Cyclic convex bodies and optimization moment problems

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Abstract

We deal with two discrete moment problems: first, deciding when a fixed element of $\mathbb{R}^d$ is the vector of $d$ first moments for some discrete probability distribution on a given interval $[a, b]$ (feasibility moment problem) and, second, maximizing (minimizing) a given linear combination of moments on the set of discrete probability distributions on $[a, b]$ whose $d$ first moments are given (optimization moment problem). These problems are linked with the cyclic body (which is the union of all cyclic polytopes on $[a, b]$). The cyclic polytopes have been extensively studied and their combinatorial and geometric properties are noteworthy. The cyclic body also has interesting geometric properties. We totally determine its facial structure and supporting hyperplanes, and we construct an external representation by means of linear inequality systems whose coefficients are symmetric polynomials depending on parameters. These tools allow us to solve the mentioned moment problems by using linear semi-infinite programming, and we obtain a representation of non-negative polynomials over $[a, b]$ as well.

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1. Introduction

We address the question of when a given point in $\mathbb{R}^d$ is the vector of the first $d$ moments of some discrete probability distribution over an interval $[a, b] \subset \mathbb{R}$, and the related optimization problem of finding the distribution that maximizes a certain linear combination of moments. This problem can be considered as a discrete version of the constrained moment problem stated by Chebyshev and later solved by A. Markov. See Shohat and Tamarkin [12, p. 77] and references therein for a historical review of moment problems.

Let $\mu_1, \mu_2, \ldots, \mu_d$ be the first moments and let $\sum_{k=1}^{p} \alpha_k \mu_k$ be the linear combination to be maximized, $p > d$. A discrete probability distribution over $[a, b]$ is an element $\lambda \in \mathbb{R}^+([a, b])$, that is, a function $\lambda : [a, b] \to \mathbb{R}^+$ which has a finite support, $\text{supp} \lambda : = \{ t \in [a, b] : \lambda_t \neq 0 \}$, and $\sum_{t \in [a, b]} \lambda_t = 1$. Since $\mu_k = \sum_{t \in [a, b]} \lambda_t t^k$, the problem can be formulated as a linear semi-infinite programming problem,

$$ (D_1) \quad \sup \sum_{t \in [a, b]} \lambda_t b_t $$

s.t. $\sum_{t \in [a, b]} \lambda_t (1, t, \ldots, t^d) = (1, \mu_1, \ldots, \mu_d)$, $\lambda \in \mathbb{R}^+([a, b])$, (1)

where $b_t := \sum_{k=1}^{p} \alpha_k t^k$. This program is called a dual problem in the Haar sense [2]. A discussion of the numerical treatment of $(D_1)$ by means of non-linear entropy optimization can be found in Tagliani [13].

Denoting by $A_1$ the feasible set of $(D_1)$, by $\mu := (\mu_1, \mu_2, \ldots, \mu_d) \in \mathbb{R}^d$ the vector of moments, and by $\Gamma_{[a, b]}$ the convex hull of $\{(t, t^2, \ldots, t^d) : t \in [a, b]\}$, that we call cyclic body, it is obvious that

$$ A_1 \neq \emptyset \iff \mu \in \Gamma_{[a, b]}. $$

Moreover, as we show in Theorem 3.8,

$$ |A_1| = 1 \iff \mu \in \text{bd } \Gamma_{[a, b]}, $$

in which case the optimization problem is trivial.

The associated primal program of $(D_1)$ is

$$ (P_1) \quad \inf \sum_{i=1}^{d} \mu_i x_i $$

s.t. $\sum_{i=1}^{d} x_i t^i \geq b_t$, $t \in [a, b]$, $x \in \mathbb{R}^{d+1}$. (4)

In general, for a duality pair $((P), (D))$ of linear semi-infinite programs, the optimal values satisfy $v(D) \leq v(P)$, but a positive duality gap $\delta(P, D) := v(P) - v(D)$ can occur, even though the values of both programs are finite (see Goberna and Lopez [5, p. 50]). However, the constraint system of $(P_1)$ is continuous on $[a, b]$ (polynomial of degree $p := \deg b_t$) and the point $(1 + \max\{b_t : t \in [a, b]\}, 0, \ldots, 0) \in \mathbb{R}^{d+1}$ satisfies all the constraints with a positive slack (this is a Slater point). Therefore, this system has the so-called Farkas–Minkowski property, $(P_1)$ is discretizable and $\delta(P_1, D_1) = 0$, [5, Theorems 5.3 and 8.2]. Moreover, if $\mu \in \text{int } \Gamma_{[a, b]}$, both
programs are solvable, i.e., there exist optimal solutions of \((P_1)\) and \((D_1)\), [5, Theorem 9.8 and Corollary 9.3.1(iv)]. Thus, an approximating optimal solution of \((P_1)\) can be found by means of different linear semi-infinite programming methods: discretization [4], local reduction, exchange, and simplex-like methods, among others (see [5, Part IV]). Now, the active inequalities of the constraint system of \((P_1)\) at an optimal solution \(x^*\) are at most \([p/2]+1\), due to the fact that the slackness function at \(x^*\) is a polynomial of degree \(p\), nonnegative over \([a, b]\). Then, by the complementary slackness property, every optimal solution \(\lambda^*\) of \((D_1)\) satisfies \(|\text{supp}(\lambda^*)| \leq [p/2]+1\), and the constraint system of \((D_1)\), which has \(d+1\) linear equations and at most \([p/2]+1\) unknowns, is consistent.

On the other hand, given real numbers \(t_1 < t_2 < \cdots < t_n\), the convex hull of \((t_1^2, t_2^2, \ldots, t_n^2) : i = 1, 2, \ldots, n\) is a cyclic polytope \([1,14]\), and its combinatorial and geometric properties are noteworthy. The cyclic body, which is the union of all cyclic polytopes on \([a, b]\), also has interesting geometric properties. The knowledge of the facial structure of the cyclic body and the characterization of its supporting hyperplanes allow us to construct an external representation by means of linear semi-infinite inequality systems whose coefficients are elementary symmetric polynomials depending on parameters. That representation permits to solve the feasibility moment problem \((2)\), together with the uniqueness question \((3)\), by finding the global minimum of a pair of multivariate polynomials on a compact interval in \(\mathbb{R}^n\) (see [8] for a specific numerical method).

This paper is organized as follows. In Section 2 the notation and some preliminary results are given. In Section 3 the facial structure, supporting hyperplanes, and boundary points of the cyclic body are studied. In Section 4 an external representation of \(\Gamma_{[a,b]}\) is given. Consequently, a necessary and sufficient condition for the feasibility moment problem and a representation of non-negative polynomials on an interval, are stated. Finally, in Section 5 illustrative optimization moment problems are solved in the way outlined in this introduction.

2. Notation and preliminary results

For general concepts (e.g. those of proper and exposed faces of a convex set) the main references are Brøndsted [1] and Schneider [10]. We denote by \([a]\) the integer part of a number \(a\). If \(y \in \mathbb{R}^d\), \(\overline{y}:=[1 \ y]^T \in \mathbb{R}^{d+1}\), and for \(\emptyset \neq M \subset \mathbb{R}^d\), \(M - y := \{ x \in \mathbb{R}^d : x + y \in M \}\) and \(M\setminus\{y\}:= \{ x \in \mathbb{R}^d : x \in M, x \neq y \}\). The convex, affine and conical convex (containing the origin) hulls of \(M\) are denoted by \(\text{conv} M\), \(\text{aff} M\), and \(\text{cone} M\), respectively. In addition, \(\text{cl} M\), \(\text{int} M\), \(\text{bd} M\), and \(\text{ri} M\) denote the closure, the interior, the boundary, and the relative interior (with regard to \(\text{aff} M\)) of \(M\). For \(C \neq \emptyset\) convex and closed, the dimension of \(C\) is \(\dim C := \dim \text{aff} C\).

For \(z \in \mathbb{C}\), \(\Gamma_z := (z, z^2, \ldots, z^d) \in \mathbb{C}^d, d \geq 2\), and if \(t \in \mathbb{R}\), \(\Gamma_t\) is the moment curve. For \(X \subset \mathbb{R}\), \(\Gamma(X):= \{ \Gamma_t \in \mathbb{R}^d : t \in X \}\) whereas \(\Gamma_X := \text{conv} \Gamma(X)\). By \(T := (t_1, t_2, \ldots, t_n)\) we denote a finite linearly ordered set of real numbers \(t_1 < t_2 < \cdots < t_n\), and \(\Gamma_T\) is a cyclic polytope. It is known that any \(d+1\) points in \(\Gamma(\mathbb{R})\) are always affinely independent \([1,14]\), so that \(\Gamma_T\) is full-dimensional if \(|T| \geq d+1\). Also, for any linearly ordered sets \(T\) and \(S\) with \(|T| = |S| \geq d+1\), \(\Gamma_T\) and \(\Gamma_S\) are combinatorially equivalent. For a non-empty subset \(S\) of a linearly ordered set \(T\), by a component of \(S\) we mean a non-empty subset \(U\) of \(S\) of the form \(U = \{ t_j, t_{j+1}, \ldots, t_k \}\) such that \(t_{j-1} \notin S\) (if \(j > 1\)) and \(t_{k+1} \notin S\) (if \(k < d\)). A component \(U\) is called a proper component if \(t_1 \notin U\) and \(t_d \notin U\). A component containing an odd number of points is called an odd component. We denote \(\text{odd}(S, T)\) the number of odd proper component of \(S\) in \(T\). The following result generalizes Gale’s evenness condition [3], regarding the facets of a cyclic polytope.
Theorem 2.1 (Shepard [11]). Let \( T \) be a linearly ordered set, \(|T| \geq d + 1\), let \( j \in \mathbb{Z} \) such that \( 0 \leq j \leq d - 1 \), and let \( S \subset T \) with \(|S| = j + 1\). Then, \( \Gamma_S \) is a \( j \)-face of \( \Gamma_T \) if and only if \( \text{odd}(S, T) \leq d - j - 1 \).

Theorem 2.1 can be proven directly from Lemma 2.3, whose meaning is illustrated in Example 2.2.

Example 2.2 (Generalized Vandermonde determinant). If \( z_1, z_2, z_3 \in \mathbb{C} \),

\[
\begin{bmatrix}
  1 & 1 & 0 & 0 & 1 & 0 \\
  z_1 & z_2 & 1 & 0 & z_3 & 1 \\
  z_1^2 & z_2^2 & 2z_2 & 2 & z_3^2 & 2z_3 \\
  z_1^3 & z_2^3 & 3z_2^2 & 6z_2 & z_3^3 & 3z_3^2 \\
  z_1^4 & z_2^4 & 4z_2^3 & 12z_2^2 & z_3^4 & 4z_3^3 \\
  z_1^5 & z_2^5 & 5z_2^4 & 20z_2^3 & z_3^5 & 5z_3^4 \\
\end{bmatrix}
\]

\[
= (z_2 - z_1)^3(z_3 - z_1)^2(z_3 - z_2)^6.
\]

Lemma 2.3. For \( i = 1, \ldots, m \), let \( z_i \in \mathbb{C} \), and let \( n_i \) be positive integers such that \( \sum_{i=1}^{m} n_i = d + 1 \). Denote by \( C_i := [\bar{T}_{z_i} \bar{T}_{z_i}^{(2)} \cdots \bar{T}_{z_i}^{(n_i-1)}] \) the \((d + 1) \times n_i\) matrix whose columns are the derivatives of \( \bar{T}_{z_i} \) at \( z_i \). Then

\[
\det[C_1 C_2 \cdots C_m] = \prod_{1 \leq i < j \leq m} (z_j - z_i)^{n_i n_j}.
\]  

Proof. We prove it by induction on \( n = \max\{n_i, i = 1, 2, \ldots, m\} \). Let \( S \) denote the left hand side of (5). For \( n = 1 \), \( S \) is the Vandermonde determinant. Suppose \( n_1 = n + 1 \leq d + 1 \) and \( n_i \leq n \) for \( i = 2, 3, \ldots, m \). Because of well-known properties of the determinant function, for \( 0 \neq z \in \mathbb{C} \),

\[
S = \frac{1}{z^n} \det \left[ \bar{T}_{z_1} \bar{T}_{z_1}^{(2)} \cdots \bar{T}_{z_1}^{(n)} \right] - \frac{1}{z^n} \det \left[ \bar{T}_{z_1}^{(2)} \bar{T}_{z_1}^{(3)} \cdots \bar{T}_{z_1}^{(n+1)} \right] + \cdots
\]

and the Taylor expansion around \( z_1 \) yields

\[
S = \frac{1}{z^n} \det \left[ \bar{T}_{z_1} \bar{T}_{z_1}^{(2)} \cdots \bar{T}_{z_1}^{(n+1)} - \sum_{i=0}^{n+1} \bar{T}_{z_1}^{(i)} \frac{z^n}{z^n} \bar{T}_{z_2} \cdots \right]
\]

\[
= \frac{1}{z^n} \det \left[ \bar{T}_{z_1} \bar{T}_{z_1}^{(2)} \cdots \bar{T}_{z_1}^{(n+1)} - \sum_{i=0}^{n+1} \bar{T}_{z_1}^{(i)} \frac{z^{n+1}}{n+1} \bar{T}_{z_2} \cdots \right]
\]

\[
= \frac{1}{z^n} \det \left[ \bar{T}_{z_1} \bar{T}_{z_1}^{(2)} \cdots \bar{T}_{z_1}^{(n+1)} - \sum_{i=0}^{n+1} \bar{T}_{z_1}^{(i)} \frac{z^{n+1}}{n+1} \bar{T}_{z_2} \cdots \right] - z \det A(z).
\]  

(6)
By induction hypothesis, the first term in (6) is equal to
\[
\frac{(z_1 + z - z_1)}{z^n} \prod_{1 < j \leq m} (z_j - z_1)^{n_j} (z_1 - z)^{n_1} \prod_{2 \leq i < j \leq m} (z_j - z_i)^{n_j n_i},
\]
whereas \( z \) \( \det A(z) \to 0 \) when \( z \to 0 \). Therefore,
\[
S = \prod_{1 < j \leq m} (z_j - z_1)^{n_j (n+1)} \prod_{2 \leq i < j \leq m} (z_j - z_i)^{n_j n_i}.
\]

Now we proceed by induction on \( k \), being the hypothesis that if the two following conditions hold,
\[
n_j \leq n + 1 \text{ for } j \leq k,
\]
and
\[
n_j \leq n \text{ for } j > k,
\]
then the result is true. □

**Lemma 2.4.** Let \( n_1, n_2, \ldots, n_m \) be positive integers such that \( \sum_{i=1}^{m} n_i = d + 1 \) and let real numbers \( t_1 < t_2 < \cdots < t_m \). Then
\[
A := \left\{ \sum_{j=0}^{n_i} p_i t_i^{(j)} : p_i = 0, 1, \ldots, n_i - 1, \ i = 1, 2, \ldots, m \right\}
\]
consists of \( d + 1 \) affinely independent points. Consequently, the subsets of \( A \) containing \( d \) points, determine hyperplanes which are linearly independent.

**Proof.** Let \( \overline{A} := \{ \overline{x} \in \mathbb{R}^{d+1} : x \in A \} \) and let \( M \) be the matrix whose columns are the elements of \( \overline{A} \). For any \( i \) such that \( n_i > 1 \), due to properties of the determinant function, we have
\[
\begin{align*}
\det M &= \det \left[ \cdots T_{t_i} T_{t_i} + \sum_{j=0}^{n_i-2} T_{t_i}^{(j)} \sum_{j=0}^{n_i-2} T_{t_i}^{(j)} + \sum_{j=0}^{n_i-2} T_{t_i}^{(n_i-1)} \cdots \right] \\
&= \det \left[ \cdots T_{t_i} T_{t_i} + \sum_{j=0}^{n_i-2} T_{t_i}^{(j)} T_{t_i}^{(n_i-1)} \cdots \right].
\end{align*}
\]

Now, if \( n_i > 2 \) we proceed again in the same form, to finally obtain
\[
\det M = \prod_{1 \leq i < j \leq m} (t_j - t_i)^{n_j n_i} \neq 0,
\]
using Lemma 2.3. Therefore, \( \overline{A} \) is a linearly independent set, then \( A \) is affinely independent, and so, conv \( A \) is a \( d \)- simplex whose facets obviously determine linearly independent hyperplanes. □

Now we fix some notation and recall some results. The set of extreme points and the set of exposed extreme points of a closed convex set \( C \) are denoted by \( \text{ext } C \) and \( \text{exp } C \), respectively. By a theorem of Minkowski [10, Corollary 1.4.5], if \( C \) is a compact convex set then \( C = \text{conv } \text{ext } C \). In this case,
\[
C = \text{cl conv } \text{exp } C
\]
(7)
by Straszewicz’s theorem [10, Theorem 1.4.7]. For a convex cone \( K \) (always \( 0 \in K \)) the **linearity subspace** of \( K \), denoted by \( \text{lin } K \), is the maximum linear space contained in \( K \). The **feasible directions cone** for \( C \) at \( x \in \text{bd } C \) is \( D(C, x) := \text{cone}(C - x) \), and the **support cone** for \( C \) at \( x \) is \( S(C, x) := \text{cl } D(C, x) \). For \( \overline{x} \in \text{bd } C \) and \( 0 \neq y \in \mathbb{R}^d \), the linear function \( H_{(\overline{x}, y)}(x) := \langle x, y \rangle - \langle \overline{x}, y \rangle \) **supports** \( C \) at \( \overline{x} \) if \( H_{(\overline{x}, y)}(C) \geq 0 \). In this case,
are supporting hyperplane and half-space for $C$ at $\tilde{x}$, respectively.

The (positive) polar set of $M \subseteq \mathbb{R}^d$, $M \neq \emptyset$, is defined as

$$M^o := \{ y \in \mathbb{R}^d : \langle x, y \rangle \geq -1 \text{ for all } x \in M \}.$$ 

Obviously, $M^o$ is always convex and closed, and $0 \in M^o$. If $C$ is convex, closed and $0 \in C$, then $C^{oo} = C$ and we have

$$0 \in \text{int } C \iff C^o \text{ is bounded},$$

a statement where $C$ and $C^o$ can be interchanged. By a convex body we mean a closed convex set $C \subseteq \mathbb{R}^d$ such that $0 \in \text{int } C$. In this case, given a non-empty convex set $F \subseteq \text{bd } C$, its conjugate set is

$$F^\Delta := \{ y \in C^o : \langle x, y \rangle = -1 \text{ for all } x \in F \}.$$ 

In fact, $F^\Delta$ is a proper exposed face of $C^o$. This $\Delta$-operation inverts the inclusion and $F^{\Delta\Delta} = F$ if and only if $F$ is a proper exposed face of $C$, [7, Theorem 2.4].

Now, we recall some results which involve new concepts. The supporting hyperplane $H^0_{(\tilde{x}, y)}$ is said to be tangent [9] to a convex body $C$ at $\tilde{x} \in \text{bd } C$, if $y$ is a unique element of $\mathbb{R}^d$ such that $\langle \tilde{x}, y \rangle = -1$ (it is called a regular hyperplane in [6]). In this case, $\tilde{x}$ is called a tangency point of $\text{bd } C$ (a regular or smooth point in Schneider [10, p. 73]), and we denote by $\Omega(C)$ the set of tangency points. Further, the tangent hyperplane and half-space at $\tilde{x} \in \Omega(C)$ are denoted by $H^0_{\tilde{x}}$ and $H^+_{\tilde{x}}$. We also say that $H^0_{\tilde{x}} \cap C$ is a regular face of $C$. Thus, a proper exposed face $F$ of $C$ is regular if and only if

$$\dim \text{ lin } S(C, x) = d - 1$$

holds for all (any) $x \in \text{ri } F$. According to Theorem 1 in [9], given a convex body $C$ the following assertions are equivalent to each other:

$$C = \bigcap_{\tilde{x} \in \Omega(C)} H^+_{\tilde{x}},$$

$$C^o = \text{cl conv exp } C^o.$$ 

As a consequence of (7) and (8), (11) and (10) are representations of $C^o$ and $C$, respectively. Moreover, there is a one-to-one correspondence between regular faces of $C$ and points in $\text{exp } C^o$, via $\Delta$-conjugation (see [6, Lemma 4.1]). Therefore, a regular face of $C$ is a maximal proper face. Note that for a full-dimensional closed convex set $C$, with or without $0 \in \text{int } C$, property (9), representation (10), and the property of the regular faces being maximal, remain true. The next lemma summarizes the preceding results.

**Lemma 2.5.** For a closed convex set $C \subseteq \mathbb{R}^d$, $\dim C = d$, the following statements hold:

(i) $C$ is the intersection of all half-spaces which are tangent to $C$.

(ii) A proper exposed face $F$ of $C$ is regular if and only if $\dim \text{ lin } S(C, x) = d - 1$ for all (any) $x \in \text{ri } F$.

(iii) If $F$ is a regular face of $C$ then $F$ is a maximal proper exposed face.
3. The boundary of the cyclic body

In this section we consider a proper interval \([a, b]\) and its corresponding cyclic body \(\Gamma_{[a,b]}\) in \(\mathbb{R}^d\), which is compact, convex and full-dimensional.

**Proposition 3.1.** The set of extreme points of \(\Gamma_{[a,b]}\) is \(\Gamma([a, b])\).

**Proof.** It is clear that \(\text{ext}\ \Gamma_{[a,b]} \subset \Gamma([a, b])\). Let \(t^* \in [a, b]\), \(y^* := \Gamma_{t^*}\) and suppose \(y^* \in [z, w]\) with \(z, w \in \Gamma_{[a,b]}\). We shall prove \(y^* = z = w\). By Carathéodory’s theorem, \(z\) and \(w\) are convex combinations of \(d + 1\) points in \(\Gamma([a, b])\). Thus, \(z = \sum_{i=1}^{d+1} \lambda_i \Gamma_{t_i}\) and \(w = \sum_{i=1}^{d+1} \mu_i \Gamma_{s_i}\). If \(W := \{t^*\} \cup \{t_1, s_1, t_2, s_2, \ldots, t_{d+1}, s_{d+1}\}\), the cyclic polytope \(\Gamma_W\) contains the points \(z\), \(w\) and \(y^*\). In addition, \(y^* \in \text{ext} \Gamma_W\) by Theorem 2.1. Therefore, \(y^* = z = w\), and thus, \(y^*\) is an extreme point of \(\Gamma_{[a,b]}\). □

**Proposition 3.2.** If \(F\) is a proper face of \(\Gamma_{[a,b]}\) then \(F\) is a simplex.

**Proof.** Since \(F\) is a proper face of \(\Gamma_{[a,b]}\), \(F \neq \emptyset\) is a compact convex set and \(j := \dim F < d\). By Minkowski’s theorem, \(F = \text{conv} \text{ext} F\), and \(\text{ext} F \subset \Gamma([a, b])\) due to Theorem 5.2 in [1], and Proposition 3.1. Let \(\{x^1, x^2, \ldots, x^{j+1}\} \subset \text{ext} F\) be affinely independent and suppose that \(|\text{ext} F| > j + 1\). Then, adding another point \(x^0 \in \text{ext} F\), the set \(\{x^0, x^1, \ldots, x^{j+1}\} \subset \Gamma([a, b])\) is affinely dependent, contradicting the inequality \(j < d\). Therefore, \(F = \text{conv}\{x^1, x^2, \ldots, x^{j+1}\}\) is a \(j\)-simplex. □

**Lemma 3.3.** Let \(T \subset [a, b]\) be a linearly ordered set and let \(\Gamma_T\) be a proper face of \(\Gamma_{[a,b]}\). For any finite set \(S\) such that \(T \subset S \subset [a, b]\) with \(|S| \geq d + 1\), \(\Gamma_T\) is a proper face of \(\Gamma_S\).

**Proof.** Let \(x \in \Gamma_T\) and suppose \(x \in [z, w]\) with \(z, w \in \Gamma_S\). Because of \(\Gamma_S \subset \Gamma_{[a,b]}\) we have \(z, w \in \Gamma_{[a,b]}\), and \([z, w] \subset \Gamma_T\) since \(\Gamma_T\) is a face of \(\Gamma_{[a,b]}\). As \(0 \leq q \leq d = \dim \Gamma_S\), \(\Gamma_T\) is a proper face of \(\Gamma_S\). □

**Proposition 3.4.** If \(F\) is a proper \(j\)-face of \(\Gamma_{[a,b]}\) then \(j \leq \frac{d}{2}\).

**Proof.** By Proposition 3.2, \(F = \Gamma_T\) for \(T = (t_1, t_2, \ldots, t_{j+1}) \subset [a, b]\), and we may assume \(j > 1\). We complete \(T\) of the form \(U := (t_1, s_1, \ldots, t_j, s_j, t_{j+1})\) to obtain \(\text{odd}(T, U) = j - 1\). By Lemma 3.3, \(F\) is a face of \(\Gamma_U\). Then \(\text{odd}(T, U) \leq d - j - 1\) according to Theorem 2.1, so that \(j \leq \frac{d}{2}\). □

**Theorem 3.5.** Let \(T := (t_1, t_2, \ldots, t_{j+1}) \subset [a, b]\) be a linearly ordered set. Then \(\Gamma_T\) is a \(j\)-face of \(\Gamma_{[a,b]}\) if any of the following conditions hold:

(i) \(d\) is even, \(j = \frac{d}{2}\), \(t_1 = a\), and \(t_{j+1} = b\),

(ii) \(d\) is odd, \(j = \frac{d+1}{2}\), and either \(t_1 = a\) or \(t_{j+1} = b\),

(iii) \(j < \frac{d-1}{2}\).

Moreover, these are all the proper faces.
**Proof.** Let \( y^* \in \Gamma_T \) and suppose \( y^* \in [z,w] \) with \( z, w \in \Gamma_{[a,b]} \). We shall prove \([z, w] \subset \Gamma_T \). By Corollary 5.11 in [1] and Proposition 3.1, \( z \) and \( w \) are convex combinations of \( d + 1 \) points of \( \Gamma([a, b]) \). Thus,

\[
z = \sum_{i=1}^{d+1} \lambda_i z_i, \quad w = \sum_{i=1}^{d+1} \mu_i \Gamma_{x_i}, \quad \text{and} \quad y^* = \sum_{i=1}^{j+1} \alpha_i \Gamma_{t_i}.
\]

Let \( W := T \cup \{s_1, u_1, s_2, u_2, \ldots, s_{d+1}, u_{d+1}\} \). Considering \( W \) as a linearly ordered set, the cyclic polytope \( \Gamma_W \) contains the points \( z, w \) and \( y^* \), and

\[
\text{odd}(T, W) \leq \begin{cases} 
j - 1 & \text{if } j = d/2, \\
j & \text{if } j = (d - 1)/2, \\
j + 1 & \text{if } j < (d - 1)/2.
\end{cases}
\]

In the three cases the condition of Theorem 2.1 holds, so \( \Gamma_T \) is a face of \( \Gamma_W \). Hence, \([z, w] \subset \Gamma_T \), and thus, \( \Gamma_T \) is a face of \( \Gamma_{[a,b]} \).

Now we consider the existence of other proper faces. Because of Propositions 3.2 and 3.4, only two cases are possible:

- Case 1. \( d \) is even, \( j = d/2 \), and either \( a < t_1 \) or \( b < t_{j+1} \),
- Case 2. \( d \) is odd, \( j = \frac{d-1}{2} \), \( a < t_1 \), and \( t_{j+1} < b \).

In the first case, suppose \( \Gamma_T \) is a face of \( \Gamma_{[a,b]} \) and \( a < t_1 \). Completing the linearly ordered set \( T \) of the form \( U := (a, t_1, s_1, \ldots, t_j, s_j, t_{j+1}) \) we obtain \( \text{odd}(T, U) = d/2 \). By Lemma 3.3 and Theorem 2.1, \( d/2 \leq d - d/2 - 1 \) must hold, a contradiction which also occurs when \( b < t_{j+1} \). In the second case, when completing \( T \) of the form \( (a, t_1, s_1, \ldots, t_j, s_j, t_{j+1}, b) \), a contradiction also occurs. Therefore, every proper face of \( \Gamma_{[a,b]} \) belongs to one of the three kinds of considered faces. □

**Theorem 3.6.** Let \( a, b \) and \( t_1, t_2, \ldots, t_m \) be real numbers, \( a < b \), and let \( n_1, n_2, \ldots, n_m \) be positive integers such that \( n_i \) is even for every \( i \) with \( a < t_i < b \), and setting \( n := \sum_{i=1}^{m} n_i \), \( d - n \) is even. Set \( q := \frac{d-n}{2} \) and let \( w_i, \overline{w}_i \in \mathbb{C} \setminus \mathbb{R}, i = 1, 2, \ldots, q \). Due to Vieta’s identities, if \( (s_1, s_2, \ldots, s_d) \) is a list which contains \( n_i \) times the point \( t_i \), for \( i = 1, 2, \ldots, m \), and \( w_i, \overline{w}_i \) for \( i = 1, 2, \ldots, q \) as well, then the elementary symmetric polynomials

\[
\sigma_0 := 1 \\
\sigma_1 := s_1 + s_2 + \cdots + s_d \\
\sigma_2 := s_1s_2 + s_1s_3 + \cdots + s_{d-1}s_d \\
\vdots \\
\sigma_d := s_1s_2 \cdots s_d,
\]

are the (real) coefficients of the polynomial

\[
f(t) := \sum_{j=0}^{d} (-1)^j \sigma_{d-j} t^j = \prod_{i=1}^{d} (s_i - t) \tag{12}
\]

Set \( x_0 := 1 \) and let \( H_f : \mathbb{R}^d \to \mathbb{R} \) be the (non-null) linear function

\[
H_f(x) := \sum_{j=0}^{d} (-1)^j \sigma_{d-j} x_j. \tag{13}
\]

Denoting by \( z_i \) and \( n_i, i = 1, 2, \ldots, m + 2q \), the roots of \( f(t) \) and their multiplicities, let \( H : \mathbb{R}^d \to \mathbb{C} \) be the linear function
Proof. By Theorem 3.5, (13) yields

$$H_f = H.$$  \hspace{1cm} (15)

Moreover, if \( p := \sum_{t_i \leq a} n_i \) then \((-1)^pf(t) \geq 0\) for all \( t \in [a,b] \), and \((-1)^p H_f(x) \geq 0\) for all \( x \in \Gamma_{[a,b]} \). In addition, the hyperplane \( H_f^0 \) supports \( \Gamma_{[a,b]} \) at the set \( \text{conv}\{\Gamma_{t_i} : a \leq t_i \leq b\} \).

**Proof.** Applying (13) to \( \Gamma_t \) yields

$$H_f(\Gamma_t) = f(t) = \prod_{1 \leq i \leq m+2q} (z_i - t)^{n_i}. \hspace{1cm} (16)$$

Multiplying both sides in (16) by

$$K := \prod_{1 \leq i < j \leq m+2q} (z_j - z_i)^{\frac{n_j}{n_i}},$$

and using Lemma 2.3 and (14), we get \( KH_f(\Gamma_t) = H(\Gamma_t) \). Since any \( d + 1 \) points in \( \Gamma(\mathbb{R}) \) are affinely independent, we conclude (15). Now, by (16),

$$f(t) = \prod_{t_i \leq a} (t_i - t)^{n_i} \prod_{a < t_i < b} (t_i - t)^{n_i} \prod_{b \leq t_i} (t_i - t)^{n_i} \prod_{i=1}^q (w_i - t)(w_i - t),$$

and because of this, \((-1)^pf(t) \geq 0\) holds for all \( t \in [a,b] \). Also, \((-1)^p H_f(x) \geq 0\) for all \( x \in \Gamma([a,b]) \), and \( H_f(\Gamma(t)) = 0 \) if and only if \( t \in \{t_1, \ldots, t_m\} \). Therefore, \( H_f^0 \cap \Gamma([a,b]) = \{\Gamma_{t_i} : a \leq t_i \leq b\} \). Taking convex hulls, the assertions are easily obtained. \( \square \)

**Corollary 3.7.** If \( F \) is a proper face of \( \Gamma_{[a,b]} \) then \( F \) is exposed.

**Proof.** By Theorem 3.5, \( F = \Gamma_T \) for some linearly ordered set \( T = (t_1, t_2, \ldots, t_{j+1}) \subset [a,b] \)
and \( 0 \leq j \leq d/2 \). For \( i = 1, 2, \ldots, j + 1 \) set the multiplicities as \( n_i := 2 \) if \( a < t_i < b \), \( n_i := 1 \) otherwise, and let \( m := d - \sum_{i=1}^{j+1} n_i \). Select \( m - (j + 1) \) real numbers such that \( b < t_{j+2} < \cdots < t_m \), and set the multiplicities as \( n_i := 1 \) for \( i = j + 2, \ldots, m \). Then, the hyperplane \( H_f^0 \) of Theorem 3.6 supports \( \Gamma_{[a,b]} \) at \( F \), so that \( F \) is exposed. \( \square \)

**Theorem 3.8.** For \( x \in \Gamma_{[a,b]} \), the following statements are equivalent to each other.

(i) \( x \in \text{bd} \Gamma_{[a,b]} \).

(ii) \( x \) is a unique positive convex combination of points in \( \Gamma([a,b]) \).

Moreover, this combination has at most \( \left\lfloor \frac{d}{2} \right\rfloor + 1 \) points.

**Proof.** (i) \( \Rightarrow \) (ii). For \( x \in \text{bd} \Gamma_{[a,b]} \), let \( F_x \) be the smallest face of \( \Gamma_{[a,b]} \) containing \( x \). Then, by Theorems 5.3 and 5.6 in [1], \( F_x \subset \text{bd} \Gamma_{[a,b]} \), \( x \in \text{ri} F_x \), and \( j := \dim F_x < d \). Because \( F_x \) is a
simplex, $x$ is a unique positive convex combination of the $j + 1$ vertices of $F_x$. Moreover, if $x$ is a positive convex combination of points in $\Gamma_{[a,b]}$, these points must be in $F_x$ because $F_x$ is exposed (Corollary 3.7). In addition, $j + 1 \leq |d/2| + 1$ by Proposition 3.4.

(ii) $\Rightarrow$ (i). If $x \in \int \Gamma_{[a,b]}$, for each $t_0 \in [a,b]$ there is $y \in \bd \Gamma_{[a,b]}$ such that $x$ is an interior point of the segment $[\Gamma_{t_0}, y]$. Applying (i) to the point $y$, a positive convex combination for $x$ is obtained, which depends on $t_0$. □

4. An external representation of the cyclic body

We continue considering given $\Gamma_{[a,b]} \subset \mathbb{R}^d$.

**Lemma 4.1.** Let $T \subset [a,b]$ be a linearly ordered set and let $\Gamma_T$ be a proper $j$-face of $\Gamma_{[a,b]}$. The support cone for $\Gamma_{[a,b]}$ at any point $x \in \ri \Gamma_T$ satisfies

$$\dim \lin S(\Gamma_{[a,b]}, x) = j + |\{t_i \in T : a < t_i < b\}|.$$  

**Proof.** Denote by $s := |\{t_i \in T : a < t_i < b\}|$, let $m := d + 1 - s$, select real numbers $b < t_{j+2} < \cdots < t_m$, and set the multiplicities $n_i$, for $i = 1, 2, \ldots, m$, as $n_i := 2$ if $a < t_i < b$ and $n_i := 1$ otherwise. Since $\sum_{i=1}^m n_i = d + 1$, removing an index $k \geq j + 2$, the corresponding linear function $(-1)^{k}H_{f_k}$ of Theorem 3.6 supports $\Gamma_{[a,b]}$ at $\Gamma_T$. Applying now Lemma 2.4, the hyperplanes $H_{f_k}^0$ for $k = j + 2, j + 3, \ldots, m$, are linearly independent. Then, for any $x \in \ri \Gamma_T$ we have

$$\dim \lin S(\Gamma_{[a,b]}, x) \leq d - (m - (j + 1)) = j + s.$$  

On the other hand, the subset

$$B := \{\Gamma_{t_i} : t_i \in T\} \cup \{\Gamma_{t_i} + \Gamma_{t_i} : t_i \in T, n_i = 2\}$$

of the affinely independent set $A$ of Lemma 2.4, contains $j + 1 + s$ points, so that, $\dim \conv B = j + s$. Since $B - x \subset S(\Gamma_{[a,b]}, x)$ holds for any $x \in \ri \Gamma_T$, we conclude that $\dim \lin S(\Gamma_{[a,b]}, x) \geq j + s$. □

**Theorem 4.2.** The hyperplanes tangent to $\Gamma_{[a,b]}$, and their corresponding regular faces, are the following: if $d = 2m$, they are defined by the linear functions $(-1)^{p}H_{f}$ of Theorem 3.6 determined by

(i) $a = t_1 < t_2 < \cdots < t_{m+1} = b$; $n_1 = n_{m+1} = 1$, $n_i = 2$ for $2 \leq i \leq m$,

or

(ii) $a < t_1 < t_2 < \cdots < t_m < b$; $n_i = 2$ for $1 \leq i \leq m$,

and the regular faces are the maximal proper faces. If $d = 2m + 1$, they are defined by the linear functions determined by

(iii) $a = t_1 < t_2 < \cdots < t_{m+1} < b$; $n_1 = 1$, $n_i = 2$ for $2 \leq i \leq m + 1$,

or

(iv) $a < t_1 < t_2 < \cdots < t_{m+1} = b$; $n_{m+1} = 1$, $n_i = 2$ for $1 \leq i \leq m$,

and there exist maximal proper faces which are not regular.

**Proof.** (i) Because of Theorem 3.6, $-H_f(\Gamma_{[a,b]}) \geq 0$, and applying Lemma 4.1, $\dim \lin S(\Gamma_{[a,b]}, x) = d - 1$ for every $x \in \ri (H_f^0 \cap \Gamma_{[a,b]})$. Then, according to Corollary 3.7 and Lemma 2.5(ii),
we conclude that $H_f^0 \cap \Gamma_{[a,b]}$ is a regular face. In addition, by Theorem 3.5, all the $m$-faces are like this.

(ii) Just as in case (i), $H_f^0 \cap \Gamma_{[a,b]}$ is a regular face of dimension $m - 1$.

We consider now the remaining faces when $d$ is even. Let $G$ be a $j$-face. When $j = m - 1$, if $\Gamma_a \in G$ and $\Gamma_b \notin G$ then $G \subsetneq \text{conv}(G \cup \{\Gamma_b\})$, which is an exposed $m$-face. The same happens if $\Gamma_a \notin G$ and $\Gamma_b \in G$. If $j < m - 1$ or both $j = m - 1$ and $\Gamma_a, \Gamma_b \in G$, we have $G \subsetneq \text{conv}(G \cup \{\Gamma_1\})$ for any $t \in [a, b]$ such that $\Gamma_t \notin G$. In these cases $\text{conv}(G \cup \{\Gamma_t\})$ is an exposed $(j + 1)$-face by Theorem 3.5 as well. Therefore, none of the remaining faces is maximal (nor regular either by Lemma 2.5(iii)).

Remark 4.3. Concerning Theorem 4.2, observe that replacing the condition $a < t_1 < \cdots < t_k < b$, with the condition $t_i \in [a, b], i = 1, 2, \ldots, k$, the evenness of the exponent $p$ of Theorem 3.6 is preserved because $n_i = 2$ for $i = 1, \ldots, k$. Also, the corresponding function $(-1)^p H_f$ keeps its supporting properties. Thus, taking into account Lemma 2.5(i), the aggregation of redundant inequalities yields the following representation.

Corollary 4.4. $\Gamma_{[a,b]}$ is the solution set of the following system $\tau$ (using the notation of Theorem 3.6). If $d = 2m$, the system $\tau = \tau_1 \cup \tau_2$ given by

$$
\tau_1 := \left\{ \sum_{j=0}^{d} (-1)^j \sigma_{d-j} x_j \geq 0 : t_1 = a, \ t_m+1 = b, \ t_i \in [a, b], \begin{array}{l}
n_1 = n_{m+1} = 1, \ n_i = 2 \text{ otherwise}\end{array} \right\},
$$

$$
\tau_2 := \left\{ \sum_{j=0}^{d} (-1)^j \sigma_{d-j} x_j \geq 0 : t_i \in [a, b], \ n_i = 2 \text{ for } i = 1, 2, \ldots, m \right\},
$$

and if $d = 2m + 1$, the system $\tau = \tau_3 \cup \tau_4$ given by

$$
\tau_3 := \left\{ \sum_{j=0}^{d} (-1)^j \sigma_{d-j} x_j \geq 0 : t_1 = a, \ t_i \in [a, b], \begin{array}{l}
n_1 = 1, \ n_i = 2 \text{ for } i = 2, 3, \ldots, m+1\end{array} \right\},
$$

$$
\tau_4 := \left\{ \sum_{j=0}^{d} (-1)^j \sigma_{d-j} x_j \geq 0 : t_{m+1} = b, \ t_i \in [a, b], \begin{array}{l}
n_{m+1} = 1, \ n_i = 2 \text{ for } i = 1, 2, \ldots, m\end{array} \right\}.
$$

Remark 4.5. Note that the system $\tau$ of Corollary 4.4 is continuous on the compact set of the parameters, so that the set of points satisfying all the constraints of $\tau$ with a positive slack (Slater points) coincides with the (non-empty) interior of $\Gamma_{[a,b]}$ (see [5, Theorem 6.1]). Therefore, we can solve the feasibility moment problem (2) together with the uniqueness question (3), by finding the global minimum of a pair of multivariate polynomials on a compact interval in $\mathbb{R}^m$. 
Now we reformulate Corollary 4.4 and Theorem 3.8 in a more classic form (see [12]). \( \mathcal{P}_d(a, b) \) denotes the set of polynomials of degree exactly \( d \), which are non-negative on \([a, b]\) and have all their roots in this interval.

**Corollary 4.6.** A necessary and sufficient condition for the existence of a discrete probability distribution having first moments \( \mu_1, \mu_2, \ldots, \mu_d \), is that \( H_f(\mu) \geq 0 \) whenever \( f \in \mathcal{P}_d(a, b) \). One such distribution is unique if and only if \( H_f(\mu) = 0 \) for some \( f \in \mathcal{P}_d(a, b) \).

We close this Section with a result on representation of polynomials.

**Theorem 4.7.** A polynomial \( g(t) \) of degree \( d \), non-negative on the interval \([a, b]\), admits a representation

\[
g(t) = \sum_{i=1}^{n} f_i(t), \quad n \leq d + 1, \quad f_i \in \mathcal{P}_d(a, b). \tag{17}
\]

Consequently, if \( d = 2m \), it admits a representation

\[
g(t) = \sum_{i=1}^{k} (g_i(t))^2 + (t - a)(b - t) \sum_{i=k+1}^{n} (h_i(t))^2, \tag{18}
\]

where \( 0 \leq k \leq n \), \( g_i \in \mathcal{P}_m(a, b) \) and \( h_i \in \mathcal{P}_{m-1}(a, b) \). If \( d = 2m + 1 \), \( g(t) \) admits a representation

\[
g(t) = (t - a) \sum_{i=1}^{k} (g_i(t))^2 + (b - t) \sum_{i=k+1}^{n} (g_i(t))^2, \quad g_i \in \mathcal{P}_m(a, b). \tag{19}
\]

**Proof.** Let \( \tau \) be the system of Corollary 4.4, \( S := \{(a_s, b_s) \in \mathbb{R}^{d+1} : (a_s x) \geq b_s \in \tau \} \), and \( u := (0, 0, \ldots, 0, -1) \in \mathbb{R}^{d+1} \). Since \( \tau \) is continuous and has a Slater point (see Remark 4.5), \( N := \text{cone } S \) and \( K := N + \text{cone } \{u\} \) are closed convex cones, [5, Theorem 5.3]. Moreover, \( \Gamma_{[a, b]} \) is bounded and \( \dim \Gamma_{[a, b]} = d \), therefore \( K = N \) and this is a pointed cone, [5, Theorem 5.8]. Because \( g(t) \geq 0 \) for all \( t \in [a, b] \), we may assume, without lost of generality, that \( g(t) := (-1)^p f(t) \), where \( f(t) \) is of the form (12), verifying all conditions of Theorem 3.6. Then \((-1)^p H_f(\Gamma_{[a, b]}) \geq 0 \), i.e.

\[
(-1)^p \sum_{j=1}^{d} (-1)^j \sigma_{d-j} x_j \geq -(-1)^p \sigma_d \tag{20}
\]

holds for every \( x \in \Gamma_{[a, b]} \). Thus, the inequality (20) is a consequence of \( \tau \), and extended Farkas lemma [5, Corollary 3.1.2] yields

\[
v := (-1)^p (-\sigma_{d-1}, \sigma_{d-2}, -\sigma_{d-3}, \ldots, \pm 1, -\sigma_d) \in K = N.
\]

By Carathéodory’s theorem, we have

\[
v = \sum_{i=1}^{n} \lambda_i (a_i, b_i), \quad n \leq d + 1, \quad \lambda_i > 0, \quad (a_i, b_i) \in S.
\]

Denoting by \( f_i(t) := \lambda_i (-b_i + (a_i, \Gamma_i)) \), \( i = 1, 2, \ldots, n \), representation (17) is obtained. The forms (18) and (19) follow because \( f_i \in \mathcal{P}_d[a, b] \). \( \square \)
Example 4.8. According to Theorem 4.7, the polynomial \(-t^4 - 2t^3 + t^2 + 2t + 1\), which is non-negative on \([-1, 1]\), can be written as

\[
t^4 + (t + 1)(1 - t)(t^2 + (t + 1)^2),
\]

whereas the following is a Pólya-Szegö representation (cited in [12, pp. 77–78]) of the same polynomial,

\[
(t^2 - t - 1)^2 + (t + 1)(1 - t)2t^2.
\]

Both representations are essentially different because the polynomial \(t^2 + (t + 1)^2\) in (21) is not a square, whereas \(t^2 - t - 1\) in (22) has a root out of the interval \([-1, 1]\).

5. An optimization moment problem

In order to illustrate the proposed procedure in Section 1, consider the following particular instances: maximize (minimize) the 4th moment, \(\mu_4\), over the set of discrete probability distributions on \([0, 1]\) whose first three moments are \(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\).

For \(\Gamma_{[0,1]} \subset \mathbb{R}^3\), the external representation of Corollary 4.4 is

\[
\begin{align*}
t_2^2 x_1 - 2t_2 x_2 + x_3 \geq 0, & \quad t_2 \in [0, 1], \\
t_1^2 - t_1(t_1 + 2)x_1 + (2t_1 + 1)x_2 - x_3 \geq 0, & \quad t_1 \in [0, 1].
\end{align*}
\]

(23)

We check conditions (2) and (3) by replacing \(\mu := (\frac{1}{2}, \frac{1}{3}, \frac{1}{4})\) in (23),

\[
\mu \in \Gamma_{[0,1]} \iff \begin{align*}
\frac{1}{2}t_2^2 - \frac{2}{3}t_2 + \frac{1}{4} & \geq 0, \\
\frac{1}{2}t_1^2 - \frac{1}{3}t_1 + \frac{1}{12} & \geq 0
\end{align*}
\]

(24)

Since

\[
\min_{t_2 \in [0, 1]} \left\{ \frac{1}{2}t_2^2 - \frac{2}{3}t_2 + \frac{1}{4} \right\} = \min_{t_1 \in [0, 1]} \left\{ \frac{1}{2}t_1^2 - \frac{1}{3}t_1 + \frac{1}{12} \right\} = \frac{1}{36} > 0,
\]

we conclude that \(\mu \in \text{int} \Gamma_{[0,1]}\). Recalling (1), the programs \((D_1)\), and \((D_2)\) for minimization, are

\[(D_1) \quad \sup_{t \in [0, 1]} \sum_{i \in [0, 1]} \lambda_i t^4 \quad \quad (D_2) \quad -\sup_{t \in [0, 1]} \sum_{i \in [0, 1]} -\lambda_i t^4 \]

s.t. \(\sum_{i \in [0, 1]} \lambda_i (1, t, t^2, t^3) = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right)\),

s.t. \(\lambda \in \mathbb{R}^+_{[0,1]}\)

and both of them are solvable via their primal programs (see (4)),

\[(P_1) \quad \inf_{\lambda \in \mathbb{R}^+_{[0,1]}} \lambda \quad \quad (P_2) \quad -\inf_{\lambda \in \mathbb{R}^+_{[0,1]}} \lambda \]

s.t. \(x_0 + \frac{1}{2}x_1 + \frac{1}{3}x_2 + \frac{1}{4}x_3 \geq t^4, \quad t \in [0, 1], \quad x \in \mathbb{R}^4, \quad x_0 + x_1 t + x_2 t^2 + x_3 t^3 \geq -t^4, \quad t \in [0, 1], \quad x \in \mathbb{R}^4.
\]

Solving \((P_1)\) by a grid discretization method [4], we obtain the underestimate value \(v(P_1) \approx .208333\), with active inequalities for \(t \in T_1 := [0, .5, 1]\). By complementary slackness, \(\text{supp}(\lambda^*) \subset T_1\), then, solving (by minimum squares) the over-determined linear system
we obtain the distribution $\lambda_0^* \approx .166667$, $\lambda_2^* \approx .666667$, and $\lambda_3^* \approx .166667$. Analogously, we obtain $v(P_2) \approx .194445$, $T_2 = \{t_1 \approx .211325, t_2 \approx .788675\}$, and $\lambda_{t_1} = \lambda_{t_2}^* \approx .5$. In addition, there is $\lambda \in \mathbb{R}_+^{3} \{0,1\}$ with first moments $\frac{1}{2}, \frac{1}{4}, \mu_4$ if and only if $v(P_2) \leq \mu_4 \leq v(P_1)$.

On the other hand, if we take $\mu' := \left(\frac{1}{2}, \frac{1}{3}, \frac{2}{5}\right)$ instead of $\mu$, the condition (24) remains

$$
\mu' \in \Gamma[0,1] \iff \left\{ \begin{array}{l}
\frac{1}{2} t_2^2 - \frac{2}{3} t_2 + \frac{2}{9} \geq 0 \\
\frac{1}{2} t_2^2 - \frac{1}{3} t_1 + \frac{1}{9} \geq 0
\end{array} \right.
$$

for all $t_1, t_2 \in [0,1]$, and we obtain $\min\{\frac{1}{2} t_2^2 - \frac{2}{3} t_2 + \frac{2}{9}\} = 0$ at $t_2 = \frac{2}{3}$. Here, $\{\frac{1}{2} t_2^2 - \frac{1}{3} t_1 + \frac{1}{9}\} > 0$ for all $t_1 \in [0,1]$. We conclude (see Remark 4.5) that $\mu' \in \text{bd}\Gamma[0,1]$ and thus, $\mu'$ is a unique convex combination of two points in $\Gamma[0,1]$ (see Theorem 3.8). Solving the non-linear system

$$
\begin{align*}
\lambda_a + \lambda_b &= 1, \\
\lambda_a t_1 + \lambda_b t_2 &= 1/2, \\
\lambda_a t_1^2 + \lambda_b t_2^2 &= 1/3, \\
\lambda_a t_1^3 + \lambda_b t_2^3 &= 2/9,
\end{align*}
$$

we get $\mu' = .25 \Gamma_0 + .75 \Gamma_{2/3}$. The distribution $\lambda^*$ for which $\text{supp}(\lambda^*) = \{0, \frac{2}{3}\}$, $\lambda_0^* = .25$, and $\lambda_{2/3}^* = .75$, is the unique element of $\mathcal{A}'$ and we obtain $v(D'_1) = v(D'_2) = \mu'_4 \approx .148148$.

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References