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Operator-valued measures and linear operators

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Abstract

We study operator-valued measures $m : \Sigma \to \mathcal{L}(X, Y)$, where $\mathcal{L}(X, Y)$ stands for the space of all continuous linear operators between real Banach spaces X and Y and Σ is a σ-algebra of sets. We extend the Bartle–Dunford–Schwartz theorem and the Orlicz–Pettis theorem for vector measures to the case of operator-valued measures. We generalize the classical Vitali–Hahn–Saks theorem to sets of operator-valued measures which are compact in the strong operator topology.

Keywords: Vector measures; Operator-valued measures; Mackey topologies; Radon–Nikodym property; Strong operator topology; Weak operator topology; Weak* operator topology; Linear operators

1. Introduction and preliminaries

In the theory of vector measures the classical theorems: the Bartle–Dunford–Schwartz theorem (about the range of countably additive vector measures) (see [4]), the Orlicz–Pettis theorem (see [12]), the Vitali–Hahn–Saks theorem (see [12,20]) are of importance.

The purpose of this paper is to extend these theorems to the case of operator-valued measures $m : \Sigma \to \mathcal{L}(X, Y)$, where $\mathcal{L}(X, Y)$ stands for the space of all linear continuous operators between Banach spaces X and Y and Σ is a σ-algebra of sets. We obtain these results as consequences of the properties of the corresponding operators $T_m : L^\infty(X) \to Y$.

We denote by $\sigma(L, K)$ and $\tau(L, K)$ the weak topology and the Mackey topology on L with respect to a dual pair $\langle L, K\rangle$. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be real Banach spaces and let B_X and S_X stand for the closed unit ball and the unit sphere in X, respectively. Let X^* and Y^* stand for Banach duals of X and Y, respectively. Denote by $\mathcal{L}(X, Y)$ the space of all continuous linear operators between Banach spaces X and Y. Recall that the *uniform operator topology* (briefly UOT) is the topology on $\mathcal{L}(X, Y)$ defined by the usual norm $\|\cdot\|_{X\to Y}$. The *weak* operator topology* (briefly W*OT) is the topology on $\mathcal{L}(X, Y)$ defined by the family of seminorms $\{p_{y*}\colon y^* \in Y^*\}$, where $p_{y*}(U) := \|y^* \circ U\|_{X^*}$ for $U \in \mathcal{L}(X, Y)$. The *weak operator topology* (briefly WOT) on $\mathcal{L}(X, Y)$ is the topology defined by the family of seminorms $\{p_{x,y^*}\colon x \in X, y^* \in Y^*\}$, where $p_{x,y^*}(U) = |y^*(U(x))|$ for $U \in \mathcal{L}(X, Y)$ (see [14]).

The following general result will be needed (see [21, Theorem 13.10.13]).

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Proposition 1.1. *The space $\mathcal{L}(X,Y)$ is* WOT*-sequentially complete.*

For terminology concerning vector lattices and function spaces we refer to [1,2,15]. Let $\mathbb{N}$ and $\mathbb{R}$ stand for the sets of all natural and real numbers. Throughout the paper we assume that (Ω, Σ, μ) is a complete σ-finite measure space. By $L^0(X)$ we denote the set of μ-equivalence classes of all strongly Σ-measurable functions $f : \Omega \to X$. For $f \in L^0(X)$ let us set $\tilde{f}(\omega) = \|f(\omega)\|_X$ for $\omega \in \Omega$. Let

$$L^\infty(X) = \Big\{ f \in L^0(X)\colon \ \|f\|_\infty = \operatorname*{ess\,sup}_{\omega\in\Omega} \tilde{f}(\omega) < \infty \Big\}.$$

In case $X = \mathbb{R}$ we simply denote L^∞. Recall that the algebraic tensor product $L^\infty \otimes X$ is the subspace of $L^\infty(X)$ spanned by the functions of the form $u \otimes x$, $(u \otimes x)(\omega) = u(\omega)x$, where $u \in L^\infty$, $x \in X$.

A linear functional F on $L^\infty(X)$ is said to be *order continuous* whenever $\tilde{f}_\alpha \xrightarrow{(o)} 0$ in L^∞ implies $F(f_\alpha) \to 0$. The set consisting of all order continuous linear functionals on $L^\infty(X)$ will be denoted by $L^\infty(X)^\sim_n$ and called the *order continuous* dual of $L^\infty(X)$.

Let $L^0(X^*, X)$ be the set of weak*-equivalence classes of all weak*-measurable functions $g : \Omega \to X^*$. Following [7,8] one can define the so-called *abstract norm* $\vartheta : L^0(X^*, X) \to L^0$ by $\vartheta(g) := \sup\{|g_x|\colon x \in B_X\}$, where $g_x(\omega) = g(\omega)(x)$ for $\omega \in \Omega$ and $x \in X$. Then for $f \in L^0(X)$ and $g \in L^0(X^*, X)$ the function $\langle f, g\rangle : \Omega \to \mathbb{R}$ defined by $\langle f, g\rangle(\omega) := \langle f(\omega), g(\omega)\rangle$ is measurable and $|\langle f, g\rangle| \leqslant \tilde{f}\,\vartheta(g)$. Moreover, $\vartheta(g) = \tilde{g}$ for $g \in L^0(X^*)$. Let

$$L^1(X^*, X) := \big\{ g \in L^0\big(X^*, X\big)\colon \ \vartheta(g) \in L^1 \big\}.$$

Due to Bukhvalov (see [7, Theorem 4.1]) $L^\infty(X)^\sim_n$ can be identified with $L^1(X^*, X)$ throughout the mapping: $L^1(X^*, X) \ni g \mapsto F_g \in L^\infty(X)^\sim_n$, where

$$F_g(f) = \int_\Omega \big\langle f(\omega), g(\omega)\big\rangle\, d\mu \quad \text{for all } f \in L^\infty(X).$$

Then $L^1(X^*) \subset L^1(X^*, X)$. Moreover, the identity $L^1(X^*, X) = L^1(X^*)$ holds whenever X^* has the Radon–Nikodym property (see [8, Theorem 3.5], [12, Chapter 3.1]).

Now we recall basic terminology concerning operator-valued measures (see [3,5,6,13,16,17]). A finitely additive mapping $m : \Sigma \to \mathcal{L}(X,Y)$ is called an *operator-valued measure*. We define the *semivariation* $\tilde{m}(A)$ of m on $A \in \Sigma$ by $\tilde{m}(A) = \sup \|\Sigma m(A_i)(x_i)\|_Y$, where the supremum is taken over all finite Σ-partitions (A_i) of A and $x_i \in B_X$ for each i. For $y^* \in Y^*$, let $m_{y^*} : \Sigma \to X^*$ be a set function defined by $m_{y^*}(A)(x) := \langle m(A)(x), y^*\rangle$ for $x \in X$. Then m_{y^*} is a finite additive measure and $\tilde{m}_{y^*}(A) = |m_{y^*}|(A)$, where $|m_{y^*}|(A)$ stands for the variation of m_{y^*} on $A \in \Sigma$. It is known that $\tilde{m}(A) < \infty$ if and only if $|m_{y^*}|(A) < \infty$ for every $y^* \in Y^*$. Moreover, $\tilde{m}(A) = \sup\{|m_{y^*}|(A)\colon y^* \in B_{Y^*}\}$ for $A \in \Sigma$ (see [3, Theorem 5]).

By $\operatorname{fasv}(\Sigma, \mathcal{L}(X,Y))$ we denote the set of all finitely additive measures $m : \Sigma \to \mathcal{L}(X,Y)$ with a finite semivariation, i.e., $\tilde{m}(\Omega) < \infty$. Recall that $m \in \operatorname{fasv}(\Sigma, \mathcal{L}(X,Y))$ is said to be *variationally semi-regular* if $\tilde{m}(A_n) \to 0$ whenever $A_n \downarrow \emptyset$, $(A_n) \subset \Sigma$. $m \in \operatorname{fasv}(\Sigma, \mathcal{L}(X,Y))$ is said to be *countably additive in* UOT if $\|m(A_n)\|_{X\to Y} \to 0$ whenever $A_n \downarrow \emptyset$, $(A_n) \subset \Sigma$. $m \in \operatorname{fasv}(\Sigma, \mathcal{L}(X,Y))$ is said to be *countably additive in* W*OT if for every $y^* \in Y^*$, $\|m_{y^*}(A_n)\|_{X^*} \to 0$ whenever $A_n \downarrow \emptyset$, $(A_n) \subset \Sigma$ (see [3, p. 921], [17, p. 382]).

From now on by $\operatorname{fasv}_\mu(\Sigma, \mathcal{L}(X,Y))$ we will denote the set of all $m \in \operatorname{fasv}(\Sigma, \mathcal{L}(X,Y))$ which vanish on μ-null sets, i.e., $m(A) = 0$ whenever $\mu(A) = 0$. It is well known that if $m \in \operatorname{fasv}_\mu(\Sigma, \mathcal{L}(X,Y))$, then $T_m(f) = \int_\Omega f(\omega)\, dm$ defines a continuous linear operator from $L^\infty(X)$ into Y, and any continuous linear operator $T : L^\infty(X) \to Y$ is given this way. Moreover, $\|T_m\|_{L^\infty(X)\to Y} = \tilde{m}(\Omega)$ (see [13, §9]).

2. Countably additive operator-valued measures

The Mackey topology $\tau(L^\infty, L^1)$ on L^∞ is of importance in the theory of vector measures (see [20]). It is well known that $\tau(L^\infty, L^1)$ is the finest locally convex-solid topology on L^∞ with the Lebesgue property. Schaefer and Xiao-Dong Zhang [20] provide a characterization of the countable additivity of vector measures $m : \Sigma \to Y$ in terms of continuity of the corresponding linear operators T_m from L^∞ (provided with $\tau(L^\infty, L^1)$) into a Banach space Y.

In this section we characterize countable additivity of measures $m : \Sigma \to \mathcal{L}(X,Y)$ in W*OT in terms of continuity of the corresponding linear operators $T_m : L^\infty(X) \to Y$.

Theorem 2.1. *For every $m \in \mathrm{fasv}_\mu(\Sigma, \mathcal{L}(X,Y))$ the following statements are equivalent*:

(i) T_m *is* $(\tau(L^\infty(X), L^\infty(X)_n^\sim), \|\cdot\|_Y)$*-continuous.*
(ii) T_m *is* $(\sigma(L^\infty(X), L^\infty(X)_n^\sim), \sigma(Y, Y^*))$*-continuous.*
(iii) m *is countably additive in* W*OT.

Proof. (i) $\Leftrightarrow$ (ii) See [22, Corollaries 11-1-3, 11-2-6].

(ii) $\Rightarrow$ (iii) Assume that T_m is $(\sigma(L^\infty(X), L^\infty(X)_n^\sim), \sigma(Y, Y^*))$-continuous. It follows that $y^* \circ T_m \in L^\infty(X)_n^\sim$ for every $y^* \in Y^*$. To show that m is countably additive in W*OT, assume that $A_n \downarrow \emptyset$, $(A_n) \subset \Sigma$ and $y^* \in Y^*$. We show that $\|m_{y^*}(A_n)\|_{X^*} \to 0$. Indeed, for every $n \in \mathbb{N}$ there exists $x_n \in S_X$, such that

$$\|m_{y^*}(A_n)\|_{X^*} = \|y^*(m(A_n))\|_{X^*} \leqslant |\langle m(A_n)(x_n), y^*\rangle| + \frac{1}{n}.$$

Let $f_n = \mathbb{1}_{A_n} \otimes x_n$ for $n \in \mathbb{N}$. Then $\tilde{f}_n = \mathbb{1}_{A_n}$ and $\mathbb{1}_{A_n} \xrightarrow{(o)} 0$ in L^∞. Hence $(y^* \circ T_m)(f_n) \to 0$, because $y^* \circ T_m \in L^\infty(X)_n^\sim$. But for $n \in \mathbb{N}$ we have

$$(y^* \circ T_m)(f_n) = \langle T_m(\mathbb{1}_{A_n} \otimes x_n), y^*\rangle = \langle m(A_n)(x_n), y^*\rangle.$$

It follows that $\|m_{y^*}(A_n)\|_{X^*} \to 0$, as desired.

(iii) $\Rightarrow$ (ii) Assume that m is countably additive in W*OT, i.e., for every $y^* \in Y^*$, the measure $m_{y^*} : \Sigma \to X^*$ is countably additive and $|m_{y^*}|(\Omega) = \tilde{m}_{y^*}(\Omega) < \infty$, $m_{y^*}(A) = 0$ if $\mu(A) = 0$. For every $k \in \mathbb{N}$ let $\Sigma_k = \{A \cap \Omega_k\colon A \in \Sigma\}$, where (Ω_k) is a pairwise disjoint sequence in Σ such that $\Omega = \bigcup_{k=1}^\infty \Omega_k$ and $\mu(\Omega_k) < \infty$ for $k \in \mathbb{N}$. Let $y^* \in Y^*$ be given. Then by the Radon–Nikodym type theorem (see [10, Theorem 1.5.3]), for every $k \in \mathbb{N}$ there exists a weak*-measurable function $g_{k,y^*} : \Omega_k \to X^*$ which satisfies the following conditions:

1. the function $\Omega_k \ni \omega \mapsto \|g_{k,y^*}(\omega)\|_{X^*} \in \mathbb{R}$ is Σ_k-measurable and integrable,
2. for every $x \in X$ and every $A \in \Sigma$,
$$m_{y^*}(A \cap \Omega_k)(x) = \int_{A\cap\Omega_k} \langle x, g_{k,y^*}(\omega)\rangle\, d\mu,$$
3. for every $A \in \Sigma$
$$|m_{y^*}|(A \cap \Omega_k) = \int_{A\cap\Omega_k} \|g_{k,y^*}(\omega)\|_{X^*}\, d\mu.$$

Define a function $g_{y^*} : \Omega \to X^*$ by setting $g_{y^*}(\omega) = g_{k,y^*}(\omega)$ for $\omega \in \Omega_k$, i.e., $\mathbb{1}_{\Omega_k} g_{y^*} = g_{k,y^*}$ for all $k \in \mathbb{N}$. Then g_{y^*} is weak*-measurable and since the measure $|m_{y^*}| : \Sigma \to [0, \infty)$ is countably additive, we have

$$|m_{y^*}|(\Omega) = |m_{y^*}|\left(\bigcup_{k=1}^\infty \Omega_k\right) = \sum_{k=1}^\infty |m_{y^*}|(\Omega_k) = \sum_{k=1}^\infty \int_{\Omega_k} \|g_{k,y^*}(\omega)\|_{X^*}\, d\mu = \int_\Omega \|g_{y^*}(\omega)\|_{X^*}\, d\mu.$$

It follows that the function $\Omega \ni \omega \mapsto \|g_{y^*}(\omega)\|_{X^*} \in \mathbb{R}$ is integrable; hence the w^*-equivalence class of g_{y^*} belongs to $L^1(X^*, X)$. Note that for every $x \in X$ and $A \in \Sigma$ we have:

$$\begin{aligned} m_{y^*}(A)(x) &= \sum_{k=1}^\infty m_{y^*}(A \cap \Omega_k)(x) = \sum_{k=1}^\infty \int_{A\cap\Omega_k} \langle x, g_{k,y^*}(\omega)\rangle\, d\mu = \int_A \langle x, g_{y^*}(\omega)\rangle\, d\mu \\ &= \int_\Omega \langle (\mathbb{1}_A \otimes x)(\omega), g_{y^*}(\omega)\rangle\, d\mu. \end{aligned}$$

Hence for every $s = \sum_{i=1}^n (\mathbb{1}_{A_i} \otimes x_i) \in \mathcal{S}(\Sigma, X)$ (= the set of X-valued Σ-simple functions) we get

$$\int_\Omega s(\omega)\, dm_{y^*} = \int_\Omega \langle s(\omega), g_{y^*}(\omega)\rangle\, d\mu.$$

Since $\mathcal{S}(\Sigma, X)$ is dense in $(L^\infty(X), \|\cdot\|_\infty)$, one can easily show that for $f \in L^\infty(X)$

$$\int_\Omega f(\omega)\, dm_{y^*} = \int_\Omega \langle f(\omega), g_{y^*}(\omega)\rangle\, d\mu$$

and moreover,

$$y^*\left(\int_\Omega f(\omega)\, dm\right) = \int_\Omega f(\omega)\, dm_{y^*}.$$

Now let $f_\alpha \to 0$ in $L^\infty(X)$ for $\sigma(L^\infty(X), L^1(X^*, X))$. Then

$$y^*\big(T_m(f_\alpha)\big) = y^*\left(\int_\Omega f_\alpha(\omega)\, dm\right) = \int_\Omega f_\alpha(\omega)\, dm_{y^*} = \int_\Omega \langle f_\alpha(\omega), g_{y^*}(\omega)\rangle\, d\mu \xrightarrow[\alpha]{} 0.$$

This means that T_m is $(\sigma(L^\infty(X), L^1(X^*, X)), \sigma(Y, Y^*))$-continuous, as desired. □

Now we are in position to generalize the classical Bartle–Dunford–Schwartz theorem concerning the range of countably additive vector measures (see [4]) to the case of operator-valued measures.

Theorem 2.2. *Let $m \in \mathrm{fasv}_\mu(\Sigma, \mathcal{L}(X, Y))$ be countably additive in* W*OT. *Then the range $\{m(A)\colon A \in \Sigma\}$ of m is a relatively* WOT*-sequentially compact subset of $\mathcal{L}(X, Y)$.*

Proof. It is well known that $\{\mathbb{1}_A\colon A \in \Sigma\}$ is a relatively $\sigma(L^\infty, L^1)$-sequentially compact subset of L^∞ (see [9, Corollary 5.2]).

Let $(m(A_n))$ be a sequence in $\mathcal{L}(X, Y)$. Then there exists a $\sigma(L^\infty, L^1)$-Cauchy subsequence $(\mathbb{1}_{A_{k_n}})$ of $(\mathbb{1}_{A_n})$. Now, let $g \in L^1(X^*, X)$ and $x \in X$. Then $|g_x| \leqslant \|x\|_X \vartheta(g)$, so $g_x \in L^1$. Then for $m, n \in \mathbb{N}$ we have

$$\begin{aligned}\big|F_g(\mathbb{1}_{A_{k_n}} \otimes x) - F_g(\mathbb{1}_{A_{k_m}} \otimes x)\big| &= \left|\int_\Omega \langle \mathbb{1}_{A_{k_n}}(\omega)x - \mathbb{1}_{A_{k_m}}(\omega)x, g(\omega)\rangle\, d\mu\right| \\ &= \left|\int_\Omega \big(\mathbb{1}_{A_{k_n}}(\omega) - \mathbb{1}_{A_{k_m}}(\omega)\big) g_x(\omega)\, d\mu\right|\end{aligned}$$

and this means that $(\mathbb{1}_{A_{k_n}} \otimes x)$ is a $\sigma(L^\infty(X), L^1(X^*, X))$-Cauchy sequence. In view of Theorem 2.1 the operator T_m is $(\sigma(L^\infty(X), L^1(X^*, X)), \sigma(Y, Y^*))$-continuous, so $(m(A_{k_n})(x))$ $(= (T_m(\mathbb{1}_{A_{k_n}} \otimes x)))$ is a $\sigma(Y, Y^*)$-Cauchy sequence in Y. This means that $(m(A_{k_n}))$ is a WOT-Cauchy sequence in $\mathcal{L}(X, Y)$, and by Proposition 1.1 there exists $U \in \mathcal{L}(X, Y)$ such that $m(A_{k_n}) \to U$ for WOT. □

3. An Orlicz–Pettis type theorem for operator-valued measures

In this section we derive an Orlicz–Pettis type theorem for operator-valued measures.

Definition 3.1. A linear operator $T : L^\infty(X) \to Y$ is said to be *smooth* (respectively σ*-smooth*) if $\tilde{f}_\alpha \xrightarrow{(o)} 0$ in L^∞ (respectively $\tilde{f}_n \xrightarrow{(o)} 0$ in L^∞) implies $\|T(f_\alpha)\|_Y \to 0$ (respectively $\|T(f_n)\|_Y \to 0$).

Proposition 3.1. *Let $m \in \mathrm{fasv}_\mu(\Sigma, \mathcal{L}(X, Y))$ and assume that $T_m : L^\infty(X) \to Y$ is σ-smooth. Then m is variationally semi-regular.*

Proof. Indeed, let $A_n \downarrow \emptyset$, $(A_n) \subset \Sigma$. Then for every n there exist a finite Σ-partition $(A_{n,i})_{i=1}^{k_n}$ of A_n and $x_{n,i} \in B_X$, $1 \leqslant i \leqslant k_n \in \mathbb{N}$ such that

$$\tilde{m}(A_n) \leqslant \left\|\sum_{i=1}^{k_n} m(A_{n,i})(x_{n,i})\right\|_Y + \frac{1}{n}.$$

Let $s_n = \sum_{i=1}^{k_n} (\mathbb{1}_{A_{n,i}} \otimes x_{n,i})$ for $n \in \mathbb{N}$. Then $\tilde{s}_n \leqslant \mathbb{1}_{A_n} \leqslant \mathbb{1}_{\Omega}$ for $n \in \mathbb{N}$, so $\tilde{s}_n \to 0$ μ-a.e., i.e., $\tilde{s}_n \xrightarrow{(o)} 0$ in L^∞. Hence

$$\left\| T_m(s_n) \right\|_Y = \left\| \sum_{i=1}^{k_n} m(A_{n,i})(x_{n,i}) \right\|_Y \xrightarrow[n]{} 0,$$

so $\tilde{m}(A_n) \to 0$, as desired. □

Now, we recall briefly some concepts and results concerning locally solid topologies on vector-valued function spaces (see [18] for more details). A subset H of $L^\infty(X)$ is said to be *solid* whenever $\tilde{f_1} \leqslant \tilde{f_2}$ and $f_1 \in L^\infty(X)$ and $f_2 \in H$ imply $f_1 \in H$. A linear topology τ on $L^\infty(X)$ is said to be locally solid if it has a local base at zero consisting of solid sets. A locally solid topology τ on $L^\infty(X)$ is said to be a *Lebesgue topology* whenever $\tilde{f}_\alpha \xrightarrow{(o)} 0$ in L^∞ implies $f_\alpha \to 0$ for τ.

Assume that ξ is a locally convex-solid topology on L^∞ that is generated by a family $\{p_\alpha\colon \alpha \in \mathcal{A}\}$ of Riesz seminorms on L^∞. By putting $\overline{p}_\alpha(f) = p_\alpha(\tilde{f})$ for $f \in L^\infty(X)$ we obtain a family $\{\overline{p}_\alpha\colon \alpha \in \mathcal{A}\}$ of seminorms on $L^\infty(X)$ that generates a locally convex-solid topology $\bar{\xi}$ on $L^\infty(X)$ (called the *topology associated* with ξ) (see [18] for more details).

The following theorem will be of importance (see [19, Theorem 4.1]).

Theorem 3.2. *Assume that X^* has the Radon–Nikodym property. Then the Mackey topology $\tau(L^\infty(X), L^\infty(X)_n^\sim)$ is a locally convex-solid topology with the Lebesgue property. Moreover, we have*:

$$\tau\big(L^\infty(X), L^\infty(X)_n^\sim\big) = \tau\big(L^\infty(X), L^1(X^*)\big) = \overline{\tau\big(L^\infty, L^1\big)}.$$

Remark. (i) One can show that if X^* has the Radon–Nikodym property then $\tau(L^\infty(X), L^\infty(X)_n^\sim)$ coincides with the natural mixed topology $\gamma_{L^\infty(X)}$ on L^∞ (see [19, Theorem 4.3] and [11,18] for more details).

(ii) In [11, Corollary 3.14] it is shown that $\gamma_{L^\infty(X)} = \tau(L^\infty(X), L^1(X^*))$ whenever μ is a positive Radon measure on a compact topological space and X is a Banach space whose dual is separable (hence X^* has the Radon–Nikodym property).

Now we are ready to state an Orlicz–Pettis type theorem for operator-valued measures.

Theorem 3.3. *Assume that X^* has the Radon–Nikodym property. Then for $m \in \mathrm{fasv}_\mu(\Sigma, \mathcal{L}(X, Y))$ the following statements are equivalent*:

(i) *T_m is smooth.*
(ii) *T_m is σ-smooth.*
(iii) *m is variationally semi-regular.*
(iv) *m is countably additive in* UOT.
(v) *m is countably additive in* W*OT.

Proof. (i) ⇒ (ii) It is obvious.

(ii) ⇒ (iii) See Proposition 3.1.

(iii) ⇒ (iv) Clearly, because $\|m(A)\|_{X\to Y} \leqslant \tilde{m}(A)$ for $A \in \Sigma$.

(iv) ⇒ (v) It is obvious because W*OT ⊂ UOT.

(v) ⇒ (i) Assume that m is countably additive in W*OT. Then by Theorem 2.1 T_m is $(\tau(L^\infty(X), L^\infty(X)_n^\sim), \|\cdot\|_Y)$-continuous. Hence, since $\tau(L^\infty(X), L^\infty(X)_n^\sim)$ is a Lebesgue topology (see Theorem 3.2), we obtain that T_m is smooth. □

4. Vitali–Hahn–Saks type theorems for operator-valued measures

Schaefer and Xiao-Dong Zhang [20] has obtained a generalization of the classical Vitali–Hahn–Saks theorem to compact sets (in the strong operator topology) of countably additive vector measures. In this section we extend this result to the operator-valued case.

Let $\mathcal{L}_\tau(L^\infty(X), Y)$ stand for the space of all $(\tau(L^\infty(X), L^\infty(X)_n^\sim), \|\cdot\|_Y)$-continuous linear operators from $L^\infty(X)$ to Y. *The strong operator topology* (briefly SOT) is a locally convex topology on $\mathcal{L}_\tau(L^\infty(X), Y)$ defined by the family of seminorms $\{p_f\colon f \in L^\infty(X)\}$, where $p_f(T) = \|T(f)\|_Y$ for all $T \in \mathcal{L}_\tau(L^\infty(X), Y)$. *The weak operator topology* (briefly WOT) is a locally convex topology on $\mathcal{L}_\tau(L^\infty(X), Y)$ defined by the family of seminorms $\{p_{f,y^*}\colon f \in L^\infty(X),\ y^* \in Y^*\}$, where $p_{f,y^*}(T) = |\langle T(f), y^*\rangle|$ for all $T \in \mathcal{L}_\tau(L^\infty(X), Y)$.

Note that for $Y = \mathbb{R}$, both SOT and WOT coincide on $L^\infty(X)_n^\sim$ with $\sigma(L^\infty(X)_n^\sim, L^\infty(X))$.

In view of [19, Corollary 5.3] we have:

Theorem 4.1. *Assume that X^* has the Radon–Nikodym property. Then the space $\mathcal{L}_\tau(L^\infty(X), Y)$ is* WOT*-sequentially complete.*

Moreover, by [19, Theorem 5.5] we have:

Theorem 4.2. *Assume that X^* has the Radon–Nikodym property and let $\mathcal{K}$ be a SOT-compact subset of $\mathcal{L}_\tau(L^\infty(X), Y)$. Then the set $\mathcal{K}$ is $(\tau(L^\infty(X), L^1(X^*)), \|\cdot\|_Y)$-equicontinuous.*

As an application of [19, Theorem 5.6] and Theorem 3.3 we have a Vitali–Hahn–Saks type theorem for operator-valued measures.

Theorem 4.3. *Assume that X^* has the Radon–Nikodym property and let $m_n \in \mathrm{fasv}_\mu(\Sigma, \mathcal{L}(X, Y))$ be countably additive in* UOT *for $n \in \mathbb{N}$. Assume that for every $f \in L^\infty(X)$,*

$$T(f) := \lim T_{m_n}(f) = \lim \int_\Omega f(\omega)\, dm_n \quad \text{exists in } \big(Y, \|\cdot\|_Y\big).$$

Then the following statements hold:

(i) *$T \in \mathcal{L}_\tau(L^\infty(X), Y)$ and the measure $m : \Sigma \to \mathcal{L}(X, Y)$, defined by*

$$m(A)(x) = T(\mathbb{1}_A \otimes x) \quad \text{for } A \in \Sigma \text{ and } x \in X,$$

belongs to $\mathrm{fasv}_\mu(\Sigma, \mathcal{L}(X, Y))$ *and is also countably additive in* UOT.

(ii) *$\{T_{m_n}\colon n \in \mathbb{N}\}$ is a $(\tau(L^\infty(X), L^1(X^*)), \|\cdot\|_Y)$-equicontinuous subset of $\mathcal{L}_\tau(L^\infty(X), Y)$.*

Now we are ready, to state a generalization of the Vitali–Hahn–Saks theorem to SOT-compact sets of $\mathcal{L}_\tau(L^\infty(X), Y)$ (see [20, Theorem 8]).

Theorem 4.4. *Assume that X^* has the Radon–Nikodym property. Let $\mathcal{M}$ be a subset of* $\mathrm{fasv}_\mu(\Sigma, \mathcal{L}(X, Y))$ *and assume that $\{T_m\colon m \in \mathcal{M}\}$ is a* SOT*-compact subset of $\mathcal{L}_\tau(L^\infty(X), Y)$. Then the set $\mathcal{M}$ is uniformly variationally semi-regular, i.e., $\sup_{m\in\mathcal{M}} \tilde{m}(A_n) \xrightarrow[n]{} 0$ as $A_n \downarrow \emptyset$. Hence $\mathcal{M}$ is uniformly countably additive in* UOT*, i.e., $\sup_{m\in\mathcal{M}} \|m(A_n)\|_{X\to Y} \xrightarrow[n]{} 0$ as $A_n \downarrow \emptyset$.*

Proof. Let $\mathcal{C}$ stand for the family of all absolutely convex, solid, $\sigma(L^1, L^\infty)$-compact subsets of L^1. In view of Theorem 3.2 the Mackey topology $\tau(L^\infty(X), L^1(X^*))$ is generated by the family $\{\overline{p}_Z\colon Z \in \mathcal{C}\}$ of seminorms on $L^\infty(X)$, where $\overline{p}_Z(f) = p_Z(\tilde{f}) = \sup\{\int_\Omega \tilde{f}(\omega)|v(\omega)|\, d\mu\colon v \in Z\}$. Let $\varepsilon > 0$ be given. Then by Theorem 4.2 there exist $Z_i \in \mathcal{C}$ for $1 \leqslant i \leqslant i_0$ for some $i_0 \in \mathbb{N}$ and $\delta > 0$ such that

$$\sup_{m\in\mathcal{M}} \|T_m(f)\|_Y \leqslant \frac{\varepsilon}{2} \quad \text{whenever} \quad \max_{1\leqslant i\leqslant i_0} \bar{p}_{Z_i}(f) = \max_{1\leqslant i\leqslant i_0} p_{Z_i}(\tilde{f}) \leqslant \delta. \tag{1}$$

Let $A_n \downarrow \emptyset$, $(A_n) \subset \Sigma$. Then $\mathbb{1}_{A_n}(\omega) \to 0$ μ-a.e. and $\mathbb{1}_{A_n} \leqslant \mathbb{1}_\Omega$ for $n \in \mathbb{N}$. Since $\tau(L^\infty, L^1)$ is a Lebesgue topology, we get $p_Z(\mathbb{1}_{A_n}) \to 0$ for every $Z \in \mathcal{C}$. Hence there exists $n_0 \in \mathbb{N}$ such that $\max_{1\leqslant i\leqslant i_0} p_{Z_i}(\mathbb{1}_{A_n}) \leqslant \delta$ for $n \geqslant n_0$.

Now let $n \in \mathbb{N}$ be fixed. Then for every $m \in \mathcal{M}$ there exist a finite Σ-partition $(A^m_{n,j})_{j=1}^{k_{m,n}}$ of A_n and $x^m_{n,j} \in B_X$ for $1 \leqslant j \leqslant k_{m,n}$ such that

$$\tilde{m}(A_n) \leqslant \left\| \sum_{j=1}^{k_{m,n}} m\big(A^m_{n,j}\big)\big(x^m_{n,j}\big) \right\|_Y + \frac{\varepsilon}{2}. \tag{2}$$

Let $s^m_n = \sum_{j=1}^{k_{m,n}} (\mathbb{1}_{A^m_{n,j}} \otimes x^m_{n,j})$ for $m \in \mathcal{M}$. Then $\tilde{s}^m_n \leqslant \mathbb{1}_{A_n}$ for every $m \in \mathcal{M}$ and $\max_{1\leqslant i\leqslant n_0} \bar{p}_{Z_i}(s^m_n) \leqslant \max_{1\leqslant i\leqslant i_0} p_{Z_i}(\mathbb{1}_{A_n})$. Hence by (1) and (2) for $n \geqslant n_0$ we get:

$$\sup_{m\in\mathcal{M}} \tilde{m}(A_n) \leqslant \sup_{m\in\mathcal{M}} \big\|T_m\big(s^m_n\big)\big\|_Y + \frac{\varepsilon}{2} \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This means that $\sup_{m\in\mathcal{M}} \tilde{m}(A_n) \xrightarrow[n]{} 0$, as desired.

It follows that $\sup_{m\in\mathcal{M}} \|m(A_n)\|_{X\to Y} \xrightarrow[n]{} 0$, because $\|m(A_n)\|_{X\to Y} \leqslant \tilde{m}(A_n)$ for $n \in \mathbb{N}$. $\square$

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