing this category from the category of finite spectra or (what is almost equivalent) from the category of generalised homology theories. A central result (to be explained in more detail below) is that the category of spectra is a non-split linear extension of the category of homology theories by a certain square-zero ideal, the ideal of phantom maps.

Many of our results hold not only for the category of spectra but also for other categories with similar formal properties. In Section 2, we give a list of axioms which are sufficient for most of the theory. Let $\mathscr{S}$ be a category satisfying these axioms, and $\mathscr{F}$ the full subcategory of finite objects. In Section 3 we study the category $\mathscr{A}$ of additive functors from $\mathscr{F}$ to the category $\mathscr{A}b$ of Abelian groups, with emphasis on the homology theories. We also study the function $h: \mathscr{S} \to \mathscr{A}$ that sends a spectrum X to the homology theory h_X it represents.

In Section 4, we consider phantom maps: a map $f: X \to Y$ is called *phantom* if $h_f: h_X \to h_Y$ is zero, and the group of phantom maps from X to Y is written $\mathscr{P}(X, Y)$. In Section 5, we show how our results about phantoms give new evidence for a conjectured axiomatic characterisation of the classical stable homotopy category, due to Margolis. In Section 6, we analyse the groups $\mathscr{P}(X, HA)$, where X is an arbitrary spectrum and HA is an Eilenberg–Mac Lane spectrum. Finally, in Section 7, we show that the functor $h: \mathscr{S} \to \mathscr{A}$ is the universal example of a homology theory on $\mathscr{S}$ with values in an Abelian category satisfying Grothendieck's axiom AB 5. We also compare this with Freyd's construction of a universal example without the AB 5 condition, and make some related remarks about pro-spectra and ind-spectra.

We next give a more detailed summary of our main results. First we show that there are several characterisations of phantom maps.

PROPOSITION 1.1. *Let* $f: X \to Y$ *be a map of spectra. Then the following conditions are equivalent:*

(i) f *is phantom, i.e.,* $h_X(W) \to h_Y(W)$ *is zero for each finite* W.

(ii) $H(f)$: $H(X) \to H(Y)$ *is zero for each homology theory* H.

(iii) *The composite* $W \to X \to Y$ *is zero for each finite spectrum* W *and each map* $W \to X$.

(iv) *The composite* $X \to Y \to IW$ *is zero for each finite spectrum* W *and each map* $Y \to IW$. (*Here* IW *denotes the Brown–Comenetz dual of* W; *see Section* 3).

Another important result is the following.

THEOREM 1.2. *The composite of two phantom maps is zero* (*and thus the phantom maps form a square-zero ideal*).

This is a result that is folklore, but as far as we are aware the only proof that works in this generality is the one presented here, which was independently discovered by Neeman [24]. Neeman also proved some parts of Propositions 1.4–1.6. Ohkawa [25] has a proof of Theorem 1.2 which works in the stable homotopy category and uses CW-structures; it is not clear whether it goes through under our axiomatic assumptions. A simpler proof that works for a stricter notion of phantom map appears in Gray's thesis [7] and is published in [9]. The two notions coincide when the source has finite skeleta.

It turns out that a number of interesting concepts can be described in terms of the homological algebra of the Albelian category $\mathscr{A}$. As usual, an object F of $\mathscr{A}$ is said to be *projective* if maps from F lift over epimorphisms, and *injective* if maps to F extend over monomorphisms. A spectrum X is $\mathscr{A}$-*projective* if h_X is projective in $\mathscr{A}$ and $\mathscr{A}$-*injective* if h_X is injective in $\mathscr{A}$.

Here are two of our main results.

THEOREM 1.3. *There is a natural isomorphism* $\mathscr{P}(\Sigma^{-1}X, Y) \cong \mathrm{Ext}_{\mathscr{A}}(h_X, h_Y)$.

PROPOSITION 1.4. *Let* $F \in \mathscr{A}$. *The following are equivalent*:

(i) F *has finite projective dimension*.

(ii) F *has projective dimension at most one*.

(iii) F *is a homology theory*.

(iv) F *has injective dimension at most one*.

(v) F *has finite injective dimension*.

In view of the above, if an object F of $\mathscr{A}$ is projective or injective, then it has the form h_X for some spectrum X (which is unique up to isomorphism). The following result describes those X for which h_X is projective or injective.

PROPOSITION 1.5. *Let* X *be a spectrum. Then the following are equivalent*:

(i) X *is* $\mathscr{A}$-*projective*.

(ii) X *is a retract of a wedge of finite spectra*.

(iii) $\mathscr{P}(X, Y) = 0$ *for each spectrum* Y.

Similarly, the following are equivalent:

(i) X *is* $\mathscr{A}$-*injective*.

(ii) X *is a retract of a product of Brown–Comenetz duals of finite spectra*.

(iii) $\mathscr{P}(Y, X) = 0$ *for each spectrum* Y.

We also prove the following facts:

PROPOSITION 1.6. 1. *The category $\mathscr{A}$ has enough injectives and projectives.*

2. *Any spectrum X sits in a cofibre sequence $P \to Q \to X \to \Sigma P$, where P and Q are $\mathscr{A}$-projective and $X \to \Sigma P$ is phantom. The sequence $h_P \to h_Q \to h_X$ is a a short exact sequence in $\mathscr{A}$. The map $X \to \Sigma P$ is weakly initial among phantom maps out of X.*

3. *Dually, any X sits in cofibre sequence $\Sigma^{-1} K \to X \to J \to K$, where J and K are $\mathscr{A}$-injective and $\Sigma^{-1} K \to X$ is phantom. The sequence $h_X \to h_J \to h_K$ is a short exact sequence in $\mathscr{A}$. The map $\Sigma^{-1} K \to X$ is weakly terminal among phantom maps into X.*

4. *IX is $\mathscr{A}$-injective for each X.*

5. *If $\pi_i Y$ is finite for each i, then Y is $\mathscr{A}$-injective.*

6. *If $\pi_i Y$ is finitely generated for each i, then $\mathscr{P}(X, Y)$ is divisible for each X.*

7. *The group $\mathscr{P}(H\mathbb{Z}/p, Y)$ is always a vector space over $\mathbb{Z}/p$, and is nonzero (and thus not divisible) for some Y.*

8. *If X is $\mathscr{A}$-projective and $[X, W] = 0$ for each finite W, then $X = 0$.*

The above material appears in Sections 3 and 4. We warn the reader that while the results are for the most part self-dual, the proofs are not.

In Section 5 we show how the stable homotopy category can be viewed as a linear extension of the category of homology theories by the bimodule of phantom maps. Our point in making this rigorous is that both the category of homology theories and the bimodule of phantom maps are determined by the category of finite spectra, and so we see that the category of spectra is determined up to extension by the category of finite spectra. Moreover, the goal of Section 6 is to prove that the extension is not split. We begin with the following result on phantom cohomology classes. Here PExt denotes the subgroup of Ext consisting of the pure or phantom extensions, HB denotes the Eilenberg–MacLane spectrum with $\pi_0 HB = B$, and H_* denotes integral homology.

THEOREM 1.7. *For any spectrum X and Abelian group B we have $\mathscr{P}(X, HB) = \mathrm{PExt}(H_{-1}X, B)$.*

After submitting this paper, we discovered that this theorem is a special case of some earlier results. One such result is due to Huber and Meier [16]. They show that if $E_*(-)$ is a homology theory of finite type, B is an Abelian group, and $F^*(-)$ is a cohomoloy theory fitting into a natural short exact sequence

$$0 \to \mathrm{Ext}(E_{n-1}(X), B) \to F^n(X) \to \mathrm{Hom}(E_n(X), B) \to 0.$$

then the subgroup of phantom cohomology classes in $F^n(X)$ is isomorphic to $\mathrm{PExt}(E_{n-1}(X), B)$. Taking $E = H$ and $F = HB$ gives our result. Pezennec [26] proves essentially the same result, while Yosimura [28] removes the finite type hypothesis on the homology theory E and concludes that the subgroup of phantom cohomology classes is isomorphic to $\varprojlim^1 F^{n-1}(X_\alpha)$, where the X_α range over the finite subspectra of X. Ohkawa [25] also comes to this conclusion, but without assuming the existence of E, B, or the short exact sequence. Another reference for this last result is [4].

Using the theorem we are able to calculate all phantom maps between Eilenberg–Mac Lane spectra.

COROLLARY 1.8. *We have*

$$\mathscr{P}(\Sigma^k HA, HB) = \begin{cases} \mathrm{PExt}(A, B) & \text{if } k = -1, \\ 0 & \text{otherwise.} \end{cases}$$

When we take $A = \mathbb{Z}/p^\infty$ and $B = \oplus_k \mathbb{Z}/p^k$ we can use the above result and an explicit calculation to show that the phantom sequence

$$0 \to \mathscr{P}(\Sigma^{-1}HA, HB) \to \mathscr{S}(\Sigma^{-1}HA, HB) \to \mathscr{A}(\Sigma^{-1}HA, HB) \to 0$$

is not split. This implies that the linear extension is also not split.

In Section 7 our main result is that $h : \mathscr{S} \to \mathscr{A}$ is the universal example of a homology theory with values in a AB 5 category.

PROPOSITION 1.9. *Let $\mathscr{C}$ be an AB 5 category, and $K : \mathscr{S} \to \mathscr{C}$ a homology theory. Then there is an essentially unique strongly additive exact functor $K' : \mathscr{A} \to \mathscr{C}$ such that $K' \circ h \simeq K$.*

We also prove that the Ind completion of the category of the finite spectra is the category of homology theories.

2. AXIOMATIC STABLE HOMOTOPY THEORY

Many of the properties of the stable homotopy category follow from a collection of axioms which we state below. These axioms are a slight generalisation of those found in [22], and a specialisation of those studied in [15] (as one sees using [15, Theorem 1.2.1]). We shall say that an object X in an additive category $\mathscr{S}$ is *small* if the functor $\mathscr{S}(X, -)$ preserves all coproducts that exist in $\mathscr{S}$.

Definition 2.1. A *monogenic Brown category* is a category $\mathscr{S}$ (whose objects are called *spectra* and whose morphism sets are denoted $[-, -]$ or $\mathscr{S}(-, -)$) satisfying the following axioms:

1. $\mathscr{S}$ is triangulated (and satisfies the octahedral axiom). Triangles are sometimes called cofibre sequences.
2. $\mathscr{S}$ has set-indexed coproducts. The coproduct is usually written $\vee$.
3. $\mathscr{S}$ is closed symmetric monoidal [20]. The multiplication is called the *smash product* and is denoted $\wedge$, the unit is denoted S^0, and the function spectra are denoted $F(X, Y)$. The smash product and function spectrum functors are required to be compatible with coproducts and the triangulated structure, and all diagrams that one would expect to commute are required to. See [15, Appendix A] for more details.
4. S^0 is small.
5. S^0 is a *graded weak generator* for $\mathscr{S}$: if $\pi_n X = 0$ for each $n \in \mathbb{Z}$ then $X = 0$, where $\pi_n X$ is defined to be $[S^n, X]$ and S^n is $\Sigma^n S^0$.
6. Homology theories and maps between them are representable—see Section 3 for an explanation of this axiom.

Note 2.2. If we replace axioms 4 and 5 with the weaker assumption that there exists a set of small graded weak generators, we get the notion of a *Brown category*. Most, if not all, of what we discuss here goes through in this more general setting; we restrict ourselves to the monogenic setting only for simplicity. In fact, one can get a long way without a symmetric monoidal structure.

The classical stable homotopy category, the derived category of a countable commutative ring, the homotopy category of G-equivariant spectra (for G a compact Lie group) and suitable categories of comodules over countable cocommutative Hopf algebras all from Brown categories, the first two being monogenic.

An important subcategory of a monogenic Brown category $\mathscr{S}$ is the category $\mathscr{F}$ of finite spectra which we define below. Its importance stems from the fact that a homology functor on $\mathscr{S}$ is determined by how it behaves of finite spectra. Later, we will see that even more of the structure of $\mathscr{S}$ is captured by $\mathscr{F}$.

We first make some auxiliary definitions.

Definition 2.3. A *thick subcategory* $\mathscr{C}$ of a triangulated category $\mathscr{S}$ is a full sub-category which is closed under cofibres and retracts. That is, if $X \rightarrow Y \rightarrow Z$ is a cofibre sequence with two of X, Y, and Z in $\mathscr{C}$, then so is the third; and if X is in $\mathscr{C}$ and Y is a retract of X, then Y is in $\mathscr{C}$. If D is a class of spectra in $\mathscr{S}$, then the *thick subcategory generated by* D is the intersection of all thick subcategories containing D.

The following definition was made and studied in [18], following work of Dold and Puppe.

Definition 2.4. Write $DX = F(X, S^0)$. A spectrum X is *strongly dualizable* if the natural map $DX \wedge Y \rightarrow F(X, Y)$ is an isomorphism for each Y.

It is not hard to see that the following conditions on a spectrum X are equivalent. For a proof, see [15, Theorem 2.1.3].

1. X lies in the thick subcategory generated by S^0.
2. X is small.
3. X is strongly dualizable.

Definition 2.5. We say that a spectrum X is *finite* if it satisfies the above conditions, and we write $\mathscr{F}$ for the category of finite spectra.

One can show that $\mathscr{F}$ has a small skeleton $\mathscr{F}'$. One can also show that $\mathscr{F}$ is closed under the functor D, and that there is a natural map $X \rightarrow D^2 X$ that is an isomorphism when X is finite, so that D gives an equivalence $\mathscr{F}^{\mathrm{op}} \simeq \mathscr{F}$. We call this equivalence *Spanier–Whitehead duality*.

In the case of the classical stable homotopy category, a spectrum is finite if and only if it is a possibly desuspended suspension spectrum of a finite CW-complex.

3. HOMOLOGY THEORIES

An additive functor from a triangulated category to an Abelian category is *exact* if it sends cofibre sequences to exact sequences. A *homology theory* on a triangulated category $\mathscr{S}$ is an exact functor to an Abelian category which preserves the coproducts that exist in $\mathscr{S}$. Unless we state otherwise, the target category will always be taken to be the category $\mathscr{A}b$ of Abelian groups. It is shown in [15, Section 4] that a homology theory defined on $\mathscr{F}$ has an essentially unique extension to a homology theory defined on all of $\mathscr{S}$, so the categories of homology theories on $\mathscr{F}$ and $\mathscr{S}$ are equivalent. More precisely, we have the following result.

PROPOSITION 3.1. *For each spectrum X there is a naturally defined small diagram $\Lambda(X) = \{X_\alpha \,|\, \alpha \in A(X)\}$ of small spectra with compatible maps $X_\alpha \to X$ such that for any homology theory H on $\mathscr{S}$, the induced map $\varinjlim_\alpha H(X_\alpha) \to H(X)$ is an isomorphism. Moreover, if K is a homology theory defined on $\mathscr{F}$ and we define $\hat{K}(X) = \varinjlim_\alpha K(X_\alpha)$ then $\hat{K}$ is the unique homology theory on $\mathscr{S}$ extending K (up to canonical isomorphism).*

COROLLARY 3.2. *If W is finite then $[W, -]$ is a homology theory so*

$$[W, X] = \varinjlim_\alpha [W, X_\alpha].$$

In particular, we see that any map $W \to X$ factors through some X_α.

In fact, if we take $\mathscr{F}'$ to be a small skeleton of $\mathscr{F}$, we can define $A(X)$ to be the category of pairs (U, u) where $U \in \mathscr{F}'$ and $u: U \to X$. The diagram $\Lambda(X)$ is then just the functor $A(X) \to \mathscr{F}$ sending (U, u) to U.

Definition 3.3. The homology theory $h_X: \mathscr{F} \to \mathscr{A}b$ represented by a spectrum X is the functor $h_X(W) = \pi_0(X \wedge W)$. We shall write $h(X)$ instead of h_X where this is typographically convenient. We use the same symbol h_X for the unique extension of this to a homology theory on all of $\mathscr{S}$, which is again given by $h_X(W) = \pi_0(X \wedge W) = h_W(X)$.

We also write $\mathscr{A}$ for the Abelian category of additive functors from $\mathscr{F}$ to $\mathscr{A}b$. This category has small Hom sets since $\mathscr{F}$ has a small skeleton. Note that h gives a functor $\mathscr{S} \to \mathscr{A}$. Note also that if W is a finite spectrum then $h_W(Z) = [DW, Z]$; it follows easily that $[V, W] = \mathscr{A}(h_V, h_W)$ when V and W are finite.

We now give a more complete statement of Axiom 6 of Definition 2.1. This follows from the other axioms if $\pi_* S^0$ is countable, but not otherwise. (See [24] and [15, Section 4] for details.)

Axiom 3.4. If H is a homology theory on $\mathscr{F}$ (taking values in $\mathscr{A}b$), then there is a spectrum Y in $\mathscr{S}$ and a natural isomorphism $h_Y \to H$. Moreover, a natural transformation from h_Y to h_Z is always induced by a map from Y to Z. (This map need not be unique. It turns out that a spectrum Y representing a given homology theory is unique up to a non-unique isomorphism.)

Note 3.5. A *cohomology theory* with values in an Abelian category $\mathscr{B}$ is a homology theory with values in $\mathscr{B}^{op}$. It *does* follow from the first five axioms that every cohomology theory on $\mathscr{S}$ with values in $\mathscr{A}b$ is of the form $[-, Y]$ for some Y. By the Yoneda lemma, natural transformations are uniquely representable.

We record some basic facts about the functor h.

PROPOSITION 3.6. *The functor $h: \mathscr{S} \to \mathscr{A}$ preserves both products and coproducts, and it sends cofibre sequences to exact sequences.*

Proof. It is easy to see that limits and colimits in a functor category such as $\mathscr{A}$ are computed pointwise. Thus, the first claim is that

$$h\left(\prod_i X_i\right)(W) = \prod_i h(X_i)(W)$$

for each small W. This follows easily using $h(Y)(W) = [DW, Y]$. The second claim is that

$$h\left(\bigvee_i X_i\right)(W) = \bigoplus_i h(X_i)(W),$$

which follows similarly using the smallness of DW. The third claim is that for any cofibre sequence $X \to Y \to Z$, the resulting sequence

$$\pi_0(X \wedge W) \to \pi_0(Y \wedge W) \to \pi_0(Z \wedge W)$$

is exact, and this is clear.   □

We can now start our study of homological algebra in the category $\mathscr{A}$.

LEMMA 3.7. *A finite spectrum W is $\mathscr{A}$-projective. Hence, a retract of a wedge of finite spectra is $\mathscr{A}$-projective.*

Proof. Let W be a finite spectrum and suppose that $\alpha: h_W = [DW, -] \to G$ is a natural transformation. By the Yoneda Lemma it corresponds to an element of GDW. If $\beta: F \to G$ is an epimorphism then $FDW \to GDW$ is as well, so α factors through β. Thus h_W is projective for W finite. But projectives are closed under coproducts and retracts, so if X is a retract of a wedge of finite spectra, then h_X is projective.   □

We can use this to show that $\mathscr{A}$ has enough projectives.

LEMMA 3.8. *The category $\mathscr{A}$ has enough projectives.*

Proof. Let $F: \mathscr{F} \to \mathscr{A}b$ be an additive functor, and choose a small skeleton $\mathscr{F}'$ of $\mathscr{F}$. Then the natural map

$$\bigoplus_{W \in \mathscr{F}'} \bigoplus_{\alpha \in FW} [W, -] \to F$$

is clearly an epimorphism. Using the fact that $[W, -] = h_{DW}$, we see that the source is projective.   □

Note that the source of the above epimorphism is just h_X, where

$$X = \bigvee_{W \in \mathscr{F}'} \bigvee_{\alpha \in FW} W.$$

Moreover, h_X is not just projective, but free in the following sense. Let $\mathscr{C}$ be the category of ob($\mathscr{F}'$)-indexed families of sets, and consider the evident forgetful functor $\mathscr{A} \to \mathscr{C}$. This has a left adjoint, whose image consists of the functors h_X, where X is a wedge of finite spectra; it is natural to regard these as the free objects of $\mathscr{A}$. As usual, an object is projective if and only if it is a retract of a free object; it follows that projective objects are homology theories.

LEMMA 3.9. *A map $f: X \to Y$ is an isomorphism if and only if $h_f: h_X \to h_Y$ is an isomorphism. The same holds with "isomorphism" replaced by "split monomorphism" or "split epimorphism".*

Proof. Suppose that $h_f: h_X \to h_Y$ is an isomorphism. As $\pi_k(X) = h_X(S^{-k})$, we see that $\pi_*(f)$ is an isomorphism, so f is an isomorphism.

If $h_f : h_X \to h_Y$ is a split monomorphism, choose a splitting, which by Brown Representability is of the form h_g. The composition $h_g \circ h_f$ is the identity, so gf is an isomorphism. By composing g with the inverse of this isomorphism we get a splitting of f.

The case when h_f is a split epimorphism is dual.   □

PROPOSITION 3.10. *A spectrum X is $\mathscr{A}$-projective if and only if it is a retract of a wedge of finite spectra.*

Proof. ⇐: This is Lemma 3.7.

⇒: If h_X is projective, it is a retract of h_Y with Y a wedge of finite spectra. By Brown Representability and the previous lemma, X is a retract of Y.   □

The dual picture. We first recall the basic facts about duality for Abelian groups.

Definition 3.11. For any Abelian group A, we write $I(A) = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$. It is well known that this is a contravariant exact functor which converts sums to products, and that the natural map $A \to I^2(A)$ is a monomorphism. Moreover, if A is finitely generated then $I^2(A)$ is the profinite completion of A; in particular, if A is finite then $I^2(A) = A$.

Given a spectrum X consider the contravariant functor from $\mathscr{S}$ to $\mathscr{A}b$ sending Y to $I(\pi_0(X \wedge Y))$; this is clearly a cohomology theory. There is thus a representing object IX such that $I(\pi_0(X \wedge Y)) \simeq [Y, IX]$; we call this the *Brown–Comenetz dual* of X [3].

PROPOSITION 3.12. *For each spectrum X, IX is $\mathscr{A}$-injective.*

Proof. Fix a spectrum X. As in Corollary 3.2, we have a diagram $\{X_\alpha\}$ of finite spectra such that $[W, X] = \varinjlim_\alpha [W, X_\alpha]$ for all finite W. We temporarily write $\mathscr{A}'$ for the category of contravariant additive functors from $\mathscr{F}$ to $\mathscr{A}b$. If F is in $\mathscr{A}$ we have

$$\mathscr{A}(F, h_{IX}) = \mathscr{A}(F, I[-, X])$$

$$= \mathscr{A}'([-, X], IF)$$

$$= \mathscr{A}'(\varinjlim [-, X_\alpha], IF)$$

$$= \varprojlim \mathscr{A}'([-, X_\alpha], IF)$$

$$= \varprojlim IFX_\alpha$$

$$= I(\varinjlim FX_\alpha).$$

Suppose now that $F \to G$ is a monomorphism in $\mathscr{A}$. We must show that the map $\mathscr{A}(F, h_{IX}) \leftarrow \mathscr{A}(G, h_{IX})$ is a surjection. Each map $FX_\alpha \to GX_\alpha$ is monic, and a filtered colimit of monomorphisms is monic, so the map $I(\varinjlim FX_\alpha) \leftarrow I(\varinjlim GX_\alpha)$ is surjective, since $\mathbb{Q}/\mathbb{Z}$ is injective. Thus $\mathscr{A}(F, h_{IX}) \leftarrow \mathscr{A}(G, h_{IX})$ is surjective.   □

COROLLARY 3.13. *If Y has finite homotopy groups, then $Y \cong I^2 Y$ and so Y is $\mathscr{A}$-injective. Moreover, for any family $\{X_i\}$ of spectra, the product $\prod_i I(X_i) = I(\bigvee_i X_i)$ is $\mathscr{A}$-injective, as is any retract of such a product.*   □

PROPOSITION 3.14. $\mathscr{A}$ has enough injectives.

Proof. For finite W a natural transformation from $G \in \mathscr{A}$ to h_{IW} corresponds to an element of $\mathbb{I}G(W)$. Let $\mathscr{F}'$ be a small skeleton of $\mathscr{F}$, so there is a natural map

$$G \to \prod_{W \in \mathscr{F}'} \prod_{\alpha \in \mathbb{I}G(W)} h_{IW}.$$

Since $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator in the category of Abelian groups, one can show that this map is a monomorphism.   □

In fact, the target of the monomorphism is the homology theory represented by a product of Brown–Comenetz duals of finite spectra. In $\mathscr{A}$, being injective is equivalent to being a retract of such a functor. In particular, injectives are homology theories.

PROPOSITION 3.15. *A spectrum X is $\mathscr{A}$-injective if and only if it is a retract of a product of Brown–Comenetz duals of finite spectra.*

Proof. $\Leftarrow$: This follows from Proposition 3.12.

$\Rightarrow$: If h_X is injective, it is a retract of $h(\prod IW_\alpha)$ with each W_α finite. As in the proof of Proposition 3.10, this implies that X is a retract of $\prod IW_\alpha$.   □

4. PHANTOM MAPS

There is a class of maps that we cannot see, at least not easily.

PROPOSITION 4.1. *The following conditions on a map $f \colon X \to Y$ are equivalent:*

(i) *The natural transformation $h_f \colon h_X \to h_Y$ is zero.*

(ii) *For each homology theory H, we have $H(f) = 0$.*

(iii) *The composite $W \to X \to Y$ is zero for each finite spectrum W and each map $W \to X$.*

(A fourth equivalent condition appears in Proposition 4.12.)

Proof. (iii) $\Rightarrow$ (ii): Let $\Lambda(X) = \{X_\alpha\}$ be as in Proposition 3.1, so that $H(X) = \varinjlim_\alpha H(X_\alpha)$. The composite $X_\alpha \to X \to Y$ is zero by (iii), so $H(X_\alpha) \to H(Y)$ is zero. It follows that $H(f) \colon H(X) \to H(Y)$ is zero.

(ii) $\Rightarrow$ (i): Suppose that $H(f) = 0$ for each homology theory H. Then for each finite spectrum W, the map $h_W(f) \colon \pi_0(X \wedge W) \to \pi_0(Y \wedge W)$ is zero. In other words, the natural map h_f is zero at W.

(i) $\Rightarrow$ (iii): Suppose that (i) holds and that W is finite. Then DW is also finite, and f induces the zero map $[W, X] = \pi_0(DW \wedge X) \to \pi_0(DW \wedge Y) = [W, Y]$.   □

Definition 4.2. A map $X \to Y$ satisfying the equivalent conditions of the proposition is called *phantom* or $\mathscr{A}$-*null*. The collection of phantom maps from X to Y is denoted $\mathscr{P}(X, Y)$ and is a subgroup of $[X, Y]$. Similarly, we say that a map $X \to Y$ is $\mathscr{A}$-*monic* or $\mathscr{A}$-*epic* if the natural transformation $h_X \to h_Y$ is monic or epic, respectively.

If $\{X_\alpha\}$ is an indexed collection of spectra, then the map $\bigvee X_\alpha \to \prod X_\alpha$ is $\mathscr{A}$-monic, and hence its fibre is phantom. As an example of this in the classical stable homotopy category,

let C be the cokernel of the map from the sum of countably many copies of $\mathbb{Z}$ to the product. The fibre of the map $H(\oplus\mathbb{Z}) \to H(\prod\mathbb{Z})$ between Eilenberg–MacLane spectra is a phantom map $\Sigma^{-1}HC \to H(\oplus\mathbb{Z})$. It is non-zero because the short exact sequence $0 \to \oplus\mathbb{Z} \to \prod\mathbb{Z} \to C \to 0$ is not split. To see that this sequence is not split, notice that the coset of the quotient containing $(1, 2, 4, 8, 16, \ldots)$ is non-zero and is divisible by 2^k for each k, since initial terms may be dropped without changing the coset. But $\prod\mathbb{Z}$ contains no such elements, so C could not be a summand. We learned this argument from Dan Dugger, who credits it to [10].

As further evidence of the ubiquity of phantom maps, it can be shown that in the classical stable homotopy category there are uncountably many phantom maps from $\mathbb{C}P^\infty$ to S^3. Gray [8] has a proof for spaces which simplifies when read stably.

Note 4.3. It is not hard to see that phantom maps form an *ideal* in $\mathscr{S}$: if f, g and h are composable and g is phantom, then fg and gh are phantom; and if f and g are parallel phantom maps, then $f + g$ is phantom. This means that there is a well-defined additive category $\mathscr{S}/\mathscr{P}$ having the same objects as $\mathscr{S}$ and with $\mathscr{S}/\mathscr{P}(X, Y) := \mathscr{S}(X, Y)/\mathscr{P}(X, Y)$. We have a natural isomorphism $\mathscr{A}(h_X, h_Y) \cong \mathscr{S}/\mathscr{P}(X, Y)$, so h gives an equivalence between $\mathscr{S}/\mathscr{P}$ and the category $\mathscr{H}$ of homology theories.

LEMMA 4.4. *For any spectrum X there is a weakly initial phantom map*

$$\delta : X \to \tilde{X}$$

from X. By 'weakly initial' we mean that any other phantom map from X factors through δ, but we don't insist upon uniqueness.

Proof. Let $\Lambda(X) = \{X_\alpha\}$ be as in Proposition 3.1. For each α we have a given map $X_\alpha \to X$, so we get a map $\bigvee_\alpha X_\alpha \to X$. Let $\delta : X \to \tilde{X}$ be the cofibre of this map. Corollary 3.2 tells us that every map from a finite spectrum W to X factors though $\bigvee_\alpha X_\alpha$, so the composite $W \to X \xrightarrow{\delta} \tilde{X}$ is zero. It follows that δ is phantom. Moreover, any phantom map from X is zero when restricted to $\bigvee_\alpha X_\alpha$ and so factors through δ. $\square$

COROLLARY 4.5. *Let X be a spectrum. Then the following are equivalent:*

(i) *X is $\mathscr{A}$-projective.*

(ii) *X is a retract of a wedge of finite spectra.*

(iii) *$\mathscr{P}(X, Y) = 0$ for each spectrum Y.*

Proof. If there are no phantom maps from X, then the weakly initial phantom map $X \to \tilde{X}$ is zero, and so X is a retract of the wedge of finite spectra $\bigvee X_\alpha$. Conversely, if X is a retract of a wedge of finite spectra, then it is clear that there are no phantoms from X.

Proposition 3.10 tells us that being a retract of a wedge of finite spectra is equivalent to being $\mathscr{A}$-projective. $\square$

PROPOSITION 4.6. *Any spectrum X sits in a cofibre sequence $P \to Q \to X \to \Sigma P$, where P and Q are $\mathscr{A}$-projective and $X \to \Sigma P$ is phantom. The sequence $h_P \to h_Q \to h_X$ is a short exact sequence in $\mathscr{A}$. The map $X \to \Sigma P$ is weakly initial among phantom maps out of X.*

Proof. Consider the diagram

$$\begin{array}{ccccc}
\bigvee_{\alpha\to\beta} X_\alpha & \xrightarrow{1-s} & \bigvee_\alpha X_\alpha & \longrightarrow & Y \\
\Big\downarrow & & \| & & \Big\downarrow \\
P & \longrightarrow & \bigvee_\alpha X_\alpha & \longrightarrow & X\;.
\end{array}$$

The map called 1 includes the $\alpha \to \beta$ summand into the α summand via the identity map, while the map s (for 'shift') sends the $\alpha \to \beta$ summand to the β summand via the map $X_\alpha \to X_\beta$. The map $\bigvee_\alpha X_\alpha \to X$ is the map considered in Lemma 4.4. The spectra Y and P are defined to make the rows cofibre sequences, so $\widetilde{X}$ (from the lemma) is ΣP. The composite $\bigvee_{\alpha\to\beta} X_\alpha \to \bigvee_\alpha X_\alpha \to X$ is null, so there is a map of cofibre sequences in the downward direction. Now consider the following natural transformation from h_X to h_Y. Let W be a finite spectrum. An element of $h_X(W)$ is a map $DW \to X$. DW is finite, so this map is $X_\gamma \to X$ for some γ. We have a map $\bigvee_\alpha X_\alpha \to Y$, so in particular we have a map $X_\gamma \to Y$. That is, we have a map $DW \to Y$, or an element of $h_Y(W)$. This defines a natural transformation $h_X \to h_Y$, and by Brown Representability this natural transformation is induced by a map $X \to Y$. By definition, the square commutes up to phantoms, but since $\bigvee_\alpha X_\alpha$ is $\mathscr{A}$-projective, the square commutes. One thus obtains a fill-in map $P \to \bigvee_{\alpha\to\beta} X_\alpha$. Also, one can check that the composite $X \to Y \to X$ is an isomorphism, and it follows that the composite $P \to \bigvee_{\alpha\to\beta} X_\alpha \to P$ is an isomorphism as well. Thus, P is a retract of a wedge of finite spectra, and we have demonstrated that X is the cofibre of a map between $\mathscr{A}$-projective spectra. We saw in Lemma 4.4 that the map $X \to \Sigma P$ is weakly initial. $\qquad\square$

We now get an easy proof of a result that is folklore. The method of proof presented in this section was independently discovered by Neeman [24]. A proof for the special case of the classical stable homotopy category was given by Ohkawa [25]. A proof assuming that the source has finite skeleta appears in [7, 9]. (See the Introduction for more detailed comments.)

COROLLARY 4.7. *The composite of two phantom maps is zero.*

Proof. Suppose that $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ are phantom. Factor f through δ:

$$\begin{array}{ccc}
X & \xrightarrow{\;\delta\;} & \Sigma P \\
\Big\downarrow{\scriptstyle f} & \nearrow{\scriptstyle f'} & \\
Y & & \\
\Big\downarrow{\scriptstyle g} & & \\
Z\;. & &
\end{array}$$

The $\mathscr{A}$-projectivity of ΣP implies tht $gf' = 0$ and so $gf = 0$. $\qquad\square$

We can now characterise homology theories in terms of the homological algebra of the category $\mathscr{A}$.

PROPOSITION 4.8. *A functor in $\mathscr{A}$ is a homology theory if and only if it has finite projective dimension if and only if it has projective dimension at most one.*

Proof. First, consider a short exact sequence $F \to G \to H$ in $\mathscr{A}$, in which two of F, G and H are homology theories. We claim that the third is also. Indeed, consider a cofibre sequence $X \to Y \to Z$. By applying F, we get a chain complex

$$\cdots \to F(\Sigma^{-1}Z) \to FX \to FY \to FZ \to F(\Sigma X) \to \cdots .$$

By doing the same with G and H, we obtain a short exact sequence of chain complexes. By assumption, two of the three chain complexes are exact; it follows easily that the third is also, as required.

We have seen that projective functors are homology theories. It follows easily from the above that functors of finite projective dimension are homology theories (by induction on dimension).

Consider a homology theory H. There exists a spectrum X such that $H = h_X$, and a cofibre sequence $P \to Q \to X \to \Sigma P$ as in Proposition 4.6. This gives a projective resolution $0 \to h_P \to h_Q \to h_X = H \to 0$, so H has projective dimension at most one. $\square$

We can now describe the phantom maps in terms of $\mathscr{A}$.

THEOREM 4.9. *The group* $\mathscr{P}(\Sigma^{-1}X, Y)$ *of phantom maps is naturally isomorphic to* $\text{Ext}_{\mathscr{A}}(h_X, h_Y)$.

Proof. Consider the usual projective resolution $0 \to h_P \to h_Q \to h_X \to 0$ of h_X in $\mathscr{A}$. The first cohomology group of the left column of

$$
\begin{array}{ccc}
0 & & 0 \\
\uparrow & & \uparrow \\
\mathscr{A}(h_P, h_Y) & === & [P, Y] \\
\uparrow & & \uparrow \\
\mathscr{A}(h_Q, h_Y) & === & [Q, Y] \\
\uparrow & & \uparrow \\
0 & & 0
\end{array}
$$

is the Ext group in question, and the left column can be identified with the right column since P and Q are $\mathscr{A}$-projective. But the first cohomology of the right column is $\mathscr{P}(\Sigma^{-1}X, Y)$ because every phantom $\Sigma^{-1}X \to Y$ extends to P, and the difference between two such extensions factors through Q.

It is easy to see that the isomorphism is natural in X and Y. $\square$

Note 4.10. The above proposition can also be proved using the definition of Ext in terms of equivalence classes of short exact sequences. The isomorphism sends a phantom map $f \colon \Sigma^{-1}X \to Y$ to the short exact sequence

$$0 \to h(Y) \to h(\text{cofibre} f) \to h(X) \to 0.$$

The dual picture. Now we prove the dual results, making use of what came above.

PROPOSITION 4.11. *For any spectra X and Y, we have $\mathscr{P}(X, IY) = 0$.*

Proof. By Theorem 4.9, $\mathscr{P}(\Sigma^{-1}X, IY) = \text{Ext}_{\mathscr{A}}(h_X, h_{IY})$. But h_{IY} is injective, so this is zero.

One can prove this directly from the definition of IY as well. $\square$

With this we can now prove our fourth characterisation of phantom maps.

PROPOSITION 4.12. *A map $X \to Y$ is phantom if and only if the composite $X \to Y \to IW$ is null for each finite W and each map $Y \to IW$.*

Proof. By the previous proposition, every phantom map is null when composed with a map $Y \to IW$.

Conversely, suppose that $X \to Y$ is such that $(X \to Y \to IW) = 0$ for all $Y \to IW$. Consider the spectrum

$$Z = \prod_{W \in \mathcal{F}'} \prod_{Y \to IW} IW.$$

The evident map $Y \to Z$ is $\mathscr{A}$-monic, as in Proposition 3.14. Since

$$h_X \to h_Y \to h_Z$$

is null by assumption, the map $h_X \to h_Y$ must also be null, so $X \to Y$ is phantom. $\qquad\square$

LEMMA 4.13. *There is a natural map $X \to I^2X$, which is $\mathscr{A}$-monic for all X.*

Proof. Consider $[X, I^2X]$. By using the definition of I twice, we find $[X, I^2X] = \mathbb{I}(\pi_0(X \wedge IX)) = [IX, IX]$, and so there is a natural map $X \to I^2X$ corresponding to the identity map in $[IX, IX]$.

We need to show that for W finite, the map $[W, X] \to [W, I^2X]$ is monic. We can calculate the latter group and we find that it is $\mathbb{I}^2[W, X]$. The map $[W, X] \to [W, I^2X]$ is the natural inclusion of $[W, X]$ into its double dual; since $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator, this is monic. $\qquad\square$

PROPOSITION 4.14. *Any spectrum X sits in a cofibre sequence $\Sigma^{-1}K \to X \to J \to K$, where J and K are $\mathscr{A}$-injective and $\Sigma^{-1}K \to X$ is phantom. The sequence $h_X \to h_J \to h_K$ is a short exact sequence in $\mathscr{A}$. The map $\Sigma^{-1}K \to X$ is weakly terminal among phantom maps into X.*

Proof. Let $J = I^2X$ and let K be the cofibre of the natural map $X \to I^2X$. Similarly, let $L = I^2K$ and form the cofibre sequence $K \to L \to M$. By Lemma 4.13 the maps $\Sigma^{-1}K \to X$ and $\Sigma^{-1}M \to K$ are phantom and so the cofibre sequences $X \to J \to K$ and $K \to L \to M$ become short exact in $\mathscr{A}$. Thus $\mathrm{Ext}^1_{\mathscr{A}}(h_M, h_K) = \mathrm{Ext}^2_{\mathscr{A}}(h_M, h_X)$, which vanishes as h_M has projective dimension at most one (Proposition 4.8). Therefore the extension $h_K \to h_L \to h_M$ splits in $\mathscr{A}$ and hence h_K is injective.

We showed in Proposition 4.11 that $\mathscr{P}(-, J) = 0$, and it follows easily that $\Sigma^{-1}K \to X$ is weakly terminal. $\qquad\square$

The following corollary is an easy consequence of the above constructions.

COROLLARY 4.15. *The following are equivalent:*

(i) *X is $\mathscr{A}$-injective.*
(ii) *X is a retract of a product of Brown–Comenetz duals of finite spectra.*
(iii) *$\mathscr{P}(Y, X) = 0$ for each spectrum Y.* $\qquad\square$

For example, this means that the completed Johnson–Wilson spectrum $\widehat{E(n)}$ is $\mathscr{A}$-injective. Indeed, if W is finite then $\widehat{E(n)}^*W$ is compact Hausdorff in the I_n-adic

topology. The inverse limit factor is exact for inverse systems of compact Haus-dorff topological groups, and one can deduce from this that there are no phantom maps to $\widehat{E(n)}$.

Summarising our homological results gives:

THEOREM 4.16. *Let* $F \in \mathscr{A}$. *Then the following are equivalent:*

 (i) *F has finite projective dimension.*
 (ii) *F has projective dimension at most one.*
 (iii) *F is a homology theory.*
 (iv) *F is in the image of* h.
 (v) *F has finite injective dimension.*
 (vi) *F has injective dimension at most one.*

Divisibility. To start with, we recall a result that is well-known to the experts.

PROPOSITION 4.17. *If* $\pi_i Y$ *is a finitely generated Abelian group for each i, then* $\mathscr{P}(X, Y)$ *is divisible for each X.*

Proof. Let Z be the cofibre of the natural map $Y \to I^2 Y$. We have seen that the resulting map $\Sigma^{-1} Z \to Y$ is a weakly terminal phantom map, so that $\mathscr{P}(X, Y)$ is a quotient of $[X, \Sigma^{-1} Z]$. It will thus be enough to show that $[X, \Sigma^{-1} Z]$ is a rational vector space.

The induced map $\pi_k(Y) \to \pi_k(I^2 Y)$ is just the inclusion of $\pi_k(Y)$ into its double dual with respect to $\mathbb{Q}/\mathbb{Z}$, which is the same as its profinite completion (as $\pi_k(Y)$ is finitely generated). It follows that $\pi_k(Z)$ is a finite direct sum of copies of $\hat{\mathbb{Z}}/\mathbb{Z}$, which is well-known to be a rational vector space. It follows that any nonzero integer n induces an isomorphism $\pi_*(Z) \to \pi_*(Z)$, and thus an isomorphism $Z \to Z$. It follows that $[X, \Sigma^{-1} Z]$ is a rational vector space, as required.   □

It is not the case that $\mathscr{P}(X, Y)$ is always divisible, however. Indeed, we have the following result.

PROPOSITION 4.18. *Let* $\mathscr{S}$ *be the classical stable homotopy category, and* $H\mathbb{Z}/p$ *the mod* p *Eilenberg–Mac Lane spectrum in* $\mathscr{S}$. *Then* $\mathscr{P}(H\mathbb{Z}/p, Y)$ *is a vector space over* $\mathbb{Z}/p$, *and there exist spectra Y for which it is nonzero (and thus not divisible).*

Proof. As p times the identity map of $H\mathbb{Z}/p$ is zero, we see that $[H\mathbb{Z}/p, Y]$ is a vector space over $\mathbb{Z}/p$, so the same is true of $\mathscr{P}(H\mathbb{Z}/p, Y)$. Next, recall that $[H\mathbb{Z}/p, W] = 0$ for each finite spectra W. Ravenel proves this in [27] by showing that $H\mathbb{Z}/p$ is E-acyclic (dissonant) and that finite spectra are E-local (harmonic), where $E = \bigvee_{p,n} K(n)$. It was also proved earlier by Margolis in [21] and by Lin in [19] using the Adams spectral sequence, and can be found in Margolis's book [22, Cor. 16.27]. If $\mathscr{P}(H\mathbb{Z}/p, Y)$ were zero for all Y, then the following proposition would imply that $H\mathbb{Z}/p = 0$, a contradiction.   □

PROPOSITION 4.19. *If X is* $\mathscr{A}$-*projective and* $[X, W] = 0$ *for each* $W \in \mathscr{F}$ *then* $X = 0$.

Proof. We can write X as a retract of a wedge of finite spectra:

$$\bigvee_\alpha X_\alpha$$
$$\left(\;\right) i$$
$$X\;.$$

Consider $X \xrightarrow{i} \bigvee X_\alpha \xrightarrow{j} \prod X_\alpha$, where j is the natural map. As X has no maps to finite spectra, the composite ji is zero. But $\pi_*(j)$: $\bigoplus_\alpha \pi_* X_\alpha \to \prod_\alpha \pi_* X_\alpha$ is monic, so we see that $\pi_*(i) = 0$. As i is a split monomorphism, we know that $\pi_*(X)$ is the image of $\pi_*(i)$. It follows that $X = 0$. □

5. MARGOLIS'S AXIOMATISATION CONJECTURE

The Spanier–Whitehead category $\mathscr{F}$ of finite spectra (in the classical, topological sense) can be constructed quite simply. However, all known constructions of the homotopy category $\mathscr{S}$ of all spectra are rather intricate. Moreover, there are number of apparently different constructions of this category, all giving the same result up to equivalence. (In this section, equivalences of categories are tacitly required to preserve triangulations and symmetric monoidal structures.) It is thus natural to look for a system of axioms that characterises $\mathscr{S}$ uniquely in terms of $\mathscr{F}$. Margolis [22] conjectured such a characterisation, which translates into our language as follows: if $\mathscr{S}'$ is a monogenic Brown category whose subcategory $\mathscr{F}'$ of finite object is equivalent to $\mathscr{F}$, then $\mathscr{S}'$ is equivalent to $\mathscr{S}$. As a first approximation to this conjecture, Margolis showed that $\mathscr{S}'/\mathscr{P}'$ is equivalent to the category $\mathscr{H}$ of homology theories on $\mathscr{F}$, or equivalently to $\mathscr{S}/\mathscr{P}$. Of course, Note 4.3 is just a generalisation of this.

We can now come somewhat closer to a proof of Margolis's conjecture. To explain this, we recall some of the theory of linear extensions of categories. Our treatment is inspired by [1], but is different in detail as we only consider additive categories. Let $\mathscr{B}$ be an additive category. A *bimodule* over $\mathscr{B}$ consists of Abelian groups $D(A, B)$ (for every pair of objects, A, B in $\mathscr{B}$) together with a trilinear composition operation

$$\mathscr{B}(A, B) \otimes D(B, C) \otimes \mathscr{B}(C, D) \to D(A, D)$$

written

$$f \otimes u \otimes g \mapsto f^* g_* u = g_* f^* u.$$

This operation is supposed to have the obvious functoriality properties. As an example, because the composite of two phantom maps is trivial, there is a well-defined composition

$$\mathscr{S}/\mathscr{P}(A, B) \otimes \mathscr{P}(B, C) \otimes \mathscr{S}/\mathscr{P}(C, D) \to \mathscr{P}(A, D).$$

This makes $\mathscr{P}$ into a bimodule over $\mathscr{S}/\mathscr{P}$.

If we have an additive functor $F: \mathscr{A} \to \mathscr{B}$ and a bimodule D over $\mathscr{B}$, then we can define a bimodule F^*D over $\mathscr{A}$ by $F^*D(A, B) = D(FA, FB)$. If F is naturally isomorphic to G then one can check that F^*D and G^*D are isomorphic as bimodules.

A *linear extension* of $\mathscr{B}$ by a bimodule D is a category $\mathscr{C}$ with the same objects as $\mathscr{B}$, together with short exact sequences

$$D(A, B) \xrightarrow{j} \mathscr{C}(A, B) \xrightarrow{p} \mathscr{B}(A, B)$$

such that p is a functor and $j(p(f)^* p(g)_* u) = g \circ j(u) \circ f$. To such extensions are considered equivalent if there is a functor $\varepsilon: \mathscr{C} \to \mathscr{C}'$ with $p'\varepsilon = p$ and $\varepsilon j = j'$ (strict equalities of functors,

not just natural isomorphisms). We write $M(\mathscr{B}, D)$ for the collection of equivalence classes of linear extensions of $\mathscr{B}$ by D. The main example of interest to us is of course the extension $\mathscr{P} \to \mathscr{S} \to \mathscr{S}/\mathscr{P}$.

Suppose again that we have an additive functor $F: \mathscr{A} \to \mathscr{B}$ and a linear extension $D \to \mathscr{C} \to \mathscr{B}$. Given objects A, B in $\mathscr{A}$ we define $F^*\mathscr{C}(A, B)$ by the pullback diagram

$$
\begin{array}{ccc}
F^*\mathscr{C}(A, B) & \longrightarrow & \mathscr{C}(FA, FB) \\
\downarrow & & \downarrow p \\
\mathscr{A}(A, B) & \xrightarrow{F} & \mathscr{B}(FA, FB).
\end{array}
$$

One can check that $F^*\mathscr{C}$ becomes a linear extension of $\mathscr{A}$ by F^*D. Moreover, if G is naturally isomorphic to F then $G^*\mathscr{C}$ is equivalent to $F^*\mathscr{C}$ as a linear extension. Thus, a natural equivalence class of functors $\mathscr{A} \to \mathscr{B}$ induces a map $M(\mathscr{B}, D) \to M(\mathscr{A}, F^*D)$. It is clear that this is essentially functorial, and thus $M(\mathscr{B}, D) \simeq M(\mathscr{A}, F^*D)$ if F is an equivalence of categories.

A procedure analogous to the Baer sum of extensions makes $M(\mathscr{B}, D)$ into an Abelian group. For any pair of objects A, B in $\mathscr{B}$, the evident map $M(\mathscr{B}, D) \to \mathrm{Ext}(\mathscr{B}(A, B), D(A, B))$ is a homomorphism. Unfortunately, this is almost all the information that we have about the group $M(\mathscr{B}, D)$ in the cases of interest. We do not even know whether $M(\mathscr{B}, D)$ is a set or a proper class.

We now return to the context of the Margolis conjecture. We have an equivalence $F: \mathscr{S}/\mathscr{P} \simeq \mathscr{S}'\mathscr{P}'$. It follows from Theorem 4.9 that there is a canonical euivalence $\mathscr{P} \simeq F^*\mathscr{P}'$ of bimodules over $\mathscr{S}/\mathscr{P}$. Thus, Margolis's conjecture is true up to an extension problem. Together with F, the above equivalence induces a canonical isomorphism $M(\mathscr{S}'/\mathscr{P}', \mathscr{P}') \simeq M(\mathscr{S}/\mathscr{P}, \mathscr{P})$. We need to know whether the class $u(\mathscr{S}')$ in $M(\mathscr{S}'/\mathscr{P}', \mathscr{P}')$ that classifies the extension $\mathscr{P}' \to \mathscr{S}' \to \mathscr{S}'/\mathscr{P}'$ maps to the analogous class $u(\mathscr{S}) \in M(\mathscr{S}/\mathscr{P}, \mathscr{P})$. This would follow from Margolis's conjecture. Conversely, it would almost imply the conjecture, apart from Possible questions about preservation of the triangulation and the monoidal structure.

We shall show in the next section that for each p we can choose spectra A and B such that the image of $u(\mathscr{S})$ in $\mathrm{Ext}(\mathscr{S}/\mathscr{P}(A, B), \mathscr{P}(A, B))$ is not divisible by p, and is not annihilated by any integer $n > 0$. It follows that the same is true of $u(\mathscr{S})$ itself. In particular, we will see that $u(\mathscr{S})$ is non-zero. This implies that there is no functorial way to choose a representing spectrum for a homology theory.

6. PHANTOM COHOMOLOGY

In this section we restrict attention to the classical stable homotopy category; a more axiomatic approach would yield only a small amount of extra generality. Recall that for each Abelian group A there is an essentially unique spectrum HA with $\pi_0 HA = A$ and $\pi_k HA = 0$ for all $k \neq 0$, and that $[X, HA] = H^0(X; A)$. These objects are called Eilenberg–MacLane spectra. We shall study phantom cohomology classes, in other words, phantom maps from arbitrary spectra to Eilenberg–MacLane spectra.

We start with some algebraic preliminaries.

Definition 6.1. A monomorphism $B \to C$ of Abelian groups is said to be *pure* if for each $n > 0$ the induced map $B/n \to C/n$ is monic. If we regard B as a subgroup of C, this says that $nC = (nB) \cap C$. A short exact sequence $B \to C \to A$ is said to be *pure* if the map $B \to C$ is.

Let $B \to C \to A$ be a short exact sequence. The six term exact sequence involving $\text{Hom}(\mathbb{Z}/n, -)$ and $\text{Ext}(\mathbb{Z}/n, -)$ reads

$$0 \to {}_nB \to {}_nC \to {}_nA \to B/n \to C/n \to A/n \to 0,$$

where we use the notation

$${}_nA := \{a \in A \mid na = 0\}$$

and the identifications ${}_nA = \text{Hom}(\mathbb{Z}/n, A)$ and $A/n = \text{Ext}(\mathbb{Z}/n, A)$. Thus it is clear that pureness of the short exact sequence is equivalent to the requirement than ${}_nA \to B/n$ be zero, or that ${}_nC \to {}_nA$ be epic.

Now we present an algebraic proposition which summarises results that cn be found, for example, in [6].

PROPOSITION 6.2. *Consider an element $u \in \text{Ext}(A, B)$, corresponding to an extension $B \to C \to A$. The following are equivalent:*

(a) *The extension is pure.*

(b) *For each map $A' \to A$ with A' finitely generated, the image of u in $\text{Ext}(A', B)$ is zero.*

(c) *For each $n > 0$, $u \in n\text{Ext}(A, B)$. That is, u is in $\bigcap_n n\text{Ext}(A, B)$, the first Ulm subgroup of $\text{Ext}(A, B)$.*

(d) *For each map $B \to B'$ with B' finite, the image of u in $\text{Ext}(A, B')$ is zero.*

We define the *phantom* Ext *group* $\text{PExt}(A, B)$ to be the subgroup of $\text{Ext}(A, B)$ consisting of all elements u satisfying the above conditions. These are the phantom maps from A to ΣB in $D(\mathbb{Z})$, the derived category of the integers. It is easy to see that PExt is a subfunctor of Ext.

Proof. (a) $\Rightarrow$ (b): It suffices to prove (b) when $A' = \mathbb{Z}/n$, as any finitely generated group is a sum of cyclic groups, and $\mathbb{Z}$ is projective. Given a map $f: \mathbb{Z}/n \to A$, the class f^*u in $\text{Ext}(\mathbb{Z}/n, B)$ is zero if and only if f factors through $C \to A$. Now f corresponds to an element of ${}_nA$, and since we are assuming u is pure, we know than ${}_nC \to {}_nA$ is epic and can therefore factor f through C. Thus $f^*u = 0$.

(b) $\Rightarrow$ (c): We will show that u is in the image of the endomorphism of $\text{Ext}(A, B)$ induced by $n: A \to A$. Consider the inclusion $i: {}_nA \to A$. By [6, Lemma 17.2], a bounded torsion group is a sum of cyclic groups. Thus $i^*u = 0$. Now in the diagram

$$
\begin{array}{c}
\text{Ext}(A, B) \\
\downarrow \quad \searrow^{n} \\
\text{Ext}(nA, B) \longrightarrow \text{Ext}(A, B) \xrightarrow{\;i^*\;} \text{Ext}({}_nA, B),
\end{array}
$$

the row is exact and the vertical map is an epimorphism (because $\text{Ext}^2 = 0$), so u is in the image of multiplication by n.

(c) $\Rightarrow$ (d): Let $f: B \to B'$ be a map with B' finite. To see that the image of u in $\text{Ext}(A, B')$ is zero, it suffices to check this when B' is $\mathbb{Z}/n$, since a finite group is a product of finite cyclic groups. But n kills $\text{Ext}(A, \mathbb{Z}/n)$, so if u is a multiple of n, then $f_*u = 0$.

(d) $\Rightarrow$ (a): Finally, assume that for any map $B \to B'$ with B' finite, the image of u in $\text{Ext}(A, B')$ is zero. Choose an element $b \in B$ with $b \notin nB$. We will show $b \notin nC$. (For notational simplicity we regard B as a subgroup of C.) Let K be a maximal subgroup of

B containing nB but not b. The quotient B/K can be shown to be "cocyclic" and so by [6, Section 3] B/K is isomorphic to $\mathbb{Z}/p^k$ for some prime p and some k with $1 \leqslant k \leqslant \infty$. Therefore, by assumption (for finite k) or since $\mathbb{Z}/p^\infty$ is divisible, the quotient map $B \to B/K$ extends over $B \to C$. By the choice of K, the image of b in B/K is non-zero, but the image of nC is zero, so $b \notin nC$.   □

Note 6.3. Clearly, if A is finitely generated, or if there is an integer n such that $nA = 0$, then $\mathrm{PExt}(A, B) = 0$ for all B.

More generally, if A is a torsion group then $A = \varinjlim_{n} {}_{n!}A$ so there is a short exact sequence

$$\bigoplus_n {}_{n!}A \to \bigoplus_n {}_{n!}A \to A$$

and a resulting short exact sequence

$$\varprojlim{}^1 \mathrm{Hom}({}_{n!}A, B) \to \mathrm{Ext}(A, B) \to \varprojlim \mathrm{Ext}({}_{n!}A, B).$$

Using part (b) of the definition of $\mathrm{PExt}(A, B)$, we see that

$$\mathrm{PExt}(A, B) = \varprojlim{}_n^1 \mathrm{Hom}({}_{n!}A, B).$$

Our reason for introducing the phantom Ext groups is the following theorem, in which H denotes the integral Eilenberg–Mac Lane spectrum. See the introduction for references to more general results.

THEOREM 6.4. *For any spectrum X and Abelian group B we have* $\mathscr{P}(X, HB) = \mathrm{PExt}(H_{-1}X, B)$.

Proof. We begin by describing a map $\mathscr{P}(X, HB) \to \mathrm{PExt}(H_{-1}X, B)$. Let $u\colon X \to HB$ be a phantom map. If Y is the cofibre of u, then we have a short exact sequence $0 \to B \to H_0Y \to H_{-1}X \to 0$, since $H_0(HB) = B$ by the Hurewicz theorem and since $H_*(u) = 0$ by Proposition 4.1. We claim that this is a phantom extension, and we prove this by showing that for each n the map ${}_n(H_0 Y) \to {}_n(H_{-1}X)$ is surjective. Let a be an element of $H_{-1}X$ with $na = 0$. This corresponds to a map $S^{-1} \to H \wedge X$ which can be extended to give a map $a'\colon S^{-1}/n \to H \wedge X$. Since phantoms form an ideal under the smash product, the composite $S^{-1}/n \to H \wedge X \to H \wedge HB$ is null and a' factors through $H \wedge Y$. Thus $S^{-1} \to S^{-1}/n \to H \wedge Y$ represents a class in ${}_n(H_0Y)$ mapping to a.

Conversely, consider the composite

$$\mathrm{PExt}(H_{-1}X, B) \to \mathrm{Ext}(H_{-1}X, B) \to [X, HB] = H^0(X; B),$$

where the first map is the inclusion and the second map comes from the universal coefficient sequence. We claim that a map u in the image of this composite is a phantom map. Indeed, if W is a finite spectrum and $W \to X$ is a map, then by naturality the restriction of u to W lies in the image of $\mathrm{PExt}(H_{-1}W, B)$, which is trivial because $H_{-1}W$ is finitely generated. It follows that u is a phantom map.

We leave it to the reader to check that the two maps we have constructed are inverse.   □

This allows us to calculate all phantom maps between Eilenberg–Mac Lane spectra.

COROLLARY 6.5. *We have*

$$\mathscr{P}(\Sigma^k HA, HB) = \begin{cases} \text{PExt}(A, B) & \text{if } k = -1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For $j < 0$ we have $H_j HA = 0$, and $H_0 HA = A$ by the Hurewicz theorem. Given this, the claim follows for $k \geq -1$ by a simple application of Theorem 6.4. For $j > 0$ we may have $H_j HA \neq 0$, but we claim that $\text{PExt}(H_j HA, B) = 0$ nonetheless; this will cover the case $k < -1$. To see this, fix $j > 0$ and let $\{A_\alpha\}$ be the directed set of finitely generated subgroups of A. The natural map $\varinjlim_\alpha (HA_\alpha)_* X \to (HA)_* X$ is an isomorphism for each X, since it is when X is a sphere, and both sides are homology theories. Taking $X = H$ we find that $H_j HA = \varinjlim_\alpha H_j HA_\alpha$. By working rationally, we see that $H_j HA$ is a torsion group, so it is the direct sum of its localisations at different primes. We claim that $H_j(HA_\alpha)_{(p)}$ is a vector space over $\mathbb{Z}/p$. Using the fact that $H_j HA_\alpha = (HA_\alpha)_j H$ and the fact that the universal coefficient sequence splits, we are reduced to proving that $H_j H$ is killed by p. This is a classical calculation; an account appears in [17]. This implies that $H_j HA$ is a direct sum of (prime) cyclic groups; it follows easily that $\text{PExt}(H_j HA, B) = 0$ as required. □

We next study a special case in which the short exact sequence

$$\mathscr{P}(\Sigma^{-1} HA, HB) \to \mathscr{S}(\Sigma^{-1} HA, HB) \to \mathscr{A}(\Sigma^{-1} HA, HB)$$

can be understood explicitly. We choose a prime p and take

$$A = \mathbb{Z}/p^\infty = \mathbb{Q}/\mathbb{Z}_{(p)} = \varinjlim_k \mathbb{Z}/p^k.$$

For the moment we consider an arbitrary Abelian group B. As in Note 6.3, we have a short exact sequence

$$\text{PExt}(A, B) \to \text{Ext}(A, B) \to \varprojlim_k \text{Ext}(\mathbb{Z}/p^k, B).$$

Note that $\text{Ext}(\mathbb{Z}/p^k, B) = B/p^k$, so the third term is just the p-completion $\hat{B}$ of B. The middle term is the Ext-p-completion of B, as studied in [2]; we shall denote it by $\tilde{B}$. And it is clear that the first term is

$$p^\infty \tilde{B} = \bigcap_k p^k \tilde{B},$$

since everything is p-local. Using the fact that $\mathscr{S}(\Sigma^{-1} HA, HB) = \text{Ext}(A, B)$, we find that our phantom sequence is just

$$p^\infty \tilde{B} \to \tilde{B} \to \hat{B}.$$

It is tempting to believe that $p^\infty \tilde{B}$ is a divisible group, but this is never true unless $p^\infty \tilde{B} = 0$. Any element of $p^\infty \tilde{B}$ is divisible by p in $\tilde{B}$ but not necessarily in $p^\infty \tilde{B}$.

Let B be a free Abelian group, say $B = \bigoplus_{k=0}^\infty \mathbb{Z}$. Then $\text{Hom}(\mathbb{Z}/p^j, B) = 0$ so $p^\infty \tilde{B} = \varprojlim_j^1 \text{Hom}(\mathbb{Z}/p^j, B) = 0$ so $\tilde{B} = \hat{B}$. Let $v(a)$ denote the p-adic valuation of a p-adic integer $a \in \mathbb{Z}_p$. It is not hard to see that

$$\text{Ext}(\mathbb{Z}/p^\infty, B) = \tilde{B} = \hat{B} = \left\{ \underline{a} \in \prod_k \mathbb{Z}_p \,\middle|\, v(a_k) \to \infty \right\}.$$

One can also see directly that $\text{Hom}(\mathbb{Z}/p^\infty, B) = 0$.

Now consider the case $B = \bigoplus_k \mathbb{Z}/p^k$. We then have a short exact sequence

$$\bigoplus_k \mathbb{Z} \xrightarrow{f} \bigoplus_k \mathbb{Z} \to B$$

where f is multiplication by p^k on the k'th factor. One can again see directly that $\text{Hom}(\mathbb{Z}/p^\infty, B) = 0$. The six-term exact sequence obtained by applying the functors $\text{Hom}(\mathbb{Z}/p^\infty, -)$ and $\text{Ext}(\mathbb{Z}/p^\infty, -)$ to the above presentation of B therefore collapses to a short exact sequence

$$\text{Ext}\left(\mathbb{Z}/p^\infty, \bigoplus_k \mathbb{Z}\right) \xrightarrow{f} \text{Ext}\left(\mathbb{Z}/p^\infty, \bigoplus_k \mathbb{Z}\right) \to \tilde{B}.$$

It follows using the previous paragraph that

$$\tilde{B} = \{\underline{a} \,|\, v(a_k) \to \infty\}/\{\underline{a} \,|\, 0 \leqslant v(a_k) - k \to \infty\}.$$

One can also see directly that

$$\hat{B} = \{\underline{a} \,|\, v(a_k) \to \infty\}/\{\underline{a} \,|\, 0 \leqslant v(a_k) - k\}.$$

It follows that $p^\infty \tilde{B}$ (which is the kernel of the map $\tilde{B} \to \hat{B}$) is given by

$$p^\infty \tilde{B} = \{\underline{a} \,|\, 0 \leqslant v(a_k) - k\}/\{\underline{a} \,|\, 0 \leqslant v(a_k) - k \to \infty\},$$

and this can also be expressed as

$$p^\infty \tilde{B} = \prod_k \mathbb{Z}_p \Big/ \left\{ \underline{b} \in \prod_k \mathbb{Z}_p \,\Big|\, v(b_k) \to \infty \right\}$$

(where $a_k = p^k b_k$). It is easy to see from this that $p^\infty \tilde{B}$ is nonzero and torsion-free.

We now return to the case of a general Abelian group B. Let $w \in \text{Ext}(\hat{B}, p^\infty \tilde{B})$ be the element classifying the canonical sequence $p^\infty \tilde{B} \to \tilde{B} \to \hat{B}$, and let

$$\delta: \text{Hom}(\mathbb{Z}/p, \hat{B}) \to \text{Ext}(\mathbb{Z}/p, p^\infty \tilde{B})$$

be the obvious connecting homomorphism.

PROPOSITION 6.6. *Let B be an Abelian group. The following are equivalent:*

(i) $p^\infty \tilde{B} = 0$.

(ii) *The natural map $\tilde{B} \to \hat{B}$ is an isomorphism.*

(iii) $w = 0$.

(iv) *w is divisible by p.*

(v) $\delta = 0$.

(vi) *δ is divisible by p.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (vi): easy.

(vi) $\Rightarrow$ (v): This is also clear, as the source and target of δ are killed by p.

(v) $\Rightarrow$ (i): The next map in the sequence is $\text{Ext}(\mathbb{Z}/p, p^\infty \tilde{B}) \to \text{Ext}(\mathbb{Z}/p, \tilde{B})$, which can be identified with the natural map $(p^\infty \tilde{B})/p \to \tilde{B}/p$. But this latter map is clearly zero, so the connecting homomorphism δ is epic. Its image is $(p^\infty \tilde{B})/p$, so this group is zero, so $p^\infty \tilde{B}$ is p-divisible. This means that $p^\infty \tilde{B} = \text{Hom}(\mathbb{Z}, p^\infty \tilde{B})$ is a quotient of $\text{Hom}(\mathbb{Z}[\frac{1}{p}], p^\infty \tilde{B})$, which is a subgroup of $\text{Hom}(\mathbb{Z}[\frac{1}{p}], \tilde{B})$. However, [2, VI.3.4] tells us that $\text{Hom}(\mathbb{Z}[\frac{1}{p}], \tilde{B}) = 0$; it follows that $p^\infty \tilde{B} = 0$.   □

If $B = \bigoplus_k \mathbb{Z}/p^k$, then we have $p^\infty \tilde{B} \neq 0$ and thus w is not divisible by p. Our next result will show that w has infinite order.

PROPOSITION 6.7. *Let B be an Abelian group such that $p^k w = 0$. Then $p^k p^\infty \tilde{B} = 0$.*

Proof. Let $i: p^\infty \tilde{B} \to \tilde{B}$ and $q: \tilde{B} \to \hat{B}$ be the usual maps. Let C be the pullback of $\tilde{B}$ along the map $p^k: \hat{B} \to \hat{B}$, so $C = \{(a, b) \in \tilde{B} \times \hat{B} | q(a) = p^k b\}$. The hypothesis $p^k w = 0$ means that the evident sequence $p^\infty \tilde{B} \to C \to \hat{B}$ is split; the splitting map $\hat{B} \to C$ necessarily has the form $c \mapsto (f(c), c)$, where $qf = p^k: \hat{B} \to \hat{B}$. The functor $A \mapsto p^\infty A$ preseves split exact sequences, and $p^\infty \hat{B} = 0$ (directly from the definitions) so $p^\infty C = p^\infty p^\infty \tilde{B}$. On the other hand, suppose that $b \in p^\infty \tilde{B}$, say $b = p^i b_i$ for each i, with $b_i \in \tilde{B}$. Then $(p^k b_i, q(b_i)) \in C$ and $p^i(p^k b_i, q(b_i)) = (p^k b, 0)$. It follows that $p^k p^\infty \tilde{B} \leqslant p^\infty C = p^\infty p^\infty \tilde{B}$. This means that $p^k p^\infty \tilde{B}$ is a divisible subgroup of $\tilde{B}$; as in the proof of Proposition 6.6, we conclude that $p^k p^\infty B = 0$.   □

Now take $B = \bigoplus_k \mathbb{Z}/p^k$ again. We saw previously that $p^\infty \tilde{B}$ is non-trivial and torsion-free. It follows easily that $p^k w \neq 0$ for all k.

We can now prove a result stated in Section 5. Recall that we defined there a group $M(\mathscr{S}/\mathscr{P}, \mathscr{P})$ and an element $u \in M(\mathscr{S}/\mathscr{P}, \mathscr{P})$ that classifies the linear extension of categories $\mathscr{P} \to \mathscr{S} \to \mathscr{S}/\mathscr{P}$. The image of u under a certain homomorphism $M(\mathscr{S}/\mathscr{P}, \mathscr{P}) \to \text{Ext}(\hat{B}, p^\infty \tilde{B})$ is w. It follows that u is not divisible by p for any prime, so u is not divisible by any integer $n > 1$. If u were annihilated by any $m > 0$ then the image of u in any p-local group (such as $\text{Ext}(\hat{B}, p^\infty \tilde{B})$) would be annihilated by some power of p. Thus, we conclude that u does not have finite order.

7. UNIVERSAL HOMOLOGY THEORIES

Although one is mostly interested in homology theories with values in the category of Abelian groups, one can also consider more general Abelian categories. In this section, we recall a construction of Freyd [5] which gives a universal example of an Abelian category $\mathscr{B}$ equipped with a homology theory $\mathscr{S} \to \mathscr{B}$. We also show that the functor h: $\mathscr{S} \to \mathscr{A}$ is the universal example of a homology theory with values in an Abelian category satisfying Grothendieck's axiom AB 5.

At one point we need a fact that holds in all monogenic Brown categories that we care about, but which we have not been able to deduce from the axioms (although we suspect that it does follow). For simplicity, we therefore restrict attention to the classical stable homotopy category.

Let $\mathscr{B}$ be the following category. The objects of $\mathscr{B}$ are just the morphisms of $\mathscr{S}$. Given a map $u: W \to X$ in $\mathscr{S}$, we shall write $I(u)$ for u thought of as an object of $\mathscr{B}$. The group $\mathscr{B}(I(u), I(v))$ is the quotient of the group of commutative squares

$$
\begin{array}{ccc}
W & \xrightarrow{f} & Y \\
u \downarrow & & \downarrow v \\
X & \xrightarrow{g} & Z
\end{array}
$$

by the subgroup of squares for which the map $vf = gu$ vanishes. This gives a category $\mathscr{B}$ in an obvious way. There is a full and faithful embedding $J: \mathscr{S} \to \mathscr{B}$ sending X to $I(1_X)$. Freyd

shows that $\mathcal{B}$ is an Abelian category and that J is a homology theory. Given a morphism u: $W \to X$ in $\mathcal{S}$, the image of the morphism $Ju: J(W) \to J(X)$ is just $I(u)$. Moreover, the image of J is the subcategory of injective objects in $\mathcal{B}$, which is the same as the subcategory of projective objects. Freyd also shows that for any Abelian category $\mathcal{C}$ and any homology theory $K: \mathcal{S} \to \mathcal{C}$, there is an essentially unique strongly additive exact functor $K': \mathcal{B} \to \mathcal{C}$ such that $K'J \simeq K$. (We say that a functor is *strongly additive* if it preserves all coproducts. Freyd actually proves the corresponding result without strong additivity but the necessary modifications are trivial.) In fact, $K'I(u)$ is just the image of the morphism Ku in $\mathcal{C}$. In particular, this construction gives a functor $\mathcal{B} \to \mathcal{A}$.

The following result is analogous to Theorem 4.16.

PROPOSITION 7.1. *In $\mathcal{B}$, any object of finite projective or injective dimension is both projective and injective, and thus lies in the image of J.*

Proof. Suppose that X has projective dimension at most $n > 0$. There is then a short exact sequence $Y \to P \to X$, where Y has projective dimension at most $n - 1$; by induction, we may assume that Y is projective. As projectives are injective, the sequence splits, so X is a retract of P and thus is projective. $\square$

While this seems a pleasant construction, the finiteness properties of the category $\mathcal{B}$ are poor. We believe that every nonzero object has a proper class of subobjects, for example.

Next, recall that an Abelian category is said to satisfy AB 5 if set-indexed colimits exist and filtered colimits are exact [11]. The category of Abelian groups satisfies AB 5, as does the functor category $\mathcal{A}$.

PROPOSITION 7.2. *If $\mathcal{C}$ is an Abelian category satisfying AB 5 and $K: \mathcal{S} \to \mathcal{C}$ is a homology theory, then $KX = \varinjlim_{\Lambda(X)} KX_\alpha$. Thus, Kf is zero for any phantom map f.*

Proof. Define $\hat{K}X = \varinjlim_{\Lambda(X)} KX_\alpha$. Here we will need to use the fact (mentioned after Corollary 3.2) that $\Lambda(X)$ is the diagram of all pairs (U, u), where U lies in some small skeleton of $\mathcal{F}$ and $u: U \to X$. Using this we see that $\hat{K}$ is an additive functor $\mathcal{S} \to \mathcal{C}$, and that there is an evident natural map $\hat{K} \to K$ (compare [15, Proposition 2.3.9]). If X is finite then $\Lambda(X)$ has a terminal object, so that $\hat{K}X = KX$. If we can show that $\hat{K}$ preserves coproducts and sends cofibre sequences to exact sequences, then the usual argument will show that $\hat{K}X = KX$ for all X. Consider a cofibre sequence $X \to Y \to Z$. We may assume that X is a CW subspectrum of Y, and that Z is the quotient. Let $\{Y_\alpha \mid \alpha \in I\}$ be the directed set of finite subspectra of Y. Write $X_\alpha = Y_\alpha \cap X$ and $Z_\alpha = Y_\alpha / X_\alpha$, so we have a cofibre sequence $X_\alpha \to Y_\alpha \to Z_\alpha$ for each α. It is easy to see that the evident functors from I to $\Lambda(X)$, $\Lambda(Y)$ and $\Lambda(Z)$ are cofinal, so that $\hat{K}X = \varinjlim_I KX_\alpha$ and so on. As direct limits are exact, we conclude that the sequence $\hat{K}X \to \hat{K}Y \to \hat{K}Z$ is exact as required.

We next verify that $\hat{K}$ preserves coproducts. Consider a family of spectra $\{X_i \mid i \in I\}$. Let Λ be the full subcategory of $\prod_I \Lambda(X_i)$ consisting of those objects $(Z_i)_{i \in I}$ such that $Z_i = 0$ for almost all i. It is not hard to see that this is a filtered category, and that the projections $\Lambda \to \Lambda(X_i)$ are cofinal functors. The functor from Λ to $\Lambda(\bigvee_i X_i)$ is also cofinal. By writing $\hat{K}(X_i)$ and $\hat{K}(\bigvee_i X_i)$ as colimits indexed by Λ, we see that $\hat{K}(\bigvee_i X_i) = \bigoplus_i \hat{K}(X_i)$ as required. $\square$

In a more general monogenic Brown category is is more difficult to prove that $\hat{K}$ is an exact functor. As obvious approach is to replace the $X_\alpha \to Y_\alpha \to Z_\alpha$ considered above by the category of all cofibre sequences of finite objects equipped with a map to the sequence $X \to Y \to Z$. However, it is not clear that this is a filtered category. The difficulty is related to the existence of maps of cofibre sequences that are not good in the sense of Neeman [23].

We can deduce from the above that h: $\mathscr{S} \to \mathscr{A}$ is the universal example of a homology theory with values in an AB 5 category.

PROPOSITION 7.3. *Let $\mathscr{C}$ be an AB5 category, and $K: \mathscr{S} \to \mathscr{C}$ a homology theory. Then there is an essentially unique strongly additive exact functor $K': \mathscr{A} \to \mathscr{C}$ such that $K' \circ h \simeq K$.*

Proof. First, let $\mathscr{H}$ be the category of homology theories, so $\mathscr{H} \subset \mathscr{A}$ and $\mathscr{H} \simeq \mathscr{S}/\mathscr{P}$. By Proposition 7.2, we know that K kills phantom maps, so it factors in an essentially unique way through h: $\mathscr{S} \to \mathscr{H}$. We write K again for the resulting functor $\mathscr{H} \to \mathscr{C}$. As the cofibre of an $\mathscr{A}$-epimorphism is phantom, we see that K sends epimorphisms of homology theories to epimorphisms, and similarly for monomorphisms.

Consider an object $F \in \mathscr{A}$. We know that $\mathscr{A}$ has enough projectives and injectives, so we can choose maps $P \xrightarrow{f} F \xrightarrow{g} I$ where f is epic, g is monic, P is projective and I is injective. In particular, P and I are homology theories, so $K(P)$ and $K(I)$ are defined. We would like to define $K'(F)$ to be the image of the map $K(gf): K(P) \to K(I)$; we need only check that this is well-defined. Indeed, if we chose a different epimorphism $f': P' \to F$ then we could use the projectivity of P and P' to show that f and f' factor through each other; it follows easily that $K(gf)$ and $K(gf')$ have the same image, regarded as a subobject of $K(I)$. A similar argument shows that our definition is essentially independent of g.

Next, consider a morphism $v: F \to G$ in $\mathscr{A}$. Choose sequences $P \to F \to I$ and $Q \to G \to J$ as above. Using the projectivity of P and the injectivity of J, we can choose maps $u: P \to Q$ and $w: I \to J$ compatible with v. These induce a map $K(F) \to K(G)$, which we would like to call $K'(v)$. We must check that this does not depend on the choice of u and w. An easy argument reduces us to the case $v = 0$; this implies that the diagonal map in the square

$$
\begin{array}{ccc}
P & \xrightarrow{u} & Q \\
\downarrow & & \downarrow \\
J & \xrightarrow{w} & J
\end{array}
$$

is zero, and thus the induced map image$(K(P \to I)) \to$ image$(K(Q \to J))$ is zero as required. Our definition of $K'(v)$ is thus unambiguous, and it is easy to see that it gives a functor.

Suppose that F is a homology theory. Then $P \to F$ is an epimorphism of homology theories, so $K(P) \to K(F)$ is epic. Similarly, $K(F) \to K(I)$ is monic. It follows directly that $K'(F) = K(F)$. Thus, K' is an extension of K.

If $v: F \to G$ is a monomorphism then we may choose $I = J$ and $w = 1$; this makes it clear that $K'(v)$ is a monomorphism. Similarly, K' preserves epimorphisms.

We next show that K' preserves kernels. Consider a map $v: F \to G$. Choose an epimorphism $f: P \to F$ and a monomorphism $g: G \to J$. As P and J are homology theories, we can choose a map of spectra inducing the map $gvf: P \to J$, and let $j: H \to P$ be its fibre. As $H \to P \to J$ is zero, we see that $H \to P \to F$ factors through ker$(gv) = $ ker(v). As K' preserves

monomorphisms and epimorphisms, we obtain a diagram as follows:

$$
\begin{array}{ccc}
K'(H) & \xrightarrow{\ K'j\ } & K'(P) \\
\downarrow & & {\scriptstyle K'f}\downarrow \\
K'(\ker(v)) \rightarrowtail K'(F) \underset{K'v}{\rightrightarrows} K'(G) & \underset{K'g}{\rightrightarrows} & K'(J).
\end{array}
$$

As $H \to P \to J$ comes from a cofibre seqence of spectra, we know that $K'(H) \to K'(P) \to K'(J)$ is exact. A diagram chase (using elements in the sense of [20], for example) now shows that $K'(\ker(v)) \to K'(F) \to K'(G)$ is exact as required.

Similarly, we see that K' preserves cokernels; it is thus an exact functor.

Finally, we need to show that K' preserves coproducts. Consider a family $\{F_i\}$ of objects of $\mathscr{A}$, and choose maps $P_i \to F_i \to I_i$ in the usual way. Write $P = \bigoplus_i P_i$ and $F = \bigoplus_i F_i$ and $I = \bigoplus_i I_i$, so we have an epimorphism $P \to F$ and a monomorphism $F \to I$ (but I need not be injective). As K' is exact, we see that $K'(F)$ is the image of $K'(P) \to K'(I)$. As K preserves coproducts of spectra, we see that K' preserves coproducts of homology theories, so $K'(P) = \bigoplus_i K'(P_i)$. Similarly, $K'(I) = \bigoplus_i K'(I_i)$. It follows that $K'(F) = \bigoplus_i K'(F_i)$ as required.

It is also clear that any extension of K that preserves images (in particular, any exact extension of K) must be equivalent to K'.   □

We conclude this section with an interesting, if somewhat disconnected result. Consider an essentially small additive category $\mathscr{F}$. Recall that there is an essentially unique category $\mathrm{Ind}(\mathscr{F})$ (the Ind completion of $\mathscr{F}$) equipped with a full and faithful embedding $\mathscr{F} \to \mathrm{Ind}(\mathscr{F})$ (thought of as an inclusion) such that

  (i) $\mathrm{Ind}(\mathscr{F})$ has colimits for all small filtered diagrams.
  (ii) Every object of $\mathrm{Ind}(\mathscr{F})$ is the colimit of a small filtered diagram of objects of $\mathscr{F}$.
  (iii) If X is an object of $\mathscr{F}$ then the functor $\mathrm{Ind}(\mathscr{F})(X, -)$ preserves filtered colimits.

The Ind completion of a category was introduced in [12] and was described in detail in [13].

This category can be constructed at (at least) two ways. The first way is to consider pairs (I, X) where I is a small filtered category and X is a functor $I \to \mathscr{F}$. We define $\mathrm{Ind}(\mathscr{F})$ to be the category of such pairs, with morphisms

$$
\mathrm{Ind}(\mathscr{F})((I, X), (J, Y)) = \varprojlim_I \varinjlim_J \mathscr{F}(X_i, Y_j).
$$

Alternatively, we can embed $\mathscr{F}$ in the category $\mathscr{B}$ of additive functors $\mathscr{F}^{\mathrm{op}} \to \mathscr{A}b$ by $X \mapsto [-, X]$. We then define $\mathrm{Ind}(\mathscr{F})$ to be the subcategory of all functors $F \in \mathscr{B}$ that can be written as a filtered colimit of a small diagram of objects of $\mathscr{F}$. It is equivalent to require that the category of pairs (X, a) (where $X \in \mathscr{F}$ and $a \in FX$) is filtered.

THEOREM 7.4. *Let $\mathscr{F}$ be the category of finite spectra. Then there is an equivalence of categories* $\mathrm{Ind}(\mathscr{F}) = \mathscr{H}$ *(where $\mathscr{H}$ is the category of homology theories on $\mathscr{F}$).*

Proof. We use the second description of $\mathrm{Ind}(\mathscr{F})$, as a subcategory of $\mathscr{B} = [\mathscr{F}^{\mathrm{op}}, \mathscr{A}b]$. Composition with the Spanier–Whitehead duality functor gives an equivalence of $\mathscr{B}$ with $\mathscr{A} = [\mathscr{F}, \mathscr{A}b]$, which sends $[-, X]$ to h_X. Thus, $\mathrm{Ind}(\mathscr{F})$ is equivalent to the category of those functors $\mathscr{F} \to \mathscr{A}b$ that can be written as small filtered colimits of functors of the form

h_X where X is small. As filtered colimits are exact, every such functor is a homology theory. Conversely, every homology theory is of the form h_Y for some Y. Since $h_Y = \varinjlim_{A(Y)} h_{Y_i}$, it follows that every homology theory lies in $\mathrm{Ind}(\mathscr{F})$. $\qquad\square$

We conclude that the Ind completion of a triangulated category need not be a triangulated category. For example, consider a monogenic Brown category with a non-zero phantom map $f: X \to Y$. If Z is the cofibre of f, then the map $h_Y \to h_Z$ is monic but not split. However in a triangulated category all monics split, so $\mathscr{H}$ is not triangulated. (Hartshorne mentions in [14] that the Ind completion may not be triangulated, but he does not indicate a proof.) We also gain some insight into the Pro completion of the category of spectra, which has been used by various people for various purposes, mainly concentrating on towers of spectra rather than more general inverse systems. The Pro completion of a category $\mathscr{C}$ is just $\mathrm{Ind}(\mathscr{C}^{\mathrm{op}})^{\mathrm{op}}$. As $\mathscr{F} \simeq \mathscr{F}^{\mathrm{op}}$ (by Spanier–Whitehead duality) we see that the Pro category of finite spectra is equivalent to the opposite of the category of homology theories. The subcategory consisting of towers of finite spectra is equivalent to the opposite of the category of homology theories with countable coefficients.

REFERENCES

1. H. J. Baues: *Algebraic Homotopy*, Cambridge Studies in Advanced Mathematics, Vol. 15, Cambridge University Press, Cambridge (1989).
2. A. K. Bousfield and D. M. Kan: *Homotopy limits, completions and localizations*, SLNM **304**, Springer, Berlin (1972).
3. E. H. Brown and M. Comenetz: Pontrjagin duality for generalized homology and cohomology theories, *Amer. J. Math.* **98** (1976), 1–27.
4. J. D. Christensen: Ideals in triangulated categories: phantoms, ghosts and skeleta, Ph.D. thesis, Massachusetts Institute of Technology (1997).
5. P. Freyd: Stable homotopy, in *Proc. Conf. on Categorical algebra*, La Jolla, California, 1965, S. Eilenberg, D. K. Harrison, S. Mac Lane and H. Röhrl, Eds, Springer, New York (1966), pp. 121–172.
6. L. Fuchs: *Infinite abelian groups*, Vol. I, Academic Press, New York (1970).
7. B. Gray: Operations and a problem of Heller, Ph.D. thesis, University of Chicago (1965).
8. B. Gray: Spaces of the same n-type, for all n, *Topology* **5** (1966), 241–243.
9. B. Gray and C. A. McGibbon: Universal phantom maps, *Topology* **32** (1993), 371–394.
10. P. Griffith: A note on a theorem of Hill, *Pacific J. Math.* **29** (1969), 279–284.
11. A. Grothendieck: Sur quelque points d'algèbre homologique, *Tôhoku Math. J.* **9** (1957), 119–221.
12. A. Grothendieck: Technique de descente et théorèmes d'existence en géométrie algébrique II: le théorème d'existence en théorie formelle des modules, *Séminaire Bourbaki* **195** (1960).
13. A. Grothendieck and J. L. Verdier: *Théorie des topos et cohomologie étale des schémas, Tome* 1: *Théorie des topos*, (SGA 4, second edition), SLNM **269**, Springer, Berlin (1972).
14. R. Hartshorne: *Residues and Duality*, SLNM 20, Springer, Berlin (1966).
15. M. Hovey, J. H. Palmieri and N. P. Strickland: Axiomatic stable homotopy theory, *Mem. Amer. Math. Soc.*, to appear.
16. M. Huber and W. Meier: Cohomology theories and infinite CW-complexes, *Comment. Math. Helv.* **53** (1978), 239–257.
17. S. O. Kochman: Integral cohomology operations, In *Current trends in algebraic toplogy*, Vol. 2, Part 1, R. M. Kane, Ed., American Mathematical Society, Providence, RI (1982), pp. 437–478.
18. L. G. Lewis, J. P. May and M. Steinberger, with contributions by J. E. McClure: *Equivariant Stable Homotopy Theory*, SLNM **1213**, Springer, New York (1986).
19. T. Y. Lin: Duality and Eilenberg–Mac Lane spectra, *Proc. Amer. Math. Soc.* **56** (1976), 291–299.
20. S. Mac Lane: *Categories for the Working Mathematician*, Graduate Texts in Mathematics, Vol. 5, Springer, Berlin (1971).
21. H. R. Margolis: Eilenberg–Mac Lane Spectra, *Proc. Amer. Math. Soc.* **43** (1974), 409–415.
22. H. R. Margolis: *Spectra and the Steenrod Algebra*, North–Holland, Amsterdam (1983).
23. A. Neeman: Some new axioms for triangulated categories, *J. Algebra* **139** (1991), 221–255.

24. A. Neeman: On a theorem of Brown and Adams, *Topology* **36** (1997), 619–645.
25. T. Ohkawa: A vanishing theorem of Araki–Yosimura–Bousfield–Kan spectral sequences. *Hiroshima Math. J.* **23** (1993), 1–14.
26. J.-P. Pezennec: Propriétés topologiques de [X, Y] et fantômes de finitude, *Bull. Soc. Math. France* **107** (1979), 113–126.
27. D. C. Ravenel: Localization with respect to certain periodic homology theories, *Amer. J. Math.* **106** (1984), 351–414.
28. Zen-ichi Yosimura: Hausdorff condition for Brown–Peterson cohomologies, *Osaka J. Math.* **25** (1988), 88–890.

Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA 02139, USA

Trinity College
Cambridge CB2 1TQ
U.K.