Existence of solutions and multiple solutions for a class of weighted $p(r)$-Laplacian system

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Abstract

This paper investigates the existence of solutions for a class of weighted $p(r)$-Laplacian ordinary system boundary value problems. The sufficient conditions for the existence of solutions have been given via Leray–Schauder’s degree, and the existence of multiple solutions has been discussed via critical point theory. As an application, we discussed the existence of the radial solutions for the $p(x)$-Laplacian partial differential system boundary value problems.

1. Introduction

In this paper, we consider the existence of solutions for the following weighted $p(r)$-Laplacian ordinary system

$$-(w(r)|u'|^{p(r)-2}u')' + f(r, u, (w(r))^{1/p(r)})u = 0, \quad r \in (T_1, T_2),$$

with one of the following boundary value conditions

$$u(T_1) = u(T_2) \quad \text{and} \quad \lim_{r \to T_1^+} w(r)|u'|^{p(r)-2}u'(r) = \lim_{r \to T_2^-} w(r)|u'|^{p(r)-2}u'(r);$$

$$\lim_{r \to T_1^+} w(r)|u'|^{p(r)-2}u'(r) = \lim_{r \to T_2^-} w(r)|u'|^{p(r)-2}u'(r) = 0;$$

$$u(T_1) = u(T_2) = 0;$$

$$\lim_{r \to T_1^+} w(r)|u'|^{p(r)-2}u'(r) = u(T_2) = 0,$$

where $p \in C([T_1, T_2], \mathbb{R})$ and $p(r) > 1$, $-\Delta_{p(r)}u := -(w(r)|u'|^{p(r)-2}u')'$ is called the weighted $p(r)$-Laplacian; $w \in C([T_1, T_2], \mathbb{R})$ satisfies $0 < w(r), \forall r \in (T_1, T_2)$, and $(w(r))^{1/p(r)} \in L^1(T_1, T_2)$; the equivalent $\lim_{r \to T_1^+} w(r)|u'|^{p(r)-2}u'(r) = \lim_{r \to T_1^+} w(r)|u'|^{p(r)-2}u'(r)$ means $\lim_{r \to T_1^+} w(r)|u'|^{p(r)-2}u'(r)$ and $\lim_{r \to T_2^-} w(r)|u'|^{p(r)-2}u'(r)$ both exist and equal.

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The study of differential equations and variational problems with variable exponent is a new and interesting topic. We refer to [2,14,20], the applied background on these problems. Many results have been obtained on these problems, for example [4–11,13–20]. The framework to deal with the $p(x)$-Laplacian problems is variable exponent Sobolev space (see [6,11]). If $p(r) \equiv p$ (a constant), (1) is the well-known $p$-Laplacian problem. But if $p(r)$ is a general function, since the $-\Delta_{p(r)}$ represents a non-homogeneity and possesses more nonlinearity, it is more complicated than $-\Delta_p$; for example:

(i) If $\Omega \subset \mathbb{R}^n$ is a bounded domain, the Rayleigh quotient

$$
\lambda_{p(x)} = \inf_{u \in W_0^{1,p(x)}(\Omega) \setminus \{0\}} \frac{\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx}{\int_\Omega \frac{1}{p(x)} |u|^{p(x)} \, dx}
$$

is zero in general, and only under some special conditions $\lambda_{p(x)} > 0$ (see [8]), but the fact that $\lambda_p > 0$ is very important in the study of $p$-Laplacian problems.

(ii) If $w(r) \equiv 1$ and $p(r) \equiv p$ (a constant) and $-\Delta_p u > 0$, then $u$ is concave, this property is used extensively in the study of one-dimensional $p$-Laplacian problems, but it is invalid for $-\Delta_{p(r)}$. This is another difference on $-\Delta_p$ and $-\Delta_{p(r)}$.

On the one-dimensional $p$-Laplacian boundary value problems, there are many papers, for example [1,3,12]. In [1], Bohr and O’Regan give the existence of solutions of one-dimensional weighted Laplacian equation boundary value problems. In [12], Manásevich and Mawhin give the existence of periodic solutions of $p$-Laplacian-like ordinary systems. On the existence of solutions for $p(x)$-Laplacian system Dirichlet problems, we refer to [9,18]. But the results on the existence of solutions for $p(x)$-Laplacian problems with Neumann or periodic boundary value conditions are rare.

In [17], the authors deal with the existence of solutions of (1) with (2), and gives the existence of solutions under the following conditions

$$
\lim_{|u| \to +\infty} \left( f(t, u, v)/(|u| + |v|)^{q(t)} \right) = 0, \quad \text{uniformly a.e. } r \in [T_1, T_2],
$$

where $q(r) \in C([T_1, T_2], \mathbb{R})$ satisfies $0 < q^- \leq q^+ < p^- - 1$, where $h^- = \min_{t \in [T_1, T_2]} h(t)$ and $h^+ = \max_{t \in [T_1, T_2]} h(t)$.

In [7], the authors deal with the existence of solutions of

$$
\begin{cases}
-\Delta_{p(x)} u = f(x, u) & \text{in } \Omega, \\
u(x) = 0 & \text{on } \partial \Omega,
\end{cases}
$$

and give the existence of solutions under the following conditions

$$
\lim_{|u| \to +\infty} \left( f(x, u)/|u|^{q(x)} \right) = 0, \quad \text{uniformly a.e. in } x \in \Omega,
$$

where $q(x) \in C(\overline{\Omega}, \mathbb{R})$, and $0 < q^- \leq q^+ < p^- - 1$, where $h^+ = \sup_{\Omega} h(x)$, $h^- = \inf_{\Omega} h(x)$, or

$$
\lim_{|u| \to +\infty} \left( f(x, u)/|u|^{q(x)-1} \right) = +\infty, \quad \text{uniformly a.e. in } \Omega,
$$

and $p^+ - 1 < q^- \leq q^+ < p^*(x) - 1 := Np(x)/(N - p(x)) - 1$.

Because of the non-homogeneity, $-\Delta_{p(x)}$ possesses a special characteristic case, that is the following typical case

$$
0 < \lim_{|u| \to +\infty} \left( f(x, u)/|u|^{q(x)-1} \right) = +\infty, \quad \text{uniformly in } \Omega,
$$

where $p^+ - 1 < q(x) \leq p^+ - 1$. In [7] and [17], the existence of solutions of $p(x)$-Laplacian problems for the typical case has not been discussed. The similar instance has occurred in [9]. It is a difficult and interesting problem to deal with the typical case of $p(x)$-Laplacian problems. The results on these problems are rare.

In this paper, we investigate the existence of solutions for (1) with Dirichlet, Neumann and periodic boundary value condition, respectively. Our results include the case of the above typical case. This paper was motivated by [7,12,17]. Our results generalized partly of [1,7,12,17] that include ordinary and partial differential systems.

Throughout the paper, we denote

$$
w(T_1)u'(p(T_1)-2)u'(T_1) = \lim_{r \to T_1^+} w(r)u'(p(r)-2)u'(r),
$$

$$
w(T_2)u'(p(T_2)-2)u'(T_2) = \lim_{r \to T_2^-} w(r)u'(p(r)-2)u'(r).
$$

Let $0 < T_1 < T_2$, $N \geq 1$ and $I = [T_1, T_2]$. The function $f : I \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is assumed to be Carathéodory, and we mean:

(i) for almost every $t \in I$ the function $f(t, \cdot, \cdot)$ is continuous;

(ii) for each $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$ the function $f(\cdot, x, y)$ is measurable on $I$;
(iii) for each $\rho > 0$ there is a $\alpha_\rho \in L^1(I, \mathbb{R})$ such that, for almost every $t \in I$ and every $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$ with $|x| \leq \rho$, $|y| \leq \rho$, one has
$$|f(t, x, y)| \leq \alpha_\rho(t).$$

The inner product in $\mathbb{R}^N$ will be denoted by $(\cdot, \cdot)$, $|\cdot|$ will denote the absolute value and the Euclidean norm on $\mathbb{R}^N$. For $N \geq 1$, we set $C = C(I, \mathbb{R}^N)$, $C^1 = \{u \in C | u' \in C((T_1, T_2), \mathbb{R}^N)\}$. For any $u(r) = (u^1(r), \ldots, u^N(r))$, we denote $|u^i|_0 = \text{sup}_{r \in (r_1, r_2)} |u^i(r)|$, $\|u\|_0 = (\sum_{i=1}^N |u^i|_0^2)^{1/2}$ and $\|u\|_1 = \|u\|_0 + \|f(r)\|^{p-1}{p-2}u'|_0$. Space $C$ will be equipped with the norm $\|\cdot\|_0$, space $C^1$ will be equipped with the norm $\|\cdot\|_1$. Then $(C, \|\cdot\|_0)$ and $(C^1, \|\cdot\|_1)$ are Banach spaces. Let $L^1 = L^1(I, \mathbb{R}^N)$ with the norm $\|\cdot\|_1 = (\sum_{i=1}^N |x^i|_1^2)^{1/2}$, $\forall x \in L^1$, where $|x^i|_1 = \int_0^1 |x^i(t)| dt$.

We say a function $u : I \to \mathbb{R}^N$ is a solution of (1) if $u \in C^1$ with $w(r)|u'|^{p(r)-2}u'$ absolutely continuous on $(T_1, T_2)$, which satisfies (1) a.e. on $I$.

As an application, we consider the existence and boundary asymptotic behavior of solutions for $p(x)$-Laplacian partial differential systems
$$-\text{div}(\nabla u|^{p(x)-2}\nabla u) + f^i(x, u, |x|^{p-1} \nabla u) = 0, \quad \forall x \in \Omega, \quad i = 1, 2, \ldots, N,$$
where $\Omega$ is a bounded symmetric domain in $\mathbb{R}^N$, and
$$|\nabla u(x)| = \left(\sum_{j=1}^n \sum_{i=1}^N \left|\frac{\partial u^i}{\partial x_j}\right|^2\right)^{1/2}.$$
$p \in C(\overline{\Omega} ; \mathbb{R})$ be radially symmetric, and satisfies $1 < p(x)$, we will write $p(x) = p(|x|) = p(r)$.

This paper is divided into four sections. In Section 2, we present some preliminary. In Section 3, we consider the existence of solutions for system (1) with one of the boundary value conditions of (2)–(5) via Leray–Schauder degree. Finally, in Section 4, we mainly consider the existence of solutions and multiple solutions for system (1) with one of the boundary value conditions of (3)–(5) via critical point theory.

2. Preliminary

In this paper, in the case of without leading to confusion, we always use $c$ and $c_1$ to denote positive constant. For any $(r, x) \in (I \times \mathbb{R}^N)$, denote $\varphi(r, x) = |x|^{p(r)-2}x$. Obviously, $\varphi$ has the following properties

**Lemma 2.1.** $\varphi$ is a continuous function and satisfies:

(i) For any $r \in [T_1, T_2]$, $\varphi(r, \cdot)$ is strictly monotone, i.e.
$$(\varphi(r, x_1) - \varphi(r, x_2))x_1 - x_2 > 0, \quad \forall x_1, x_2 \in \mathbb{R}^N, \quad x_1 \neq x_2.$$

(ii) There exists a function $\alpha : [0, +\infty) \to [0, +\infty)$, $\alpha(s) \to +\infty$ as $s \to +\infty$, such that
$$(\varphi(r, x), x) \geq \alpha(|x|)|x|, \quad \text{for all } x \in \mathbb{R}^N.$$

It is well known that $\varphi(r, \cdot)$ is a homeomorphism from $\mathbb{R}^N$ to $\mathbb{R}^N$ for any fixed $r \in I$. For any $r \in I$, denote by $\varphi^{-1}(r, \cdot)$ the inverse operator of $\varphi(r, \cdot)$, then
$$\varphi^{-1}(r, x) = |x|^{2-p(r)}x, \quad \text{for } x \in \mathbb{R}^N \setminus \{0\}, \quad \varphi^{-1}(r, 0) = 0.$$

It is clearly that $\varphi^{-1}(r, \cdot)$ is continuous and send bounded sets into bounded sets. Let us now consider the following problem with boundary value condition (2)
$$(w(r)\varphi(r, u'(r)))' = f(r),$$
where $f \in L^1$ satisfies $\int_{T_1}^{T_2} f(r) dr = 0$. If $u$ is a solution of (7) with (2), by integrating (7) from $T_1$ to $r$, we find that
$$w(r)\varphi(r, u'(r)) = w(T_1)\varphi(T_1, u'(T_1)) + \int_{T_1}^{r} f(t) dt, \quad \forall r \in I.$$

Denote $a = w(T_1)\varphi(T_1, u'(T_1))$. It is easy to see that $a$ is dependent on $f$. Define operator $F$ from $L^1$ to $C$ as
$$F(f)(r) = \int_{T_1}^{r} f(t) dt, \quad \forall r \in I.$$
The boundary conditions imply that

\[ \frac{1}{T} \int_{T_1}^{T_2} \varphi^{-1} \left[ r, \ (w(r))^{-1} [a + F(f(r))] \right] \, dr = 0, \quad \text{where} \ T = T_2 - T_1. \]

For any fixed \( h \in C \), we denote

\[ \Lambda_h(a) = \frac{1}{T} \int_{T_1}^{T_2} \varphi^{-1} \left[ r, \ (w(r))^{-1} [a + h(r)] \right] \, dr. \]

**Lemma 2.2.** (See [17, Lemma 2.2].) The function \( \Lambda_h(\cdot) \) has the following properties:

(i) For any fixed \( h \in C \), the equation \( \Lambda_h(a) = 0 \) has a unique solution \( \tilde{a}(h) \in \mathbb{R}^N \).

(ii) The function \( \tilde{a} : C \rightarrow \mathbb{R}^N \), defined in (i), is continuous and sends bounded sets to bounded sets. Moreover \( |\tilde{a}(h)| \leq 2N\|h\|_0 \).

Now we define mapping \( a : L^1 \rightarrow \mathbb{R}^N \) as

\[ a(u) = \tilde{a}(F(u)). \]

It is clear that \( a \) is a continuous mapping which send bounded sets of \( L^1 \) into bounded sets of \( \mathbb{R}^N \), and hence it is a compact continuous mapping.

We continue now with our argument previous to Lemma 2.2. By solving for \( u' \) in (8) and integrating we find

\[ u(r) = u(T_1) + F \left[ \varphi^{-1} \left[ r, \ (w(t))^{-1} (a(f) + F(f)) \right] \right] (r), \quad \forall r \in I. \]

Let us define

\[ P : C^1 \rightarrow C^1, \quad u \mapsto u(T_1); \quad Q : L^1 \rightarrow L^1, \quad h \mapsto \frac{1}{T} \int_{T_1}^{T_2} h(r) \, dr, \]

where \( T = T_2 - T_1 \), and we denote

\[ K_1(h)(r) = F \left[ \varphi^{-1} \left[ r, \ (w(t))^{-1} (a(I - Q)h) + F((I - Q)h) \right] \right] (r), \quad \forall r \in [T_1, T_2], \]

\[ K_2(h)(r) = F \left[ \varphi^{-1} \left[ r, \ (w(t))^{-1} F((I - Q)h) \right] \right] (r), \quad \forall r \in [T_1, T_2], \]

\[ K_3(h)(r) = F \left[ \varphi^{-1} \left[ r, \ (w(t))^{-1} (a(h) + F(h)) \right] \right] (r), \quad \forall r \in [T_1, T_2], \]

\[ K_4(h)(r) = \int_{T_1}^{r} \varphi^{-1} \left[ s, \ (w(s))^{-1} \int_{T_1}^{s} h(\tau) \, d\tau \right] \, ds, \quad \forall r \in [T_1, T_2]. \]

**Lemma 2.3.** (See [17, Lemma 2.3].) The operators \( K_i \ (i = 1, 2, 3, 4) \) are continuous and send equi-integrable sets in \( L^1 \) into relatively compact sets in \( C^1 \).

We denote \( N_f(u) : C^1 \times [T_1, T_2] \rightarrow L^1 \) the Nemytski operator associated to \( f \) defined by

\[ N_f(u)(r) = f \left( r, u(r), (w(r))^{-1} u'(r) \right), \quad \text{on} \ I. \]

**Lemma 2.4.** (See [17, Lemma 2.4].) \( u \) is a solution of (1) with boundary condition (2), (3), (4) or (5) if and only if \( u \) is a solution of the following abstract equation respectively

\[ u = Pu + QN_f(u) + K_1(N_f(u)), \]

\[ u = Pu + QN_f(u) + K_2(N_f(u)), \]

\[ u = K_3(N_f(u)), \]

\[ u = K_4(N_f(u)). \]
Lemma 2.5. (See [17, Theorem 3.1].) Assume that $\Omega$ is an open bounded set in $C^1$ such that the following conditions hold.

\((1^0)\) For each $\lambda \in (0, 1)$ the problem

$$\left( w(r) \right) u^{(p(r)-2)} u' = \lambda f (r, u, \left( w(r) \right)^{\frac{1}{p(r)}} u'),$$

with boundary condition (2) or (3) has no solution on $\partial \Omega$.

\((2^0)\) The equation

$$\omega (a) := \frac{1}{T} \int_{T_1}^{T_2} f (t, a, 0) dt = 0,$$

has no solution on $\partial \Omega \cap \mathbb{R}^N$.

\((3^0)\) The Brouwer degree $d_B [\omega, \Omega \cap \mathbb{R}^N, 0] \neq 0$.

Then system (1) with (2) or (3) has a solution on $\overline{\Omega}$.

According to the homotopy invariant property of Leray–Schauder degree, we have

Lemma 2.6. (See [17, Theorem 3.3].) Assume that $\Omega$ is an open bounded set in $C^1$ such that the following conditions hold.

\((1^0)\) For each $\lambda \in [0, 1)$ the problems

$$u = \Psi_1 (u, \lambda) = K_3 (\lambda N_f (u)),
\quad u = \Psi_2 (u, \lambda) = K_4 (\lambda N_f (u)),$$

have no solution on $\partial \Omega$.

\((2^0)\) The Leray–Schauder degree

$$d_{LS} [I - \Psi_i (\cdot, 0), \Omega, 0] \neq 0, \quad i = 1, 2.$$

Then the system (1) with (4) or (5) has at least one solution on $\overline{\Omega}$.

Denote

$$L^{p(r)} (I) = L^{p(r)} (I, \mathbb{R}^N) = \left\{ u \mid u \text{ is measurable, } \int_{T_1}^{T_2} |u|^{p(r)} dr < \infty \right\},$$

endowed with the norm

$$|u|_{p(r)} = \inf \left\{ \lambda > 0 \mid \int_{T_1}^{T_2} \frac{|u|^{p(r)}}{\lambda} dr \leq 1 \right\}.$$

We denote

$$W^{1, p(r)}_w = \left\{ u \in L^{p(r)} (I) \mid \left( w(r) \right)^{\frac{1}{p(r)}} u' (r) \in L^{p(r)} (I) \right\}.$$

$W^{1, p(r)}_w$ is endowed with the norm $\|u\|_{p(r)} = |u|_{p(r)} + \left| \left( w(r) \right)^{\frac{1}{p(r)}} u' \right|_{p(r)}$.

Lemma 2.7.

\((i)\) $W^{1, p(r)}_w$ is a Banach space.

\((ii)\) If there exists a constant $s \in (1, p^-)$ such that $\left( w(r) \right)^{\frac{1}{p(r)}} \in L^1 (I)$, then $W^{1, p(r)}_w \hookrightarrow C (I)$ is compact.

**Proof.** (i) It is easy to see that $W^{1, p(r)}_w$ is a normed lineared space. Next, we will prove that $W^{1, p(r)}_w$ is completed.

If $\{u_n\}$ is a Cauchy sequence of $W^{1, p(r)}_w$, then $\{u_n\}$ and $\{(w(r))^{\frac{1}{p(r)}} u_n'\}$ are Cauchy sequence of $L^{p(r)} (I)$. Since $L^{p(r)} (I)$ is a Banach space, there exists $v_0, v_1 \in L^{p(r)} (I)$ such that $u_n \rightharpoonup v_0$ and $\left( w(r) \right)^{\frac{1}{p(r)}} u_n' \rightharpoonup v_1$ in $L^{p(r)} (I)$. It only remains to prove that $v_1 = \left( w(r) \right)^{\frac{1}{p(r)}} v_0'$. a.e. on $I$. 

Denote $c_m = T_1 + \frac{T_2 - T_1}{4m}$, $d_m = T_2 - \frac{T_2 - T_1}{4m}$, then $[c_m, d_m] \subset (T_1, T_2)$. Denote $W_m^{1,p(r)}(T_1, T_2)$. Since $w(r)|_{[c_m, d_m]} > 0$ for any $r \in [c_m, d_m]$, we can see $W_m^{1,p(r)}$ is a Banach space (see [6]), then $u_n \rightharpoonup v_0$ and $(w(r))^{\frac{1}{p(r)}} u_n' \rightharpoonup (w(r))^{\frac{1}{p(r)}} v'_0$ in $L^p(I)$ for any $m = 1, 2, \ldots$. Thus $(w(r))^{\frac{1}{p(r)}} u_n' \rightharpoonup (w(r))^{\frac{1}{p(r)}} v'_0$ in measure in $[c_1, d_1]$, so there exists a subsequence of $\{ (w(r))^{\frac{1}{p(r)}} u_n' \}$ (we still denote it by $\{ (w(r))^{\frac{1}{p(r)}} u_n' \}$), such that $(w(r))^{\frac{1}{p(r)}} u_n' \rightarrow (w(r))^{\frac{1}{p(r)}} v'_0$, a.e. on $[c_1, d_1]$. Similarly, there exists a subsequence of $\{ (w(r))^{\frac{1}{p(r)}} u_n' \}$ (we still denote it by $\{ (w(r))^{\frac{1}{p(r)}} u_n' \}$), such that $(w(r))^{\frac{1}{p(r)}} u_n' \rightarrow (w(r))^{\frac{1}{p(r)}} v'_0$, a.e. on $[c_2, d_2]$. Thus, there exists a subsequence of $\{ (w(r))^{\frac{1}{p(r)}} u_n' \}$ (we still denote it by $\{ (w(r))^{\frac{1}{p(r)}} u_n' \}$), such that $(w(r))^{\frac{1}{p(r)}} u_n' \rightarrow (w(r))^{\frac{1}{p(r)}} v'_0$ in measure in $(T_1, T_2)$. Thus $(w(r))^{\frac{1}{p(r)}} u_n' \rightarrow (w(r))^{\frac{1}{p(r)}} v'_0$ in measure in $(T_1, T_2)$. Since $(w(r))^{\frac{1}{p(r)}} u_n' \rightarrow v_1$ in $L^p(I)$, we have $v_1 = (w(r))^{\frac{1}{p(r)}} v'_0$.

(ii) Similar to the definition of $W_w^{1,p(r)}$, we define $W_w^{1,p(r)}$. It is easy to see that there is a continuous imbedding $W_w^{1,p(r)} \hookrightarrow W_w^{1,p(r)}$.

Let $u \in W_w^{1,p(r)}$. Since $s \in (1, p^-)$, such that $(w(r))^{\frac{s}{s-1}} \in L^1(T_1, T_2)$, we have

$$|u'|_{p^-/s} = \left( \int_{T_1}^{T_2} |u'|^{p^-/s} dr \right)^{s/p^-} \leq \left( \int_{T_1}^{T_2} |u'|^{p^-/s} (w(r))^{\frac{1}{p^-}} dr \right)^{s/p^-} \leq c_1 (|w(r)|^{\frac{1}{p^-}} u'|_{p^-}).$$

Obviously, $|u'|_{p^-/s} \leq c_2 |u|_{p^-}$, then $W_w^{1,p^-} \hookrightarrow W^{1,p^-/s}$ is continuous. Notice that $p^-/s > 1$, according to the classical Sobolev embedding theorem $W^{1,p^-/s} \hookrightarrow C(I)$, we get a compact imbedding $W_w^{1,p(r)} \hookrightarrow C(I)$.

This completes the proof. □

$W_w^{1,p(r)}$ is called the weighted variable exponent Sobolev spaces. Similarly, we have

**Lemma 2.8.** $(C^1, \| \cdot \|_1)$ is a Banach space.

### 3. Existence of solutions

In this section, under the condition that $f = (f^1, \ldots, f^N)$ satisfies

$$f^i(r, x, y) = \sigma^i(r)(|x|^{q_i(r)-1}x + \mu^i(r)|y|^{q_i(r)}) + e^i(r),$$

where $q_1, q_2 \in L^\infty(I, \mathbb{R})$, we will apply Leray–Schauder’s degree to deal with the existence of solutions for (1) with boundary value conditions. Our results are consequences of Lemmas 2.5 and 2.6.

In this section, we assume that

(A1) $q_1, q_2 \in L^\infty(I, \mathbb{R})$ are nonnegative, $\sigma, \mu, e \in L^\infty(I, \mathbb{R}^N)$, we will apply Leray–Schauder’s degree to deal with the existence of solutions for (1) with boundary value conditions. Our results are consequences of Lemmas 2.5 and 2.6.

In this section, we assume that

(A1) $q_1, q_2 \in L^\infty(I, \mathbb{R})$ are nonnegative and satisfying $\text{essinf}(q_1(r) - q_2(r)) > 0$ or $\text{essinf}(q_2(r) - q_1(r)) > 0$,

(A2) $\mu = (\mu^1, \ldots, \mu^N) \in L^\infty(I, \mathbb{R}^N)$, $\sigma = (\sigma^1, \ldots, \sigma^N) \in L^\infty(I, \mathbb{R}^N)$, $e = (e^1, \ldots, e^N) \in L^\infty(I, \mathbb{R}^N)$. For any $i = 1, \ldots, N$, $\sigma^i$ keeps sign on $I$, and satisfies

$$\sigma^i \leq \min_{1 \leq i \leq N} \text{essinf} |\sigma^i| \leq \max_{1 \leq i \leq N} \text{esssup} |\sigma^i| \leq \sigma_2,$$

where $\sigma_1$ and $\sigma_2$ are positive constants.

For any $h \in L^\infty(I, \mathbb{R}^N)$, denote $|h'|_0 = \text{esssup} r \in (T_1, T_2)|h'(r)|$ $|i = 1, \ldots, N|$, $\|h\|_0 = (\sum_{i=1}^N |h'|_0^2)^{1/2}$. Denote $M = \int_{T_1}^{T_2} (w(r))^{\frac{1}{p(r)}} dr$, $\theta = \frac{\epsilon}{2 + \frac{1}{\theta}}$. According to (A1), there exists a positive constant $\epsilon$ satisfies

$$b_1 := \text{essinf}_{t \in I} \left( \frac{1}{N(2 + \frac{1}{M})} |\theta|^{q_1(t)} - \|\mu\|_0 |N e|^{q_2(t)} \right) > 0.$$

We also assume

(A3) $e = (e^1, \ldots, e^N)$ satisfies

$$|e^i|_0 < \sigma_1 b_1, \quad i = 1, \ldots, N.$$
Lemma 2.5 are satisfied, then we can conclude that the system (1) with (2) has a solution on \( \Omega \). We only prove the existence of solutions for (1) with (2), the rest is similar. We will prove that the conditions of

\[
\begin{align*}
\sigma_2 & < b_2 := \frac{\inf_{\theta \in \Omega} \left( | \theta |^{q(r)} - 1 \right)}{N(2M + 1) \int_{T_1}^{T_2} (|N\theta|^{q(r)} + |\theta|^{q(r)}) \, dt}. 
\end{align*}
\]

**Note 3.1.** Let \( f^i(r, x, y) = \lambda |\sigma(r)(x^{q(r)} - 1) + 1 + \mu(r) y^{q(r)} + \delta e^i(r) | \), and (A1)–(A2) are satisfied. If \( \lambda \) and \( \delta \) are positive small enough, then it is easy to see that (A3)–(A4) are satisfied.

Denote

\[
\Omega_{\varepsilon} = \left\{ u \in C^1 \mid \sup_{1 \leq i \leq N} \left( | u_i |^2 + | u'(r) |^{q(r)} \right) < \varepsilon \right\}.
\]

It is easy to see that \( \Omega_{\varepsilon} \) is an open bounded domain in \( C^1 \).

**Theorem 3.1.** If \( f \) satisfies (9), and (A1)–(A4) are satisfied, then the system (1) with (2) or (3) has a solution on \( \Omega_{\varepsilon} \).

**Proof.** We only prove the existence of solutions for (1) with (2), the rest is similar. We will prove that the conditions of Lemma 2.5 are satisfied, then we can conclude that the system (1) with (2) has a solution on \( \Omega_{\varepsilon} \).

(1) We only need to show that for each \( \lambda \in (0, 1) \) the problem

\[
\begin{align*}
(w(r)u')^{(q(r)) - 2}(u')'' &= \lambda f(r, u, (w(r))^{\frac{1}{p(r) - 1}} u'),
\end{align*}
\]

with boundary condition (2) has no solution on \( \partial \Omega_{\varepsilon} \).

If it is false, then there exists \( \lambda \in (0, 1) \) and \( u \in \partial \Omega_{\varepsilon} \) is a solution of (10) with (2).

Since \( u \in \partial \Omega_{\varepsilon} \), there exists an \( i \) such that \( | w(r) |^2 + | u'(r) |^2 \). Since \( u \in C \), there exists \( \varepsilon > 0 \) such that

\[
| u'(r_0) | < \varepsilon.
\]

For any \( r \in I \), we have

\[
| u'(r) - u'(r_0) | = \left| \int_{r_0}^{r} u''(t) \, dt \right| < \int_{r_0}^{r} | u''(t) | \, dt \leq \int_{r_0}^{r} (w(t))^{\frac{1}{p(r) - 1}} | u'(t) |^{q(r)} \, dt \leq M \cdot \frac{\varepsilon}{M} = \varepsilon.
\]

This implies that \( | u'(r) | > \varepsilon \) for all \( r \in I \). Since \( u \in C \), we can see that \( u'(r) \) keeps sign on \( I \). Since \( \sigma(r) \) keeps sign, we can see that \( \sigma(r) u'(r) \) also keeps sign on \( I \).

Assume that \( \sigma(r) u'(r) \) is positive, then

\[
| u'(r) | > \varepsilon \]
Since $\sigma_2 < b_2$, combining (11) and the above equation, we have
\[
\frac{|\theta|^p(t) - 1}{N(2M + 1)} \leq \frac{1}{N(2M + 1)} w(r_1) |(u')^p(t)|^{p(t) - 1} \leq \frac{1}{N(2M + 1)} w(r_1)|u'(r_1)|^{p(t) - 1} \\
\leq w(r_1)|u'(r_1)|^{p(t) - 2} |(u')^p(t)| \leq \lambda \int_{r_0}^{r_2} |f^i(t, u, (w(t))^{p(t) - 1} u')| dt \\
\leq T_2 \sigma_2^i(N\xi^q_{1(t)} + |\mu||N\xi^q_{2(t)}| + |e|^i) dt \\
\leq T_2 \sigma_2^i \left( |N\xi^q_{1(t)}| + \|\mu\|_0 |N\xi^q_{2(t)}| + |\theta|^{q(t)} \right) dt \\
< \frac{1}{N(2M + 1)} \inf_{t \in I} \frac{|\theta|^p(t) - 1}{M} \leq \frac{1}{N(2M + 1)} \|\theta|^p(t) - 1 \| _M .
\]

It is a contradiction.

Summarizing this argument, for each $\lambda \in (0, 1)$, the problem (10) with (2) has no solution on $\partial \Omega_\varepsilon$.

(2') For any $u \in \partial \Omega_\varepsilon \cap \mathbb{R}^N$, without loss of generality, we may assume that $a^i = \varepsilon$ and $\sigma^i(t) > 0$, then we have
\[
\int_{T_1}^{T_2} f^i(t, a, 0) dt = \int_{T_1}^{T_2} \left( \sigma^i(t)\varepsilon^{q(t)-1} + e^i(t) \right) dt \geq \int_{T_1}^{T_2} \left( \frac{\sigma_1}{N} |\varepsilon|^{q(t)-1} \varepsilon + e^i(t) \right) dt > 0.
\]

It means that $\omega(a) = 0$ has no solution on $\partial \Omega_\varepsilon \cap \mathbb{R}^N$.

(3') Let
\[
h^i(t, a, \lambda) = \lambda \left[ \sigma^i(t)\varepsilon^{q(t)-1} - a^i + e^i(t) \right] + (1 - \lambda)a^i \text{sgn} \sigma^i(t),
\]
\[
h(t, a, \lambda) = (h^1(t, a, \lambda), \ldots, h^N(t, a, \lambda)).
\]

Define
\[
\Phi(a, \lambda) = \int_{T_1}^{T_2} h(t, a, \lambda) dt.
\]

According to (A3), it is easy to see that, for any $\lambda \in [0, 1]$, $\Phi(a, \lambda) = 0$ does not have solution on $\partial \Omega_\varepsilon \cap \mathbb{R}^N$, then the Brouwer degree
\[
d_B[\omega, \Omega_\varepsilon \cap \mathbb{R}^N, 0] = d_B[\Phi(a, 1), \Omega_\varepsilon \cap \mathbb{R}^N, 0] = d_B[\Phi(a, 0), \Omega_\varepsilon \cap \mathbb{R}^N, 0] \neq 0.
\]

This completes the proof. □

Next, we will deal with the existence of solutions of (1) with (4) or (5). We assume
(A5) $q_1, q_2 \in L^\infty(I, \mathbb{R})$ are nonnegative. $\mu, \sigma, \varepsilon \in L^\infty(I, \mathbb{R}^N)$.
(A6) $e = (e^1, \ldots, e^N)$ satisfies
\[
|e|^i_0 < \|\sigma\|_0 \text{ ess inf}_{t \in I} |\theta|^{q(t)}, \quad i = 1, \ldots, N,
\]
(A7) $\sigma$ satisfies
\[
\|\sigma\|_0 \leq b_2 := \frac{\inf_{t \in I} \|\theta^p(t) - 1\|_M}{N(2M + 1)} \int_{T_1}^{T_2} \left( |N\xi^q_{1(t)}| + \|\mu\|_0 |N\xi^q_{2(t)}| + |\theta|^{q(t)} \right) dt.
\]

Note 3.2. Let $f^i(r, x, y) = \lambda|\sigma^i(r)(x)^{q(t)-1}x^i + \mu^i(r)y^{q^i(r)} + \delta e^i(r)|$, and (A5) are satisfied. If $\lambda$ and $\delta$ are positive small enough, then, it is easy to see that (A6)–(A7) are satisfied.

**Theorem 3.2.** If $f$ satisfies (9), and (A5)–(A7) are satisfied, then the system (1) with (4) or (5) has a solution on $\overline{\Omega_\varepsilon}$. 


Proof. We will prove that the conditions of Lemma 2.6 are satisfied, then we can conclude that the system (1) with (4) or (5) has a solution on $\bar{\Omega}_\varepsilon$. In the following, we only prove the existence of solutions for (1) with (4), the existence of solutions for (1) with (5) is similar.

(1) We only need to proof that for each $\lambda \in [0, 1]$ the problem

$$
(w(r)|u_t|^{p(r)-2}u_t)' = \lambda f(r, u, (w(r))^{\frac{1}{p(r)-1}}u'),
$$

(12) with boundary condition (4) has no solution on $\partial \Omega_\varepsilon$.

If it is false, then there exists a $\lambda \in [0, 1]$ and $u \in \partial \Omega_\varepsilon$ is a solution of (12) with (4).

Since $u \in \partial \Omega_\varepsilon$, there exists a $i$ such that $|u_i'|_0 + |(w(r))^{\frac{1}{p(r)-1}}(u_i')|_0 = \varepsilon$. Then, for some $r \in l$. On the other hand,

$$
|u_i'(r)| = \int_{T_1}^{T_2} (u_i'(t)) dt \leq \int_{T_1}^{T_2} ((w(t))^{\frac{1}{p(r)-1}}(u_i'(t)) dt \leq M \cdot \frac{\varepsilon}{M} < 2\theta, \; \forall r \in l.
$$

It is a contradiction.

(ii) Suppose that $|u_0| \geq 2\theta$, then $|(w(r))^{\frac{1}{p(r)-1}}(u_i')(r)|_0 \leq \varepsilon - 2\theta = \frac{\theta}{M}$. This implies that $|u_i'(r)| \geq 2\theta$ for some $r \in l$. On the other hand,

$$
|(w(r))^{\frac{1}{p(r)-1}}(u_i')(r)|_0 \geq \frac{\theta}{M} = \frac{N\varepsilon}{N(2M + 1)} \geq \frac{1}{N(2M + 1)} u_i'(r_2).
$$

According to the boundary value condition, there exists a $r_0^i \in l$ such that

$$
w(r_0^i)|u_i'(r_0^i)|^{p(r)-2}(u_i')(r_0) = 0,
$$

then

$$
w(r)|u_t|^{p(r)-2}(u_t)'(r) = \lambda \int_{r_0^i}^r f_i(t, u, (w(t))^{\frac{1}{p(r)-1}}u') dt, \; r \in l.
$$

Since $\sigma_2 < b_2$, combining (13), we have

$$
\left| \frac{\theta}{M} \right|^{p(r_2)-1} \leq \frac{1}{N(2M + 1)} w(r_2) \left| (u_i')^{p(r_2)-1} \right| \leq \frac{1}{N(2M + 1)} w(r_2) \left| u_i'(r_2) \right|^{p(r_2)-1}
$$

$$
\leq w(r_2) \left| u_i'(r_2) \right|^{p(r_2)-2} \left| (u_i')^{p(r_2)-1} \right| \leq \lambda \left| \int_{r_0^i}^{T_2} f_i(t, u, (w(t))^{\frac{1}{p(r)-1}}u') dt \right|
$$

$$
\leq \int_{T_1}^{T_2} \left| \sigma_0 \left[ |N\varepsilon|^{q_1(t)} + \|\mu\|_0 N\varepsilon|^{q_1(t)} \right] + |\theta| \right| dt
$$

$$
\leq \|\sigma\|_0 \int_{T_1}^{T_2} \left( \left| N\varepsilon|^{q_1(t)} + \|\mu\|_0 N\varepsilon|^{q_1(t)} \right| + \left| \theta \right| \right) dt
$$

$$
\leq \frac{1}{N(2M + 1)} \inf_{\varepsilon \in \mathbb{R}} \left| \theta \right|^{p(r)-1} \leq \frac{1}{N(2M + 1)} \left| \frac{\theta}{M} \right|^{p(r_2)-1}.
$$

It is a contradiction.

Summarizing this argument, for each $\lambda \in [0, 1]$, the problem (12) with (4) has no solution on $\partial \Omega_\varepsilon$.

(2) Since $0 \in \partial \Omega_\varepsilon$, the Leray–Schauder degree

$$
d_{LS}[I - \Psi_i, 0, \Omega_\varepsilon, 0] = d_{LS}[I, \Omega_\varepsilon, 0] = 1 \neq 0, \; i = 1, 2.
$$

This completes the proof.  \(\square\)
Corollary 3.3. If $f$ satisfies (9), and $(A_1)$–$(A_2)$ are satisfied, and $(p^+ - 1) < q_1(t) < q_2(t)$ for any $t \in I$, then there exists a positive small enough constant $\delta$ such that, if $e \in L^\infty(1, \mathbb{R}^N)$ satisfies $|e'|_0 < \delta$ $(i = 1, \ldots, N)$, then (1) with one of (2)–(5) has at least a solution.

Corollary 3.4. If $f$ satisfies (9), and $(A_1)$–$(A_2)$ are satisfied, and $(p^- - 1) > q_1(t) > q_2(t)$ for any $t \in I$, then (1) with one of (2)–(5) has at least a solution.

Corollary 3.5. If $f$ satisfies (9), and $q_2(t) = 0$ and $(p^- - 1) \leq q_1(t) \leq (p^+ - 1)$ for any $t \in I$, under the conditions of Theorems 3.1 and 3.2, then (1) with one of (2)–(5) has at least a solution.

Corollary 3.6. If $w(T_1) = w(T_2) \neq 0$ and $p(T_1) = p(T_2)$, under the conditions of Theorem 3.1, then (1) with (2) has at least a periodic solution.

Let $\Omega = \{x \in \mathbb{R}^n \mid 0 < T_1 < |x| < T_2 \} \subset \mathbb{R}^n$ is an open bounded domain. As an application, let us now consider the system (6) with one of the following boundary value conditions

\begin{align}
\frac{u}{\partial \Omega} &= 0; \\
\forall u_i = 0, \quad \forall x \in \partial \Omega, \quad i = 1, 2, \ldots, N. \tag{14}
\end{align}

Theorem 3.7. If $f$ satisfies (9), in each of the following cases:

1. $0 < T_1 < T_2$, $\Omega = \{x \in \mathbb{R}^n \mid T_1 < |x| < T_2\}$.
2. $0 = T_1 < T_2$, $\Omega = \{x \in \mathbb{R}^n \mid T_1 < |x| < T_2\} = B(0; T_2)\setminus\{0\}$, and $p^- > n$.
3. $T_2 > 0$, $\Omega = \{x \in \mathbb{R}^n \mid |x| < T_2\} = B(0; T_2)$, and $p^- > n$.

If $(A_1)$–$(A_4)$ are satisfied, then system (6) with (15) has at least one weak radially symmetric solution $u$, if $(A_5)$–$(A_7)$ are satisfied, then system (6) with (14) has at least one weak radially symmetric solution $u$.

Proof. If $u$ is a radial solution of (6), then it can be transformed into

\begin{equation}
-(r^{n-1}|u'|^{p(r)-2}u')' + r^{n-1}f(r, u, r^{\frac{n-1}{p(r)}}u') = 0, \quad r \in (T_1, T_2), \quad \text{where } T_1 \geq 0, \tag{16}
\end{equation}

and the boundary value condition will be transformed into (3), (4) or (5), respectively. Notice that $(r^{n-1})^{\frac{1}{p(r)-1}} \in L^1(0, T_2)$ and satisfies $0 < r^{n-1}$, $\forall r \in (0, T_2)$; we can conclude the existence of solutions for (16) with (3), (4) or (5) from Theorems 3.2 and 3.1.

If $\lim_{r \to 0} r^{n-1}|u'|^{p(r)-2}u'(r) = 0$, notice that

$$
||u'|^{p(r)-2}u'(r)|| \leq r^{1-n} \int_0^r t^{n-1} |f(t, u, t^{\frac{n-1}{p(r)}}u')| dt \leq \int_0^r |f(t, u, t^{\frac{n-1}{p(r)}}u')| dt \to 0 \quad \text{(as } r \to 0),
$$

then we have $u'(0) = 0$. This completes the proof. \Box

4. Existence of solutions and multiple solutions

In this section, under the condition that

\begin{equation}
(H_1) \quad f(r, u, (w(r))^{p(r)-1}u') = g(r, u) + h(r, u)c(r, u, (w(r))^{p(r)-1}u') \quad \text{where } c(\cdot, \cdot, \cdot) \text{ is bounded, } g \text{ and } h \text{ are continuous},
\end{equation}

we will consider the existence of solutions and multiple solutions recur to weighted variable exponent Sobolev space.

Theorem 4.1. If $f$ satisfies $(H_1)$ and $\lim_{|u| \to \infty} \frac{h(r, u)}{|u|^{\beta^-}} = 0$, $\lim_{|u| \to \infty} \frac{|g(r, u)|}{|u|^{\beta^+}} = +\infty$, where $\beta \in C(I, \mathbb{R}^+)$ and $\beta^+ < \frac{(r^-)^2}{p}$, then the system (1) with (2), (3), (4) or (5) has a solution.

Proof. We only prove the existence of solutions for (1) with (2), the rest is similar. We will prove that the conditions of Lemma 2.5 are satisfied, then we can conclude that the system (1) with (2) has a solution.
(1°) We only need to prove that for each \( \lambda \in [0, 1] \) solutions of the problem
\[
(w(r)|u|^{|p(r)-2}u')' = \lambda f(r, u, (w(r))^{\frac{1}{p(r)-1}}u') + (1 - \lambda)Q\, N_f(u).
\]
with boundary condition (2) are uniformly bounded in \( C^1 \).
When \( \lambda = 0 \), if \( u \) is a solution of (17) with (2), it is easy to see that \( u \) is a constant and
\[
\int_{T_1}^{T_2} f(t, u, 0) \, dt = 0 \quad \text{and} \quad \int_{T_1}^{T_2} f(t, u, 0) \, dt = 0.
\]
Obviously, when \( u \) is a constant and \( |u| \) is large enough, it is easy to see that
\[
\int_{T_1}^{T_2} \{|f(t, u, 0)| \, dt = +\infty.
\]
Thus, when \( \lambda = 0 \), all the solutions of (17) are uniformly bounded in \( C^1 \). Next, we shall prove that all the solutions of (17) are uniformly bounded in \( C^1 \) when \( \lambda \in (0, 1] \). We only need to prove that all the solutions of the following system are uniformly bounded in \( C^1 \)
\[
(w(r)|u|^{|p(r)-2}u')' = \lambda f(r, u, (w(r))^{\frac{1}{p(r)-1}}u') \quad \forall r \in (T_1, T_2).
\]
If \( u \) is a solution of (18), multiplying (18) by \( u \), and integrating on \((T_1, T_2)\), according to (2), we can deduce
\[
\int_{T_1}^{T_2} w(r)|u|^{|p(r)} \, dr = -\lambda \int_{T_1}^{T_2} \{ f(r, u, (w(r))^{\frac{1}{p(r)-1}}u') \} \, dr \leq c_1 + c_2 \int_{T_1}^{T_2} |u|^{|\beta(r)} \, dr \leq c_1 + c_3|u|^{|\beta(r)}.
\]
We can confirm that there exists a positive constant \( C_\ast \) such that for each \( i \in \{1, \ldots, N\} \), there exists \( r_i \in I \) such that \( u^i(r_i) \leq C_\ast \). Suppose the contrary, then there exists an \( i \in \{1, \ldots, N\} \) such that \( |u^i(r)| > C_\ast \) for any \( r \in I \) (where \( C_\ast \) is large enough). Combining (H1), we have \( f^i(r, u, (w(r))^{\frac{1}{p(r)-1}}u') \) keep sign on \( I \), it is a contradiction to boundary (2). Thus, for any \( i \in \{1, \ldots, N\} \), we have
\[
|u^i(r)| = |u^i(r_1) + \int_{r_1}^{r} (u^i(t))' \, dt| \leq C_\ast + \int_{T_1}^{T_2} (w(r))^{\frac{1}{p(r)-1}} \{ |u^i(r)|' \} \, dr \leq C_\ast + c_4 \int_{T_1}^{T_2} w(r)^{|p(r)} \, dr \, |u^i(r)| \, p(r). \quad \forall r \in I.
\]
From (20), we have
\[
\int_{T_1}^{T_2} |u|^{|p(r)} \, dr \leq c_5 + c_6 \int_{T_1}^{T_2} \{ |u(r)| \, p(r) \}^{|p(r)} \, dr \leq c_7 + c_8 \int_{T_1}^{T_2} |u(r)| \, p(r) \, p(r) \, p(r) = 1.
\]
Combining (19) and (21), when \( \int_{T_1}^{T_2} |u|^{|p(r)} \, dr > 1 \), we have
\[
(|u(r)| \, p(r))^p \leq \int_{T_1}^{T_2} |u|^{|p(r)} \, dr \leq c_9 + c_{10}(|u(r)| \, p(r))^{|p(r)} \, p(r) + 1.
\]
Since \( |p(r)| > \frac{p(r)}{p(r)-1} \), from (22), we obtain that \( |u| \, p(r) \) is uniformly bounded, and then \( |(w(r))^{\frac{1}{p(r)}}(u(r))'| \, p(r) \) is uniformly bounded. Combining (20), we can see \( |u| \, p(r) \) is uniformly bounded on \( I \). From (18), we can see \( |(w(r))^{\frac{1}{p(r)}}(u(r))'| \, p(r) \) is uniformly bounded. Then we can conclude that all the solutions of (17) with (2) are uniformly bounded in \( C^1 \) when \( \lambda \in [0, 1] \). Let \( R \) be a large enough number such that all the solutions of (17) with (2) are belong to \( B(0, R) \).
(2°) From the proof of (1°), we can see that \( \omega(a) = 0 \) has no solution on \( \partial B(0, R) \cap \mathbb{R}^N \).
(3°) Define
\[
S_\mu(d) = \mu d + (1 - \mu)\omega(d),
\]
where \( \omega \) is defined in Lemma 2.5.
Obviously, $S_{\mu}(d) \neq 0$ on $\partial B(0, R) \cap \mathbb{R}^N$ for any $\mu \in [0, 1]$. Then we have

$$d_{\theta}[\omega, B(0, R) \cap \mathbb{R}^N, 0] = d_{\theta}[S_0, B(0, R) \cap \mathbb{R}^N, 0] = d_{\theta}[S_1, B(0, R) \cap \mathbb{R}^N, 0] = 1.$$ This completes the proof. \(\square\)

In the following, we always assume that there exists a constant $s \in (1, p^-)$ such that $(w(r))^{-\frac{1}{p}} \in L^1(T_1, T_2)$. We will use critical point theory to discuss the existence of multiple solutions of (1). We assume

(H2)$_c$, (H2)$_p$ = –1,

(H3) for any $i = 1, \ldots, N$, $g^i(r, u)u^i \geq 0$, $\forall u \in \mathbb{R}^N$, and $h^i(r, u) = 0$ when $u^i = 0$, and there exists $C^1$ functions $G(r, u), H(r, u): I \times \mathbb{R}^N \to \mathbb{R}$ such that

$$G(r, 0) = H(r, 0) = 0, \quad \frac{\partial}{\partial u} G(r, u) = g(r, u), \quad \frac{\partial}{\partial u} H(r, u) = h(r, u),$$

where $\frac{\partial}{\partial u} = (\frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_N})$.

Denote

$$E_3 = \left\{ u \in W^{1,p(r)}_w \mid \lim_{r \to T_1^+} w(r) \frac{1}{p(r)} u'(r) = \lim_{r \to T_2^-} w(r) \frac{1}{p(r)} u'(r) = 0 \right\},$$

$$E_4 = \left\{ u \in W^{1,p(r)}_w \mid u(T_1) = u(T_2) = 0 \right\},$$

$$E_5 = \left\{ u \in W^{1,p(r)}_w \mid u(T_2) = 0, \lim_{r \to T_2^-} w(r) \frac{1}{p(r)} u'(r) = 0 \right\}.$$

Let $X = E_3, E_4, E_5$, when we discuss the existence of solutions of (1) with (3), (4) or (5), respectively. Then the integral functional associated with (3), (4) or (5) is

$$\psi(u) = \int_{T_1}^{T_2} w(r) |u|^p dr + \int_{T_1}^{T_2} G(r, u) dr - \int_{T_1}^{T_2} H(r, u) dr, \quad \forall u \in X,$$

and $\psi$ is a weak semicontinuous $C^1$ functional. $u$ is called a weak solution of (1) with (3), (4) or (5) if

$$\int_{T_1}^{T_2} w(r) |u|^p dr + \int_{T_1}^{T_2} G(r, u) v dr - \int_{T_1}^{T_2} H(r, u) v dr = 0, \quad \forall v \in X.$$

We only need to prove that $\psi$ has a critical point $u$. The critical point $u$ of $\psi$ is the weak solution of (1) with (3), (4) or (5). According to the regularity of solutions of ODE, $u$ is a solution of (1) with (3), (4) or (5).

Denote $H^+(r, u) = H(r, S(u))$, where $S(u) = (S_1(u^1), \ldots, S_N(u^N))$, where $S_n(t) = \max[0, t]$.

**Note 4.1.** Under the conditions of (H3), $h^i(r, u) = 0$ when $u^i = 0$, then $H^+(r, u)$ is a $C^1$ function and $\frac{\partial}{\partial u} H^+(r, u) = h(r, S(u))$.

**Theorem 4.2.** If $f$ satisfies (H1)–(H3), and the following

(H4) There exists a constant $q > 1$ such that $\lim_{|u| \to +\infty} \frac{H(r, u)}{|u|^q} = 0, \lim_{|u| \to +\infty} \frac{G(r, u)}{|u|^q} = +\infty$;

(H5) There exists $u_\ast = (u_1^\ast, \ldots, u_N^\ast) \in X$ such that $\psi(u_\ast) < 0$, where $u_\ast$ is nonnegative, namely, $u_\ast$ satisfies $u_\ast^i(r) \geq 0$ for any $r \in I$ and $i = 1, 2, \ldots, N$.

then the system (1) with (3), (4) or (5) has a nontrivial nonnegative solution.

**Proof.** Let us consider

$$\psi_+(u) = \int_{T_1}^{T_2} w(r) \frac{1}{p(r)} |u'|^p dr + \int_{T_1}^{T_2} G(r, u) dr - \int_{T_1}^{T_2} H^+(r, u) dr, \quad \forall u \in X.$$

According to (H4), then there exists a positive constant $C_\theta > 1$, such that

$$\int_{T_1}^{T_2} [G(r, u) + C_\theta^q] dr - \int_{T_1}^{T_2} H^+(r, u) dr > 1, \quad \forall u \in X.$$
For any $C_r > (1 + T)C_\theta^q$ is large enough, let $R = 2p^+ [C_r + C_\theta^q]$. For any $u \in X$ and $\|u\|_p > R$, if $\|w(r)\|^{1/p} |u'(r)|_{p(r)} > 2p^+ C_r$, then we have $\psi_+(u) > C_r$.

If $\|w(r)\|^{1/p} |u'(r)|_{p(r)} \leq 2p^+ C_r$, then $|u(r)|_{p(r)} > 2C_\theta^q$, then there exists a point $r_0 \in I$ such that $|u(r_0)| > \frac{2}{r_0+1} C_\theta^q$. It is easy to see that

$$|u(r_1) - u(r_2)| = \left| \int_{r_1}^{r_2} u' \, dr \right| \leq \int_{r_1}^{r_2} |u'| \, dr \leq \int_{r_1}^{r_2} \left[ w(r) \right]^{1/p} |u'| \, dr \leq \left[ \left[ w(r) \right]^{\frac{1}{p(r)}} \right]_{p(r)} |u'|_{p(r)} \leq \frac{2}{r_0+1} C_\theta^q. \quad (23)$$

Since $C_r$ is large enough, from (23), we have $|u(r)| > \frac{1}{r_0+1} C_\theta^q$ for any $r \in I$, then we have $\psi_+(u) > C_r$. It means that $\psi_+$ is coercive on $X$. Since $\psi_+$ is a weak semicontinuous $C^1$ functional, then $\psi_+$ can attain an infimum on $X$. According to condition (H5), we can see that $\inf_{v \in X} \psi_+(v) < 0$. If $\psi_+(u) = \inf_{v \in X} \psi_+(v)$, taking $u^- = ((u^1)^-, \ldots, (u^N)^-)$ as a test function, where $(u^i)^- = \max(0, -u^i)$, from (H3) we can find that $u^- = 0$, we can obtain that $u$ is nonnegative and nontrivial. Then $u$ is a solution of (1) with (3), (4) or (5). \hfill $\square$

**Theorem 4.3.** If $f$ satisfies (H1)–(H3), and the following

(H4) $\lim_{u \to 0} \frac{H(r, u)}{|u|^{p(r)}} = 0 \text{ and } G(r, u) = \rho(r)|u|^{\theta(r)}$, where $\theta \in C(I, (1, +\infty))$ and $0 < \rho$ is continuous;

(H7) There exists $\beta > \max\{p^+, \theta^+\}$ and $D > 0$ such that $\langle h(r, u), u \rangle \geq \beta H(r, u)|u| \geq D$, then the system (1) with (3), (4) or (5) has at least three solution, including a trivial solution, a nonpositive solution, and a nonnegative solution.

In order to prove Theorem 4.3, we need to do some preparations.

Define $L, J : X \to X^*$ as

$$(L(u), v) = \int_{T_1}^{T_2} w(r)|u'|^{p(r)-2} u' v' \, dr + \int_{T_1}^{T_2} g(r, u, v) \, dr, \quad \forall u, v \in X,$$

$$(J(u), v) = \int_{T_1}^{T_2} \langle h(r, u), v \rangle \, dr, \quad \forall u, v \in X.$$

Similar to the proof of Lemma 2.6 of [5], we have

**Lemma 4.4.** Under the condition of Theorem 4.3,

(i) $L : X \to X^*$ is a continuous, bounded and strictly monotone operator;

(ii) $L$ is a mapping of type $(S_+)$, i.e., if $u_n \to u$ in $X$ and $\lim_{n \to +\infty} (L(u_n) - L(u), u_n - u) \leq 0$, then $u_n \to u$ in $X$;

(iii) $L : X \to X^*$ is a homeomorphism.

Similar to the proof of Lemma 2.6 of [5], we have

**Lemma 4.5.** The norm on $X$ is equal to $|u|_\theta = |(w(r))^{1/p} u'|_p$.

**Lemma 4.6.** Under the condition of Theorem 4.3, $\psi$ satisfies (PS) conditions, namely, if $\{u_n\} \subset X$ satisfies $\psi(u_n) \to c$ and $\psi'(u_n) \to 0$ when $n \to +\infty$, then $\{u_n\}$ possesses a convergent subsequence.

**Proof.** If $\{u_n\} \subset X$ satisfies $\psi(u_n) \to c$ and $\psi'(u_n) \to 0$ when $n \to +\infty$, and $\|u_n\| > 1$. Since the norm on $X$ is equal to $|u|_\theta + |(w(r))^{1/p} u'|_p$, we have

$$1 + c + o(1)\|u_n\| \geq \psi(u_n) - \frac{1}{\beta} \psi'(u_n), u_n$$

$$= \int_{T_1}^{T_2} \frac{1}{p(r)} |\nabla u_n|^p \, dr + \int_{T_1}^{T_2} \left( G(r, u_n) - \frac{1}{\beta} g(r, u_n, u_n) \right) \, dr - \int_{T_1}^{T_2} \left( H(r, u_n) - \frac{1}{\beta} h(r, u_n, u_n) \right) \, dx$$
Mountain Pass Lemma, then \(J\) Lemma 2.7 and the continuity of Nemytsky operator, we get enough constant.

**Proof of Theorem 4.3.** Obviously, \(u = 0\) is a trivial solution, and \(\psi(0) = 0\).

Similar to the proof of Theorem 4.2, let us consider

\[
\psi_+(u) = \int_{\Omega} \frac{w(r)|u|^m}{p(r)} dr + \int_{\Omega} g(r, u) dr - \int_{\Omega} H^+(r, u) dr, \quad u \in X.
\]

Since \(W^{m,p(r)}_{w} \hookrightarrow C(\Omega)\) is compact, then \(\max_{e \in J} |u(r)| \leq c\|u\|\). From \((H_6)\), it is easy to see that there exists a positive small enough constant \(\delta\) such that \(H^+(r, u) \leq \frac{1}{2} |u|^m \min_{e \in J} |p(r)|\) for any \(\|u\| \leq \delta\), then there exists a positive constant \(\sigma\) such that

\[
\psi_+(u) \geq \frac{1}{2} \int_{\Omega} \frac{w(r)}{p(r)} |u|^m dr + \frac{1}{2} \int_{\Omega} G(r, u) dr \geq \sigma > 0, \quad \forall u \in X \text{ with } \|u\| = \delta.
\]

Similar to the proof of Theorem 5 of [9], we have \(H(r, u) \geq C|u|^\beta\) when \(|u| \geq D\).

For any positive \(u \in X\) (each \(u^i > 0\) a.e. on \(I\)), from \((H_7)\), we have

\[
\psi_+(tu) = \int_{\Omega} \frac{w(r)}{p(r)} |tu|^m dr + \int_{\Omega} G(r, tu) dr - \int_{\Omega} H^+(r, tu) dr \to -\infty, \quad \text{as } t \to +\infty.
\]

Similar to the proof of Lemma 4.6, we can see that \(\psi_+\) satisfies (PS) conditions. Thus \(\psi_+\) satisfies the conditions of Mountain Pass Lemma, then \(\psi_+\) has a critical point \(u\) such that \(\psi_+(u) > 0\). Similar to the proof of Theorem 4.2, we get that \(u\) is a nontrivial nonnegative solution of (1) with (3), (4) or (5). Similarly, (1) with (3), (4) or (5) has a nontrivial nonpositive solution. \(\square\)

**References**


