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Spaces with sharp bases and with other special bases of countable order

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Dedicated, with deep respect and admiration, to the memory of Professor Jun-iti Nagata

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ABSTRACT

We study spaces with sharp bases and bases of countable order. A characterization of spaces with external bases of countable order is established (Theorem 2.7). Some necessary and sufficient conditions for a space $X \times S$, where S is the convergent sequence, to have a sharp base are given (Theorem 3.2). It follows that a pseudocompact space X is metrizable iff $X \times S$ has a sharp base (Corollary 3.3). It is proved that a sharp base of finite rank is a uniform base (Theorem 4.4). Some other new results are also obtained, and some open questions are formulated.

1. Introduction

One of the important problems in general topology is to establish concrete connections between various classes of spaces in terms of continuous mappings satisfying certain additional conditions. There is a long and fruitful tradition in general topology to use natural restrictions on bases and to characterize or to introduce various important classes of spaces. In particular, this was done by J. Nagata [25], R. Bing [14] and Ju. Smirnov [28] (see [16]). The work in this direction has lead to discovery of important kinds of bases (like uniform bases, point-countable bases, bases of countable order, ortho-bases, bases of finite and sub-infinite rank and others). Some new properties of spaces with bases of countable order and of spaces with sharp bases are studied. In particular, we introduce some new kinds of sharp bases and of bases of countable order. Some open questions are formulated.

We use the terminology from [16]. Any space we consider is assumed to be a T_1-space, $\omega = \{0, 1, 2, \ldots\}$, the closure of a set A in a space X is denoted by $cl_X A$ or $cl\, A$.

2. Bases of countable order

A *monotonic* collection of sets is a family such that for every two of its elements one is a subset of the other. An indexed family of sets is *perfectly monotonic* if for every two of its elements one is a proper subset of the other.

A base $\mathcal{B}$ of a space X is called:

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– a base of *countable order* if for any infinite perfectly decreasing sequence $\{U_n \in \mathcal{B}:\ n \in \omega\}$ and any point $x \in \bigcap\{U_n:\ n \in \omega\}$ the sequence $\{U_n:\ n \in \omega\}$ is a base for X at x (see [5,9]);
– a *strong base of countable order* if for any infinite perfectly monotonic sequence $\{U_n \in \mathcal{B}:\ n \in \omega\}$ and a point $x \in \bigcap\{U_n:\ n \in \omega\}$ the sequence $\{U_n:\ n \in \omega\}$ is a base for X at the point x.

Now we introduce a relative version of the concept of a base of countable order and a corresponding to it concept of co-embedding.

Let X be a topological space and Y a subspace of X. Recall that an *external base* of Y in X is a family $\mathcal{B}$ of open subsets of X such that for each $y \in Y$ and each open neighbourhood U of y in X there exists $V \in \mathcal{B}$ satisfying the condition $y \in V \subset U$ (see [3]). If an external base $\mathcal{B}$ of Y in X has the property that every strictly decreasing sequence η of elements of $\mathcal{B}$ such that $\bigcap \eta$ contains a point y of Y is a base of the space X at y, then we call $\mathcal{B}$ an *external base of countable order* of Y in X. If there exists an external base of countable order of Y in X, then we say that Y is *co-embedded* in X.

Let $\mathcal{B}$ be a strong base of countable order of the space X. If $\{U_i \in \mathcal{B}:\ i \in \omega\}$ is an increasing sequence of sets, i.e. $U_i \subseteq U_{i+1}$ for any $i \in \omega$, then there exists $m \in \omega$ such that $U_i = U_m$ for each $i \geqslant m$.

Let X be a subspace of a space Z, $\{\gamma_n = \{U_\alpha:\ \alpha \in A_n\}:\ n \in \omega\}$ be a sequence of families of open subsets of a space Z, and let $\{\pi_n : A_{n+1} \to A_n:\ n \in \omega\}$ be a sequence of mappings. A sequence $\alpha = \{\alpha_n:\ n \in \omega\}$ is a *c-sequence* if $\alpha_n \in A_n$ and $\pi_n(\alpha_{n+1}) = \alpha_n$ for every n.

Consider the following conditions:

$(B1)$ $X \subseteq \bigcup\{U_\alpha:\ \alpha \in A_0\}$.
$(B2)$ $X \cap U_\alpha \subseteq \bigcup\{U_\beta:\ \beta \in \pi_n^{-1}(\alpha)\} \subseteq U_\alpha$ for all $\alpha \in A_n$ and $n \in \omega$.
$(B3)$ For any c-sequence $\alpha = \{\alpha_n \in A_n:\ n \in \omega\}$ and any point $x \in \bigcap\{X \cap U_{\alpha_n}:\ n \in \omega\}$, the sequence $\{U_{\alpha_n}; n \in \omega\}$ is a base for Z at the point x.

If the sequences $\{\gamma_n:\ n \in \omega\}$ and $\{\pi_n:\ n \in \omega\}$ have Properties $(B1)$, $(B2)$ and $(B3)$, then they form *a sieve-base* of X in the space Z (see [15]). If $X = Z$, then the sequences $\{\gamma_n:\ n \in \omega\}$ and $\{\pi_n:\ n \in \omega\}$ form *a sieve-base* on X.

Remark 2.1. If a space X has a sieve-base in a T_3-space Z, then there exists a sieve-base $\{\gamma_n = \{U_\alpha:\ \alpha \in A_n\}:\ n \in \omega\}$, $\{p_n : A_{n+1} \to A_n:\ n \in \omega\}$ on X in the space Z with the next property:

$(B4)$ $\bigcup\{cl_Z\, U_\beta:\ \beta \in \pi_n^{-1}(\alpha)\} \subseteq U_\alpha$ for all $\alpha \in A_n$ and $n \in \omega$.

Proposition 2.2. *Let X be a subspace of a space Z. Then the following conditions are equivalent*:

1. *X has a base of countable order in Z, i.e. X is co-embedded in Z*;
2. *X has a sieve-base in Z.*

Proof. $1 \to 2$. Let $\mathcal{B}$ be an external base of countable order of X in Z. Assume that $U \cap X \neq \varnothing$ for any $U \in \mathcal{B}$.

Fix $U \in \mathcal{B}$. If U is a singleton, then $A(U)$ is a singleton, $U_\beta = U$ for any $\beta \in A(U)$, and $\gamma(U) = \{U_\beta:\ \beta \in A(U)\}$. If U is not a singleton, then $\gamma(U) = \{U_\beta:\ \beta \in A(U)\}$ is the family $\{V \in \mathcal{B}:\ V \subseteq U, V \neq U\}$.

Now we assume that $A(U) \cap A(V) = \varnothing$ for any distinct U and V.

Let $\gamma_1 = \mathcal{B} = \{U_\alpha:\ \alpha \in A_1\}$. If the family $\gamma_n = \{U_\alpha:\ \alpha \in A_n\}$ is constructed and $\gamma_n \subseteq \mathcal{B}$, then $A_{n+1} = \bigcup\{A(U_\alpha):\ \alpha \in A_n\}$, $\gamma_{n+1} = \{U_\beta:\ \beta \in A_{n+1}\} = \bigcup\{\gamma(U):\ U \in \gamma_n\}$, and $\pi_n^{-1}(\alpha) = A(U_\alpha)$ for any $\alpha \in A_n$. Clearly, the sequences $\{\gamma_n:\ n \in \omega\}$ and $\{\pi_n:\ n \in \omega\}$ form a sieve-base of X in the space Z.

$2 \to 1$. Let $\{\gamma_n = \{U_\alpha:\ \alpha \in A_n\}:\ n \in \omega\}$ and $\{p_n : A_{n+1} \to A_n:\ n \in \omega\}$ be a sieve-base of X in the space Z. There exist some well-orderings on the sets $\{A_n:\ n \in \omega\}$ such that $p_n(\alpha) \leqslant p_n(\beta)$ provided $\alpha, \beta \in A_{n+1}$, $n \in \omega$ and $\alpha \leqslant \beta$. We assume that if $\alpha \in A_n$, $n \in \omega$, and U_α is not a singleton, then U_β is a proper subset of U_α for any $\beta \in p_n^{-1}(\alpha)$.

Put $B_n = \{\alpha \in A_n:\ (X \cap U_\alpha) \setminus \bigcup\{U_\beta:\ \beta \in A_n, \beta < \alpha\} \neq \varnothing\}$ and $\mathcal{B} = \{U_\alpha:\ \alpha \in B_n, n \in \omega\}$.

Let $x \in X$. Denote by $\alpha_n(x)$ the first element of A_n such that $x \in U_{\alpha_n(x)}$. Clearly, $\alpha_n(x) \in B_n$ and $p_n(\alpha_{n+1}(x)) = \alpha_n(x)$ for each $n \in \omega$. Therefore $\{U_{\alpha_n(x)}:\ n \in \omega\}$ is a base of Z at the point x and $\mathcal{B}$ is an external base for the space X in the space Z.

Let $m > n$ and $p_{(m,n)} = p_n \circ p_{n-1} \circ \cdots \circ p_{m-1} : A_m \to A_n$. In this case $p_{(m,n)}(B_m) = B_n$. Assume that $p_{(n,n)} : A_n \to A_n$ is the identical mapping.

Claim 1. *If $n \in \omega$, $\alpha, \beta \in B_n$ and $U_\alpha \subseteq U_\beta$, then $\alpha \leqslant \beta$.*

Really, if $\alpha > \beta$, then $U_\alpha \setminus U_\beta \neq \varnothing$.

Claim 2. *If $n \in \omega$ and $\{U_{\alpha_m}:\ \alpha_m \in B_n, m \in \omega\}$ is a decreasing sequence of sets, then there exists $k \in \omega$ such that $\alpha_m = \alpha_k$ for any $m \geqslant k$.*

This statement follows from Claim 1.

Claim 3. *If $n, m \in \omega$, $m \leqslant n$, $\alpha \in B_n$, $\beta \in B_m$, and $U_\alpha \subseteq U_\beta$, then $p_{(n,m)}(\alpha) \leqslant \beta$.*

Really, if $\mu = p_{(m,n)}(\alpha) > \beta$, then $U_\alpha \setminus U_\beta \supseteq U_\mu \setminus U_\beta \neq \varnothing$.

Fix a perfectly decreasing sequence $\{V_m \in \mathcal{B}\colon m \in \omega\}$ and a point $b \in \bigcap\{X \cap V_m\colon m \in \omega\}$. Then $V_n = U_{\alpha_{v(n)}}$ for some $v(n) \in \omega$ and $\alpha_{v(n)} \in B_{v(n)}$.

It follows from Claim 2 that the set $\{m\colon \alpha_m \in B_n\}$ is finite for each $n \in \omega$. Thus, without less of generality, we can assume that $|\{m\colon \alpha_m \in B_n\}| \leqslant 1$ and $v(n+1) > v(n)$ for each $n \in \omega$. We put $N = \{v(n)\colon n \in \omega\}$.

Claim 4. *There exist a c-sequence $\{\beta_n \in B_n\colon n \in \omega\}$ and a sequence $\{c(n) \in N\colon n \in N\}$ such that*:

(i) *$n < c(n)$ and $\beta_n \leqslant \alpha_n$ for each $n \in N$;*
(ii) *$p_{(m,n)}(\alpha_m) = p_{(c(n),n)}(\alpha_{c(n)})$ for each $m \geqslant c(n)$ and $m \in N$;*
(iii) *$b \in U_{\alpha_{c(n)}} \subseteq U_{\beta_n} \cap U_{\alpha_n}$ for each $n \in N$.*

Fix $n \in N$. By Claim 3, the sequence $\{\mu_{(n,m)} = p_{(m,n)}(\alpha_m)\colon m \in N, n \leqslant m\}$ is decreasing and there exist $\beta_n \in B_n$ and $c(n) \in N$ such that $n < c(n)$ and $\beta_n = p_{(m,n)}(\alpha_m)$ for each $m \geqslant c(n)$, and $m \in N$. Moreover, we assume that $c(n)$ is the first element of N with such properties. In this case, $c(n) \leqslant c(m)$ and $p_{(m,n)}(\beta_m) = p_{(k,n)}(\alpha_k) = \beta_n$ for all $n, m, k \in N$ and $c(n) \leqslant k$, $n < m$. Clearly, $\{\beta_n \in B_n\colon n \in N\}$ is a subsequence of some c-sequence $\{\beta_n \in B_n\colon n \in \omega\}$, where $\beta_n = p_{(m,n)}(\beta_m)$ for all $n \in \omega$, $m \in N$ and $m > n$.

Clearly, the sequence $\{U_{\beta_n}\colon n \in N\}$ is a base for Z at the point b. Statement (ii) of Claim 4 implies that the sequence $\{U_{\alpha_n}\colon n \in N\} = \{V_m\colon m \in \omega\}$ is a base for Z at b too. Hence, $\mathcal{B}$ is an external base of countable order for X in Z. □

Remark 2.3. The concept of a sieve-base provides an effective tool for constructing bases of countable order. Moreover, from Proposition 2.2 it follows that the class of spaces with a base of countable order has the following properties:

– it is countable multiplicative;
– it contains any subspace of an arbitrary space with a base of countable order;
– it contains any space that has a base of countable order locally.

This was established in [19,30].

Remark 2.4. Let X be a subspace of a space Z, $\{W_n\colon n \in \omega\}$ be a sequence of non-empty pairwise disjoint open subsets of Z, $\{V_n\colon n \in \omega\}$ be a sequence of open subsets of Z, V be an open subset of X, $b \in \bigcap\{V \cap V_n\colon n \in \omega\} \subseteq X$, $\{X \cap V_n\colon n \in \omega\}$ be a perfectly decreasing sequence, $\mathcal{B}$ be a base of countable order of the space Z, $V_1 \cap W_n = \varnothing$ and $(X \cap V_n) \setminus V \neq \varnothing$ for all $n \in \omega$ (i.e. $\{X \cap V_n\colon n \in \omega\}$ is not a base for X at the point b). Put $\mathcal{B}' = \mathcal{B} \cup \{V_n \cup W_n\colon n \in \omega\}$. Then $\mathcal{B}'$ is a base of countable order of the space Z, is an external base of countable order of X in Z, and $\{U \cap X\colon U \in \mathcal{B}'\}$ is not a base of countable order of the space X. Thus, bases of countable order, in general, are not stable with respect to subspaces. On the contrary, sieve-bases are stable in this sense.

Corollary 2.5. *Suppose that X is a co-embedded subspace of a space Z. Then the space X has a base of countable order.*

Proof. Let $\{\gamma_n = \{U_\alpha\colon \alpha \in A_n\}\colon n \in \omega\}$ and $\{p_n : A_{n+1} \to A_n\colon n \in \omega\}$ be a sieve-base on X in the space Z. Then $\{\eta_n = \{V_\alpha = X \cap U_\alpha\colon \alpha \in A_n\}\colon n \in \omega\}$ and $\{p_n : A_{n+1} \to A_n\colon n \in \omega\}$ is a sieve-base of X. Proposition 2.2 implies that X has a base of countable order. □

Corollary 2.6. *A subspace of a space with a base of countable order has a base of countable order.*

A mapping $f : X \longrightarrow Y$ is open at a point $b \in X$ if for any open subset $U \ni b$ there exists an open subset V of Y such that $f(b) \in V \subseteq f(U)$. The mapping f is uniformly complete if on X there exists a metric d which generates the topology of X, and the fibers $f^{-1}(y)$, $y \in Y$, are complete subspaces of the metric space (X, d) [15,31].

The external characterization of spaces with bases of countable order from [31] admits a similar characterization of co-embedded subspaces.

Theorem 2.7. *Let Y be a subspace of a space Z. The following statements are equivalent*:

1. *Y has a base of countable order in Z, i.e. Y is co-embedded in Z.*
2. *There exist a metric space (X, d), a subspace M of X and a continuous mapping $f : X \longrightarrow Z$ of X onto Z such that $M = f^{-1}(Y)$, f is open at every point of M, and the fibers $f^{-1}(y)$, $y \in Y$, are complete subspaces of the metric space (X, d).*

3. *There exist a metric space (X,d), a subspace M of X and a continuous mapping $f: X \longrightarrow Z$ of X onto Z such that $f(M) = Y$, f is open at every point of M, the mapping $g = f|M : M \longrightarrow Y$ is open and $(M \cap f^{-1}(y), d)$, $y \in Y$, are complete subspaces of the metric space (X,d).*

Proof. $1 \to 2$. Assume that $\mathcal{B}$ is an external base of countable order of Y in Z. Let $\mathcal{L} = \mathcal{B} \cup \{\{z\}: z \in Z \setminus Y\}$. Denote by Z_1 the set Z with the topology generated by the open base $\mathcal{L}$. Then the identity mapping $\theta : Z_1 \longrightarrow Z$ is continuous and open at the points of the subspace Y. Moreover, $\theta|Y$ is the identical homeomorphism of Y onto Y.

On $\mathcal{L}$ we introduce the discrete topology, and on $\mathcal{L}^\omega$ we introduce the Baire metric $d((U_n: n \in \omega), (V_n: n \in \omega)) = \Sigma\{2^{-n}: U_n \neq V_n\}$. A point $(U_n: n \in \omega) \in \mathcal{L}^\omega$ will be called *correct* if the following conditions are satisfied:

– $U_{n+1} \subseteq U_n$ for any $n \in \omega$;
– $f(U_n: n \in \omega) = \bigcap\{U_n: n \in \omega\}$ is a singleton, and $\{U_n: n \in \omega\}$ is a base of the space Z_1 at the point $f(U_n: n \in \omega)$;
– if $n \in \omega$ and U_n is not a singleton, then U_{n+1} is a proper subset of U_n.

Denote by (X,d) the subspace consisting of all correct points of the metric space $(\mathcal{L}^\omega, d)$. Then $f: X \longrightarrow Z_1$ is a single-valued mapping of X onto Z_1.

Let $x = (U_n: n \in \omega) \in X$. For any $n \in \omega$ we put $O(x,n) = \{(V_n: n \in \omega) \in X: V_i = U_i \text{ for any } i \leqslant n\}$. Then:

– $\{O(x,n): n \in \omega\}$ is a base for X at the point x;
– $f(O(x,n)) = U_n$ for each $n \in \omega$.

Therefore, f is an open continuous mapping of X onto Z_1.

Fix $y \in Y$. If $U \in \mathcal{L}$ and $y \in U$, then $U \in \mathcal{B}$. Put $h(U) = \{U\}$ if U is a singleton, and $h(U) = \{V \in \mathcal{B}: y \in V \subseteq U, V \neq U\}$ if U is not a singleton. Let $h(U) = \{U_\alpha: \alpha \in A(U)\}$. Assume that $A(U) \cap A(V) = \varnothing$ for $U \neq V$. Let $\{U_\alpha: \alpha \in A_0\} = \{U \in \mathcal{L}: y \in U\}$. If $\{U_\alpha: \alpha \in A_n\}$ is constructed, then $A_{n+1} = \bigcup\{A(U_\alpha): \alpha \in A_n\}$. The sequence $(\alpha_n \in A_n: n \in \omega)$ we identify with the point $(U_{\alpha_n}: n \in \omega)$ from $(\mathcal{L}^\omega, d)$. Consider the mapping $p_n : A_{n+1} \longrightarrow A_n$, where $p_n^{-1}(\alpha) = A(U_\alpha)$. A point $(\alpha_n \in A_n: n \in \omega)$ is *y-marked* if $p(\alpha_{n+1}) = \alpha_n$ for all $n \in \omega$. Then $f^{-1}(y)$ is the set of all y-marked points, and this set is closed in $(\mathcal{L}^\omega, d)$, i.e. $(f^{-1}(y), d)$ is a complete metric subspace.

Since θ is the identity mapping, we can assume that $f = \theta \circ f$. Clearly, the mapping $f = \theta \circ f : X \longrightarrow Z$ is open at the points of the subspace $M = f^{-1}(Y)$. The implication $1 \to 2$ is proved.

The implication $2 \to 3$ is obvious.

$3 \to 1$. Let (X,d) be a metric space, M be a subspace of X, and $f: X \longrightarrow Z$ be a continuous mapping of X onto Z such that $f(M) = Y$, f is open at every point of M, the mapping $g = f|M : M \longrightarrow Y$ is open and $(g^{-1}(y), d) = (M \cap f^{-1}(y), d)$, $y \in Y$, are complete metric subspaces of the metric space (X,d). If U is an open set of X, then we put $f^*(U) = Z \setminus cl_Z(Z \setminus f(U))$. Obviously, $f^*(U) \subseteq f(U)$ and $f(M \cap U) \subseteq Y \cap f^*(U)$.

We fix two sequences $\{\eta_n = \{U_\alpha: \alpha \in A_n\}: n \in \omega\}$ and $\{p_n : A_{n+1} \to A_n: n \in \omega\}$ with the following properties:

– η_n is a family of open subsets of X, $M \cap U_\alpha \neq \varnothing$ and $\operatorname{diam}(U_\alpha) \leqslant 2^{-n}$ for any $n \in \omega$ and $\alpha \in A_n$;
– $M \subseteq \bigcup \gamma_0$ and $M \cap U_\alpha \subseteq \bigcup\{U_\beta: \beta \in p_n^{-1}(\alpha)\} \subseteq \bigcup\{cl_X U_\beta: \beta \in p_n^{-1}(\alpha)\} \subseteq U_\alpha$ for any $n \in \omega$ and $\alpha \in A_n$.

Let $\gamma_n = \{V_\alpha = f^*(U_\alpha): \alpha \in A_n\}$. Assume that $\alpha = \{\alpha_n \in A_n: n \in \omega\}$, $y \in Y$, $p_n(\alpha_{n+1}) = \alpha_n$ for any $n \in \omega$ and $y \in \bigcap\{V_{\alpha_n}: n \in \omega\}$. Then $y \in V_{\alpha_{n+1}} \subseteq V_{\alpha_n}$, $cl_X U_{\alpha_{n+1}} \subseteq U_{\alpha_n}$ and $M \cap f^{-1}(y) \cap U_{\alpha_n} \neq \varnothing$ for any $n \in \omega$. Since the metric space $(M \cap f^{-1}(y), d)$ is complete, there exists a unique point $x \in \bigcap\{U_{\alpha_n}: n \in \omega\}$. Clearly, $x \in M$ and $f(x) = y$. Thus $\{U_{\alpha_n}: n \in \omega\}$ is a base of X at the point x, and $\{V_{\alpha_n}: n \in \omega\}$ is a base of Z at the point $y = f(x)$. Therefore, $\{\gamma_n: n \in \omega\}$ and $\{p_n: n \in \omega\}$ is a sieve-base on Y in the space Z. Proposition 2.2 completes the proof of the implication $3 \to 1$. □

Remark 2.8. The assumption in condition 3 of Theorem 2.7 that the mapping $g = f|M \longrightarrow Y$ is open is essential. Assume that $f: X \longrightarrow Y$ is an open continuous mapping of a metric space (X,d) onto a space $Z = Y$. For any $y \in Y$ fix a point $x(y) \in f^{-1}(y)$. Let $M = \{x(y): y \in Y\}$. Then $f(M) = Y$, f is open at every point of M, $M \cap f^{-1}(y)$ is a singleton, and $(M \cap f^{-1}(y), d)$ is a complete subspace of the metric space (X,d) for any $y \in Y$. If Y does not have a base of countable order, then the mapping $g = f|M$ is not open.

In [4] an open base $\mathcal{B}$ of the space $X = \omega^\omega$ of the space of irrational numbers was constructed such that if $\gamma \subseteq \mathcal{B}$ is an open cover of X by elements of $\mathcal{B}$, then γ contains some infinite perfectly increasing sequence $\{V_n \in \gamma: n \in \omega\}$. Thus the base $\mathcal{B}$ does not contain any strong base of countable order of X. The situation with bases of countable order is quite different. We have the following remarkable result.

Theorem 2.9. *Let X be a subspace with an external base of countable order in the space Z. Then for each open external base $\mathcal{B}$ of the subspace X in Z there exists a sieve-base $\{\gamma_n = \{U_\alpha: \alpha \in A_n\}: n \in \omega\}$, $\{p_n : A_{n+1} \to A_n: n \in \omega\}$ of the subspace X in Z with the following properties*:

1. $\mathcal{B}' = \{U_\alpha\colon \alpha \in A_n, n \in \omega\} \subseteq \mathcal{B}$.
2. *$\mathcal{B}'$ is an external base of countable order of the subspace X in Z.*

Proof. In the proof of Proposition 2.2 we have established the following two statements:

Statement 1. If $\{\eta_n = \{V_\alpha\colon \alpha \in A_n\}\colon n \in \omega\}$, $\{p_n : A_{n+1} \to A_n\colon n \in \omega\}$ is a sieve-base of the subspace X in Z, then there exists an external base of countable order $\mathcal{B} \subseteq \{V_\alpha\colon \alpha \in A_n, n \in \omega\}$ of X in Z.

Statement 2. If $\mathcal{B}$ is an external base of countable order of X in Z, then there exists a sieve-base $\{\eta_n = \{V_\alpha\colon \alpha \in A_n\}\colon n \in \omega\}$, $\{p_n : A_{n+1} \to A_n\colon n \in \omega\}$ of X in Z such that $\mathcal{B} = \{V_\alpha\colon \alpha \in A_n, n \in \omega\}$.

Assume that X is a subspace with an external base of countable order in Z. Fix some sieve-base $\{\eta_n = \{V_\alpha\colon \alpha \in A_n\}\colon n \in \omega\}$, $\{p_n : A_{n+1} \to A_n\colon n \in \omega\}$ of X in Z.

Let $\mathcal{B}$ be a given open external base of the subspace X in Z. For any open subset U of Z fix a family $\gamma(U) = \{V_\beta\colon \beta \in B(U)\} \subseteq \mathcal{B}$ such that $U \cap X \subseteq \bigcup\{V_\beta\colon \beta \in B(U)\} \subseteq U$ and each V_β is a proper subset of U provided U is not a singleton. We assume that $B(V) \cap B(W) = \varnothing$ for $V \neq W$.

Now we construct the open families $\{\gamma_n = \{U_\beta\colon \beta \in B_n\}\colon n \in \omega\}$ of the space Z, the sequence of mappings $\{q_n : B_{n+1} \longrightarrow B_n\colon n \in \omega\}$ and the sequence of mappings $\{h_n : B_n \longrightarrow A_n\colon n \in \omega\}$ such that:

– $X \subseteq \bigcup\{U_\beta\colon \beta \in B_0\}$;
– $\gamma_n \subseteq \mathcal{B}$ for any $n \in \omega$;
– $h_n \circ q_n = p_n \circ h_{n+1}$ for any $n \in \omega$;
– if $n \in \omega$ and $\beta \in B_n$, then $U_\beta \subseteq V_{h_n(\beta)}$ and $U_\beta \cap X \subseteq \bigcup\{U_\mu\colon \mu \in q_n^{-1}(\beta)\} \subseteq U_\beta$ for all $n \in \omega$ and $\beta \in B_n$.

We put $B_0 = \bigcup\{B(V_\alpha)\colon \alpha \in A_0\}$, $\gamma_0 = \bigcup\{\gamma(V_\alpha)\colon \alpha \in A_0\}$ and $h_0^{-1}(\alpha) = B(V_\alpha)$ for any $\alpha \in A_0$. The objects γ_0, h_0 are constructed. Assume that $n \geqslant 0$ and the objects $\gamma_n = \{U_\beta\colon \beta \in B_n\}$ and $h_n : B_n \longrightarrow A_n$ are constructed. Fix $\beta \in B_n$ and put $\alpha = h_n(\beta)$. Then $U_\beta \subseteq V_\alpha$. We put $B_\beta = \bigcup\{B(U_\beta \cap V_\mu)\colon \mu \in p_n^{-1}(\alpha)\}$, $\gamma(\beta) = \bigcup\{\gamma(U_\beta \cap V_\mu)\colon \mu \in p_n^{-1}(\alpha)\} = \{U_\lambda\colon \lambda \in B_\beta\}$, $B_{n+1} = \bigcup\{B_\beta\colon \beta \in B_n\}$ and $\gamma_{n+1} = \{U_\lambda\colon \lambda \in B_{n+1}\}$. If $\mu \in p_n^{-1}(\alpha)$, then $h_{n+1}(\mu) = B(U_\beta \cap V_\mu)$. Let $q_n^{-1}(\beta) = B_\beta$. Thus, the objects γ_{n+1}, q_n, h_{n+1} are constructed.

Fix a sequence $\{\beta_n \in B_n\colon n \in \omega\}$ for which $q_n(\beta_{n+1}) = \beta_n$ for any $n \in \omega$. Put $\alpha_n = h_n(\beta_n)$. Then $U_{\beta_n} \subseteq V_{\alpha_n}$ and $p_n(\alpha_{n+1}) = \alpha_n$ for any $n \in \omega$. If $x \in X$ and $x \in \bigcap\{U_{\beta_n}\colon n \in \omega\}$, then $x \in \bigcap\{V_{\alpha_n}\colon n \in \omega\}$ and $\{V_{\alpha_n}\colon n \in \omega\}$ is a base of Z at the point x. Therefore, $\{U_{\beta_n}\colon n \in \omega\}$ is a base for Z at the point x too.

Therefore $\{\gamma_n\colon n \in \omega\}$, $\{q_n\colon n \in \omega\}$ is a sieve-base of X in Z. Statements 1 and 2 complete the proof. □

3. Sharp bases

A base $\mathcal{B}$ for a space X is said to be a *sharp base* [1] if, whenever $x \in X$, $\{U_n\colon n \in \omega\}$ is a sequence of distinct elements of $\mathcal{B}$ and $x \in \bigcap\{U_n\colon n \in \omega\}$, then $\{\bigcap\{U_i\colon i \leqslant n\}\colon n \in \omega\}$ is a base for X at the point x.

If γ is a family of subsets of a space X, and $L \subseteq X$, then $St(L, \gamma) = \bigcup\{H \in \gamma\colon L \cap H \neq \varnothing\}$ is the *star* of L with respect to γ. We put $St(x, \gamma) = St(\{x\}, \gamma)$.

Let $X \subseteq Z$, where Z is an arbitrary space. A countable family $\mathcal{F}$ of families of open subsets of Z is said to be a *plumage* of X in Z if $X \subset \bigcup \gamma$ for each $\gamma \in \mathcal{F}$ and $x \in \bigcap\{St(x, \gamma)\colon \gamma \in \mathcal{F}\} \subseteq X$ for each point $x \in X$.

A space with a plumage in some compact space is called a p-space or a *feathered space* (this concept was introduced in [7], see also [12] where a plumage is called a *feathering*).

Each Tychonoff space with a sharp base is a hereditarily p-space with a G_δ-diagonal [1]. Distinct properties of spaces with sharp bases are established in [1,11,13,20,23].

A base $\mathcal{B}$ for a space X is said to be *a strong sharp base* if it is a σ-point-finite sharp base for X.

A base $\mathcal{B}$ for a space X is said to be *a uniform base* [2] if whenever $x \in X$, $\{U_n\colon n \in \omega\}$ is a sequence of distinct elements of $\mathcal{B}$ and $x \in \bigcap\{U_n\colon n \in \omega\}$, then $\{U_n\colon n \in \omega\}$ is a base for X at x. Any space with a uniform base is perfect, i.e. each closed subset of it is a G_δ-subset.

Each uniform base is a strong sharp base.

Given a space X, we denote by X' the subspace of X consisting of all non-isolated points of X. If X is a space with a sharp base, then X has a point-countable sharp base which is point-finite on the set $X \setminus X'$ of isolated points ([13], Theorem 3.1) (in particular, X' is a G_δ-subset of the space X). Obviously, any sharp base is point-countable on the set X' of non-isolated points.

A base $\mathcal{B}$ of a space X is said to be a *fibering base* if there exists a sequence $\{n(\mathcal{B}) = \{n(U)\colon U \in \mathcal{B}\}\colon n \in \omega\}$ of point-finite families of open subsets of X such that the following conditions are satisfied:

– $U \cap X' \subseteq \bigcup\{n(U)\colon n \in \omega\} \subseteq U$ for each $U \in \mathcal{B}$;
– if $U, V \in \mathcal{B}$, $n, m \in \omega$, $n \leqslant m$ and $U \subseteq V$, then $n(U) \cap X' \subseteq m(V)$;

– if $x \in X'$, $\mu = \{U_n \in \mathcal{B}\colon n \in \omega\}$ is a sequence of distinct elements and $x \in \bigcap\{U_n\colon n \in \omega\}$, then for any $n \in \omega$, there exists $m = m(\mu) \in \omega$ such that $\bigcap\{n(U_i)\colon i \leqslant m\} = \varnothing$.

We say that the sequence $\{n(\mathcal{B}) = \{n(U)\colon U \in \mathcal{B}\}\colon n \in \omega\}$ is the fibering of the family $\mathcal{B}$.

If a space has a fibering base, then it has a σ-point-finite open base.

We need the following property of spaces with fibering sharp bases.

Lemma 3.1. *If X is a space with a fibering sharp base $\mathcal{B}$, then X has a point-countable fibering sharp base $\mathcal{L}$ which is point-finite on the set F of isolated points. Moreover, if the base $\mathcal{B}$ is σ-point-finite, then the base $\mathcal{L}$ is σ-point-finite too.*

Proof. Assume that $\mathcal{B}$ is a fibering sharp base of the space X with the fibering $\{n(\mathcal{B}) = \{n(U)\colon U \in \mathcal{B}\}\colon n \in \omega\}$.

Since the set $F = X \setminus X'$ is an F_σ-subset of X ([13], Theorem 3.1), we can fix a sequence $\{F_n\colon n \in \omega\}$ of closed subsets of X such that $F = \bigcup\{F_n\colon n \in \omega\}$ and $F_n \subseteq F_{n+1}$ for any $n \in \omega$.

We put $\mathcal{B}' = \{U \in \mathcal{B}\colon U \cap X' \neq \varnothing\}$.

If $n \in \omega$ and $x \in F_n$, then $U_x = \{x\}$, $m(U_x) = \varnothing$ for $m > n$ and $m(U_x) = U_x$ for $m \leqslant n$.

If $U \in \mathcal{B}'$, then we put $n(s(U)) = n(U) \setminus F_n$ for each $n \in \mathbb{N}$ and $s(U) = \bigcup\{n(s(U))\colon n \in \mathbb{N}\}$.

Clearly, $\mathcal{L} = \{s(U)\colon U \in \mathcal{B}'\} \cup \{\{x\}\colon x \in F\}$ is a fibering sharp base of X. If $n \in \mathbb{N}$, $U \in \mathcal{B}'$ and $x \in F_n \cap s(U)$, then $x \in m(U)$ for some $m < n$. Hence, since the families $m(\mathcal{B})$ are point-finite, the point x is contained in finitely many elements of the family $\mathcal{L}$. If $\gamma \subseteq \mathcal{B}'$ is point-finite, then the family $s(\gamma) = \{s(U)\colon U \in \gamma\}$ is point-finite too. □

In the proof of Lemma 3.1, a procedure for modifying any base $\mathcal{B}$ of a space X, in which the isolated points form an F_σ-subset, is described. This procedure eliminates from the family $\mathcal{B}$ each non-singleton set $U \subseteq X \setminus X'$ and reduces the isolated part of other infinite sets $U \in \mathcal{B}$. As a result, we obtain a new base $\mathcal{L}$ which is point-finite at any isolated point. This procedure is applicable to fibering bases and to σ-pint-finite bases, and it preserves these properties.

It is important to observe that the base $\mathcal{B}$, to which the procedure is applied, need not be point-countable or point-finite.

Indeed, let X be an infinite space with a sharp base $\mathcal{B}$, and let $X \neq X'$. Fix an infinite disjoint family $\gamma \subseteq \mathcal{B}$ and a point $z \in X \setminus X'$. Now put $\mathcal{B}_1 = \mathcal{B} \cup \{U \cup \{z\}\colon U \in \gamma\}$. Then:

– $\mathcal{B}_1$ is a sharp base which is not point-finite at the point z;
– if the set γ is uncountable, then $\mathcal{B}_1$ is a sharp base which is not point-countable at the point z;
– if γ is countable, and $\mathcal{B}$ is σ-point-finite, then $\mathcal{B}_1$ is σ-point-finite too;
– if $\mathcal{B}$ is a fibering base, then $\mathcal{B}_1$ is a fibering base too. It is sufficient to put $n(U \cup \{z\}) = n(U)$ for all $U \in \gamma$ and $n \in \omega$.

Let $S = \{0\} \cup \{2^{-n}\colon n \in \omega\}$ be a subspace of the space of real numbers, and $S_n = \{0\} \cup \{2^{-m}\colon m \in \omega, m \geqslant n\}$.

Theorem 3.2. *For any space X, the following statements are equivalent*:

1. *X has a strong sharp base which is point-finite at any isolated point $x \in X \setminus X'$.*
2. *X has a sharp base and a σ-point finite base.*
3. *X has a fibering sharp base.*
4. *$X \times S$ has a sharp base.*
5. *$X \times S$ has a strong sharp base.*
6. *$Y \times X$ has a strong sharp base, for any space Y with a uniform base.*
7. *$X \times Y$ has a sharp base for some non-discrete space Y.*

Proof. Implications $6 \to 5 \to 4$, $3 \to 2$, $6 \to 7$ and $1 \to 2$ are obvious.

$7 \to 4$. If Y is a non-discrete space, and the space $X \times Y$ has a sharp base, then $X \times S$ can be embedded in $X \times Y$ as a subspace, and consequently, $X \times S$ has a sharp base.

$4 \to 1$. Consider the projection $p : X \times S \longrightarrow X$.

Assume that $\mathcal{B}$ is a sharp base for the space $X \times S$.

Let $X_\omega = X \times \{0\}$ and $X_n = X \times \{2^{-n}\}$ for any $n \in \omega$.

Since the set $F = X \setminus X'$ is an F_σ-subset of X ([13], Theorem 3.1), we can fix a sequence $\{F_n\colon n \in \omega\}$ of closed subsets of X such that $F = \bigcup\{F_n\colon n \in \omega\}$ and $F_n \subseteq F_{n+1}$, for any $n \in \omega$.

If $n \in \omega$ and $x \in F_n$, then $U_x = \{x\}$, $m(U_x) = \varnothing$ for $m > 0$ and $0(U_x) = U_x$.

If U is an open subset of the space $X \times S$, and $n \in \omega$, then $n(U) = \bigcup\{V \subseteq X \setminus F_n\colon V \times S_n \subseteq U, V \text{ is open in } X\}$, $d(U) = \omega$, if $U \cap X_\omega = \varnothing$, and $d(U) = \min\{n\colon n(U) \neq \varnothing\}$, if $U \cap X_\omega \neq \varnothing$. The number $d(U)$ is called the *depth* of the set U.

Put $\mathcal{B}' = \{U \in \mathcal{B}\colon U \cap X_\omega \neq \varnothing\}$ and $\mathcal{B}_n = \{U \in \mathcal{B}'\colon d(U) = n\}$ for any $n \in \omega$.

Let $\gamma_n = \{l(U) = p(U \cap X_\omega) \setminus F_n\colon U \in \mathcal{B}_n\} \cup (\{U_x\colon x \in F_n\})$, $\gamma'_n = \{l(U) = p(U \cap X_\omega) \setminus F_n\colon U \in \mathcal{B}_n\}$ and $\mathcal{L} = \bigcup\{\gamma_n\colon n \in \mathbb{N}\}$.

Claim 1. *The family γ_n is point-finite for any $n \in \omega$.*

If $x \in F_n$ and $x \in V \in \gamma_n$, then $V = U_x$. If $x \in X \setminus F_n$ and $x \in l(U) \in \gamma_n$, then $d(U) = n$ and $\{x\} \times S_n \subseteq U$.

Let $n \in \omega$, $x \in X \setminus F_n$, $\{l(U_m) \in \gamma_n\colon m \in \omega\}$ be a sequence of distinct elements and $x \in \bigcap\{l(U_m)\colon m \in \omega\}$. Then, clearly, $\{x\} \times S_n \subseteq \bigcap\{U_m\colon m \in \omega\}$. The sharpness of $\mathcal{B}$ implies that $\bigcap\{U_m\colon m \in \omega\}$ is at most a singleton and $\{\bigcap\{U_i\colon i \leqslant m\}\colon m \in \omega\}$ is a base of $X \times S$ at each point of the set $\{x\} \times S_n$, a contradiction. Therefore the family γ_n is point-finite.

Claim 2. *The family $\mathcal{L}$ is point-finite at any isolated point $x \in X \setminus X'$.*

There exists $m \in \omega$ such that $x \in F_m$. Then $x \notin \bigcup \gamma_n'$, for each $n > m$. A reference to Claim 1 completes the proof.

Claim 3. *The family $\mathcal{L}$ is a strong sharp base for the space X.*

Assume that $x \in X$, M is an infinite subset of ω, $V_n = l(U_n) \in \gamma_n$ for any $n \in M$, and that $x \in \bigcap\{V_n\colon n \in M\}$. Then $x \in X'$, $\mu = \{U_n \in \mathcal{B}\colon n \in \omega\}$ is a sequence of distinct elements and $(x, 0) \in \bigcap\{U_n \in \mathcal{B}\colon n \in \omega\}$.

If V is an open subset of X and $x \in V$, then $(x, 0) \in V \times S$ and there exists $n \in M$ such that $(x, 0) \in \bigcap\{U_m\colon m \in M, m \leqslant n\} \subseteq V \times S$. Clearly, $x \in \bigcap\{V_m\colon m \in M, m \leqslant n\} \subseteq V$. Claim 3 and implication $4 \to 1$ are proved.

$2 \to 1, 3$. Suppose that $\mathcal{B}$ is a sharp base of the space X, $F = X \setminus X'$, $\{F_n\colon n \in \omega\}$ is a sequence of closed subsets of X such that $F = \bigcup\{F_n\colon n \in \omega\}$ and $F_n \subseteq F_{n+1}$ for any $n \in \omega$. By Theorem 3.1 from [13], we can assume that the base $\mathcal{B}$ is point-finite at the points of the set F.

Since X is a space with a σ-point-finite base, there exists a sequence $\{\gamma_n\colon n \in \omega\}$ of point-finite systems of open sets in the space X such that the next two conditions are satisfied:

- if $n \in \omega$ and $V \in \gamma_n$, then $V \cap X' \neq \varnothing$ and $V \cap F_n = \varnothing$;
- $\gamma = \bigcup\{\gamma_n\colon n \in \omega\}$ is a base for X at the points of the set X'.

If $x \in F$, then $U_x = \{x\}$ and $n(U_x) = \varnothing$ for any $n \in \omega$.

Let $\mathcal{B}' = \{U \in \mathcal{B}\colon U \cap X' \neq \varnothing\}$. For any $U \in \mathcal{B}'$ and $n \in \omega$ put $n(U) = \bigcup\{V \setminus F_n\colon V \in \bigcup\{\gamma_i\colon i \leqslant n\}, V \subseteq U\}$, $\mu(U) = \bigcup\{n(U)\colon n \in \omega\}$ and $n(\mu(U)) = n(U)$. Since $U \cap X' = \mu(U) \cap X'$, the family $\mathcal{M}' = \{\mu(U)\colon U \in \mathcal{B}'\}$ is a base for X at the points of the set X'. Hence $\mathcal{M} = \mathcal{M}' \cup \{U_x\colon x \in F\}$ is a sharp base for the space X which is point-finite at the points of the set F.

Now let $n(\mathcal{M}) = \{n(\mu(U))\colon U \in \mathcal{B}'\}$.

Claim 4. *Let $\{U_n \in \mathcal{B}'\colon n \in \omega\}$ be a sequence of distinct elements, and let $x \in \bigcap\{\mu(U_n)\colon n \in \omega\}$. Then:*

1) *$\{\bigcap\{\mu(U_i)\colon i \leqslant n\}\colon n \in \omega\}$ is a base for X at the point x;*
2) *for any $n \in \omega$ there exists $m \in \omega$ such that $\bigcap\{n(\mu(U_i))\colon i \leqslant m\} = \varnothing$.*

Since $\mathcal{B}$ is a sharp base which is point-finite at any isolated point, $x \in X'$ and $\{\bigcap\{\mu(U_i)\colon i \leqslant n\}\colon n \in \omega\}$ is a base for X at the point x.

Fix $n \in \omega$. Assume that $\bigcap\{n(\mu(U_i))\colon i \in \omega\} \neq \varnothing$. Then $x \in \bigcap\{n(\mu(U_i))\colon i \in \omega\}$. Since any γ_i is point-finite, the set $\{V \in \bigcup\{\gamma_i\colon i \leqslant n\}\colon x \in V\}$ is finite. Let $\{V \in \bigcup\{\gamma_i\colon i \leqslant n\}\colon x \in V\} = \{V_1, V_2, \ldots, V_k\}$. There exists an open subset W of X such that $x \in W$ and W is a proper subset of the set $\bigcap\{V_i \setminus F_n\colon i \leqslant k\}$.

Since $x \in n(\mu(U_m))$, there exists $i(m) \leqslant k$ such that $V_{i(m)} \subseteq U_m$. We put $N_i = \{m \in \omega\colon i(m) = i\}$. Since $\bigcup\{N_i\colon i \leqslant k\} = \omega$, there exists $i \leqslant k$ for which the set N_i is infinite. Since $\mathcal{B}$ is a sharp base, there exists $m \in N_i$ such that $V_i \subseteq \bigcap\{U_j\colon j \in N_i, j \leqslant m\} \subseteq W$, a contradiction. Claim 4 is proved.

Claim 5. *The family $n(\mathcal{M})$ is point-finite for any $n \in \omega$.*

This follows from assertion 2 of Claim 4.

Claim 6. *The family $\mathcal{M}$ is a fibering sharp base for the space X.*

Since $\mathcal{B}$ is a sharp base, $\mathcal{M}$ is a sharp base too. Claims 4 and 5 imply that $\{n(\mathcal{M})\colon n \in \omega\}$ is a fibration of the base $\mathcal{M}$. Claim 6 is proved.

Let $n \in \omega$ and $\mathcal{M}_n = \{W \in \mathcal{M}\colon n(W) \neq \varnothing\}$.

Claim 7. *The family $\mathcal{M}_n$ is point-finite for any $n \in \omega$.*

The proof is similar to the proof of Claim 4.

From Claim 7 it follows that the family $\mathcal{M}$ is σ-point-finite. The implications $2 \to 1, 3$ are proved.

$1 \to 6$. Fix a sequence of open point-finite covers $\{\gamma_n = \{U_\alpha\colon \alpha \in A_n\}\colon n \in \omega\}$ of the space Y such that:

– $\mathcal{B}_1 = \bigcup\{\gamma_n\colon n \in \omega\}$ is a uniform base of Y which is point-finite at all isolated points;
– if $n, m \in \omega$, $n \neq m$ and $U \in \gamma_n \cap \gamma_m$, then U is a singleton;
– $\{\{y\}\colon y \in Y \setminus Y'\} \subseteq \gamma_n$ for each $n \in \omega$;
– if $U \in \mathcal{B}_1$ and $V \cap Y' = \varnothing$, then U is a singleton;
– if $n, m \in \omega$ and $m \leqslant n$, then γ_n is a refinement of γ_m.

Since X has a strong sharp base, it follows from Theorem 3.1 in [13] and Lemma 3.1 that there exist a sequence $\{F_n\colon n \in \omega\}$ of open-and-closed subsets of X and a sequence of point-finite families $\{\xi_n = \{V_\beta\colon \beta \in B_n\}\colon n \in \omega\}$ of open subsets of the space X such that:

– $\mathcal{B}_2 = \bigcup\{\xi_n\colon n \in \omega\}$ is a sharp base of X which is point-finite at all isolated points;
– if $V \in \mathcal{B}_2$ and $V \cap X' = \varnothing$, then V is a singleton;
– $F = \bigcup\{F_n\colon n \in \omega\}$ is the set of all isolated points of X and $F_n \subseteq F_{n+1}$ for each $n \in \omega$;
– if $n, m \in \omega$, $n \neq m$, then $\xi_n \cap \xi_m = \varnothing$;
– $\xi_0 = \{\{x\}\colon x \in F\}$;
– $\bigcup\{V_\beta\colon \beta \in B_n\} \subseteq X \setminus F_n$ for each $n \in \omega\}$ and $n \geqslant 1$.

We put $\eta_n = \{W_{(\alpha,\beta)} = U_\alpha \times V_\beta\colon (\alpha, \beta) \in C_n = A_n \times B_n\}$ and $\mu_n = \{W_{(\alpha,x)} = U_\alpha \times \{x\}\colon (\alpha, x) \in D_n = A_n \times F\}$ for each $n \in \omega$.

Let $\mathcal{B}' = \bigcup\{\eta_n\colon n \in \omega\}$, $\mathcal{B}'' = \bigcup\{\mu_n\colon n \in \omega\}$ and $\mathcal{B} = \mathcal{B}' \cup \mathcal{B}''$.

For every $n \in \omega$, $x \in X$ and $y \in Y$, the sets $A_n(y) = \{\alpha \in A_n\colon y \in U_\alpha\}$ and $B_n(x) = \{\beta \in B_n\colon x \in V_\beta\}$ are finite. Moreover, if $x \in F_n$, then $\bigcup\{\{B_i\colon i \in \omega, i \geqslant n\} = \varnothing\}$. We put $N(x) = \{n \in \omega\colon B_n(x) \neq \varnothing\}$.

Clearly, $\mathcal{B}$ is σ-point-finite and point-finite at any point of the set $Y' \times F$. Moreover, the family $\mathcal{B}'$ is point-finite at any point of the set $Y \times F$.

Claim 8. *The family $\mathcal{B}$ is a σ-point-finite base for $Y \times X$.*

Fix a point $z = (y, x) \in Y \times X$.

For each $n \in \omega$ fix $U_n \in \gamma_n$ such that $y \in U_n$ and $\{U_n\colon n \in M\}$ is a base for Y at the point y for any infinite subset $M \subseteq \omega$. We can assume that $U_{n+1} \subseteq U_n$ for any n and $U_0 = \{y\}$ if y is an isolated point of the space Y.

If $x \in F$, then $U_n \times \{x\} \in \mathcal{B}''$ and $\{U_n \times \{x\}\colon n \in \omega\}$ is a base for $Y \times X$ at the point z. Let $x \notin F$. In this case the set $N(x)$ is infinite and the sequence $\{U_n\colon n \in N(x)\}$ is a base for Y at y. Then the subfamily $\{U_n \times V_\beta\colon n \in N(x), \beta \in B_n(x)\} \subseteq \mathcal{B}'$ is a base for $Y \times X$ at z. Thus $\mathcal{B}$ is a σ-point-finite base for $Y \times X$.

Claim 9. *The family $\mathcal{B}$ is a sharp base for $Y \times X$.*

Fix a sequence $\{W_n \in \mathcal{B}\colon n \in \omega\}$ of pairwise distinct elements. Let $z = (y, x) \in \bigcap\{W_n\colon n \in \omega\}$.

For every $n \in \omega$ there exist $i(n) \in \omega$, $U_n \in \gamma_{i(n)} \subseteq \mathcal{B}_1$, and $V_n \in \xi_{i(n)} \subseteq \mathcal{B}_2$ such that $W_n = U_n \times V_n \in \eta_{i(n)}$.

Since the families η_n are point-finite, we can assume that $i(n) \neq i(m)$ for $n \neq m$. Hence the set $M(z) = \{i(n)\colon n \in \omega\}$ is infinite.

Case 1. Let $x \in F$.

In this case, the set $\{n \in \omega\colon W_n \in \mathcal{B}'\}$ is finite, and therefore, we can assume that this set is empty. Hence, we can assume that $W_n = U_n \times \{x\}$ for some $U_n \in \mathcal{B}_1$. Since $\mathcal{B}_1$ is a uniform base, $\{W_n\colon n \in \omega\}$ is a base for $Y \times X$ at z.

Case 2. $x \in X'$ and y is an isolated point of Y.

Since $x \notin F$, the set $M(z) = \{i(n)\colon n \in \omega\}$ is infinite and the elements $\{V_n\colon n \in \omega\}$ are distinct. Thus the sequence $\{\bigcap\{V_j\colon j \leqslant n\}\colon n \in \omega\}$ is a base for X at x.

Since y is isolated in Y and the base $\mathcal{B}_1$ is a uniform base for the space Y which is point-finite at all isolated points, there exists $m \in \omega$ such that $U = \{y\}$ provided $y \in U \in \gamma_n$ and $n \geqslant m$. Since the set $M(z)$ is infinite, we can suppose that $U_n = \{y\}$ for each $n \in \omega$.

Hence, $\{\bigcap\{W_m\colon m \leqslant n\}\colon n \in \omega\}$ is a base for $Y \times X$ at the point z.

Case 3. $x \in X'$ and $y \in Y'$.

Since $x \notin F$, the set $M(z) = \{i(n)\colon n \in \omega\}$ is infinite, and the elements $\{V_n\colon n \in \omega\}$ are distinct. Thus, the sequence $\{\bigcap\{V_m\colon m \leqslant n\}\colon n \in \omega\}$ is a base for X at x.

Since y is not isolated in X and the set $M(z) = \{i(n)\colon n \in \omega\}$ is infinite, the elements $\{U_n\colon n \in \omega\}$ are distinct. Since $\mathcal{B}_1$ is a uniform base, the sequence $\{U_n\colon n \in \omega\}$ is a base for Y at y.

Hence, $\{\bigcap\{W_m\colon m \leqslant n\}\colon n \in \omega\}$ is a base for $Y \times X$ at z.

Therefore $\mathcal{B}$ is a sharp base for the space $Y \times X$.

The implication $1 \to 6$ is proved. The proof of the theorem is complete. □

Theorem 3.2 contains a positive answer to Question 6 from [23]: If $X \times [0,1]$ has a sharp base, does X have a σ-point-finite sharp base?

In [29] V.V. Uspenskii has proved that a pseudocompact space with a σ-point-finite base is metrizable. Thus from Theorem 3.2 the next statement follows.

Corollary 3.3. *Let X be a pseudocompact space. Then the space X is metrizable if and only if $X \times S$ is a space with a sharp base.*

We mention that a regular locally countably compact space with a sharp base is metrizable [1].

The following theorem improves Proposition 8 from [23].

Theorem 3.4. *For any space X, the following conditions are equivalent*:

1. *X has a uniform base*;
2. *X is countably metacompact and has a strong sharp base*;
3. *X is a perfect space with a σ-point-finite base*;
4. *There exist a sharp base $\mathcal{B}$ of X and a sequence $\gamma_n = \{U_\alpha\colon \alpha \in A_n\}$, $n \in \omega$, of point-finite open covers of X such that $\bigcup\{\gamma_n\colon n \in \omega\} \subseteq \mathcal{B}$ and $U \in \gamma_n \cap \gamma_m$ for $n \neq m$ and $n, m \in \omega$ if and only if U is a singleton.*

Proof. Implications $1 \to 2, 3, 4$ are obvious.

$4 \to 1$. We put $A_n(x) = \{\alpha_n \in A_n\colon x \in U_{\alpha_n}\}$. Consider $A = \Pi\{A_n\colon n \in \omega\}$ as a metric space with the Baire distance $d((\alpha_n\colon n \in \omega), (\beta_n\colon n \in \omega)) = \Sigma\{2^{-n}\colon \alpha_n \neq \beta_n\}$. For any $\alpha = (\alpha_n\colon n \in \omega) \in A$ we put $g(\alpha) = \bigcap\{U_{\alpha_n}\colon n \in \omega\}$. If $Z = \{\alpha \in A\colon g(\alpha) \neq \varnothing\}$, then $g : Z \longrightarrow X$ is a single-valued mapping. Let $\alpha = (\alpha_n\colon n \in \omega) \in Z$, $m \in \omega$ and $m[\alpha] = \{(\beta_n\colon n \in \omega) \in Z\colon \beta_i = \alpha_i$ for any $i \leqslant m\}$. Clearly, $g(m[\alpha]) = \bigcap\{U_{\alpha_i}\colon i \leqslant m\}$ and $g^{-1}(x) = \Pi\{A_n(x)\colon n \in \omega\}$. Hence, g is an open continuous mapping with compact fibers of the metric space Z onto X. By the theorem from [8], X is a space with a uniform base.

$3 \to 1$. Let $\eta_n = \{U_\alpha\colon \alpha \in C_n\}$ be a sequence of point-finite families of open subsets of the perfect space X such that $\bigcup\{\eta_n\colon n \in \omega\}$ is a base of X.

Fix $n \in \omega$. For each $m \geqslant 1$ we put $C_{nm} = \{H \subseteq C_n\colon |H| = m\}$. If $\alpha = (\alpha_1, \ldots, \alpha_m) \in C_{nm}$, then $U_\alpha = \bigcap\{U_{\alpha_i}\colon i \leqslant m\}$. There exists a sequence $\{F_{nmk}\colon k \in \omega\}$ of closed subsets of X such that $\bigcup\{F_{nmk}\colon k \in \omega\} = \bigcup\{U_\alpha\colon \alpha \in C_{nm}\}$ and $F_{nmi} \subseteq F_{mnj}$ for $i < j$. Let $U_{nmk} = X \setminus F_{nmk}$ and $C_{nmk} = \{(nmk)\} \cup C_{nm}$. Now assume that a sequence $\{\gamma_n = \{U_\alpha\colon \alpha \in A_n\}\colon n \in \omega\}$ of point-finite open covers of X is the sequence $\{\eta_{nmk} = \{U_\alpha\colon \alpha \in C_{nmk}\}\colon n, m, k \in \omega, m \geqslant 1\}$. As in the case $4 \to 1$, we construct an open continuous mapping g with compact fibers of a metric space Z onto X.

$2 \to 1$. A countably metacompact space with a σ-point-finite base is metacompact, and a metacompact space with a base of countable order is a space with a uniform base [15,30]. □

Corollary 3.5. *For any space X, the following conditions are equivalent*:

1. *X has a uniform base*;
2. *X is a countably metacompact space with a strong sharp base*;
3. *X is countably metacompact, and $X \times S$ has a sharp base*;
4. *$X \times Y$ is a space with a uniform base for any space Y with a uniform base.*

The next statement practically was proved in [20], Lemma 1. Our proof is simpler than the argument in [20].

Proposition 3.6. *Let $X = Y \cup Z$ be a pseudocompact non-compact space, and $\mathcal{B}$ be a sharp base for X such that $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ and the following conditions are satisfied*:

(a) *$\mathcal{B}_1$ is a σ-point-finite base for X at the points of the set Y.*

(b) *For each $x \in L$ there exists a local base $\mathcal{B}(x) = \{U_n(x)\colon n \in \omega\}$ so that*:
  - *$n < m$ implies $U_m(x) \subseteq U_n(x)$*;
  - *if $x, y \in L$ and $x \neq y$, then $U_n(x) \neq U_m(y)$ for all $n, m \in \omega$*;
  - *$\mathcal{B}_2 = \{U_n(x)\colon x \in L, n \in \omega\}$.*

Suppose further that Z is any non-discrete first-countable space. Then $X \times Z$ does not have a sharp base.

Proof. It is enough to prove that $X \times S$ does not have a sharp base. Assume, by way of contradiction, that $\mathcal{W}$ is a sharp base for $X \times S$. Then by Theorem 3.2, X is a pseudocompact non-metrizable space with a σ-point-finite base. However, any pseudocompact space with a σ-point-finite base is metrizable [29]. This contradiction completes the proof. □

Example 3.7. Let X be the pseudocompact space with a sharp base constructed in [20]. Then, by Proposition 3.6, the space $X \times S$ does not have a sharp base. Moreover, for any non-discrete first-countable space Z, the space $X \times Z$ does not have a sharp base. In particular, the square $X \times X$ does not have a sharp base. On the other hand, the space $X \times S$ has a strong base of countable order.

4. On bases of finite rank

In his talk at the First International Prague Symposium on General Topology and its Applications in Algebra and Analysis, in 1961, J. Nagata introduced the concept of the rank of a family of sets. The general idea was to use bases of finite rank to study and characterize the dimension of spaces (see [6,26]).

A family γ of subsets of the set X is called *independent* if $U \setminus V \neq \varnothing$ and $V \setminus U \neq \varnothing$ for any two distinct elements $U, V \in \gamma$. Otherwise, it is called dependent. The number $r(\gamma, x) = \sup\{|\xi|\colon\ \xi \subseteq \gamma, x \in \bigcap \xi, \xi \text{ is independent}\}$, if it is defined, is called the *rank* of the family γ at the point $x \in X$. the number $r(\gamma) = \sup\{r(\gamma, x)\colon\ x \in X\}$, if it is defined, is called the *rank* of the family γ. The family γ is of *point-finite rank* if $r(\gamma, x)$ is finite for any point $x \in X$. The family γ is of *sub-infinite rank* if any independent subfamily $\eta \subseteq \gamma$ with a non-empty intersection is finite.

A family γ of sets is *Noetherian*, if for any infinite increasing sequence $\{H_n \in \gamma\colon\ n \in \omega\}$ there exists $m \in \omega$ such that $H_n = H_m$ for each $n \geqslant m$.

Any uniform base is a Noetherian family of sub-infinite rank.

Spaces with a Noetherian base have been considered by O. Förrster, G. Grabner, G. Gruenhage, W.F. Lindgren, V.I. Malykhin, P.J. Nyikos, S.A. Peregudov (see [17,18,21,22,24,27]).

Some interesting properties of spaces with a Noetherian base of sub-infinite rank are presented in the next three statements. In particular, the next statement and its proof is similar to some results and arguments in [6,10].

Proposition 4.1. *Let γ be a Noetherian family of sub-infinite rank of subsets of a space X. Then there exists a point-finite subfamily $\xi \subseteq \gamma$ such that $\bigcup \xi = \bigcup \gamma$.*

Proof. An element $H \in \gamma$ is called *maximal in* γ if H is not a proper subset of any other element of γ. Since the family γ is bounded, for any $x \in \bigcup \gamma$ there exists a maximal element $H_x \in \gamma$ such that $x \in H_x$. Let ξ be the family of all maximal elements of γ. Then $x \in H_x \subseteq \bigcup \xi \subseteq \bigcup \gamma$, for any $x \in \bigcup \gamma$. In particular, $\bigcup \xi = \bigcup \gamma$. The maximal elements always form an independent family of sets. Therefore, the families $\mu \subseteq \xi$ are independent. If $\mu \subseteq \xi$ and $x \in \bigcap \mu$, then $|\mu| \leqslant r(\gamma, x)$. Hence the family ξ is point-finite. □

Corollary 4.2. *(O. Förrster and G. Grabner [17], W.F. Lindgren and P.J. Nyikos [22]) A space X with a Noetherian base of sub-infinite rank is hereditarily metacompact.*

Corollary 4.3. *(W.F. Lindgren and P.J. Nyikos [22]) A space X with a Noetherian base $\mathcal{B}_1$ of sub-infinite rank and a base $\mathcal{B}_2$ of countable order has a uniform base $\mathcal{B}$.*

Proof. Indeed, any metacompact space with a base of countable order has a uniform base (see [15,30]). □

The next theorem follows from the O. Förrster's theorem (see [18], Theorem 1.4) which affirms that a Noetherian base of countable order and of sub-infinite rank is uniform. For the completeness we have presented another proof.

Theorem 4.4. *Let $\mathcal{B}$ be a base of a space X. Then the following statements are equivalent*:

1. *$\mathcal{B}$ is a uniform base for X.*
2. *$\mathcal{B}$ is a sharp base of sub-infinite rank for X.*
3. *$\mathcal{B}$ is a strong base of countable order of sub-infinite rank for X.*

Proof. Implications $1 \to 2 \to 3$ are obvious.

Assume that $\mathcal{B}$ is a strong base of countable order for X. In particular, this means that $\mathcal{B}$ is Noetherian.

We now fix an infinite subfamily $\mu \subseteq \mathcal{B}$ and a point $x \in X$ such that $x \in \bigcap \mu$.

Claim 1. *There exists $U \in \mu$ such that the family $\mu(U) = \{V \in \mu\colon\ V \subseteq U, V \neq U\}$ is infinite.*

Claim 1 will be proved with the help of the next statement:

Claim 2. *Assume that the family $\mu(U)$ is finite for any $U \in \mu$. Then there exists a sequence $\eta = \{U_n \in \mu\colon n \in \omega\}$ such that $U_n \setminus U_m \neq \varnothing$, for all distinct $n, m \in \omega$.*

We construct η by induction. Let U_0 be any maximal element of μ. Suppose that elements U_i of μ are already defined for $i \in \{0, \dots, k\}$, for some $k \in \omega$. By the assumption, the family $\mu(U_i)$ is finite for any $i \leqslant k$. Put $\mu_k = \mu \setminus (\bigcup\{\mu(U_i) \cup \{U_i\}\colon i = 0, \dots, k\})$. Since μ is infinite, the family μ_k is an infinite subfamily of μ. Let U_{k+1} be some maximal element of the family μ_k. The inductive construction is complete. It is easy to verify that the sequence η so constructed satisfies the required condition. Claim 2 is established. Thus, the family η is infinite and independent subfamily of $\mathcal{B}$. This is a contradiction, since the rank of $\mathcal{B}$ is finite in any point. Claim 1 is proved.

We continue to discuss properties of the family μ fixed before Claim 1.

Claim 3. *The family μ is a base of the space X at the point x.*

This statement is true if $U = \{x\}$, for some $U \in \mu$. Assume that $U \neq \{x\}$ for any $U \in \mu$. By virtue of Claim 1, there exists a sequence $\xi = \{U_n \in \mu\colon n \in \omega\}$ such that:

- $U_{n+1} \subseteq U_n$ for any $n \in \omega$;
- $\mu(U_n)$ is infinite for any $n \in \omega$;
- $U_{n+1} \in \mu(U_n)$ for any $n \in \omega$.

Since $\mathcal{B}$ is a base of countable order, ξ is a base of the space X at the point x. Thus, $\mu \supseteq \xi$ is a base of the space X at the point x too. Claim 3 is proved.

It follows from Claim 3 that $\mathcal{B}$ is a uniform base of the space X. Implication $3 \to 1$ is proved. The proof is complete. □

In ([10], Theorem 2) it was proved that a normal space with a base of finite big rank is paracompact. We recall that a family γ has a finite big rank if there exist a positive integer $n \in \omega$ and a finite sequence $\{\gamma_i\colon i \in \{1, 2, \dots, n\}\}$ of families of the rank 1 such that $\gamma = \bigcup\{\gamma_i\colon \in \{1, 2, \dots, n\}\}$. Hence from Theorem 4.4 it follows:

Corollary 4.5. *A normal space X with a strong base of countable order $\mathcal{B}$ of a big finite rank is metrizable.*

5. Examples and problems

The union of two metric spaces is not, as a rule, a p-space. For instance, if $\omega \subseteq X \subseteq \beta\omega$ and $X \setminus \omega$ is a discrete non-empty space, then X is not a p-space and X is the union of two discrete subspaces.

Proposition 5.1. *Let $X = Y \cup Z$, Z be a discrete closed subspace of X, and $\chi(z, X) \leqslant \aleph_0$ for each $z \in Z$. Then*:

1. *If Y has a base of countable order, then X has a base of countable order too.*
2. *If Y has a strong base of countable order, then X has a strong base of countable order too.*
3. *If Y is a (hereditarily) p-space and Z is a G_δ-subset of X, then X is a (hereditarily) p-space.*

Proof. Let $\{W_n\colon n \in \omega\}$ be a sequence of open subsets of X and $Z \subseteq U_{n+1} \subseteq U_n$ for any $n \in \omega$. We put $Z' = Z \setminus cl_X Y$. The set Z' is open-and-closed in X. Fix a compactification B of X such that $cl_B Z'$ is open-and-closed in B.

Now for any point $z \in Z$ fix a countable base $\{U_n z\colon n \in \omega\}$ for the space B at the point z such that $Z \cap U_n z = \{z\}$ and $X \cap U_n z \subseteq W_n$ for each $n \in \omega$. Let $\{\eta_n = \{U_n z\colon z \in Z\}\colon n \in \omega\}$ and $\mathcal{B}_1 = \{X \cap U_n z\colon z \in Z, n \in \omega\}$.

Clearly, $\{\eta_n\colon n \in \omega\}$ is a plumage of Z in B. Statement 1 is proved.

Let $\mathcal{B}_2$ be a base of the space Y, and $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$. Obviously, $\mathcal{B}$ is a base for X. If $\mathcal{B}_2$ is a (strong) base of countable order for Y, then $\mathcal{B}$ is a (strong) base of countable order for X. Statements 2 and 3 are proved.

Suppose that $Z = \bigcap\{W_n\colon n \in \omega\}$ and $\{\zeta_n = \{U_\alpha\colon \alpha \in A_n\}\colon n \in \omega\}$ is a plumage of Y in B. We put $A_{nm} = A_n \cup Z$ and $\gamma_{nm} = \zeta_n \cup \eta_m = \{U_\alpha\colon \alpha \in A_{nm}\}$. Then $\{\gamma_{nm}\colon n, m \in \omega\}$ is a plumage of X in B. Statement 4 is proved. □

Proposition 5.2. *Let $X = Y \cup Z$, Z be a discrete closed subspace of X, $\chi(z, X) \leqslant \aleph_0$ for every $z \in Z$, $\zeta = \{W_\alpha\colon \alpha \in A\}$ is a family of open subsets of the space X and $n \in \omega$ such that*:

- *$|\{\alpha \in A\colon x \in W_\alpha\}| \leqslant n$ for every $x \in X$;*
- *the set $W_\alpha \cap Z$ is finite for each $\alpha \in A$ and $Z \subseteq \bigcup\{W_\alpha\colon \alpha \in A\}$.*

Then:

1. *Z is a G_δ-subset of the Stone–Čech compactification $B = \beta X$ of the space X.*

2. *If Y has a sharp base, then X has a sharp base too.*
3. *If Y is a Čech-complete space, then X is a Čech-complete space too.*

Proof. Since Z is a discrete closed subspace, we can assume that $Z = A$ and $Z \cap W_z = \{z\}$ for any $z \in Z$. In the Stone–Čech compactification $B = \beta X$ we fix the open sets V_z such that $X \cap V_z = W_z$ for each $z \in Z$. Then $|\{\alpha \in A\colon x \in V_\alpha\}| \leqslant n$ for any $x \in B$.

Now for any point $z \in Z$ fix a countable base $\{U_n z\colon n \in \omega\}$ for the space B at the point z such that $Z \cap U_n z = \{z\}$ and $U_{n+1} z \subseteq U_n z \cap V_z$ for each $n \in \omega$.

Let $U_n = \bigcup\{U_n z\colon z \in Z\}$. Then U_n is an open subset of B and $\bigcap\{U_n\colon n \in \omega\} = Z$. Assertion 1 is proved. Statement 3 follows from assertion 1.

We put $\mathcal{B}_1 = \{X \cap U_n z\colon z \in Z, n \in \omega\}$. Let $\mathcal{B}_2$ be a base of the space Y and $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$. Obviously, $\mathcal{B}$ is a base for X. If $\mathcal{B}_2$ is a sharp base for Y, then $\mathcal{B}$ is a sharp base for X. Statement 2 is proved. □

Applying Example 2.1 from [13] and Proposition 5.1, we derive the following curious example.

Example 5.3. There exist a Tychonoff space Z and a perfect mapping $f : Z \to Y$ onto a Tychonoff space Y such that the following conditions are satisfied:

– Z is a hereditarily p-space with a sharp base;
– Y is a space with a strong base of countable order;
– Y is not a p-space and does not have a sharp base;
– $Z = \bigcup\{Z_n\colon n \in \omega\}$, where $Z_n \subseteq Z_{n+1}$ and Z_n is a Čech-complete open subspace of Z for each $n \in \omega$;
– $Y = \bigcup\{Y_n\colon n \in \omega\}$, where $Y_n \subseteq Y_{n+1}$ and Y_n is a Čech-complete open subspace of Y with a sharp base for each $n \in \omega$;
– Z and Y are locally complete metrizable spaces.

Really, in ([1], Example 1) an example of a Tychonoff space X with a sharp base and a closed discrete subspace M which is not a G_δ-set in X was constructed. The space X has a σ-disjoint sharp base and $B = X \setminus M$ is a complete metrizable zero-dimensional space. Moreover, the set M contains no isolated points of X. Let $S = \{0\} \cup \{2^{-n}\colon n \in \omega\}$ and $A = \{2^{-n}\colon n \in \omega\}$ be the subspaces of the space of real numbers. As in [13], we take the subspace $Z = (X \times A) \cup (M \times S)$ of $X \times S$ and define a quotient space Y of Z by identifying $\{x\} \times S$, $x \in M$, to a point. Clearly, we have:

– $X' = X \times A$ is an open subspace of Y with a σ-disjoint sharp base;
– $P = M \times \{0\}$ is a closed discrete G_δ-subset of the space Y.

From these facts it follows that Y has a sharp base (see [13]).

Since X is a space with a σ-disjoint base and the set M is closed and discrete in X, there exist a sequence $\{M_n\colon n \in \omega\}$ of subsets of M and a family $\{V_z\colon z \in M\}$ of open subsets of X such that the following conditions hold:

– $M = \bigcup\{M_n\colon n \in \omega\}$;
– the family $\{V_z\colon z \in M_n\}$ is disjoint for any $n \in \omega$;
– $V_z \cap M = \{z\}$ for each $z \in M$.

Fix $n \in \omega$. We put $X_n = B \cup M_n$, $Z'_n = X_n \times A$, $P_n = M_n \times \{0\}$ and $Z''_n = Z'_n \cup P_n$. From Propositions 5.1 and 5.2 it follows that X_n, Z'_n and Z''_n are Čech-complete. Obviously, the sets Z'_n and Z''_n are open in Z. Thus, the subspace $Z_n = \bigcup\{Z''_i\colon i \leqslant n\}$ is Čech-complete and open in Z.

Since the subspace M is discrete and closed, the resulting mapping $f : Z \to Y$ is perfect. In [13] it was proved that Y is not a p-space and it does not have a sharp base.

Since $f^{-1}(f(Z_n)) = Z_n$, the subspace $Y_n = f(Z_n)$ is Čech-complete and open in Y.

Clearly, $H = f(M \times S)$ is a closed discrete subspace of Y. Since Y is first countable, $Y \setminus H$ is with a sharp base and H is discrete in Y, the space Y is with a strong base of countable order.

By Proposition 5.2, the subspaces $Y''_n = f(Z''_n)$ and Y_n are spaces with sharp bases.

Since X is a locally complete metrizable space, Z and Y are locally complete metrizable spaces.

Example 5.4. Let $X = \mathbb{N} \cup S$ be the Mrowka space ([16], Exercise 3.6.I). Thus, X is pseudocompact, $\mathbb{N} = \{1, 2, 3, \ldots\}$ is an open discrete subspace of X, and S is a closed discrete subspace of X. Besides, X is a locally compact locally metrizable Moore space. In X any subset is a G_δ-set. Notice that the space X has a strong base of countable order and does not have a sharp base.

Example 5.5. Let X be the space of all countable ordinal numbers in the topology induced by the natural linear order. The space X has a base of countable order and does not have a strong base of countable order.

Problem 5.6. Does the square $X \times X$ of an arbitrary space X with a σ-point-finite sharp base have a sharp base?

Problem 5.7. Is every collectionwise normal space with a sharp base metrizable?

A space X is *perfect* if every closed subset of X is a G_δ-set. We recall that every space with a development is perfect. In particular, every space with a uniform base is perfect. Every σ-space, that is, a space with a σ-discrete network, is perfect.

Problem 5.8. Let X be a Čech-complete space with a sharp base, or with a fibering sharp base. Is it true that X is developable?

Some very interesting open question were posed in [23]:

Problem 5.9. ([23], Question 3) Let X be a perfect Tychonoff space with a sharp base. Is it true that X has a uniform base?

Problem 5.10. ([23], Question 3) Let X be a perfect Tychonoff space with a sharp base. Is it true that X is developable?

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