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Asymptotics of solutions to the Laplace–Beltrami equation on a rotation surface with a cusp [☆]

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ABSTRACT

In this work we study an asymptotic behaviour of solutions to the Laplace–Beltrami operator on a rotation surface near a cuspidal point. To this end we use the WKB-approximation. This approach describes the asymptotic behaviour of the solution more explicitly than abstract theory for operators with operator-valued coefficients.

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1. Introduction

We consider the Laplace–Beltrami operator Δ on a rotation surface S with a cuspidal point. This operator plays an important role in physics and in geometry. This is a natural operator which contains information about geometry of the surface in its coefficients. Many mathematicians studied an asymptotic behaviour of solutions to elliptic boundary value problems in different domains of $\mathbb{R}^n$ (cf. [2,3] and references in [5]). Often a neighbourhood of singularity is blown up by a suitable diffeomorphism to an infinite cylinder with singularity at infinity. Then the results about asymptotics of solutions to a differential equation with operator-valued coefficients on infinite interval are applied [4]. There are some restrictions on operator coefficients to use this theory. Therefore one has to make assumptions about coefficient behaviour of original operator near the singularity for applying the mentioned method. Moreover, such approach is not always convenient in practice to get asymptotics in particular cases.

In this work the solutions for the Laplace–Beltrami operator are studied on a surface S with a cusp. Cuspidal singularities considered here are quite natural (for example, power-like and exponential cusps). For such singularities the theory of ordinary differential equations with operator-valued coefficients is applicable. However, we construct asymptotics of solutions using the classical Liouville–Green approximation. It is possible, because considering a rotation surface S allows one to separate variables in the equation and reduce the problem to constructing asymptotics for ordinary differential equations. To determine unknown constants in the asymptotic solution we pose Dirichlet conditions on the edge of S and control the behaviour of solution near the singularity. At the end it remains only a finite number of unknown constants in the solution.

We give also two typical examples for illustration of our approach. Namely, asymptotics of solutions to the Laplace–Beltrami operator on a surface with a power-like cusp and with an exponential cusp are constructed.

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The main achievement of this work is a way to study cuspidal singularities for solutions of the Laplace–Beltrami operator using WKB-asymptotics.

Let us dwell upon the contents of the paper. In Section 1 we describe a surface with a cusp, on which we consider our problem. In Section 2 we compute explicitly the Laplace–Beltrami operator on this surface. In Section 3 we find a general solution to the homogeneous Laplace–Beltrami equation by means of the WKB-approximation. In order to determine some constants in general solution in Section 4 we pose Dirichlet conditions on the edge of our surface. In Section 5 we formulate the main result of the paper writing asymptotics of solution near the cusp. In Section 6 we present two typical examples (powerlike and exponential cusps) demonstrating our result.

2. Manifolds with cuspidal singularities

Consider in $\mathbb{R}^n$ the surface

$$S=\left\{x\in\mathbb{R}^n:\ f(x_n)=\sqrt{x_1^2+\cdots+x_{n-1}^2},\ x_n>0\right\}.$$

We assume that $f:R_+\to R_+$ is an infinitely smooth function with the property $f(0+)=0$. In this case the type of the origin is completely determined by the behaviour of the function f near the point zero. If $f'(0+)=\infty$ the surface S is once differentiable at 0, if $f'(0+)=\mathrm{const}\neq 0$ we have a conical singularity at the origin. And the origin is a cuspidal singular point if $f'(0+)=0$.

Example 1. Let $f(r)=r^p$ and $p>0$. If $0<p<1$ then the origin is a regular point. If $p=1$ then the origin is a conical point. If $p>1$ then the origin is a cusp.

Let us parametrise the surface S by means of spherical coordinates

$$\begin{cases} x_1=f(r)\cos\varphi_1\cdots\cos\varphi_{n-3}\cos\varphi_{n-2},\\ x_2=f(r)\cos\varphi_1\cdots\cos\varphi_{n-3}\sin\varphi_{n-2},\\ x_3=f(r)\cos\varphi_1\cdots\cos\varphi_{n-4}\sin\varphi_{n-3},\\ \quad\vdots\\ x_{n-2}=f(r)\cos\varphi_1\sin\varphi_2,\\ x_{n-1}=f(r)\sin\varphi_1,\\ x_n=r,\end{cases}$$

where r runs over $[0,+\infty)$, $\varphi_1,\ldots,\varphi_{n-3}\in[-\frac{\pi}{2},\frac{\pi}{2}]$, $\varphi_{n-2}\in[0,2\pi)$.

We consider an $(n-1)$-dimensional manifold which is smooth away from a singular point. We assume that a neighbourhood of this singularity is diffeomorphic to a neighbourhood of the cuspidal point $r=0$ on the surface S.

The Riemannian metric on the smooth part of S which is induced from $\mathbb{R}^n$

$$dS^2=\left(dx_1^2+\cdots+dx_n^2\right)\big|_S=\left(1+\left(f'(r)\right)^2\right)dr^2+f^2(r)\,ds^2,$$

where $ds^2=d\varphi_1^2+\cos^2\varphi_1\,d\varphi_2^2+\cdots+\cos^2\varphi_1\cdots\cos^2\varphi_{n-3}\,d\varphi_{n-2}^2$ is the metric on the unit sphere S^{n-2}. If we present this metric as $dS^2=\sum_{i,j=1}^{n-1}g_{ij}(r,\varphi)\,d\varphi_i\,d\varphi_j$ then $(g_{ij})_{i,j=1,\ldots,n-1}$ is a corresponding metric tensor. In our case

$$(g_{ij})=\begin{pmatrix} 1+(f'(r))^2 & 0 & 0 & \cdots & 0\\ 0 & f^2(r) & 0 & \cdots & 0\\ 0 & 0 & f^2(r)\cos^2\varphi_1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & f^2(r)\cos^2\varphi_1\cdots\cos^2\varphi_{n-3}\end{pmatrix}.$$

The corresponding area form on S is

$$\begin{aligned} d\sigma&=\sqrt{\det(g_{ij})}\,dr\,d\varphi_1\cdots d\varphi_{n-2}\\ &=\sqrt{1+\left(f'(r)\right)^2}\,f^{n-2}(r)\cos^{n-3}\varphi_1\cos^{n-4}\varphi_2\cdots\cos\varphi_{n-3}\,dr\,d\varphi_1\cdots d\varphi_{n-2}\\ &=\sqrt{1+\left(f'(r)\right)^2}\,f^{n-2}(r)\,dr\,ds,\end{aligned}$$

where $ds=\cos^{n-3}\varphi_1\cos^{n-4}\varphi_2\cdots\cos\varphi_{n-3}\,d\varphi_1\cdots d\varphi_{n-2}$ is the area form on the unit sphere S^{n-2}.

3. The Laplace–Beltrami operator

Let (g^{ij}) be the inverse of the matrix (g_{ij}), then the Laplace–Beltrami operator on the surface endowed with the metric tensor (g_{ij}) is given by the known formula (see, for instance, [8])

$$\Delta = -\frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n-1} \partial_i \big(\sqrt{\det g}\, g^{ij} \partial_j \big).$$

In our case $\partial_1 = \frac{\partial}{\partial r}$, $\partial_j = \frac{\partial}{\partial \varphi_{j-1}}$, $j = 2, \ldots, n-1$. Diagonality of the metric tensor allows us to simplify this expression.

Lemma 1. *The Laplace–Beltrami operator on S is given by*

$$\Delta = -\frac{1}{g_{22}} \left(\frac{(\sqrt{g_{22}}\partial_1)^2}{g_{11}} - \frac{\sqrt{g_{22}}}{2g_{11}} \left(\frac{\partial_1 g_{11}}{g_{11}} - \frac{\partial_1 g_{33}}{g_{33}} - \cdots - \frac{\partial_1 g_{n-1n-1}}{g_{n-1n-1}} \right) (\sqrt{g_{22}}\partial_1) - \Delta_{S^{n-2}} \right),$$

where $\Delta_{S^{n-2}}$ is the Laplace–Beltrami operator on the $(n-2)$-dimensional unit sphere S^{n-2}.

Proof. We have

$$\begin{aligned}
\Delta &= -\frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n-1} \partial_i \big(\sqrt{\det g}\, g^{ij} \partial_j \big) \\
&= -\frac{1}{\sqrt{\det g}} \sum_{i=1}^{n-1} \partial_i \big(\sqrt{\det g}\, g^{ii} \partial_i \big) \\
&= -\frac{1}{\sqrt{\det g}} \left(\partial_1 \big(\sqrt{\det g}\, g^{11} \partial_1 \big) + \sum_{i=2}^{n-1} \partial_i \big(\sqrt{\det g}\, g^{ii} \partial_i \big) \right).
\end{aligned}$$

We want to extract the powers of derivative $f(r)\partial_1 = \sqrt{g_{22}}\partial_1$ in the first summand because it is the typical derivative near such a singularity. We have

$$\begin{aligned}
-\frac{1}{\sqrt{\det g}} \partial_1 \big(\sqrt{\det g}\, g^{11} \partial_1 \big) &= -\frac{1}{\sqrt{\det g}} \big(\sqrt{\det g}\, g^{11} \partial_1^2 + \partial_1 \big(\sqrt{\det g}\, g^{11} \big) \partial_1 \big) \\
&= -\frac{1}{g_{22}} \left(g^{11} g_{22} \partial_1^2 + \frac{g_{22}}{\sqrt{\det g}} \partial_1 \big(\sqrt{\det g}\, g^{11} \big) \partial_1 \right).
\end{aligned}$$

Since

$$g_{22}\partial_1^2 = (\sqrt{g_{22}}\partial_1)^2 - (\partial_1 \sqrt{g_{22}})(\sqrt{g_{22}}\partial_1)$$

and

$$\begin{aligned}
\frac{g_{22}}{\sqrt{\det g}} \partial_1 \big(\sqrt{\det g}\, g^{11} \big) \partial_1 - g^{11} (\partial_1 \sqrt{g_{22}})(\sqrt{g_{22}}\partial_1) &= \left(\sqrt{\frac{g_{22}}{\det g}} \partial_1 \left(\frac{\sqrt{g_{22} \cdots g_{n-1n-1}}}{\sqrt{g_{11}}} \right) - \frac{\partial_1 \sqrt{g_{22}}}{g_{11}} \right) (\sqrt{g_{22}}\partial_1) \\
&= \frac{\sqrt{g_{22}}}{2g_{11}} \left(\frac{\partial_1 (g_{22} \cdots g_{n-1n-1})}{g_{22} \cdots g_{n-1n-1}} - \frac{\partial_1 g_{11}}{g_{11}} - \frac{\partial_1 g_{22}}{g_{22}} \right) (\sqrt{g_{22}}\partial_1) \\
&= -\frac{\sqrt{g_{22}}}{2g_{11}} \left(\frac{\partial_1 g_{11}}{g_{11}} - \frac{\partial_1 g_{33}}{g_{33}} - \cdots - \frac{\partial_1 g_{n-1n-1}}{g_{n-1n-1}} \right) (\sqrt{g_{22}}\partial_1),
\end{aligned}$$

we get

$$-\frac{1}{\sqrt{\det g}} \partial_1 \big(\sqrt{\det g}\, g^{11} \partial_1 \big) = -\frac{1}{g_{22}} \left(\frac{(\sqrt{g_{22}}\partial_1)^2}{g_{11}} - \frac{\sqrt{g_{22}}}{2g_{11}} \left(\frac{\partial_1 g_{11}}{g_{11}} - \frac{\partial_1 g_{33}}{g_{33}} - \cdots - \frac{\partial_1 g_{n-1n-1}}{g_{n-1n-1}} \right) (\sqrt{g_{22}}\partial_1) \right).$$

It is easy to see that the term containing the derivatives with respect to angles in the Laplace–Beltrami operator is actually the Laplace–Beltrami operator on the $(n-2)$-dimensional unit sphere S^{n-2}, that is

$$-\frac{g_{22}}{\sqrt{\det g}} \sum_{i=2}^{n-1} \partial_i \big(\sqrt{\det g}\, g^{ii} \partial_i \big) = -\sqrt{\frac{f^{2n-4}}{\det g}} \sum_{i=2}^{n-1} \partial_i \left(\sqrt{\frac{\det g}{f^{2n-4}}} \big(f^2 g^{ii} \big) \partial_i \right) = \Delta_{S^{n-2}}.$$

Hence, the Laplace–Beltrami operator gains the form as in the assertions of lemma. □

Substituting the metric tensor $(g_{ij})_{i,j=1,\dots,n-1}$ into Δ we finally obtain in our case

$$\Delta=-\frac{1}{(f(r))^2}\left(\frac{1}{1+(f'(r))^2}\left(f(r)\frac{\partial}{\partial r}\right)^2-\left(\frac{f(r)f'(r)f''(r)}{(1+(f'(r))^2)^2}-\frac{(n-3)f'(r)}{1+(f'(r))^2}\right)\left(f(r)\frac{\partial}{\partial r}\right)-\Delta_{S^{n-2}}\right).$$

Note that the coefficients of this differential operator have a singularity at $r=0$. Using a change of variables $t=\delta(r)=\int_r^{r_0}\frac{ds}{f(s)}$ we push forward this operator from $[0,+\infty)$ to the whole real axis and a neighbourhood of $r=0$ to that of $t=+\infty$. By this change $f(r)\frac{\partial}{\partial r}$ transforms to $\frac{\partial}{\partial t}$ and we can think of this operator as a linear differential operator of the second-order with operator-valued coefficients on the entire axis. A general method for constructing asymptotics of a solution to the equation with such an operator at infinity was developed in the articles of Agmon and Nirenberg [1], Kondrat'ev [2], Maz'ya and Plamenevskii [4], Schulze and Tarkhanov [7]. However there are some restrictions on coefficients of the operator under study for applicability of this method.

Our aim is to write asymptotics of a solution to the Laplace–Beltrami operator Δ on S near the origin in another way. Namely we don't push forward the singularity to $+\infty$. Keeping the singularity at the point 0 we apply the Liouville–Green approximation for a solution to the second-order equation to derive asymptotics of a solution to $\Delta u=0$ in a neighbourhood of the origin. Theoretical physicists often refer to this method as the WKB-approximation in recognition of papers by Wentzel, Kramers and Brillouin.

To do this procedure it is convenient to simplify the expression $(f(r)\frac{\partial}{\partial r})^2$ and rewrite the Laplace–Beltrami operator Δ in the form

$$\begin{aligned}\Delta&=-\frac{1}{1+(f'(r))^2}\frac{\partial^2}{\partial r^2}+\left(\frac{f'(r)f''(r)}{(1+(f'(r))^2)^2}-\frac{(n-2)f'(r)}{f(r)(1+(f'(r))^2)}\right)\frac{\partial}{\partial r}+\frac{1}{(f(r))^2}\Delta_{S^{n-2}}\\&=-\frac{1}{1+(f'(r))^2}\left(\frac{\partial^2}{\partial r^2}-\left(\frac{f'(r)f''(r)}{1+(f'(r))^2}-\frac{(n-2)f'(r)}{f(r)}\right)\frac{\partial}{\partial r}-\frac{1+(f'(r))^2}{(f(r))^2}\Delta_{S^{n-2}}\right).\end{aligned}$$

4. Separation of variables and approximation of the solution

We find a solution to the equation $\Delta u=0$ in the form $u(r,\varphi)=v(r)w(\varphi)$, $\varphi=(\varphi_1,\dots,\varphi_{n-2})$. Substituting $u(r,\varphi)$ into the equation $\Delta u=0$ we obtain

$$\left(v''-\left(\frac{f'(r)f''(r)}{1+(f'(r))^2}-\frac{(n-2)f'(r)}{f(r)}\right)v'\right)w-\frac{1+(f'(r))^2}{(f(r))^2}v\Delta_{S^{n-2}}w=0.$$

Separating variables r and φ we have

$$\frac{v''(r)-\left(\frac{f'(r)f''(r)}{1+(f'(r))^2}-\frac{(n-2)f'(r)}{f(r)}\right)v'(r)}{\frac{1+(f'(r))^2}{(f(r))^2}v(r)}=\frac{\Delta_{S^{n-2}}w(\varphi)}{w(\varphi)}.$$

Since the equality is possible only if the both sides are constant $(=\lambda)$, we get a Sturm–Liouville problem

$$\Delta_{S^{n-2}}w(\varphi)=\lambda w(\varphi).$$

The solutions to this problem are well understood (cf. [8]). They are homogeneous harmonic polynomials $w_{\nu,k}$ on the sphere S^{n-2} in $\mathbb{R}^{n-1}$, where $\nu=1,2,\dots$ is the degree of homogeneity, $k=1,\dots,J(\nu)$ is an index labelling the polynomials of degree ν. The size $J(\nu)$ of the index set for k is known, namely, $J(\nu)=\frac{(n+2\nu-3)(n+\nu-4)!}{\nu!(n-3)!}$ if $n-1>2$ and $\nu\geqslant 0$ (see [8]). If $n-1=2$ then $J(0)=1$, $J(\nu)=2$ for $\nu\geqslant 1$. The eigenvalues $\lambda_\nu=\nu(\nu+n-3)$ correspond to the eigenfunctions $w_{\nu,k}$, moreover, the multiplicity of the eigenvalue $\nu(\nu+n-3)$ is equal to $J(\nu)$.

In the sequel we investigate the obtained equation with respect to r

$$v_\nu''(r)-\left(\frac{f'(r)f''(r)}{1+(f'(r))^2}-\frac{(n-2)f'(r)}{f(r)}\right)v_\nu'(r)-\lambda_\nu\frac{1+(f'(r))^2}{(f(r))^2}v_\nu(r)=0. \tag{1}$$

This is an equation of the second-order with variable coefficients. There is no general methods to find explicit solutions to such equations. We are interested in an asymptotic behaviour of a solution near the origin. To obtain the asymptotic we use the WKB-approximation. It is convenient to substitute the form $v_\nu(r)=(1+(f'(r))^2)^{\frac14}(f(r))^{-\frac{n-2}{2}}z_\nu(r)$ and to reduce our differential equation to

$$z_\nu''(r)=\alpha(r)z_\nu(r), \tag{2}$$

where

$$\alpha(r)=\frac14\left(\frac{f'(r)f''(r)}{1+(f'(r))^2}-(n-2)\frac{f'(r)}{f(r)}\right)^2-\frac12\left(\frac{f'(r)f''(r)}{1+(f'(r))^2}-(n-2)\frac{f'(r)}{f(r)}\right)'+\lambda_\nu\frac{1+(f'(r))^2}{(f(r))^2}.$$

To apply the WKB-approximation to Eq. (2) one has to represent the function $\alpha(r)$ as a sum $\alpha(r) = \alpha_1(r) + \alpha_2(r)$ with properly chosen summands. In most cases

$$\alpha_1(r) = \frac{\lambda_\nu}{(f(r))^2}$$

plays the role of a leading coefficient near $r = 0$ and

$$\alpha_2(r) = \frac{1}{4}\left(\frac{f'f''}{1+f'^2} - (n-2)\frac{f'}{f}\right)^2 - \frac{1}{2}\left(\frac{f'f''}{1+f'^2} - (n-2)\frac{f'}{f}\right)' + \lambda_\nu \frac{f'^2}{f^2}$$

behaves better than $\alpha_1(r)$ here.

Lemma 2. *If function $f(x)$ is analytic in a neighbourhood of zero then the WKB method is applicable by specified choice of $\alpha_1(r)$ and $\alpha_2(r)$.*

Proof. Indeed, since $f(0) = f'(0) = 0$, we can represent the function $f(r)$ as $r^p a(r)$, $p \geqslant 2$, where $a(r)$ is an analytic function such that $a(0) \neq 0$. Substituting this expression in $\alpha_1(r)$ and in $\alpha_2(r)$ we can easy obtain that $\alpha_1(r) \sim \frac{c}{r^{2p}}$ and $\alpha_2(r) = O(\frac{1}{r^2})$. Sufficient conditions for applicability of the WKB method have the form [6, p. 200]

$$\alpha_1(r) \sim \frac{c}{r^{2d+2}}, \qquad \alpha_2(r) = O\left\{\frac{1}{r^{d-h+2}}\right\} \quad \text{as } r \to 0+, \tag{3}$$

where c, d and h are positive constants. Obviously they are fulfilled in our case. $\square$

According to Theorem 2.1 in [6, p. 193], Eq. (2) has on interval $(0,1)$ the linear independent solutions $z_\nu^+(r)$, $z_\nu^-(r)$ with the following behaviour near $r = 0$

$$z_\nu^+(r) = \alpha_1^{-\frac{1}{4}}(r)\exp\left\{\int \alpha_1^{\frac{1}{2}}(r)\,dr\right\}\bigl(1+\varepsilon_1(r)\bigr),$$
$$z_\nu^-(r) = \alpha_1^{-\frac{1}{4}}(r)\exp\left\{-\int \alpha_1^{\frac{1}{2}}(r)\,dr\right\}\bigl(1+\varepsilon_2(r)\bigr),$$

where

$$\varepsilon_1(r) \to 0, \qquad \varepsilon_2(r) \to 0 \quad \text{as } r \to 0+.$$

It is clear that these functions $z_\nu^+(r)$ and $z_\nu^-(r)$ form a fundamental system in the solution space of Eq. (2), i.e. any solution of (2) can be represented as a linear combination of these functions with arbitrary constants. Returning the functions $v_\nu(r)$, we write the general solution of (1) as a linear combination of functions $v_\nu^+(r)$, $v_\nu^-(r)$ where

$$v_\nu^+(r) = \bigl(1+\bigl(f'(r)\bigr)^2\bigr)^{\frac{1}{4}}\bigl(f(r)\bigr)^{-\frac{n-2}{2}}\alpha_1^{-\frac{1}{4}}(r)\exp\left\{\int_r^{r_0} \alpha_1^{\frac{1}{2}}(s)\,ds\right\}\bigl(1+\varepsilon_1(r)\bigr),$$
$$v_\nu^-(r) = \bigl(1+\bigl(f'(r)\bigr)^2\bigr)^{\frac{1}{4}}\bigl(f(r)\bigr)^{-\frac{n-2}{2}}\alpha_1^{-\frac{1}{4}}(r)\exp\left\{-\int_r^{r_0} \alpha_1^{\frac{1}{2}}(s)\,ds\right\}\bigl(1+\varepsilon_2(r)\bigr)$$

in a neighbourhood of zero. We now look for an asymptotic solution to the equation $\Delta u(r,\varphi) = 0$ of the form

$$u(r,\varphi) = \sum_{\nu=0}^{\infty}\sum_{k=1}^{J(\nu)} v_{\nu,k}(r) w_{\nu,k}(\varphi) = \sum_{\nu=0}^{\infty}\sum_{k=1}^{J(\nu)} \bigl(C_{\nu,k}^+ v_\nu^+(r) + C_{\nu,k}^- v_\nu^-(r)\bigr) w_{\nu,k}(\varphi). \tag{4}$$

Let us give an argument for this choice. It is well known that the eigenfunctions $w_{\nu,k}$ form a basis in the Lebesgue space $L^2(S^{n-2})$. Therefore for each $r \in (0,1]$ we can decompose any solution into a series

$$u(r,\varphi) = \sum_{\nu=0}^{\infty}\sum_{k=1}^{J(\nu)} v_{\nu,k}(r) w_{\nu,k}(\varphi) \quad \text{in } L^2\bigl(S^{n-2}\bigr).$$

5. Dirichlet conditions

The expression (4) contains an infinite number of unknown constants $C^+_{\nu,k}$, $C^-_{\nu,k}$. To determine some constants we pose Dirichlet conditions on the edge of S corresponding to $r = 1$. So let

$$u(1,\varphi) = u^0(\varphi) \quad \text{for } \varphi \in S^{n-2}, \tag{5}$$

where $u^0(\varphi) \in L^2(S^{n-2})$ is a given function. Decompose the function $u^0(\varphi)$ into a series with respect to the basis $\{w_{\nu,k}\}$ in $L^2(S^{n-2})$

$$u^0(\varphi) = \sum_{\nu=0}^{\infty} \sum_{k=1}^{J(\nu)} u^0_{\nu,k} w_{\nu,k}(\varphi).$$

Let us substitute the last expression and (4) into the Dirichlet condition (5). Since the system $\{w_{\nu,k}\}$ is linear independent we find $u^0_{\nu,k} = v_{\nu,k}(1)$. From this we get a connection between the constants $C^+_{\nu,k}$ and $C^-_{\nu,k}$,

$$u^0_{\nu,k} = C^+_{\nu,k} v^+_\nu(1) + C^-_{\nu,k} v^-_\nu(1).$$

It is well known that if the Wronsky determinant of a fundamental system $v^+_\nu(r)$, $v^-_\nu(r)$ vanishes in a point then it vanishes identically. Hence, $v^+_\nu(1)$ and $v^-_\nu(1)$ are not zero simultaneously. We can assume that, for example, $v^+_\nu(1) \neq 0$. Then substituting

$$C^+_{\nu,k} = \frac{u^0_{\nu,k} - C^-_{\nu,k} v^-_\nu(1)}{v^+_\nu(1)}$$

into expression (4) yields

$$v_{\nu,k}(r) = \frac{u^0_{\nu,k}}{v^+_\nu(1)} v^+_\nu(r) + C^-_{\nu,k}\left(v^-_\nu(r) - \frac{v^-_\nu(1)}{v^+_\nu(1)} v^+_\nu(r)\right).$$

6. An asymptotic solution

We see that now only half of constants remain undetermined. We cannot pose the same conditions at the cuspidal point 0 because the functions $v^+_\nu(r)$, $v^-_\nu(r)$ have a singularity here. However one can pose conditions on behaviour of a solution near the singularity. A condition on growth can be fit in this case. Our general solution (4) contains both growing and decreasing exponents. If we require the solution to grow slower than a fixed power of exponent, then an infinite number of summands should vanish. Let us realise this idea. If we can take $\alpha_1(r) = \frac{\lambda_\nu}{(f(r))^2}$ then this growth condition has a clear form and can be written as

$$u(r,\varphi) = O\left(f^{-\frac{n-2}{2}}\left(\frac{\lambda_\nu}{(f(r))^2}\right)^{-\frac{1}{4}} \exp\left\{-\gamma \int_r^{r_0} \frac{ds}{f(s)}\right\}\right) \quad \text{as } r \to 0+,$$

where $\gamma < 0$ is a constant. To meet this requirement we have to choose constants $C^-_{\nu,k}$ so that the terms

$$v^+_\nu(r) = f^{-\frac{n-2}{2}}\left(\frac{\lambda_\nu}{(f(r))^2}\right)^{-\frac{1}{4}} \exp\left\{\sqrt{\lambda_\nu} \int_r^{r_0} \frac{ds}{f(s)}\right\}\bigl(1 + \varepsilon_1(r)\bigr)$$

are not contained in the solution for $\sqrt{\lambda_\nu} > -\gamma$. It is the case, if $C^-_{\nu,k} = \frac{u^0_{\nu,k}}{v^-_\nu(1)}$ for the indices ν, such that $\sqrt{\lambda_\nu} > -\gamma$. So we see that the posed condition "cuts off" an infinite part of the series with growing exponents. Moreover, only a finite number of constants remain undetermined in our solution. We thus arrive at the following result.

Theorem 1. *A solution to the problem*

$$\begin{cases} \Delta u(r,\varphi) = 0 & \text{on } S, \\ u(1,\varphi) = u_0(\varphi) & \text{for } \varphi \in S^{n-2}, \end{cases}$$

such that

$$u(r,\varphi) = O\left(f^{-\frac{n-3}{2}} \lambda_\nu^{-\frac{1}{4}} \exp\left\{-\gamma \int_r^{r_0} \frac{ds}{f(s)}\right\}\right) \quad \text{as } r \to 0+,$$

for $\gamma < 0$, has the form

$$u(r,\varphi)=\sum_{\nu,k:\ \sqrt{\lambda_\nu}>-\gamma}\sum \frac{u^0_{\nu,k}}{v^-_\nu(1)}v^-_\nu(r)w_{\nu,k}(\varphi)+\sum_{\nu,k:\ \sqrt{\lambda_\nu}\leqslant-\gamma}\sum\left(\frac{u^0_{\nu,k}}{v^+_\nu(1)}v^+_\nu(r)+C^-_{\nu,k}\left(v^-_\nu(r)-\frac{v^-_\nu(1)}{v^+_\nu(1)}v^+_\nu(r)\right)\right)w_{\nu,k}(\varphi),$$

moreover, in a neighbourhood of zero

$$v^+_\nu(r)=\left(1+\left(f'(r)\right)^2\right)^{\frac14}\left(f(r)\right)^{-\frac{n-3}{2}}\lambda_\nu^{-\frac14}\exp\left\{\sqrt{\lambda_\nu}\int_r^{r_0}\frac{ds}{f(s)}\right\}\left(1+\varepsilon_1(r)\right),$$

$$v^-_\nu(r)=\left(1+\left(f'(r)\right)^2\right)^{\frac14}\left(f(r)\right)^{-\frac{n-3}{2}}\lambda_\nu^{-\frac14}\exp\left\{-\sqrt{\lambda_\nu}\int_r^{r_0}\frac{ds}{f(s)}\right\}\left(1+\varepsilon_2(r)\right).$$

One can see that this result is coherent with abstract theory where often weighted spaces are used. The behaviour of solutions is controlled by belonging to these spaces. A typical weight in this theory is $e^{\gamma\int_r^{r_0}\frac{ds}{f(s)}}$. In our case asymptotics of solutions are controlled also by the function $e^{\gamma\int_r^{r_0}\frac{ds}{f(s)}}$ and even contain this function.

7. Examples

Let us illustrate the described idea by two typical examples.

Example 2. Let $f(r)=r^p$, $p>1$, i.e. the point 0 is a powerlike cusp. In this case the Laplace–Beltrami operator on S has the form

$$\Delta=\frac{-1}{1+p^2r^{2p-2}}\partial_r^2+\left(\frac{p^2(p-1)r^{2p-3}}{(1+p^2r^{2p-2})^2}-\frac{(n-2)pr^{-1}}{1+p^2r^{2p-2}}\right)\partial_r+\frac{1}{r^{2p}}\Delta_{S^{n-2}}.$$

Using Theorem 1 one obtains that the solution to the problem

$$\begin{cases}\Delta u(r,\varphi)=0 & \text{on } S,\\ u(1,\varphi)=u_0(\varphi) & \text{for } \varphi\in S^{n-2},\end{cases}$$

such that

$$u(r,\varphi)=O\left(r^{\frac{-p(n-1)}{2}}\exp\left\{-\gamma\frac{r^{1-p}}{p-1}\right\}\right)\quad \text{as } r\to 0+,$$

$\gamma<0$, has the form

$$u(r,\varphi)=\sum_{\nu,k:\ \sqrt{\lambda_\nu}>-\gamma}\sum \frac{u^0_{\nu,k}}{v^-_\nu(1)}v^-_\nu(r)w_{\nu,k}(\varphi)+\sum_{\nu,k:\ \sqrt{\lambda_\nu}\leqslant-\gamma}\sum\left(\frac{u^0_{\nu,k}}{v^+_\nu(1)}v^+_\nu(r)+C^-_{\nu,k}\left(v^-_\nu(r)-\frac{v^-_\nu(1)}{v^+_\nu(1)}v^+_\nu(r)\right)\right)w_{\nu,k}(\varphi),$$

where

$$v^+_\nu(r)=r^{-\frac{p(n-3)}{2}}\left(1+p^2r^{2p-2}\right)\lambda_\nu^{-\frac14}\exp\left\{\sqrt{\lambda_\nu}\frac{r^{1-p}}{p-1}\right\}\left(1+\varepsilon_1(r)\right),$$

$$v^-_\nu(r)=r^{-\frac{p(n-3)}{2}}\left(1+p^2r^{2p-2}\right)\lambda_\nu^{-\frac14}\exp\left\{-\sqrt{\lambda_\nu}\frac{r^{1-p}}{p-1}\right\}\left(1+\varepsilon_2(r)\right)$$

near the origin.

Example 3. The second example is the Laplace–Beltrami operator on the surface S given by the function $f(r)=r^2e^{-\frac1r}$. The cusp at the origin is of exponential type. In this case the Laplace–Beltrami operator on S has the form

$$\Delta=\frac{-1}{1+(1+2r)^2e^{-\frac2r}}\partial_r^2+\left(\frac{(1+2r)(\frac{1}{r^2}+\frac{2}{r}+2)e^{-\frac2r}}{(1+(1+2r)^2e^{-\frac2r})^2}-\frac{(n-2)(1+2r)r^{-2}}{1+(1+2r)^2e^{-\frac2r}}\right)\partial_r+\frac{1}{r^4e^{-\frac2r}}\Delta_{S^{n-2}}.$$

It is easy to check that sufficient conditions (3) for applicability of the WKB method are fulfilled. Theorem 1 gives that the solution to the problem

$$\begin{cases}\Delta u(r,\varphi)=0 & \text{on } S,\\ u(1,\varphi)=u_0(\varphi) & \text{for } \varphi\in S^{n-2},\end{cases}$$

such that

$$u(r,\varphi)=O\big(r^{-(n-3)}e^{\frac{n-3}{2r}}\exp\{-\gamma e^{-\frac{1}{r}}\}\big)\quad \text{as } r\to 0+,$$

$\gamma<0$, has the form

$$u(r,\varphi)=\sum_{\nu,k:\ \sqrt{\lambda_\nu}>-\gamma}\frac{u^0_{\nu,k}}{v^-_\nu(1)}v^-_\nu(r)w_{\nu,k}(\varphi)+\sum_{\nu,k:\ \sqrt{\lambda_\nu}\leqslant-\gamma}\left(\frac{u^0_{\nu,k}}{v^+_\nu(1)}v^+_\nu(r)+C^-_{\nu,k}\left(v^-_\nu(r)-\frac{v^-_\nu(1)}{v^+_\nu(1)}v^+_\nu(r)\right)\right)w_{\nu,k}(\varphi),$$

where

$$v^+_\nu(r)=\big(1+(1+2r)^2e^{-\frac{2}{r}}\big)^{\frac{1}{4}}r^{-(n-3)}e^{\frac{n-3}{2r}}\lambda_\nu^{-\frac{1}{4}}\exp\{\sqrt{\lambda_\nu}e^{\frac{1}{r}}\}\big(1+\varepsilon_1(r)\big),$$
$$v^-_\nu(r)=\big(1+(1+2r)^2e^{-\frac{2}{r}}\big)^{\frac{1}{4}}r^{-(n-3)}e^{\frac{n-3}{2r}}\lambda_\nu^{-\frac{1}{4}}\exp\{-\sqrt{\lambda_\nu}e^{\frac{1}{r}}\}\big(1+\varepsilon_2(r)\big)$$

in a neighbourhood of zero.

8. Conclusions

In this work we obtain a simple formula for asymptotic behaviour of solutions to the Laplace–Beltrami equation near a cusp (see Theorem 1). This result enables us to study the Laplace–Beltrami operator on radially symmetric surfaces with cuspidal singularities.

While we study a local situation near a cusp the result obtained plays an important role in studying solution properties on a manifold with singularities. Indeed, the behaviour of solutions to an elliptic equation on the smooth part of the manifold and near conical points is rather well understood. A partition of the unity allows one to pass to a global situation, if local information about the solution is available. Namely, one can construct a global parametrix if we know inverse operators in local neighbourhoods. On the other hand, the Laplace–Beltrami operator is a typical and natural example of elliptic operators. So, our result sheds some light upon behaviour of solutions to an elliptic equation on a manifold with conical and cuspidal singularities.

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