SPLITTING CERTAIN $M$ Spin-MODULE SPECTRA

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§ 1. INTRODUCTION

This homotopy theoretic paper is motivated by two geometric problems. The first one is to find necessary and sufficient conditions for the existence of a Riemannian metric of positive scalar curvature on a manifold $M$. If $M$ is a spin manifold of dimension $n \geq 5$ with finite fundamental group there is a conjecture of Jonathan Rosenberg [17] saying that $M$ has such a metric if and only if all ($KO_n$-valued) index obstructions associated to Dirac operators with coefficients in flat bundles vanish. In the simply connected case this conjecture is known as the Gromov–Lawson conjecture and was recently proved by the author [19]. In general the conjecture implies that $M$ has a metric of positive scalar curvature if and only if the covering of $M$ corresponding to the 2-Sylow subgroup of $\pi_1(M)$ has such a metric. Thus it seems most interesting to study the question for manifolds whose fundamental group $\pi$ is a finite 2-group.

In this paper we give a sufficient (but not necessary) condition for the existence of positive scalar curvature metrics on such manifolds. To describe this result recall that there is a natural transformation

$$\alpha: \Omega^*_{\text{Spin}}(X) \to kO_n(X)$$

from the spin bordism groups of a space $X$ to the connective real $K$-theory of $X$ (it is induced by the Atiyah-Bott-Shapiro orientation $D: M \text{Spin} \to kO$ [3], where $M \text{Spin}$ is the Thom spectrum associated to the classifying space $B\text{Spin}$ and $kO$ is the connective real $K$-theory spectrum; cf. §8).

**Theorem A.** Let $M$ be a closed spin manifold of dimension $n \geq 5$ whose fundamental group $\pi$ is a finite 2-group and let $g: M \to B\pi$ be the classifying map of its universal covering. Assume that the bordism class $[M, g] \in \Omega^*_{\text{Spin}}(B\pi)$ is in the kernel of $\alpha$. Then $M$ has a positive scalar curvature metric.

Using a Baas–Sullivan description of connective real $K$-theory, Rainer Jung has very recently proved an analogue of Theorem A at odd primes [8]. Combining our result with his, it can be shown that Theorem A holds without restrictions on the fundamental group $\pi$.

We note that if $\pi$ is the trivial group then conversely the existence of a positive scalar curvature metric implies by a result of Hitchin [6] that the bordism class of $M$ is in the kernel of $\alpha$. In general this is not true; counter examples are provided by lens spaces [16]. Still, Theorem A could be a first step towards a proof of Rosenbergs conjecture. E.g. for

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\[ \pi = \mathbb{Z}/2 \] the conjecture has been proved recently in joint work of Rosenberg and the author using Theorem A [18]. In the simply connected case Theorem A was proved in [19]. The proof in the general case follows the same pattern (see §8); the new ingredient is that there is a suitable (2-local) splitting of the spectrum \( M \text{Spin} \wedge BPSp(3)_+ \), where \( BPSp(3)_+ \) is the classifying space of the projective symplectic group furnished with a disjoint base-point \( (\pi_0(M \text{Spin} \wedge BPSp(3)_+)) \) is isomorphic to \( \Omega^8_{\text{spin}}(BPSp(3)) \) by the Pontrjagin-Thom construction.

Theorem A is a geometric corollary of the following result which provides a geometric description of \( ko \)-homology (at the prime 2). Let \( T_n(X) \) be the subgroup of \( \Omega^8_{\text{spin}}(X) \) consisting of bordism classes \([E, f] \), where \( p: E \to B \) is an \( \mathbb{H}P^2 \)-bundle over a closed spin manifold \( B \) of dimension \( n - 8 \) and \( f \) is a map from \( B \) to \( X \). Here an \( \mathbb{H}P^2 \)-bundle is a fibre bundle with fibre \( \mathbb{H}P^2 \) and structure group the projective symplectic group \( PSp(3) \) (which is the isometry group of \( \mathbb{H}P^2 \) with its standard metric).

**Theorem B.**

1. \( T_n(X) \) is in the kernel of \( \alpha \).
2. The homomorphism \( \Omega^8_{\text{spin}}(X)/T_n(X) \to ko_n(X) \) induced by \( \alpha \) is a 2-local isomorphism.

We remark that \( \Omega^8_{\text{spin}}(X)/T_n(X) \) is not isomorphic to \( ko_n(X) \) at odd primes, in fact, it is not a homology theory. But inverting a suitable element \( B \in \Omega^8_{\text{pin}} \) we get an isomorphism \( (\Omega^8_{\text{spin}}(X)/T_n(X))[B^{-1}] \cong KO_8(X) \) [9, Thm. C].

The second problem which motivates this paper is a geometric construction of elliptic homology. In joint work with Matthias Kreck [9] we define groups \( E_n(X) \) geometrically and show that \( E_n(X) \) are the (odd primary) elliptic homology groups of Landweber-Ravenel-Stong [10, §3]. For our proof that \( E_n(X) \) is a homology theory at the prime 2 we need to know that the spectrum \( M \text{Spin} \wedge BPSp(3) \) splits into ‘nice’ pieces after localizing at 2.

This motivates our interest in splitting 2-local spectra of the form \( M \text{Spin} \wedge X \). Note that \( Y = M \text{Spin} \wedge X \) is an \( M \text{Spin} \)-module spectrum. From now on we assume tacitly that all spectra under consideration are localized at 2 and all homology groups have \( \mathbb{Z}/2 \) coefficients. The original idea for splitting \( M \text{Spin} \)-module spectra was to construct a ring spectrum map \( s: ko \to M \text{Spin} \) and to prove a splitting result for \( ko \)-module spectra. The ring spectrum map \( s \) gives an \( M \text{Spin} \)-module spectrum \( Y \) the structure of a \( ko \)-module spectrum and allows us to apply those splitting results to \( Y \).

Unfortunately, the attempts to construct \( s \) failed, and, in fact, we show in §7 below that there is no ring spectrum map \( ko \to M \text{Spin} \), thus giving a negative answer to a question of Mark Mahowald. But we get the following weaker statement.

**Theorem C.** There is a 2-local map \( s: ko \to M \text{Spin} \) which is a right inverse of the Atiyah–Bott–Shapiro orientation \( D: M \text{Spin} \to ko \) and which induces an algebra homomorphism in \( \mathbb{Z}/2 \)-homology.

Let \( \mu_{ko} : ko \wedge ko \to ko \) be the multiplication of the ring spectrum \( ko \) (\( \mu_{ko} \) is induced by the tensor product of vector bundles) and let \( \eta: S^0 \to ko \) be the unit of \( ko \) (the inclusion of the bottom cell). Recall that a \( ko \)-module spectrum is a spectrum \( Y \) together with a multiplication map \( \mu: ko \wedge Y \to Y \) such that the following diagrams are homotopy commutative

\[
\begin{array}{ccc}
S^0 \wedge Y & \xrightarrow{\eta \wedge 1} & ko \wedge Y \\
| & \downarrow \mu & | \\
Y & \xrightarrow{\eta} & Y
\end{array}
\qquad
\begin{array}{ccc}
ko \wedge ko \wedge Y & \xrightarrow{1 \wedge \mu} & ko \wedge Y \\
| & \downarrow \mu & | \\
ko \wedge Y & \xrightarrow{\mu} & Y
\end{array}
\]
Now let $Y$ be an $M\text{Spin}$-module spectrum with multiplication map $\mu' : M\text{Spin} \wedge Y \to Y$ and let $\mu : ko \wedge Y \to Y$ be the composition

$$
ko \wedge Y \xrightarrow{\wedge 1} M\text{Spin} \wedge Y \xrightarrow{\mu'} Y.
$$

(1.2)

Note that with this choice of $\mu$ the first diagram in (1.1) is homotopy commutative, but the second diagram only induces a commutative diagram in homology. This shows that an $M\text{Spin}$-module spectrum is in general not a $ko$-module spectrum, but only a homology $ko$-module spectrum, by which we mean a spectrum $Y$ together with a map $\mu : ko \wedge Y \to Y$ such that the diagrams (1.1) induce commutative diagrams in homology. The main technical result of this paper is a splitting theorem for certain homology $ko$-module spectra whose hypotheses are too technical to be stated in the introduction (see §4). These hypotheses are in particular satisfied for $Y = M\text{Spin} \wedge B\text{PSp}(3)$ and that gives the desired splitting of this spectrum. As another application we get a more direct proof of Mahowald's splitting of $ko \wedge ko$ (Corollary 4.3), a result which is used at various points in this paper, e.g. in the proof of Theorem C.

Using the same Adams spectral sequence techniques as in the proof of the splitting result we also obtain a calculation of the connective $KO$-theory of certain spaces $X$ in terms of the homology and (filtration subgroups of) the $KO$-theory of $X$. This result is used in [9] for the geometric construction of elliptic homology and might also be of independent interest.

The sections are organized as follows. In §2 we show that for a homology $ko$-module spectrum $Y$ its homology $H_* Y$ (considered as a comodule over the dual Steenrod algebra $A_*$) is determined by the $A(1)_*$-comodule $H_* Y$, where $H_* Y$ denotes the indecomposables of $H_* Y$ with respect to the $H_* ko$-action and $A(1)_*$ is the dual of the subalgebra of the Steenrod algebra generated by $Sq^1$ and $Sq^2$. In §3 we review some results of Adams and Priddy [1] about $A(1)_*$-comodules and their Ext-groups. These are used in §4 to prove splitting results for homology $ko$-module spectra by Adams spectral sequence techniques and in §5 to compute the connective $KO$-theory of certain spectra in terms of a pull back diagram. In §6 we prove Theorem C and in §7 we show that there is no ring spectrum map $ko \to M\text{Spin}$. §8 contains the proofs of Theorems A and B.

## §2. HOMOLOGY $ko$-MODULE SPECTRA

In this section we discuss the connective real $K$-theory spectrum $ko$ and homology $ko$-module spectra. We show that if $Y$ is a homology $ko$-module spectrum then $H_* Y$ is isomorphic to the cotensor product $A_* \boxtimes A(1)_* H_* Y$ (cf. [15, A1.1.4]), where $H_* Y$ denotes the indecomposables of $H_* Y$ with respect to the $H_* ko$-action.

Let $KO$ be the real $K$-theory spectrum and for $k \in \mathbb{Z}$ let $ko<k>$ be its $(k - 1)$-connected cover. It is characterized up to homotopy equivalence by the property that $\pi_n(ko<k>) = 0$ for $n < k$ and that there is a map $p<k> : ko<k> \to KO$ inducing an isomorphism in homotopy groups in degrees $\geq k$. The tensor product of vector bundles gives a map

$$
\mu_{ko} : KO \wedge KO \to KO
$$

which makes $KO$ a ring spectrum. By obstruction theory there is a unique map $\mu$ making the diagram

$$
ko<k> \wedge ko<k'> \xrightarrow{\mu} ko<k + k'>
$$

$$
\downarrow p<k> \quad \quad \quad \quad \downarrow p<k + k'>
$$

$$
ko<k> \wedge ko<k'> \xrightarrow{\mu_{ko}} KO
$$
homotopy commutative. This multiplication gives \( ko = ko \langle 0 \rangle \) the structure of a ring spectrum and makes \( ko \langle k \rangle \) a \( ko \)-module spectrum. We note that \( p \langle 0 \rangle : ko \to KO \) is a ring spectrum map and \( p \langle k \rangle : ko \langle k \rangle \to KO \) is a \( ko \)-module map.

Let \( i : ko \to H \) be the map into the \( \mathbb{Z}/2 \)-Eilenberg–MacLane spectrum corresponding to the non-trivial element of \( H^0 ko \). It turns out that \( i \) induces a monomorphism in homology. To describe its image recall that the homology of \( H \) (known as the ‘dual Steenrod algebra’ \( A_\ast \)) is the polynomial algebra \( \mathbb{Z}/2[\xi_1, \xi_2, \ldots] \), where the ‘Milnor generators’ \( \xi_i \) are elements of degree \( 2^i - 1 \) [15, Thm. 3.1.1]. To describe the image of \( i_* \) it is better to describe \( A_\ast = H_\ast H \) as the polynomial algebra \( \mathbb{Z}/2[\xi_1, \xi_2, \ldots] \), where \( \xi_i = c(\xi_i) \) is the conjugate of \( \xi_i \). In terms of the \( \xi_i \)'s the coproduct \( \Psi : A_\ast \to A_\ast \otimes A_\ast \) is given by
\[
\Psi(\xi_i) = \sum_{j=0}^{i} \xi_j \otimes \xi_{i-j} \tag{2.1}
\]
and the map \( i \) induces an isomorphism [15, Thm. 3.1.17]
\[
H_\ast ko \cong \mathbb{Z}/2[\zeta_1, \zeta_2, \zeta_3, \zeta_4, \ldots] \tag{2.2}
\]
Let \( A(1) \) be the subalgebra of the Steenrod algebra \( A \) generated by \( Sq^1 \) and \( Sq^2 \). Its dual \( A(1)_\ast \) is the quotient \( A_\ast / (\xi_1, \xi_2, \xi_3, \xi_4, \ldots) \). We note that (2.1) implies that the augmentation ideal of \( H_\ast ko \) is an \( A(1)_\ast \)-comodule.

Next we recall the notion of ‘extended’ \( A(1)_\ast \)-comodules. Note that \( A_\ast \) is a (right) \( A(1)_\ast \)-comodule by composing the coproduct (2.1) with the projection map on \( A_\ast \otimes A(1)_\ast \).

Let \( M \) be a (left) \( A(1)_\ast \)-comodule. Recall that the cotensor product \( A_\ast \square_{A(1)_\ast} M \) is defined by the exact sequence
\[
0 \to A_\ast \otimes_{A(1)_\ast} M \to A_\ast \otimes M \xrightarrow{\psi \otimes 1 - 1 \otimes \psi} A_\ast \otimes A(1)_\ast \otimes M,
\]
where \( \psi \) denotes the \( A(1)_\ast \)-comodule structure maps for both, \( A_\ast \) and \( M \). We note that the (left) \( A_\ast \)-comodule structure on \( A_\ast \otimes M \) induces a \( A_\ast \)-comodule structure on \( A_\ast \square_{A(1)_\ast} M \). Such \( A_\ast \)-comodules are referred to as ‘extended’ \( A(1)_\ast \)-comodules.

Let \( Y \) be a homology \( ko \)-module spectrum. The map on homology induced by the multiplication map \( \mu : ko \otimes Y \to Y \) makes \( H_\ast Y \) a module over \( H_\ast ko \). We denote by \( \pi : H_\ast Y \to H_\ast ko \otimes H_\ast ko \) the projection onto the indecomposables of this module. Note that \( H_\ast ko \otimes H_\ast Y \) is an \( A(1)_\ast \)-comodule since the augmentation ideal of \( H_\ast ko \) is an \( A(1)_\ast \)-comodule.

**Proposition 2.3.** [19, Prop. 6.7]: Let \( Y \) be a homology \( ko \)-module spectrum whose homology is bounded below and locally finite. Then the composition
\[
\Phi_Y : H_\ast Y \xrightarrow{\psi} A_\ast \otimes H_\ast Y \xrightarrow{1 \otimes \pi} A_\ast \otimes H_\ast Y
\]
is an \( A_\ast \)-comodule isomorphism onto \( A_\ast \square_{A(1)_\ast} H_\ast Y \).

**Remark 2.4.** Let \( Y \) be an \( M \text{ Spin} \)-module spectrum. Recall from (1.2) that the map \( s : ko \to M \text{ Spin} \) gives \( Y \) the structure of a homology \( ko \)-module spectrum. In that case Proposition 2.3 is precisely Corollary 6.8 of [19], since the image of \( s_* : H_\ast ko \to H_\ast M \text{ Spin} \) is the subalgebra \( R \) (cf. §6).

Recall that a map \( f : Y \to Z \) between \( ko \)-module spectra is called a \( ko \)-module map if the diagram
\[
\begin{array}{ccc}
ko \wedge Y & \xrightarrow{\cdot f} & ko \wedge Z \\
\mu_Y & \downarrow & \mu_Z \\
Y & \xrightarrow{f} & Z
\end{array}
\]
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is homotopy commutative. If $Y$ and $Z$ are homology $ko$-module spectra we say that $f$ is a homology $ko$-module map if the above diagram is commutative in homology. For a homology $ko$-module map the induced map $f_*$: $H_* Y \to H_* Z$ is a $H_* ko$-module map and hence it induces a map $\Phi_f: H_* Y \to H_* Z$. Note that the construction of $\Phi_f$ is natural; i.e. the following diagram is commutative:

$$
\begin{array}{ccc}
H_* Y & \xrightarrow{f_*} & H_* Z \\
\phi_f \downarrow \cong & & \phi_f \downarrow \cong \\
A_\* \square A(1)_* H_* Y & \xrightarrow{id \square f_*} & A_\* \square A(1)_* H_* Z \\
\end{array}
$$

Let $g$ be a map from some spectrum $X$ into a homology $ko$-module spectrum $Y$. We call the composition

$$
\hat{g}: ko \wedge X \xrightarrow{\eta \wedge 1_*} ko \wedge Y \xrightarrow{\mu} Y
$$

the $ko$-extension of $g$. Note that $\hat{g}$ is a homology $ko$-module map.

If $Y$ is a $ko$-module spectrum $\hat{g}$ is in fact a $ko$-module map and sending $g$ to $\hat{g}$ gives an isomorphism

$$
[X, Y] \xrightarrow{\cong} [ko \wedge X, Y]_{ko}
$$

between homotopy classes of maps from $X$ to $Y$ and homotopy classes of $ko$-module maps from $ko \wedge X$ to $Y$. An explicit inverse is given by sending a $ko$-module map $f: ko \wedge X \to Y$ to the composition

$$
X = S^0 \wedge X \xrightarrow{\eta \wedge 1} ko \wedge X \xrightarrow{f} Y.
$$

In particular, for $f = \hat{g}$ the above composition is $g$.

If $Y$ is just a homology $ko$-module spectrum then the composition (2.7) with $f = \hat{g}$ still induces the same map in homology as $g$. Consider the commutative diagram

$$
\begin{array}{ccc}
H_* X = H_* S^0 \wedge X & \xrightarrow{(\eta \wedge 1)_*} & H_* ko \wedge X \\
\pi \downarrow & & \downarrow \pi \\
H_* ko \wedge X & \xrightarrow{\hat{g}_*} & H_* Y \\
\end{array}
$$

The composition $\pi(\eta \wedge 1)_*$ is an isomorphism which we will use to identify $H_* X$ with $H_* ko \wedge X$. Composition with $\pi$ induces the change of rings isomorphism

$$
\text{Hom}_{A_*}(H_* X, H_* Y) \cong \text{Hom}_{A(1)_*}(H_* X, H_* Y).
$$

(2.8)

We conclude:

**Lemma 2.9.** Under the change of rings isomorphism (2.8) $g_*$ corresponds to $(\hat{g})_*$.

In particular, if $Y$ is a $ko$-module spectrum, we have the following commutative diagram

$$
\begin{array}{ccc}
[X, Y] & \xrightarrow{\cong} & [ko \wedge X, Y]_{ko} \\
\downarrow h & & \downarrow H \\
\text{Hom}_{A_*}(H_* X, H_* Y) & \xrightarrow{\cong} & \text{Hom}_{A(1)_*}(H_* ko \wedge X, H_* Y) \\
\end{array}
$$

(2.10)

where $h$ is the Hurewicz homomorphism and $H$ sends a $ko$-module map $f$ to the induced map $f_*$. 
§3. SOME $A(1)_*$-COMODULES AND THEIR EXT-GROUPS

In this section we review the structure theory of invertible $A(1)_*$-comodules and the calculation of their Ext-groups due to Adams and Priddy [1]. In addition we discuss certain $A(1)_*$-comodules which play a prominent role in later sections.

We note that the vector space dual of an $A(1)_*$-comodule is an $A(1)$-module and vice versa. Hence for each result concerning $A(1)$-modules there is a corresponding result about $A(1)_*$-comodules. Below we give the $A(1)_*$-comodule version of some well known results on $A(1)$-modules. It is often helpful to visualize the structure of $A(1)$-modules. For example, let $I$ be the augmentation ideal of $A(1)$, i.e. the elements of positive degree in $A(1)$ and let $J$ be the module $\Sigma^{-2}A(1)/A(1)Sq^1Sq^2$. Then the corresponding pictures look as follows:

Here a dot represents a basis element in the degree indicated by the number below and the action of $Sq^1$ and $Sq^2$ is described by the lines; i.e. if $x, y$ are two basis elements such that $\text{deg}(y) - \text{deg}(x)$ is equal to one (resp. two) then the corresponding dots are connected by a line if and only if the coefficient of $y$ in $Sq^1x$ (resp. $Sq^2x$) is non-zero.

Recall that the square of $Q_0 = Sq^1 \in A(1)$ and $Q_1 = Sq^1Sq^2 + Sq^2Sq^1 \in A(1)$ is zero. Hence we can regard $Q_1$ as a (degree increasing) differential on any $A(1)$-module. Dually, if $M$ is a $A(1)_*$-comodule, $Q_1$ acts on it as a (degree decreasing) differential and we denote the corresponding homology groups by $H_n(M; Q_1)$. It is easy to read off the $Q_1$-homology groups of an $A(1)$-comodule $M$ from the picture of $M$ (that is the picture of the $A(1)$-module dual to $M$). For example, for $M = I_*$ (resp. $M = J_*$) we get:

$$H_n(I_*; Q_1) = \begin{cases} \mathbb{Z}/2 & \text{for } i = 0, n = 1 \text{ or } i = 1, n = 3 \\ 0 & \text{otherwise} \end{cases} \quad (3.2)$$

$$H_n(J_*; Q_1) = \begin{cases} \mathbb{Z}/2 & \text{for } i = 0, 1 \text{ and } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Direct sum and tensor product give the stable isomorphism classes of finitely generated $A(1)_*$-comodules the structure of a semi ring $R$. Adams and Priddy study the invertible comodules; i.e. those comodules whose stable class is a unit in $R$. They prove that a finitely generated $A(1)_*$-comodule $M$ is invertible if and only if $H_*(M; Q_1)$ is 1-dimensional for $i = 0, 1$ [1, Lemma 3.5]. It follows that $I_*$ and $J_*$ are invertible by (3.2).

**Proposition 3.3** [1, Thm. 3.7]. The group of units in $R$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2$, generated by $\Sigma, I_*$ and $J_*$, where $\Sigma$ is the $A(1)_*$-comodule which is $\mathbb{Z}/2$ in degree one and trivial in the other degrees.

This isomorphism can be made explicit as follows. For an invertible comodule $M$ let $d_i(M) \in \mathbb{Z}$ for $i = 0, 1$ be the degree in which $H_*(M; Q_1)$ is non-zero. The K"unneth formula implies that $d_i: R^\times \rightarrow \mathbb{Z}$ is a homomorphism ($R^\times$ denotes the group of units in $R$). Let $d: R^\times \rightarrow \mathbb{Z}/8^\times$ be the map which sends a comodule $M$ to its total dimension modulo eight. Note that this is well defined since the total dimension of $A(1)_*$ is eight. Recall that $\mathbb{Z}/8^\times \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and let $\rho: \mathbb{Z}/8^\times \rightarrow \mathbb{Z}/2$ be the homomorphism which sends 1, 7 to 0 and 3,
For an $\mathcal{A}(l)_\ast$-comodule $M$ define $a(M) = d_0(M)$, $b(M) = \frac{1}{2}(d_1(M) - d_0(M))$, $c(M) = \rho(d(M))$, and $e(M) = (a(M), b(M), c(M)) \in \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2$. Then
\[ e(\Sigma_\ast) = (1, 0, 0) \quad e(I_\ast) = (1, 1, 0) \quad e(J_\ast) = (0, 0, 1), \tag{3.4} \]
which implies:

**Corollary 3.5.** The map $e: \mathbb{R}^5 \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2$ is an isomorphism.

For $(a, b, c) \in \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2$ let $M(a, b, c)$ be the $\mathcal{A}(l)_\ast$-comodule with $e(M(a, b, c)) = (a, b, c)$ which is minimal, i.e. $M(a, b, c)$ does not contain free summands. Alternatively, $M(a, b, c)$ is the minimal comodule stably isomorphic to $\Sigma^{a-b} I_\ast J_\ast$.

The following examples of $\mathcal{A}(l)_\ast$-comodules are relevant for us in the following sections.

**Example 3.6.** Recall from §2 that $H_\ast ko \langle k \rangle \cong A_\ast \square_{\mathcal{A}(l)_\ast} H_\ast ko \langle k \rangle$, where $H_\ast ko \langle k \rangle$ is a certain $\mathcal{A}(l)_\ast$-comodule. It follows from [20] that the $\mathcal{A}(1)$-module dual to $H_\ast ko \langle k \rangle$ is isomorphic to
\[
\begin{aligned}
\text{if } k &\equiv 0 \mod 8 \\
\text{if } k &\equiv 1 \mod 8 \\
\text{if } k &\equiv 2 \mod 8 \\
\text{if } k &\equiv 4 \mod 8.
\end{aligned}
\]

The corresponding pictures are

- \( k = 0 \mod 8 \)
- \( k = 1 \mod 8 \)
- \( k = 2 \mod 8 \)
- \( k = 4 \mod 8 \)

Here the bottom class of these modules is always in degree $k$. This allows us to identify the $\mathcal{A}(1)_\ast$-comodule $H_\ast ko \langle k \rangle$ using (3.5). The result is
\[
\begin{aligned}
H_\ast ko \langle k \rangle \cong & \\
M(k, 0, 0) & \text{ for } k \equiv 0 \mod 8 \\
M(k + 3, -1, 1) & \text{ for } k \equiv 1 \mod 8 \\
M(k + 2, 0, 1) & \text{ for } k \equiv 2 \mod 8 \\
M(k, 1, 1) & \text{ for } k \equiv 4 \mod 8.
\end{aligned}
\]

**Example 3.7.** The following discussion follows [11, §§2, 3]. We define a new grading on $H_\ast ko \cong \mathbb{Z}/2[\zeta^1_4, \zeta^2_2, \zeta_3, \ldots]$ by letting the degree of $\zeta_i$ be $2^{i-1}$ and by letting the degree of a product be the sum of the degrees of the factors. Let $N_{4l}$ be the $\mathbb{Z}/2$ vector space generated by the monomials of degree $4l$. It is an easy consequence of the formula (2.1) for the coaction that $N_{4l}$ is in fact a sub $\mathcal{A}(1)_\ast$-comodule. Using (2.1) it is also not hard to compute the $Q_\ast$-homology of $H_\ast ko$ for $i = 0, 1$. This turns out to be the polynomial algebra $\mathbb{Z}/2[\zeta^i_4]$ for $i = 0$, and the exterior algebra $E(\zeta^2_2, \zeta^3_4, \ldots)$ for $i = 1$. Note that for a given $l$ there is exactly one element of (new) degree $4l$ in $H_\ast ko \langle Q_\ast \rangle$, namely $\zeta^{4l}_i$ for $i = 0$ and $\zeta^{2i}_2, \zeta^{2i}_3, \ldots$ for $i = 1$, where $l = \sum i_j 2^j$ is the 2-adic expansion of $l$. Furthermore, the 'usual' degree (the 'usual' degree of $\zeta_i$ is $2^i - 1$) of this element is $4l$ for $i = 0$ and $8l - 2\alpha(l)$ for $i = 1$, where $\alpha(l)$ denote the number of 1's in the 2-adic expansion of $l$. It follows that $N_{4l}$ is an invertible $\mathcal{A}(1)_\ast$-comodule with $a(N_{4l}) = 4l$ and $b(N_{4l}) = 2l - \alpha(l)$. A counting argument shows
that \( c(N_{4l}) \equiv l \mod 2 \) and hence \( N_{4l} \) is stably isomorphic to \( M(4l, 2l - \alpha(l), l) \). We remark that there are finite Thom spectra \( B(l) \), known as (integral) Brown–Gitler spectra, such that \( H_* \Sigma^{4l} B(l) \) is isomorphic to \( N_{4l} \) as \( A(1)_* \)-comodule.

Now we turn to the computation of the Ext-groups of the modules \( M(a, b, c) \). It follows from [1, Lemma 3.8] that

\[
\text{Ext}_*^{A(1)_*}(\mathbb{Z}/2, M(a, b, c)) \cong \text{Shom}_{A(1)_*}(\Sigma^s - a + b, I^s - b N) \quad (3.8)
\]

for \( s > 0 \), where \( N = \mathbb{Z}/2 \) (resp. \( N = J_* \)) for \( c = 0 \) (resp. \( c = 1 \)), and \( \text{Shom}_{A(1)_*}(M, M') \) denotes the group of stable \( A(1)_* \)-comodule maps, i.e. the quotient of \( \text{Hom}_{A(1)_*}(M, M') \) modulo those which factor through a free comodule. It turns out that the minimality of \( M(a, b, c) \) implies that also for \( s = 0 \) we have the isomorphism (3.8).

We note that the direct sum

\[
\bigoplus_{s, t \in \mathbb{Z}} \text{Shom}_{A(1)_*}(\Sigma^s, I^t N),
\]

is a bigraded ring for \( N = \mathbb{Z} \) with multiplication given by the tensor product of homomorphisms. For \( N = J_* \) this sum is a bigraded module over that ring. The isomorphism (3.8) is multiplicative in the sense that Yoneda product with an element of \( \text{Ext}_*^{A(1)_*}(\mathbb{Z}/2, \mathbb{Z}/2) \) on the left hand side corresponds to the tensor product with the corresponding element of \( \text{Shom}_{A(1)_*}(\Sigma^s, I^t N) \) on the right hand side.

The groups \( \text{Shom}_{A(1)_*}(\Sigma^s, I^t N) \) for \( N = \mathbb{Z}/2 \) (resp. \( N = J_* \)) are tabulated in [1, 3.10 (resp. 3.11)]. To obtain information about the multiplicative structure we note that the ring \( \text{Ext}_*^{A(1)_*}(\mathbb{Z}/2, \mathbb{Z}/2) \) is well known (e.g. [15, Thm. 3.1.25]). Since \( \mathbb{Z}/2 = M(0, 0, 0) \) this by (3.8) determines the multiplicative structure of \( \text{Shom}_{A(1)_*}(\Sigma^s, I^t N) \) for \( s \geq 0 \). For \( N = J_* \) we use the following argument. Let \( M \) be the \( A(1)_* \)-comodule \( A(1)_* \boxtimes_{A(0)_*} \mathbb{Z}/2 \). Its picture is

![Diagram](image)

which shows that \( \mathbb{Z}/2 \) is a subcomodule of \( M \) with quotient \( M(5, -1, 1) \) (the identification of the quotient is done using (3.5)). The corresponding long exact Ext-sequence allows us to calculate the multiplicative structure on \( \text{Ext}_*^{A(1)_*}(\mathbb{Z}/2, M(5, -1, 1)) \) which by (3.8) determines the multiplicative structure on \( \text{Shom}_{A(1)_*}(\Sigma^s, I^t N) \) for \( s \geq -1 \). The multiplicative structure for all \( s \) as shown in the tables below can be obtained by more sophisticated homological arguments. We don’t present those here since for our applications we only need the multiplicative structure for \( s \geq 0 \).

Let \( h_0, h_1, b \in \text{Ext}_*^{A(1)_*}(\mathbb{Z}/2, \mathbb{Z}/2) \) be the unique non-trivial elements in bidegree (1, 1) (resp. (1, 2) resp. (4, 12)). In the following tables a dot means that the corresponding group is \( \mathbb{Z}/2 \); all other groups are zero. A (non-trivial) product with \( h_0 \) (resp. \( h_1 \)) is indicated by a vertical (resp. slanted) line. In each quadrant of the tables, the obvious periodicity which is given by multiplication by \( b \) continues.

The isomorphism (3.8) can then be rephrased as follows.

**Lemma 3.12.** The diagram for \( \text{Ext}_*^{A(1)_*}(\mathbb{Z}/2, M(a, b, c)) \) is obtained by shifting the diagram 3.10 (resp. 3.11) for \( c = 0 \) (resp. \( c = 1 \)) a units to the right, \( b \) units down and by deleting the parts below the \((t - s)\)-axis.
Let $X$ be a spectrum, let $Y$ be a homology $\mathbb{K}$-module spectrum and let $f: \mathbb{K} \wedge X \rightarrow Y$ be a homology $\mathbb{K}$-module map. Recall from §2 that $f$ induces an $\mathbb{A}(1)_{\mathbb{K}}$-comodule map

$$\overline{f}_{\mathbb{K}} : H_{\mathbb{K}} X = H_{\mathbb{K}}^* \mathbb{K} \wedge X \rightarrow \overline{H}_{\mathbb{K}} Y$$

whose $\mathbb{A}_{\mathbb{K}}$-extension can be identified with $f_{\mathbb{K}}$. The main result of this section is that any $\mathbb{A}(1)_{\mathbb{K}}$-comodule map $F : H_{\mathbb{K}} X \rightarrow \overline{H}_{\mathbb{K}} Y$ is induced by some homology $\mathbb{K}$-module map $f$, provided the $\mathbb{A}(1)_{\mathbb{K}}$-comodules $H_{\mathbb{K}} X$ and $\overline{H}_{\mathbb{K}} Y$ satisfy certain conditions. Recall from §3 that the stable class of an invertible $\mathbb{A}(1)_{\mathbb{K}}$-comodule $M$ is determined up to multiplication by $J_{\mathbb{K}}$ by the two integers $a(M)$ and $b(M)$. We call $M$ admissible, if $M$ is either free or $M$ is an invertible $\mathbb{A}(1)_{\mathbb{K}}$-comodules with $a(M) \equiv 0 \mod 4$ (i.e. the $Q_0$-homology of $M$ is concentrated in degrees divisible by 4).

**Theorem 4.1.** Let $Y$ be a homology $\mathbb{K}$-module spectrum such that $\overline{H}_{\mathbb{K}} Y$ is isomorphic to the (possibly infinite) sum of admissible $\mathbb{A}(1)_{\mathbb{K}}$-comodules $M_i$. Let $X$ be a finite spectrum such...
that $H_\ast X$ is an admissible comodule over $A(1)_\ast$. Assume that $a(M_1) - a(H_\ast X) \leq 0$ or $b(M_1) - b(H_\ast X) \geq -2$ if $M_1$ and $H_\ast X$ are invertible modules. Let $F: H_\ast X \rightarrow H_\ast Y$ be a $A(1)_\ast$-comodule map. Then there is a homology ko-module map $f: ko \wedge X \rightarrow Y$ such that $\overline{f}_\ast = F$. Moreover, if $Y$ is a ko module spectrum then $f$ is a ko-module map.

Corollary 4.2. Let $Y$ be a homology ko-module spectrum such that there is an $A(1)_\ast$-comodule isomorphism

$$F: \bigoplus_i H_\ast X_i \rightarrow H_\ast Y,$$

where $X_i$ is a (possibly infinite) family of finite spectra such that $H_\ast X_i$ are admissible $A(1)_\ast$-comodules. Assume that $a(H_\ast X_i) - a(H_\ast X\Sigma)$ $\leq 0$ or $b(H_\ast X_i) - b(H_\ast X\Sigma) \geq -2$ for all $i$ for which $H_\ast X_i$ and $H_\ast X\Sigma$ are invertible modules. Then there is a homotopy equivalence

$$f: ko \wedge \left( \bigvee_i X_i \right) \rightarrow Y$$

which is a homology ko-module map with $\overline{f}_\ast = F$. Moreover, $f$ is a ko-module map if $Y$ is a ko-module spectrum.

Remark. This corollary is not true without the condition $a(H_\ast X_i) - a(H_\ast X\Sigma) \leq 0$ or $b(H_\ast X_i) - b(H_\ast X\Sigma) \geq -2$. For example, there is a map $f: S^{-5} \rightarrow DB(2)$ to the Spanier–Whitehead dual of the Brown–Gitler spectrum $B(2)$ (cf. (3.7)) which has Adams filtration 2 and induces a non-trivial map in ko-homology. Then the spectrum $ko \wedge C_f$, where $C_f$ is the cofibre of $f$ is a ko-module spectrum which satisfies all the other conditions of corollary 4.2, but is not homotopy equivalent to $ko \wedge DB(2) \vee ko \wedge S^{-4}$.

Corollary 4.2 can be used to give a different proof of Mahowald’s splitting of $ko \wedge ko$ [11], which we need in later sections (cf. proofs of (5.2), (6.7) and (7.3)).

Corollary 4.3. There is a splitting of ko-module spectra

$$ko \wedge ko \sim ko \wedge \left( \bigvee_{l \geq 0} \Sigma^{4l} B(l) \right)$$

Here the $B(l)$’s are Thom spectra known as (integral Brown–Gitler spectra (cf. (3.7)).

Proof. The spectrum $ko \wedge ko$ is a ko-module spectrum via left multiplication. The indecomposables $H_\ast ko \wedge ko$ can be identified with $H_\ast ko$. Our discussion (3.7) shows that there is an isomorphism of $A(1)_\ast$-comodules

$$H_\ast ko \cong \bigoplus_{l \geq 0} H_\ast \Sigma^{4l} B(l).$$

Moreover, $H_\ast \Sigma^{4l} B(l)$ is an invertible $A(1)_\ast$-comodule with

$$a(H_\ast \Sigma^{4l} B(l)) = 4l \quad \text{and} \quad b(H_\ast \Sigma^{4l} B(l)) = 2l - \varepsilon(l).$$

We note that $2l - \varepsilon(l)$ is an increasing function of $l$ so that for all $k$, $l$ the numerical condition of Corollary 4.2 is satisfied and hence we obtain the desired homotopy equivalence of ko-module spectra.

For the proof of (4.1) we use the Adams spectral sequence

$$\text{Ext}^n_{\lambda}(H_\ast X, H_\ast Y) \Rightarrow [\Sigma^{-r} X, Y].$$

(4.5)
We find the following terminology useful:

**Definition 4.6.** The Adams spectral sequence **collapses in degree** \( n \) **if all differentials with domain** \( E_{r,s}^{t} \) **are trivial for all** \( r, s, t \) **with** \( t - s = n \).

The following result is the main step in the proof of (4.1) and useful to us in the next section.

**Proposition 4.1.** Let \( X \) and \( Y \) be spectra satisfying the hypotheses of Theorem 4.1. Then the Adams spectral sequence (4.5) collapses in degree \( n \) for \( n \geq 0 \), \( n \equiv 0 \mod 4 \).

**Proof of Theorem 4.1.** Recall [22, Thm. 19.9] that the Hurewicz homomorphism

\[
h : [X, Y] \to \text{Hom}_{A}(H_{*}X, H_{*}Y) \cong \text{Hom}_{A(1)}(H_{*}X, H_{*}Y)
\]

is the edge homomorphism of the Adams spectral sequence. Hence (4.7) implies that \( h \) is surjective and we can pick a map \( g : X \to Y \) with \( h(g) = F \). Let \( f : ko \wedge X \to Y \) be the \( ko \)-extension \( \hat{g} \). Then \( f \) is a homology \( ko \)-module map—in fact a \( ko \)-module map if \( Y \) is a \( ko \)-module spectrum—and \( f_{*} = F \) by Lemma 2.9.

**Proof of Proposition 4.7.** By Spanier–Whitehead duality the spectral sequence (4.5) is isomorphic to the Adams spectral sequence

\[
\text{Ext}_{A}^{*,*}(\mathbb{Z}/2, H_{*}(Y \wedge DX)) \cong \pi_{t-s}(Y \wedge DX), \tag{4.8}
\]

where \( DX \) is the dual of \( X \). Note that \( Y \wedge DX \) is again a homology \( ko \)-module spectrum with \( H_{*}(Y \wedge DX) \cong H_{*}Y \otimes H_{*}DX \) and hence

\[
\text{Ext}_{A}^{*,*}(\mathbb{Z}/2, H_{*}(Y \wedge DX)) \cong \text{Ext}_{A(1)}^{*,*}(\mathbb{Z}/2, H_{*}Y \otimes H_{*}DX).
\]

The assumptions of Theorem 4.1 imply that \( H_{*}Y \otimes H_{*}DX \) is of the form \( M \oplus F \), where \( F \) is a free \( A(1)_{*} \)-comodule and \( M \) is a sum of comodules \( M(a, b, c) \) with

\[
a \equiv 0 \mod 4 \quad \text{and} \quad a \leq 0 \text{ or } b \geq -2. \tag{4.9}
\]

Hence \( \text{Ext}_{A}^{*,*}(\mathbb{Z}/2, H_{*}(Y \wedge DX)) \) is a direct sum of the groups \( \text{Ext}_{A(1)}^{*,*}(\mathbb{Z}/2, M(a, b, c)) \) for \( s \geq 0 \). By Lemma 3.12 we obtain the picture for these \( \text{Ext} \)-groups by shifting the pictures of Table (3.10) (resp. (3.11)) \( a \) units to the right and \( b \) units down. A glance at the tables shows that the conditions (4.9) imply that these \( \text{Ext} \)-groups vanish for \( t - s = -1 \mod 4 \), \( t - s \geq -1 \) and \( s \geq 2 \). It follows that the Adams spectral sequence (4.8) collapses in degrees \( n \equiv 0 \mod 4 \), \( n \geq 0 \) since the \( d_{r} \)-differential decreases the \((i - s)\)-degree by 1 and increases the \( s \)-degree by \( r \).

**§5. \text{ko}(k)$$\text{-COHOMOLOGY}$$**

In this section we show that for suitable spectra \( X \) the group \([X, \text{ko}(k)]\) of homotopy classes of maps from \( X \) to \( \text{ko}(k) \) can be computed in terms of a pull back diagram.

Let \( p(k) : \text{ko}(k) \to KO \) be the projection map of the \((k - 1)\)-connected cover of \( KO \) (cf. §2). It induces a homomorphism

\[
p(k)_{*} : [X, \text{ko}(k)] \to [X, KO] = KO(X)
\]

whose image we denote by \( F_{k}KO(X) \).
The goal is to show that for certain spectra $X$ the diagram

\[ [X, ko \langle k \rangle] \xrightarrow{p(k)_*} F_k KO(X) \]

\[ h \downarrow \]

\[ \text{Hom}_{A(l)_*}(H_* X, H_* ko \langle k \rangle) \xrightarrow{S} \text{Shom}_{A(l)_*}(H_* X, H_* ko \langle k \rangle) \]

\[ (5.1) \]

can be extended to a pull back diagram. Here $h$ is the Hurewicz homomorphism (we identify $\text{Hom}_{A(l)_*}(H_* X, H_* ko \langle k \rangle)$ with $\text{Hom}_{A(l)_*}(H_* X, H_* ko \langle k \rangle)$ via the change of rings isomorphism (2.8)) and $S$ is the stabilization map which sends an $A(l)_*$-comodule homomorphism to its stable class (cf. (3.8)).

A typical assumption on $X$ is that $X$ is a finite spectrum such that $H_* X$ is a sum of free $A(l)_*$-comodules and invertible $A(l)_*$-comodules $M_i$ with $a(M_i) \equiv 0 \mod 4$. Recall from §2 that the last condition is equivalent to $H_*(H_* X; Q_0) = 0$ for $n \neq 0 \mod 4$. We call such spectra admissible.

**Theorem 5.2.** Let $X$ be a (possibly infinite) wedge of spectra $X_i$ which are admissible or of the form $ko \wedge Z_i$, with $Z_i$ admissible. Then there is a homomorphism

\[ F_k KO(X) \to \text{Shom}_{A(l)_*}(H_* X, H_* ko \langle k \rangle) \]

making (5.1) a pull back diagram.

**Proof.** We show first that it suffices to prove the theorem for admissible $X$.

Assume that $X$ is of the form $X = \bigvee_i X_i$, where $X_i$ is a (possibly infinite) family of spectra for which (5.2) holds. Then (5.2) holds for $X$, since the diagram (5.1) is a direct product of pull back diagrams and hence a pull back diagram itself. We claim that this still is true if we make the weaker assumption that the $ko$-module spectrum $ko \wedge X$ is homotopy equivalent (as module spectrum) to $ko \wedge (\bigvee_i X_i)$. To prove this recall from §2 that $p(k)_*: ko \langle k \rangle \to KO$ is a $ko$-module map. Hence by (2.10) the diagram (5.1) is isomorphic to the diagram

\[ [ko \wedge X, ko \langle k \rangle]_{ko} \xrightarrow{p(k)_*} F_k [ko \wedge X, KO]_{ko} \]

\[ h \downarrow \]

\[ \text{Hom}_{A(l)_*}(H_* ko \wedge X, H_* ko \langle k \rangle) \xrightarrow{S} \text{Shom}_{A(l)_*}(H_* ko \wedge X, H_* ko \langle k \rangle), \]

where $F_k [ko \wedge X, KO]_{ko}$ denotes the image of the map $[ko \wedge X, ko \langle k \rangle]_{ko} \to [ko \wedge X, KO]_{ko}$ induced by $p(k)_*$. This shows that a splitting of the $ko$-module spectrum $ko \wedge X$ induces a direct product decomposition of the diagram (5.1).

Now we assume that (5.2) holds for admissible $X$ and we want to show that it holds for $X = ko \wedge Z$ with $Z$ admissible. The splitting (4.3) of $ko \wedge ko$ shows that $ko \wedge X$ is homotopy equivalent (as $ko$-module spectrum) to $ko \wedge (\bigvee_{i \geq 0} \Sigma^{4i} B(l) \wedge Z)$. It follows from (3.7) that $\Sigma^{4i} B(l)$ and hence $\Sigma^{4i} B(l) \wedge Z$ are admissible spectra. This implies by the above remarks that (5.2) holds for $X = ko \wedge Z$ and hence for a wedge of spectra $X_i$ which are admissible or of the form $X_i = ko \wedge Z_i$, with $Z_i$ admissible.

From now on we assume that $X$ is an admissible spectrum. By definition the maps $p(k)_*$ and $S$ in diagram (5.1) are subjective. Hence it suffices to show

1. $h(\ker p(k)_*) \subseteq \ker S$
2. $h(\ker p(k)_*) \supseteq \ker S$
3. $\ker h \cap \ker p(k)_* = 0$
Property (1) implies that there is a well defined homomorphism making (5.1) a commutative diagram, and it follows from (2) and (3) that this diagram is a pull back square.

Before proving (1), (2) and (3) we first give an alternative description of the kernel of \( p(k)_* \). Recall that \( KO \) and \( ko(k) \) are \( ko \)-module spectra. This makes \( [\Sigma^* X, KO] \) and \( [\Sigma^* X, ko(k)] \) modules over \( \pi_\ast(ko) \). In particular, we can multiply with the Bott element \( \mu \in \pi_\ast(ko) \) (cf. 7.1).

\textbf{Claim.} \( \text{ker} \ p(k)_* = \{ x \in [X, ko(k)]: \mu^d x = 0 \text{ for some } d \} \)

\textbf{Proof.} Consider the following diagram which is commutative since \( p(k)_*: ko(k) \to KO \) is a \( ko \)-module map.

\[
\begin{array}{ccc}
[X, ko(k)] & \xrightarrow{\mu^d} & [\Sigma^{bd} X, ko(k)] \\
p(k)_* \downarrow & & \downarrow p(k)_* \\
[X, KO] & \xrightarrow{\mu^d} & [\Sigma^{bd} X, KO]
\end{array}
\]

The bottom horizontal map is an isomorphism by Bott periodicity. Since \( X \) is bounded below it follows from obstruction theory that the vertical map on the right is an isomorphism for \( d \) sufficiently large. This implies the claim.

To prove (1) and (3) we use the Adams spectral sequence

\[
\text{Ext}_{H_*}^{2i}(H_*X, H_* ko(k)) = [\Sigma^{-s} X, ko(k)].
\]

(5.3)

By the change of rings isomorphism its \( E_2 \)-term is isomorphic to

\[
\text{Ext}_{A(k)}^{2i}(H_*X, H_* ko(k)) = \text{Ext}_{A(k)}^{2i}(Z/2, H_* ko(k) \otimes DH_* X),
\]

where \( DH_* X \) is the \( A(1)_* \)-comodule (Spanier-Whitehead) dual to \( H_* X \). Note that the \( Q_0 \)-homology of \( H_* X \) and \( H_* ko(k) \) is concentrated in dimensions \( \equiv 0 \text{ mod } 4 \). By the Künneth formula the same is true for \( DH_* X \) (since it is the stable inverse of \( H_* X \) with respect to tensor product) and \( H_* ko(k) \otimes DH_* X \). It follows from the classification result (3.5) that

\[
H_* ko(k) \otimes DH_* X \cong \bigoplus M(a_k, b_k, c_k) \otimes F,
\]

(5.4)

where \( a_k \equiv 0 \text{ mod } 4 \) and \( F \) is a free \( A(1)_* \)-comodule. Using Lemma 3.12 we can read off the Ext-groups of \( M(a_k, b_k, c_k) \) from the Tables (3.10) resp. (3.11), including the structure of these Ext-groups as modules over \( \text{Ext}_{A(k)}^{1/2}(Z/2, Z/2) \). We are in particular interested in the action of the periodicity element \( b \in \text{Ext}_{A(k)}^{1/2}(Z/2, Z/2) \). Under the change of rings isomorphism

\[
\text{Ext}_{A(k)}^{1/2}(Z/2, Z/2) \cong \text{Ext}_{H_*}^{1/2}(Z/2, H_* ko)
\]

\( b \) maps to the element in the \( E_2 \)-term of the Adams spectral sequence of \( ko \) corresponding to the Bott element \( \mu \). A glance at the Tables (3.10) and (3.11) leads to the following conclusions

The Adams spectral sequence (5.3) collapses in degrees \( \equiv 1 \text{ mod } 4 \) (cf. (4.6))

(5.5)

Multiplication by \( b \) is injective on \( \text{Ext}_{A(k)}^{4t}(Z/2, H_* ko(k)) \otimes DH_* X \) for \( t - s \equiv 0 \text{ mod } 4, s > 0 \)

(5.6)

The kernel of \( S: \text{Hom}_{A(k)}(H_* X, H_* ko(k)) \to \text{Shom}_{A(k)}(H_* X, H_* ko(k)) \) consists of those elements annihilated by \( b \).

(5.7)
Comments: (5.5) follows since the elements in $E_2^{t,s}$ for $t - s \equiv 1 \mod 4$ are annihilated by $h_0$, while multiplication by $h_0$ is injective on $E_2^{t,s}$ for $t - s = 0 \mod 4, s > 0$. For (5.7) note that we can identify $S$ with the stabilization map $\text{Hom}_{A_1}(\mathbb{Z}/2, M) \to \text{Shom}_{A_1}(\mathbb{Z}/2, M)$ for $M = H_*ko(k) \otimes DH_*X$. Hence it suffices to check that the kernel of the stabilization map equals the annihilator of $b$ for all summands in the direct sum decomposition (5.4).

To prove (1) let $x \in \ker p <k>_*$. Then for some $d$ the product $\mu^d x \in [\Sigma^d X, ko(k)]$ vanishes. By (5.5) all differentials with range $E_2^{t,s}$ are trivial for $t - s \equiv 0 \mod 4$. This implies that the product $b^d h(x) \in \text{Ext}_{A_1}(\mathbb{Z}/2, ko(k))$ (which corresponds to $\mu^d x$ in the $E_2$-term of the spectral sequence) is trivial. Hence $bh(x) = 0$ by (5.6) and $h(x)$ is in the kernel of $S$ by (5.7).

To prove (3) we note that (5.5) and (5.6) imply that multiplication by $\mu$ is injective when restricted to elements of Adams filtration $\geq 1$, i.e. when restricted for ker $h$. Thus ker $h \cap \ker p <k>_* = 0$.

To prove (2) let $g$ be an element in the kernel of $S$; i.e. $g$ is an $A_1$-comodule homomorphism which factors in the form

$$H_*X \xrightarrow{g_1} F \xrightarrow{g_2} H_*ko(k),$$

where $F$ is a free $A_1$-comodule. These $A_1$-comodule maps induce $A_1$-comodule maps

$$H_*X \xrightarrow{\hat{g}_1} A_1 \boxtimes_{A_1} F \xrightarrow{\hat{g}_2} A_1 \boxtimes_{A_1} H_*ko(k) \cong H_*ko(k),$$

where $\hat{g}_1$ is the $A_1$-comodule extension of $g_1$ and $\hat{g}_2 = id \boxtimes g_2$. Note that $A_1 \boxtimes_{A_1} F$ is a free $A_1$-comodule and hence there is a (generalized) $\mathbb{Z}/2$-Eilenberg-MacLane spectrum $HV$ with $H_*HV \cong A_1 \boxtimes_{A_1} F$ and maps

$$X \xrightarrow{g_1} HV \xrightarrow{g_2} ko(k),$$

such that $(G_1)_* = g_1$ (this is obvious for $G_1$ and follows from [12] for $G_2$). Then $h(G) = g$, where $G$ is the composition of $G_1$ and $G_2$. We note that $p <k>* (G) = 0$ since the (real) $K$-theory of Eilenberg–MacLane spectra vanishes.

\section{Construction of the Splitting Map $s$}

In this section we prove Theorem B of the introduction; i.e. we construct a map $s: ko \to M Spin$ which is a right inverse of the orientation class $D: M Spin \to ko$ and induces a ring homomorphism in homology. As a warm up we consider first the case of the Thom spectrum $MO$, which is easier since $MO$ is a (generalized) Eilenberg–MacLane spectrum corresponding to a graded $\mathbb{Z}/2$-vector space. Let $H$ be the $\mathbb{Z}/2$-Eilenberg–MacLane spectrum and let $U: MO \to H$ be the orientation class (Thom class). Let $p_i \in H_{2^i-1} MO$ be the image of the coalgebra primitive in $H_{2^i-1} BO$ under the Thom isomorphism (compare [4, Lemma 1]).

\textbf{Proposition 6.1.} There is a unique ring spectrum map $s_H: H \to MO$ with $(s_H)_*(\zeta_i) = p_i$. A map with these properties is a right inverse of $U$.

\textbf{Proof.} The Hurewicz map

$$[H, MO] \to \text{Hom}_{A_1}(H_*H, H_*MO),$$

...
which sends a map $f$ to the induced map $f_*$ in homology is an isomorphism, since $MO$ at the prime 2 is a (generalized) $\mathbb{Z}/2$-Filtenberg–MacLane spectrum. Moreover, $f$ is a ring spectrum map, i.e. the diagram

$$
\begin{array}{ccc}
H \wedge H & \xrightarrow{\mu_H} & H \\
\downarrow{s_H \wedge s_H} & & \downarrow{s_H} \\
MO \wedge MO & \xrightarrow{\mu_{MO}} & MO
\end{array}
$$

is homotopy commutative if and only if the diagram is commutative in homology, i.e. if $f_*$ is an algebra homomorphism.

It is well known that $H_*MO$ is a polynomial algebra $\mathbb{Z}/2[z_1, z_2, \ldots]$ with generators $z_i$ of degree $i$. Pengelley [13, Thm. 2.1] has shown that one can choose $z_2, _j = p_j$ and Brown and Peterson [4, Thm. 2.7] proved

$$
\Psi(p_j) = \sum_{i=0}^{j} \zeta_i \otimes p_{j-i}^{2i-1}.
$$

(6.2)

Comparison with the coproduct (2.1) of $H_*H$ shows that the algebra map

$$
\zeta_i \rightarrow z_{2i-1}
$$

is an $A_*$-comodule map and hence is induced by a map $s_H : H \rightarrow MO$. The composition $U s_H$ is homotopic to the identity, since it induces the homology on homology in degree 0.

Now we turn to the construction of the map $s : ko \rightarrow M Spin$. The homotopy type of the spectrum $M Spin$ was determined by Anderson, Brown and Peterson [2]. To state their result we need some notation. For an oriented real vector bundle $E$ over a space $X$ let $n_j(E) \in KO(X)$ be the $j$-th KO-theory Pontrjagin class of $E$ defined by

$$
\sum_j \pi_j(E) u^j = \bigwedge_i (E - \text{dim } E),
$$

where $u = t/(1 + t)^2$ (cf. [21, p. 303]). It follows directly from the definition that Pontrjagin classes are multiplicative; i.e.

$$
\pi_j(E \oplus E') = \sum_{i=0}^{j} \pi_i(F) \pi_{j-i}(F')
$$

Let $\pi_j \in KO(B Spin)$ be the $j$-th Pontrjagin class of the universal bundle over $B Spin$. Let $B$ be a partition: that is, a possibly empty unordered tuple $\{j_1, \ldots, j_k\}$ of positive integers. Let $n(J)$ be the sum $\sum_{i=1}^{k} j_i$ and let $m(J) = 4n(J)$ for $n(J)$ even and $m(J) = 4n(J) - 2$ for $n(J)$ odd. Let $\pi_J = \pi_{j_1} \ldots \pi_{j_k} \in KO(B Spin)$ and let $\pi' \in KO(M Spin)$ be the image of $\pi_j$ under the Thom isomorphism ($\pi_{e}$ is the multiplicative unit and $\pi_0$ is the KO-theory Thom class).

It follows from the multiplicativity of the $\pi_J$’s that there is a formula describing how the maps $\pi_J$ interact with multiplication on $M Spin$ and $KO$. We follow the description in [7].

Let $P$ be the set of partitions, and let $\mathbb{Z}[P]$ be the set of formal linear combinations. We can make this into a ring by defining a multiplication on $P$ by set union, and then into a Hopf algebra by defining $\Lambda(n) = \sum_{k=0}^{n} \{n - k\} \otimes \{k\}$. Here we use the convention that $\{0\}$ stands for the empty set. Then

$$
\pi^J \mu - \sum \mu_{KO}(\pi^J' \otimes \pi^J'') \quad \text{if} \quad \Delta(J) = \sum J' \otimes J''.
$$

(6.3)

Here $\mu$ (resp. $\mu_{KO}$) is the multiplication map of the ring spectrum $M Spin$ (resp. $KO$).
It turns out that $\pi_J$ and hence $\pi^J$ have (skeletal) filtration $m(J)$ for $1 \notin J$; i.e. $\pi^J$ considered as a map $M Spin \to KO$ factors in the form

$$M Spin \xrightarrow{f^J} ko \langle m(J) \rangle \xrightarrow{\phi(m(J))} KO.$$  \hspace{1cm} (6.4)

We observe that $f^0$ is the Atiyah–Bott–Shapiro orientation $D: M Spin \to ko$. Anderson–Brown–Peterson make a judicious choice of the $f^J$'s, construct a map $f': M Spin \to HV$ into the Eilenberg–MacLane spectrum corresponding to a graded $\mathbb{Z}/2$ vector space $V$ and show that the map

$$f: M Spin \to \prod_{J \notin J} ko \langle m(J) \rangle \times HV$$  \hspace{1cm} (6.5)

given by $f = \prod f^J \times f'$ is a homotopy equivalence.

We want to construct a map $s: ko \to M Spin$ which is a right inverse of the Atiyah–Bott–Shapiro orientation $D: M Spin \to ko$ and which is compatible with the splitting $s_\mathbb{H}$ constructed in (6.1) in the sense that the diagram

$$\begin{array}{ccc}
ko & \xrightarrow{s} & M Spin \\
\downarrow \iota & & \downarrow Mp \\
H & \xrightarrow{\iota_\mathbb{H}} & \mathbb{H}M Spin \\
\end{array}$$  \hspace{1cm} (6.6)

is commutative, where $Mp$ is the map of Thom spectra induced by the natural map $\nu: B Spin \to BO$. Note that $M_{\mathbb{H}*}: \mathbb{H}_* M Spin \to H_\mathbb{H} MO$ is an injective ring homomorphism and hence commutativity of (6.6) implies that $s_*$ is an algebra homomorphism. Hence Theorem C of the introduction follows from the next proposition.

**Proposition 6.7.** There is a map $s: ko \to M Spin$ which is a right inverse of the Atiyah–Bott–Shapiro orientation $D: M Spin \to ko$ which makes the diagram (6.6) commutative.

**Proof.** We claim that there is a unique $A_*$-comodule map $S$ making the following diagram commutative.

$$\begin{array}{ccc}
H_* ko & \xrightarrow{s} & H_* M Spin \\
\downarrow \iota_* & & \downarrow Mp_* \\
H_* H & \xrightarrow{(\iota_\mathbb{H})_*} & H_* \mathbb{H}M Spin \\
\end{array}$$

Uniqueness follows from the injectivity of $M_{\mathbb{H}*}$. The image of $\iota_*$ is the polynomial subalgebra $\mathbb{Z}/2[\xi_1^J, \xi_2^J, \xi_3^J, \xi_4^J, \ldots]$. Recall that $(s_\mathbb{H}^*)_*(\xi_i^J) = p_i$. According to [14] the elements $p_1^J, p_2^J, p_3, p_4, \ldots$ are in the image of $M_{\mathbb{H}*}$ which shows the existence of $S$.

We note that a map $s: ko \to M Spin$ makes the diagram (6.6) commutative if and only if $s_* = S$, since $MO$ is a (generalized) $\mathbb{Z}/2$-Eilenberg–Maclane spectrum. Hence we have to construct a map $s: ko \to M Spin$ which is a right inverse of $D$ such that $s_* = S$.

Using the splitting (6.5) this corresponds to proving the existence of maps $s^J: ko \to ko \langle m(J) \rangle$ and $s^J: ko \to HV$ such that $s^0 = \text{id}$ (this corresponds to $s$ being the right inverse of $D$) and such that $s_*^J = f_*^J S$ and $s_*^J = f_*^J S$. The existence of $s^J$ with these properties is clear since the range of $s^J$ is a (generalized) Eilenberg–Maclane spectrum. Also we see that $s^0 = \text{id}$ satisfies the condition $s_*^0 = f_*^0 S$ since $S(\xi_i^J) = p_i$ and $f_*^0(p_i) = \xi_i^J$ (the latter follows from the formula (6.2)). To prove the existence of the other $s^J$’s it suffices to know that the Hurewicz map

$$h: [ko, ko \langle k \rangle] \to \text{Hom}_{A_*}(H_* ko, H_* ko \langle k \rangle)$$
is surjective for all $k$. Using the splitting (4.3) of the $ko$-module spectrum $ko \wedge ko$ into pieces of the form $ko \wedge \Sigma^4lB(l)$ and the diagram (2.10) this is equivalent to the surjectivity of the Hurewicz homomorphism

$$h : \{\Sigma^4lB(l), ko \langle k \rangle \} \to \text{Hom}_A(H_\ast \Sigma^4lB(l), H_\ast ko \langle k \rangle)$$

(6.8)

for all $k, l$. It follows from (3.6) and (3.7) that the assumptions of Proposition 4.7 are satisfied which implies surjectivity of (6.8).

\section{There is No Ring Spectrum Map $s : ko \to MSpin$}

Assume that $s : ko \to MSpin$ is a ring spectrum map. We'll show that this is enough information to determine the induced map $s_\ast : \pi_\ast(ko) \to \pi_\ast(MSpin)$ modulo some indeterminacy. In particular, it is enough to deduce Lemma 7.2 below. Recall (cf. [15, Thm. 3.1.26]) that

$$\pi_\ast(ko) = \mathbb{Z}[\eta, \omega, \mu]/(2\eta, \eta^3, \eta\omega, \omega^2 - 2^2\mu) \quad \text{with} \quad \eta \in \pi_1, \omega \in \pi_3, \mu \in \pi_8 \quad (7.1)$$

Abusing notation we use the same symbol for the image of $\eta$ (resp. $\omega$ resp. $\mu$) under the map $p \langle 0 \rangle_\ast : \pi_\ast(ko) \to \pi_\ast(KO)$. Recall from §6 that for each partition $J = \{j_1, \ldots, j_k\}$ there are maps $\pi^J : MSpin \to KO$.

**Lemma 7.2.** $(\pi^{[2]}s)_\ast(\mu) = 2^k(\text{odd})\mu$

Recall from (6.4) that $\pi^{[2]}$ factors in the form $MSpin \to ko \langle 8 \rangle \to KO$. Hence the next lemma leads to a contradiction which proves that there is no ring spectrum map $s : ko \to MSpin$.

**Lemma 7.3.** Let $g : ko \to ko \langle 4n \rangle$ be a map and let $e_\ast \in \pi_{4n}(ko)$ be the generator; i.e. $e_\ast = \mu^k$ if $n = 2k$ and $e_\ast = \omega\mu^k$ if $n = 2k + 1$. Then $g_\ast(e_\ast)$ is divisible by $2^{4n-\alpha(n)}$, where $\alpha(n)$ denotes the number of 1's in the 2-adic expansion of $n$.

**Proof of Lemma 7.2.** The ring spectrum map $ko \longrightarrow MSpin \longrightarrow ko$ induces the identity on $\mathbb{Z}/2$ homology, since $i_* : H_\ast ko \to H_\ast ko$ is injective and $Ds$ is homotopic to $i$ (a ring spectrum map from a connected ring spectrum into itself induces the identity on the 0-th cohomology group). It follows from the Anderson–Brown–Peterson splitting (6.5) that $D$ is an 8-equivalence (the bottom cell of $HV$ is in dimension 20) and hence $s$ is a 7-equivalence (2-locally). In particular, $s_\ast(\omega) = k[K]$ for some odd integer $k$, where $K$ is the Kummer surface, a 4-dimensional closed spin manifold with signature 16 whose spin bordism class $[K]$ is a generator of $\Omega^S_{\text{Spin}} \cong \pi_\ast(MSpin) \cong \mathbb{Z}$.

Note that $s_\ast(\omega) = k[K]$ implies $4s_\ast(\mu) = k^2[K \times K]$. Hence to prove the lemma it suffices to show $\pi^{[2]}([K \times K]) = 2^k(\text{odd})\mu$. The formula (6.3) implies

$$(\pi^J)_\ast([M \times N]) = \sum (\pi^J)_\ast([M])(\pi^J)_\ast([N]) \quad \text{if} \quad \Delta(J) = \sum J' \otimes J''$$

for spin manifolds $M, N$. In particular, we get

$$(\pi^{[2]})_\ast([K \times K]) = (\pi^{[1]})_\ast([K])(\pi^{[1]})_\ast([K]), \quad (7.4)$$

since $(\pi^{[2]})([K])$ is trivial for dimensional reasons ($\pi^{[2]}$ factors through $ko \langle 8 \rangle$).

Let $ph : \pi_\ast(KO) = KO(S^4) \to H_\ast(S^4; \mathbb{Q})$ be the Pontrjagin character; i.e. the composition of the complexification map and the Chern character. The Pontrjagin character of the generator $\omega$ can be evaluated on the fundamental class $[S^4] \in H_4(S^4; \mathbb{Z})$ and it is well known that $\langle ph(\omega), [S^4] \rangle = 2$. 

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A standard calculation shows that for a $4n$-dimensional spin manifold $M$

$$\langle ph(\pi^J_*([M])), [S^{4n}] \rangle = \langle A(M) ph(\pi_J(TM)), [M] \rangle,$$

where $\hat{A}(M)$ is the $\hat{A}$-genus of $M$ and, abusing notation, $[M]$ is the bordism class of $M$ in $\Omega^{2n}_{4n} = \pi_{4n}(M, \text{Spin})$ on the left hand side and the fundamental class of $M$ in $H_{4n}(M; \mathbb{Z})$ on the right hand side. In particular, for $J = \{1\}$ we have $\pi_J(TM) = TM - 4n$, where $4n$ is the trivial $4n$-dimensional bundle and we conclude

$$\langle ph((\pi^{(1)}_*)_*([K])), [S^4] \rangle = \left\langle \left( 1 - \frac{p_1(K)}{24} + \ldots \right) (p_1(K) + \ldots), [K] \right\rangle = \langle p_1(K), [K] \rangle = 3 \text{sign}(K) = 2^4 3.$$

Hence $\langle \pi^{(1)}_*([K]) \rangle = 2^3 3^{\omega}$ and by (7.4) $\langle \pi^{(2)}_*([K \times K]) \rangle = 2^6 3^2 \omega^2 = 2^9 3^2 \mu$ which proves the lemma. 

**Proof of Lemma 7.3.** The idea of the proof is to show that the Adams filtration of the element $g_*\langle e_n \rangle$ is sufficiently high. For technical reasons we prefer to work with the 'stable Adams filtration' which is defined as follows. Let $f: X \to Y$ be a map between spectra. We say that $f$ is stably trivial if $f$ factors through a (generalized) $\mathbb{Z}/2$-Eilenberg-MacLane spectrum. We call two maps $f, f'$ stably equivalent if $f - f'$ is stably trivial. We define the stable Adams filtration of $f$ as the maximum of the Adams filtration of maps $f'$ stably equivalent to $f$ and we use the notation $\text{SAF}(f)$. The stable Adams filtration inherits useful properties of the Adams filtration; e.g. if $g: Y \to Z$ is a map then $\text{SAF}(gf) \geq \text{SAF}(g) + \text{SAF}(f)$ or, if $X$ is a finite spectrum with Spanier Whitehead dual $DX$ and $Df: S^0 \to DX \wedge Y$ is the map dual to $f$, then $\text{SAF}(Df) = \text{SAF}(f)$.

First we discuss the Adams filtration of the generator $e_n \in \pi_{4n}(ko)$. The $t_2$-term of the Adams spectral sequence converging to $\pi_*(ko)$ is

$$E^1_{t2} = \text{Ext}_{t2}^t(\mathbb{Z}/2, H_*ko) \cong \text{Ext}_{t2}^t(\mathbb{Z}/2, \mathbb{Z}/2).$$

Lemma 3.12 and a glance at the table (3.10) show that $E^1_{t2} = 0$ for $t - s = 4n$ and $s < 2n + \delta$, where $\delta = 0$ for $n$ even and $\delta = 1$ for $n$ odd. This shows that the Adams filtration of $e_n$ is $2n + \delta$.

**Claim.** The stable Adams filtration of $g$ is greater or equal to $2n - \omega(n) - \delta$.

It follows from the claim that the stable Adams filtration of the composition $g_*\langle e_n \rangle = \alpha \in [S^{4n}, ko\langle 4n \rangle] \cong \mathbb{Z}$ is at least $4n - \omega(n)$. Note that a stably trivial map $f: S^{4n} \to ko\langle 4n \rangle$ is in fact trivial. Hence $g_*\langle e_n \rangle$ has Adams filtration at least $4n - \omega(n)$. Since the generator of $\pi_{4n}(ko\langle 4n \rangle)$ has Adams filtration zero, this implies that $g_*\langle e_n \rangle$ is divisible by $2^{4n - \omega(n)}$.

To prove the claim we note that $g$ factors in the form

$$ko = S^0 \wedge ko \xrightarrow{\eta - 1} ko \wedge ko \xrightarrow{\hat{g}} ko\langle 4n \rangle,$$

where $\hat{g}$ is the $ko$-extension of $g$ (cf. (2.7)). Using the splitting (4.3) of $ko \wedge ko$ we can regard $\hat{g}$ as $ko$-module map

$$\hat{g}: \bigvee_{i \geq 0} ko \wedge \Sigma^{4i}B(1) \to ko\langle 4n \rangle.$$

It follows from (2.6) that $\hat{g} = \bigvee \hat{g}_i$, where the maps $\hat{g}_i$ are $ko$-module extensions of maps
\( g_l : \Sigma^{4l} B(l) \to ko \langle 4n \rangle \). Hence it suffices to show that for all \( l \geq 0 \) the stable Adams filtration of the map \( Dg_l : S^0 \to ko \langle 4n \rangle \wedge D\Sigma^{4l} B(l) \) Spanier–Whitehead dual to \( g_l \) is greater or equal to \( 2n - \alpha(n) - \delta \). Note that \( ko \langle 4n \rangle \wedge D\Sigma^{4l} B(l) \) is a \( ko \)-module spectrum with

\[
H_* ko \langle 4n \rangle \wedge D\Sigma^{4l} B(l) = H_* ko \langle 4n \rangle \otimes H_* D\Sigma^{4l} B(l) \cong M(a, b, c) \oplus F,
\]

where \( F \) is a free \( A(1)_* \)-comodule, \( a = 4(n - l) \), \( b = -(2l - \alpha(l)) + \delta \), and \( c \equiv n - l \mod 2 \) (this follows from (3.6), (3.7) and the stable classification Theorem (3.5)). The decomposition (7.5) induces a decomposition of \( A_\ast \)-comodules

\[
H_* ko \langle 4n \rangle \wedge D\Sigma^{4l} B(l) \cong A_\ast \square_{A(1)_*} M(a, b, c) \oplus A_\ast \square_{A(1)_*} F.
\]

We note that \( A_\ast \square_{A(1)_*} F \) is a free \( A_\ast \)-comodule and by [12] this decomposition is induced by a splitting of spectra

\[
ko \langle 4n \rangle \wedge D\Sigma^{4l} B(l) \sim Z \vee HV,
\]

where \( HV \) is a (generalized) \( Z/2 \)-Eilenberg–MacLane spectrum and \( H_* Z \) is isomorphic to \( A_\ast \square_{A(1)_*} M(a, b, c) \). By Lemma 3.12 the picture of the \( E_2 \)-term of the Adams spectral sequence of \( Z \) is obtained by shifting the pictures of the Table (3.10) (resp. (3.11)) by \( a \) units to the right and \( b \) units down. A glance at the tables shows that \( E_2^{t,s} = 0 \) in degree \( t - s = 0 \) if \( n > 1 \). The same holds for \( n \leq l \) and \( s < 2n - \alpha(n) - \delta \). It follows that

\[
SAF(Dg_l) \geq 2n - \alpha(n) - \delta,
\]

which proves lemma 7.3. \( \square \)

§8. PROOF OF THEOREMS A AND B

For the proof of theorem A we need the following proposition due to Rosenberg based on results of Gromov–Lawson and Schoen–Yau.

*Proposition 8.1* [16, Thm. 2.2]. *Let \( M_1, M_2 \) be closed spin manifolds of dimension \( n \geq 5 \) and let \( g_i : M_i \to B\pi \) be maps into the classifying space of a group \( \pi \). Assume that \( g_i \) induces an isomorphism of fundamental groups, that \( M_2 \) has a metric of positive scalar curvature and that \( (M, g_i) \) for \( i = 1, 2 \) represent the same bordism class in \( \Omega^{spin}_n (B\pi) \). Then \( M_1 \) also admits a metric of positive scalar curvature.*

*Remark.* Our assumptions are slightly weaker than Rosenbergs assumptions. He assumes that both, \( g_1 \) and \( g_2 \) induce isomorphisms on \( \pi_1 \) (this is another way of saying that \( g_i \) is the classifying map of the universal covering of \( M_i \)). An analysis of his proof shows that the assumption on \( g_2 \) is superfluous.

*Proof of Theorem A.* Theorem A follows from the above proposition and Theorem B of the introduction. Recall that according to Theorem B the homomorphism \( \alpha \) induces a homomorphism

\[
\Omega^{spin}_n(X)/T_\alpha(X) \to ko_\alpha(X)
\]

which is an isomorphism localized at the prime 2. If \( X \) is point this is an isomorphism without localizing at 2 [9, Prop. 3.3]. It follows that (8.2) is an isomorphism if \( X \) is the classifying space \( B\pi \) of a finite 2-group. Hence if a bordism class \([M, g] \in \Omega^{spin}_n(B\pi)\) is in the kernel of \( \alpha \) then \((M, g)\) is spin bordant to \((E, fp)\), where \( p : E \to B \) is an \( \mathbb{HP}^2 \)-bundle over a closed spin manifold \( B \) of dimension \( n - 8 \) and \( f \) is a map from \( B \) to \( B\pi \). The total space \( E \) of an \( \mathbb{HP}^2 \)-bundle carries a metric of positive scalar curvature [19] and hence \( M \) carries
a metric of positive scalar curvature by (8.1), provided $g : M \to B\pi$ is the classifying map of the universal covering.

**Proof of Theorem B.** Part (1) of theorem B is a corollary of proposition 2.1 of [19] if we describe $a$ and $T_a(X)$ in homotopy theoretic terms. The homomorphism $a$ is the following composition

$$\Omega^\text{Spin}_n(X) = \pi_n(M\text{Spin} \wedge X_+) \xrightarrow{(D \wedge 1)_*} \pi_n(k_0 \wedge X_+) = k_0 n(X),$$

where $D : M\text{Spin} \to k_0$ is the Atiyah–Bott–Shapiro orientation and we identify $\Omega^\text{Spin}_n(X)$ with $\pi_n(M\text{Spin} \wedge X_+)$ via the Pontrjagin–Thom construction.

The subgroup $T_a(X)$ can be described as the image of a map

$$\Psi : \Omega^\text{Spin}_{n-8}(BG \times X) \to \Omega^\text{Spin}_n(X),$$

where $BG$ is the classifying space of projective symplectic group $G = \text{PSp}(3)$. The homomorphism $\Psi$ is defined as follows. Given a manifold $N$ and maps $f : N \to BG$, $g : N \to X$ let $p : \tilde{N} \to N$ be the pull back of the fibre bundle

$$\mathbb{H}P^2 \to EG \times_G \mathbb{H}P^2 \xrightarrow{\pi} BG$$

via $f$. A spin structure on $N$ induces a spin structure on $\tilde{N}$ and $\Psi$ is defined by mapping the bordism class of $(N, f \times g)$ to the bordism class of $(\tilde{N}, gp)$.

Via Pontrjagin–Thom construction $\Psi$ can be identified with

$$\pi_n(M\text{Spin} \wedge \Sigma^8 BG_+ \wedge X_+) \xrightarrow{(T \wedge 1)_*} \pi_n(M\text{Spin} \wedge X_+),$$

where $T : M\text{Spin} \wedge BG_+ \to M\text{Spin}$ is the map described in [19, §3]. By Proposition 2.1 of [19] the composition of $T$ and $D$ is trivial which implies part (1). To prove part (2) we note that the triviality of $DT$ implies that $T$ factors in the form

$$M\text{Spin} \wedge \Sigma^8 BG_+ \xrightarrow{f} M\text{Spin} \xrightarrow{i} M\text{Spin},$$

where $M\text{Spin}$ is the homotopy fibre of $D$, i.e. $M\text{Spin}$ fits in a cofibre sequence

$$M\text{Spin} \xrightarrow{i} M\text{Spin} \xrightarrow{D} k_0.$$

Part (2) of Theorem B is then a corollary of the following result.

**Proposition 8.3.** Localized at 2 the map $\hat{T}$ is a split surjection of spectra, i.e. there is a splitting $M\text{Spin} \wedge \Sigma^8 BG_+ = Y_1 \vee Y_2$ such that $\hat{T}$ restricted to $Y_1$ is a homotopy equivalence.

**Proof.** As shown in [19, §8] we can identify the map induced by $\hat{T}$ on $\mathbb{Z}/2$-homology with

$$A_\bullet \square_{A(1)_\bullet} (H_\bullet M\text{Spin} \otimes H_\bullet \Sigma^8 BG_+) \xrightarrow{\text{Id} \square \hat{T}_*} A_\bullet \square_{A(1)_\bullet} \ker D_\bullet.$$

According to [19, Prop. 8.5] $\hat{T}_*$ is a split surjection of $A(1)_\bullet$-comodules, i.e. there is an isomorphism of $A(1)_\bullet$-comodules

$$F : N_1 \oplus N_2 \xrightarrow{\sim} H_\bullet M\text{Spin} \otimes H_\bullet \Sigma^8 BG_+$$

such that $\hat{T}_*F$ maps $N_1$ isomorphically onto $\ker D_\bullet$. This proves 8.3 since the following corollary of our main splitting result 4.2 shows that the isomorphism $F$ is realized geometrically.

$\square$
Proposition 8.5. Let $N_i$ be $A(1)_*$-comodules and let

$$F: \bigoplus_i N_i \to H_* M \text{Spin} \otimes H_* \Sigma^8 B G_+$$

be an isomorphism of $A(1)_*$-comodules. Then there are homology $ko$-module spectra $Y_i$ and a homology $ko$-module map

$$f: \bigvee_i Y_i \to M \text{Spin} \wedge \Sigma^8 B G_+$$

such that $f_* = F$.

Proof. According to [19, Cor. 7.6] $H_* M \text{Spin} \otimes H_* \Sigma^8 B G_+$ is a direct sum of $A(1)_*$-comodules $M_k$, where $M_k$ is a suspension of $A(1)_*$ or an 8-fold suspension of one of the following $A(1)_*$-comodules: $\mathbb{Z}/2, \Sigma^{-1} I_*, \Sigma^4 J_*$ or $\Sigma^4 K_*$. Here $K = M(0, 1, 1)$ in the terminology of §3 which is easy to see from the picture of $K_*$ [19, §7] using the classification (3.5).

We note that in this situation the hypotheses of the splitting result (4.2) are satisfied, since the $M_k$'s are admissible, $b(M_k)$ is equal to 0 or 1 for the invertible $M_k$'s and that there are finite spectra $X_k$ with $H_* X_k \cong M_k$ ($A(1)_*$ is realized as the homology of a finite spectrum $X$ by [5, Prop. 2.1]), $I_*$ and $\Sigma^4 J_*$ are realized as quotients of $X$ and $\Sigma^4 K_*$ is realized by the second suspension of complex projective plane $\mathbb{C}P^2$ with a 7-cell attached via a map $S^6 \to \Sigma^2 \mathbb{C}P^2$ of degree 2.

Then (4.2) implies that there is a homotopy equivalence

$$f: \bigvee_k ko \wedge X_k \to M \text{Spin} \wedge \Sigma^8 B G_+$$

which is a homology $ko$-module map such that $f_*: \bigoplus_k M_k \to H_* M \text{Spin} \otimes H_* \Sigma^8 B G_+$ is a given isomorphism $F$. In particular, after decomposing $N_i$ as a direct sum $N_i = \bigoplus_{k \in K} M_k$, we can choose $F$ to be the given isomorphism. The corresponding map $f$ is then the desired homology $ko$-module map if we let $Y_i = \bigvee_{k \in K} ko \wedge X_k$.

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