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On the number of permutatively inequivalent basic sequences in a Banach space

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Abstract

Let X be a Banach space with a Schauder basis $(e_n)_{n\in\mathbb{N}}$. The relation E_0 is Borel reducible to permutative equivalence between normalized block-sequences of $(e_n)_{n\in\mathbb{N}}$ or X is c_0 or ℓ_p saturated for some $1 \leqslant p < +\infty$. If $(e_n)_{n\in\mathbb{N}}$ is shrinking unconditional then either it is equivalent to the canonical basis of c_0 or ℓ_p, $1 < p < +\infty$, or the relation E_0 is Borel reducible to permutative equivalence between sequences of normalized disjoint blocks of X or of X^*. If $(e_n)_{n\in\mathbb{N}}$ is unconditional, then either X is isomorphic to ℓ_2, or X contains 2^ω subspaces or 2^ω quotients which are spanned by pairwise permutatively inequivalent normalized unconditional bases.

Keywords: Borel reducibility; Permutative equivalence; Block basis; Homogeneous space; Dichotomy

1. Introduction

In the 1990s, W.T. Gowers and R. Komorowski–N. Tomczak-Jaegermann solved the so-called Homogeneous Banach Space Problem. A Banach space is said to be homogeneous if it is isomorphic to its infinite-dimensional closed subspaces; it is a consequence of two theorems proved by these authors that a homogeneous Banach space must be isomorphic to ℓ_2 [14,21].

It is then natural to ask how many non-isomorphic subspaces a given Banach space must contain when it is not isomorphic to ℓ_2. This question was first asked the author by G. Godefroy,

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and not much was known until recently about it in the literature, even concerning the classical spaces c_0 and ℓ_p.

The correct setting for this question is the classification of analytic equivalence relations on Polish spaces by Borel reducibility. This area of research originated from the works of H. Friedman and L. Stanley [13] and independently from the works of L.A. Harrington, A.S. Kechris and A. Louveau [18]. It may be thought of as an extension of the notion of cardinality in terms of complexity, when one counts equivalence classes.

A topological space is Polish if it is separable and its topology may be generated by a complete metric. Its Borel subsets are those belonging to the smallest σ-algebra containing the open sets. An analytic subset is the continuous image of a Polish space, or equivalently, of a Borel subset of a Polish space.

If R (respectively S) is an equivalence relation on a Polish space E (respectively F), then it is said that (E, R) is Borel reducible to (F, S) if there exists a Borel map $f : E \to F$ such that $\forall x, y \in E, xRy \Leftrightarrow f(x)Sf(y)$. An important equivalence relation is the relation E_0: it is defined on 2^ω by

$$\alpha E_0 \beta \quad \Leftrightarrow \quad \exists m \in \mathbb{N},\ \forall n \geqslant m, \quad \alpha(n) = \beta(n).$$

The relation E_0 is a Borel equivalence relation with 2^ω classes and which, furthermore, admits no Borel classification by real numbers, that is, there is no Borel map f from 2^ω into $\mathbb{R}$ (equivalently, into a Polish space), such that $\alpha E_0 \beta \Leftrightarrow f(\alpha) = f(\beta)$; such a relation is said to be *non-smooth*. In fact E_0 is the $\leqslant_B$ minimum non-smooth Borel equivalence relation [18].

There is a natural way to equip the set of subspaces of a Banach space X with a Borel structure (see, e.g., [20]), and the relation of isomorphism is analytic in this setting [2]. The relation E_0 then appears as a natural threshold for results about isomorphism between separable Banach spaces. A Banach space X was defined in [11] to be *ergodic* if E_0 is Borel reducible to isomorphism between subspaces of X; in particular, an ergodic Banach space has continuum many non-isomorphic subspaces, and isomorphism between its subspaces is non-smooth.

The question of the complexity of isomorphism between subspaces of a given Banach space X is related to results and questions of Gowers about the structure of the relation of embedding between subspaces of X [14]. In that article, Gowers proves the following structure theorem.

Theorem 1.1 *(W.T. Gowers). Any Banach space contains a subspace Y satisfying one of the following properties, which are mutually exclusive and all possible*:

(a) *Y is hereditarily indecomposable* (*i.e. contains no direct sum of infinite-dimensional subspaces*);
(b) *Y has an unconditional basis and no disjointly supported subspaces of Y are isomorphic*;
(c) *Y has an unconditional basis and is strictly quasi-minimal* (*i.e. any two subspaces of Y have further isomorphic subspaces, but Y contains no minimal subspace*);
(d) *Y has an unconditional basis and is minimal* (*i.e. Y embeds into any of its subspaces*).

Note that these properties are preserved by passing to block-subspaces (in the associated natural basis). Furthermore, knowing that a space belongs to one of the classes (a)–(d) gives a lot of informations about operators and isomorphisms defined on it (see [14] about this).

C. Rosendal proved that any Banach space satisfying (a) is ergodic [30]. The author and Rosendal noticed that a result of B. Bossard adapts easily to obtain that a space satisfying (b) is ergodic [11]. Finally by [7], using a result of [30], a space with (c) must be ergodic as well.

It is furthermore known that a non-ergodic space Y satisfying (d) must be isomorphic to its hyperplanes and to its square [30], must be reflexive, by [9] and the classical theorem of James, and that it must contain a block-subspace Y_0 such that $Y_0 \simeq Y_0 \oplus Z$ for any block-subspace Z of Y [11].

Note that the class (d) contains the classical spaces c_0 and ℓ_p, $1 \leqslant p < +\infty$, the dual T^* of Tsirelson's space [6], and Schlumprecht's space S [1]. Concerning those spaces, it is known that c_0 and ℓ_p, $1 \leqslant p < 2$ [9] are ergodic. By [29], the space T is ergodic, and the proof holds to show that T^* is ergodic as well. For $2 < p < +\infty$, it is only known that there exist ω_1 non-isomorphic subspaces of ℓ_p (see [23, Theorem 2.d.9]). The case of S is also unsolved.

These results suggest the following conjecture.

Conjecture 1.2. *Every separable Banach space is either isomorphic to ℓ_2 or ergodic.*

Now the spaces c_0 or ℓ_p, $p \neq 2$, are also very homogeneous in some sense, since they are isomorphic to any of their block-subspaces (with respect to their canonical basis).

It also turns out that all the mentioned results about ergodic Banach spaces (except of course [9]), as well as Gowers' theorem, can be proved using block-subspaces of a given basis. So it is natural to study the homogeneity question restricted to block-subspaces of a Banach space X with a Schauder basis. Block-subspaces can be thought of as "regular" subspaces in this context, for example, they will have a canonical unconditional basis, whenever the basis of X is unconditional.

In fact, classical results show that we can get a lot of information about the properties of a space with a basis from the properties of its block-subspaces. For example, recall that two basic sequences (x_n) and (y_n) are said to be equivalent if the linear map T defined on the closed linear span of (x_n) by $Tx_n = y_n, \forall n \in \mathbb{N}$, is an isomorphism onto the closed linear span of (y_n). The canonical bases of c_0 and ℓ_p are characterized, up to equivalence of basis, by the property of being equivalent to all their normalized block-bases (this is Zippin's theorem, [23, Theorem 2.a.9]).

If the basis is unconditional, it will also be natural to consider sequences of blocks (i.e. finitely supported vectors) whose supports are disjoint, but not necessarily successive (equivalently, block-sequences of permutations of the basis). This distinction is relevant as some classical results require considering such basic sequences instead of block-sequences: for example, [23, Theorem 2.10], according to which c_0 and ℓ_p are characterized by unconditionality and the property that every subspace with a basis of disjointly supported blocks is complemented.

We also note that the theorem of Komorowski and Tomczak-Jaegermann [21] is totally irrelevant in this context: it shows the existence of an "exotic" subspace of a Banach space X spanned by an unconditional basis, which has an unconditional finite-dimensional decomposition but which fails to have an unconditional basis, so it will give no information whatsoever on block-sequences or disjointly supported blocks of X.

The natural question concerning the spaces c_0 and ℓ_p is as follows.

Question 1.3. *If X is a Banach space with an* (*unconditional*) *basis, is it true that either X is isomorphic to its block-subspaces or E_0 is Borel reducible to isomorphism between the block-subspaces of X? Is it true that if X is isomorphic to its block-subspaces then X is isomorphic to*

c_0 or ℓ_p? Are these assertions true when one replaces block-subspaces by subspaces supported by disjointly supported blocks?

Note that by an easy result of [10] using the theorem of Zippin, the answer to our question is positive if one replaces isomorphism by equivalence: if X is a Banach space with a normalized basis $(e_n)_{n\in\mathbb{N}}$, then either $(e_n)_{n\in\mathbb{N}}$ is equivalent to the canonical basis of c_0 or ℓ_p, $1 \leqslant p < +\infty$, or E_0 is Borel reducible to equivalence between normalized block-sequences of X.

Some remarks and partial answers to these conjectures may be found in [8]. As solving these questions seems to be out of reach for the moment, in this paper we shall concentrate our efforts on the corresponding conjectures obtained by replacing isomorphism by permutative equivalence. As it turns out, we shall get results which are very close to positive answers in that case. Two basic sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are said to be permutatively equivalent if there is a permutation σ on $\mathbb{N}$ such that $(x_n)_{n\in\mathbb{N}}$ is equivalent to $(y_{\sigma(n)})_{n\in\mathbb{N}}$, in which case we write $(x_n) \sim^{\text{perm}} (y_n)$. Permutative equivalence between Schauder bases is implied by equivalence and implies isomorphism of the closed linear spans.

It is common to study permutative equivalence between normalized unconditional basic sequences, since then any permutation of the basis is again a basic sequence. However some of our results will concern the general case of permutative equivalence between normalized basic sequences which are not necessarily unconditional.

We list several reasons for which studying permutative equivalence is relevant. First, some classical results which are false or unknown for isomorphism can be proved for permutative equivalence. The theorem of Zippin admits a generalization to permutative equivalence, due to Bourgain et al. [3]: if an unconditional basis is permutatively equivalent to all its normalized block-sequences, then it must be equivalent to the canonical basis of c_0 or ℓ_p [3, Proposition 6.2]. Also, a Cantor–Bernstein result is valid for permutative equivalence: if $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are unconditional basic sequences such that each one is permutatively equivalent to a subsequence of the other, then $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are permutatively equivalent (apparently first proved by Mityagin [26], and [32,33]). Note that this is false without the unconditionality assumption, by the example of Gowers and Maurey of a space isomorphic to its subspaces of codimension 2, by a double shift of its natural basis, but not isomorphic to its hyperplanes [17]. The Schroeder–Bernstein problem for Banach spaces, which asks whether two Banach spaces which are isomorphic to complemented subspaces of each other must be isomorphic, is unsolved in the case of them having an unconditional basis, and solved by the negative in the general case, by Gowers [15] and the examples of [17].

On the other hand, permutative equivalence is already a complex relation. As isomorphism, it is analytic non-Borel, as we shall prove in Proposition 1.5, while equivalence of basic sequences is only K_σ [31]. In fact, as far as we know, permutative equivalence between basic sequences could well be as complex as isomorphism between Banach spaces with a Schauder basis, or between separable Banach spaces in general.

Also, some results of uniqueness of unconditional bases (see [3–5,19]) make it possible, in some special cases, to deduce permutative equivalence of basic sequences from isomorphism of the Banach spaces they span. For example, the results of [9] about the complexity of isomorphism, are essentially results about the complexity of permutative equivalence: indeed, their constructions always realize a reduction of equivalence relations to isomorphism between subspaces equipped with canonical unconditional bases, and these subspaces are isomorphic exactly when these canonical bases are permutatively equivalent [9, Theorems 2.6, 3.3]. The same holds in [29], where it is used that subsequences of the basis of Tsirelson's space are (permutatively) equivalent if and only if they span isomorphic subspaces.

In this article, we investigate the complexity of permutative equivalence between normalized basic sequences of a given Banach space; in particular, if a Schauder basis is not equivalent to c_0 or ℓ_p, we ask how many permutatively inequivalent normalized block-sequences (respectively sequences of disjointly supported blocks) it must contain.

Conjecture 1.4. *Let X be a Banach space with a* (*respectively unconditional*) *basis which is not equivalent to the canonical basis of c_0 or ℓ_p, $1 \leqslant p < +\infty$. Then E_0 is Borel reducible to permutative equivalence between normalized block-sequences* (*respectively sequences of disjointly supported blocks*) *of X.*

In Section 1, we extend the results of [2] to prove that the relation of permutative equivalence is non-Borel, and the results of [9] to show that it reduces the relation E_{K_σ}, and thus is not reducible to the orbit equivalence relation induced by the Borel action of a Polish group on a Polish space (Proposition 1.5).

In Section 2, we prove several lemmas to obtain a result which is very close to a positive answer to Conjecture 1.4. If X is a Banach space with a Schauder basis such that E_0 is not Borel reducible to permutative equivalence between normalized block-sequences of X, then there exists $p \in [1, +\infty]$ such that X is ℓ_p-saturated (or c_0-saturated if $p = +\infty$), Theorem 2.8. If the basis is unconditional, then in fact any normalized block-sequence of X has a subsequence which is equivalent to the canonical basis of ℓ_p (or c_0 if $p = +\infty$), Theorem 2.9. If the basis is unconditional and E_0 not Borel reducible to permutative equivalence between normalized sequences of disjointly supported blocks, then we also have that p is unique such that l_p is finitely disjointly representable on X, and that X satisfies an upper p estimate, Theorem 2.9.

Our main tools for this result are a technical lemma (Lemma 2.1); a result of Rosendal about reductions of E_0 to equivalence relations between subsequences of a given basis [30, Proposition 22], which uses the result of Bourgain et al. [3, Proposition 6.2]; Krivine's theorem [22] about finite block representability of the spaces ℓ_p, and a result of stabilization of Lipschitz functions, by Odell et al. [27].

In Section 3, we deduce that if X is a Banach space with a shrinking normalized unconditional basis (e_n), then either (e_n) is equivalent to the canonical basis of c_0 or some ℓ_p, $1 < p < +\infty$, or E_0 is Borel reducible to permutative equivalence between normalized disjointly supported sequences of blocks on X, or on X^* (Theorem 3.1). It follows that if X is a Banach space with an unconditional basis, then either X is isomorphic to ℓ_2, or X contains 2^ω subspaces or 2^ω quotients spanned by unconditional bases which are mutually permutatively inequivalent (Theorem 3.2).

1.1. Notation

Let us fix or recall some notation. For the reader interested in more details, we refer to [23].

A sequence $(e_n)_{n\in\mathbb{N}}$ with closed linear span X is said to be basic (or a Schauder basis of X) if for any $x \in X$, there exists a unique scalar sequence $(\lambda_n)_{n\in\mathbb{N}}$ such that $x = \sum_{n\in\mathbb{N}} \lambda_n e_n$. This is equivalent to saying that there exists $C \geqslant 1$ such that for any $x = \sum_{n\in\mathbb{N}} \lambda_n e_n$, any integer m, $\|\sum_{n\leqslant m} \lambda_n e_n\| \leqslant C\|x\|$. An interval of integers E is the intersection of an interval of $\mathbb{R}$ with $\mathbb{N}$; it will also denote the canonical projection on the span of $(e_n)_{n\in E}$, called interval projection. A Schauder basis is said to be bimonotone if every non-zero interval projection on its span is of norm 1. A Banach space with a Schauder basis may always be renormed with an equivalent norm so that the basis is bimonotone in the new norm.

Let X be a Banach space with a Schauder basis $(e_n)_{n\in\mathbb{N}}$. We shall use some standard notation about blocks on $(e_n)_{n\in\mathbb{N}}$, i.e. finitely supported non-zero vectors, for example, we shall write $x < y$ and say that x and y are successive when $\max(\mathrm{supp}(x)) < \min(\mathrm{supp}(y))$.

The set of normalized block-sequences, i.e. infinite sequences of successive normalized blocks, in X is denoted $bb(X)$. The set of normalized sequences of disjointly supported blocks in X is denoted $dsb(X)$. Both are seen here as metric spaces as subspaces of X^ω with the product of the norm topology, and this turns them into Polish spaces.

If $(x_n)_{n\in I}$ is a finite or infinite sequence in X then $[x_n]_{n\in I}$ will stand for its closed linear span. We recall that two basic sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are said to be *equivalent* if the map $T:[x_n]_{n\in\mathbb{N}} \to [y_n]_{n\in\mathbb{N}}$ defined by $T(x_n) = y_n$ for all $n \in \mathbb{N}$ is an isomorphism, in which case we write $(x_n) \sim (y_n)$; if $\|T\|\|T^{-1}\| \leqslant C$, then they are *C-equivalent*, and we write $(x_n) \sim^C (y_n)$. A basic sequence is said to be (C-)*subsymmetric* if it is (C-)equivalent to all its subsequences. Note that a subsymmetric sequence need not be unconditional. A Banach space with a subsymmetric Schauder basis may always be renormed to become 1-subsymmetric. Two basic sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are said to be *permutatively equivalent* if there is a permutation σ of $\mathbb{N}$ such that $(x_n)_{n\in\mathbb{N}}$ is equivalent to $(y_{\sigma(n)})_{n\in\mathbb{N}}$, in which case we write $(x_n) \sim^{\mathrm{perm}} (y_n)$.

Let c_{00} denote the set of eventually null scalar sequences. If $(x_n)_{n\in I}$ and $(y_n)_{n\in I}$ are finite or infinite basic sequences, we shall say that (y_n) *C-dominates* (x_n), and write $(x_n) \leqslant^C (y_n)$, to mean that for all $(\lambda_i)_{i\in I}$ in c_{00},

$$\Big\|\sum_{i\in I}\lambda_i x_i\Big\| \leqslant C\Big\|\sum_{i\in I}\lambda_i y_i\Big\|.$$

A basic sequence $(u_i)_{i\in\mathbb{N}}$ is said to be C-unconditional if for any sequence of signs $(\epsilon_i)_{i\in\mathbb{N}} \in \{-1,1\}^\omega$, any sequence $(\lambda_i)_{i\in\mathbb{N}} \in c_{00}$, we have

$$\Big\|\sum_{i\in\mathbb{N}}\epsilon_i\lambda_i u_i\Big\| \leqslant C\Big\|\sum_{i\in\mathbb{N}}\lambda_i u_i\Big\|.$$

In particular, any canonical projection on the closed linear span of some subsequence of a 1-unconditional basis is of norm 1. We may always assume by renorming that an unconditional basis is 1-unconditional. If in addition the basis is subsymmetric, we may ensure that it is also 1-subsymmetric in the new norm.

1.2. General results about permutative equivalence

In this section, we recall the setting defined by B. Bossard for studying the complexity of equivalence relations between basic sequences, and notice that his results about isomorphism easily extend to permutative equivalence [2].

Let u be the normalized universal basic sequence of Pełczyński [28] and U be its closed linear span. The sequence u is defined by the following property: any normalized basic sequence in Banach space is equivalent to a subsequence u' of u such that the canonical projection from U onto the span of u' is bounded.

Bossard defined a natural coding of basic sequences by considering the subsequences of u (identified with infinite subsets of $\mathbb{N}$). Thus a property of basic sequences is Borel (respectively analytic...) if the set of subsequences of $\mathbb{N}$, canonically identified with subsequences of u, with this property is a Borel (respectively analytic...) subset of $[\omega]^\omega$ (the set of increasing sequences of integers).

The sequence u also has an unconditional version $v = (v_n)_{n\in\mathbb{N}}$, i.e. v is a normalized unconditional basic sequence and any normalized unconditional basic sequence in a Banach space is equivalent to a subsequence of v. We may represent v as a subsequence of u.

The relation E_{K_σ} is defined as the maximum K_σ relation on a Polish space for the order $\leqslant_B$ of Borel reducibility [31]. For details about $\leqslant_B$ in the Banach space context we refer to [9]; let us just note here that E_{K_σ} cannot (and thus neither can a relation to which it reduces) be reduced to the orbit equivalence relation induced by the Borel action of a Polish group on a Polish space.

Proposition 1.5. *The relation of permutative equivalence between normalized basic sequences is analytic non-Borel and it Borel reduces E_{K_σ}. In particular it cannot be Borel reducible to the orbit equivalence relation induced by the Borel action of a Polish group on a Polish space.*

Proof. By [9], the relation E_{K_σ} is Borel reducible to isomorphism between Banach spaces. In the list of equivalence of [9, Theorem 2.6], we may obviously add the condition: "is permutatively equivalent to," since equivalence of bases implies permutative equivalence which in turn implies isomorphism of the closed linear spans. This implies that E_{K_σ} is Borel reducible to permutative equivalence. Note that the reduction of E_{K_σ} is obtained using unconditional sequences in ℓ_p, $1 \leqslant p < 2$ (respectively c_0), and so E_0 is Borel reducible to permutative equivalence between unconditional sequences in ℓ_p, $1 \leqslant p < 2$ (respectively c_0), and in particular ℓ_p, $1 \leqslant p < 2$ (respectively c_0) contains 2^ω permutatively inequivalent unconditional basic sequences. This fact will be used at the end of this article.

It is immediate that permutative equivalence is analytic (this was already observed in [10]). To prove that it is not Borel, we now define an unconditional version of a family of basic sequences indexed by the set $\mathcal{T}$ of trees on ω, which was considered in [2]. We also refer to [2] for more details about the proof or the notation, in particular concerning trees.

Let $T = \omega^{<\omega}$ denote the set of finite sequences of integers. Let $c_{00}(T)$ be the space of finitely supported functions from T to $\mathbb{R}$ and let $\phi_s : T \to \{0, 1\}$ be the characteristic function of $\{s\}$ for every $s \in T$. An admissible choice of intervals is a finite set $\{I_j,\ 0 \leqslant j \leqslant k\}$ of intervals of T such that every branch of T meets at most one of these intervals. We consider the ℓ_2-James tree space $\tilde{v}(T)$ on v, i.e. the completion of $c_{00}(T)$ under the norm defined by

$$\|y\| = \sup\left(\left(\sum_{j=0}^{k}\left\|\sum_{s\in I_j} y(s)v_{|s|}\right\|^2\right)^{1/2}\right),$$

where $|s|$ is the length of $s \in T$ and the sup is taken over $k \in \mathbb{N}$ and all admissible choices of intervals $\{I_j,\ 0 \leqslant j \leqslant k\}$.

If $A \subset T$, we let $\tilde{v}(A)$ be the subspace of $\tilde{v}(T)$ generated by $\{\phi_s,\ s \in A\}$. We thus have defined a map $\tilde{v}$ from $\mathcal{T}$ to subsequences of v and thus of u. We claim that $\tilde{v}$ satisfies the following properties:

(a) $\tilde{v}$ is Borel;
(b) for all θ, $\tilde{v}(\theta)$ is unconditional;
(c) if θ is well founded then $\tilde{v}(\theta)$ spans a reflexive space;

(d) if θ is ill founded then some subsequence of $\tilde{v}(\theta)$ (corresponding to a branch of θ) is equivalent to v.

The facts (a), (c) and (d) are valid for an ℓ_2-James space on any Schauder basis instead of (v_n). The proof of (a) is essentially the same as [2, Lemma 2.4]. Reproduce [2, Lemma 1.5] and the Fact in the proof of [2, Theorem 1.2] for (c), and [2, Lemma 1.4] for (d).

To prove (b), we write an unconditional version of [2, Lemma 1.3]. Consider a real sequence $(\lambda_i)_{i\in\mathbb{N}}$, I an interval of T, an integer $n\in\mathbb{N}$ and a subset J of $[0,n]$. We denote by c an upper bound for the norms of canonical projections on subsequences of v. As in [2], let $(s_n)_{n\in\mathbb{N}}$ be a fixed enumeration of T. Moreover, let for each $t\in T$, $t=s_{\bar{t}}$.

For $s\in T$, $(\sum_{i\in J}\lambda_i\phi_{s_i})(s)$ is equal to $\lambda_{\bar{s}}$ if $\bar{s}\in J$ and to 0 otherwise. Therefore,

$$\Big\|\sum_{s\in I}\Big(\sum_{i\in J}\lambda_i\phi_{s_i}\Big)(s)v_{|s|}\Big\| = \Big\|\sum_{s\in I,\bar{s}\in J}\lambda_{\bar{s}}v_{|s|}\Big\| \leqslant c\Big\|\sum_{s\in I,\bar{s}\leqslant n}\lambda_{\bar{s}}v_{|s|}\Big\| = \Big\|\sum_{s\in I}\Big(\sum_{i\leqslant n}\lambda_i\phi_{s_i}\Big)(s)v_{|s|}\Big\|.$$

Let $\{I_j,\ 0\leqslant j\leqslant k\}$ be an admissible choice of intervals. We have

$$\sum_{j=0}^{k}\Big\|\sum_{s\in I_j}\Big(\sum_{i\in J}\lambda_i\phi_{s_i}\Big)(s)v_{|s|}\Big\|^2 \leqslant c^2\sum_{j=0}^{k}\Big\|\sum_{s\in I_j}\Big(\sum_{i\leqslant n}\lambda_i\phi_{s_i}\Big)(s)v_{|s|}\Big\|^2.$$

Thus

$$\Big\|\sum_{i\in J}\lambda_i\phi_{s_i}\Big\| \leqslant c\Big\|\sum_{i\leqslant n}\lambda_i\phi_{s_i}\Big\|,$$

and $(\phi_{s_i})_{i\in\omega}$ is an unconditional basic sequence. The fact (b) follows.

We note the following fact about v. If v is equivalent to a subsequence of some normalized unconditional basic sequence w, then v is permutatively equivalent to w; indeed w is equivalent to a subsequence of v by definition of v and the result follows by the Cantor–Bernstein's principle for permutative equivalence mentioned in the introduction [26,32,33]. So it follows from (b) and (d):

(d′) if θ is ill-founded then $\tilde{v}(\theta)$ is permutatively equivalent to v.

By (c), $v(\theta)$ and v are never permutatively equivalent when θ is well founded. If A is the $\sim^{\text{perm}}$-class of v, it follows from this and from (d′) that $\mathcal{T}\setminus WF = v^{-1}(A)$, where WF denotes the set of ill-founded trees on ω. So by (a) and the well-known fact that WF is non-Borel, A is non-Borel, and it follows that $\sim^{\text{perm}}$ is non-Borel. □

We note here that the relations $=^+$, and the product $E_{K_\sigma}\otimes =^+$, defined as in [9], may, by similar observations as in the E_{K_σ} case, be reduced to permutative equivalence between basic sequences.

2. Reducing E_0 to permutative equivalence

2.1. Reducing E_0 to permutative equivalence between block-sequences

Our initial and important technical result bares similarity with [23, Lemma 2.a.11]: from an hypothesis on block-sequences of a Banach space, we already get a lot of information by looking at those block-sequences of the form $((1-\lambda_n)x_n + \lambda_n y_n)_{n\in\mathbb{N}}$, for some fixed sequences (x_n) and (y_n) and choices of sequences $(\lambda_n)_{n\in\mathbb{N}} \in [0,1]^{\mathbb{N}}$.

Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be normalized basic sequences generating spaces X and Y. We equip $X \oplus Y$ with its canonical normalized basis $(e_n)_{n\in\mathbb{N}}$, that is, for any $(\mu_n)_{n\in\mathbb{N}} \in c_{00}$,

$$\left\| \sum_{n\in\mathbb{N}} \mu_n e_n \right\| = \left\| \sum_{n\in\mathbb{N}} \mu_{2n-1} x_n \right\| + \left\| \sum_{n\in\mathbb{N}} \mu_{2n} y_n \right\|.$$

We shall identify vectors in X (respectively Y) with their image in $X \oplus Y$. Given a sequence $(a_n)_{n\in\mathbb{N}} \in [0,1]^{\mathbb{N}}$, the sequence $(a_i x_i + (1-a_i)y_i)_{i\in\mathbb{N}}$ is a normalized block-sequence of $X \oplus Y$. We denote by $bb_2(X \oplus Y)$ the set of such infinite block-sequences.

Let $(I_k)_{k\in\mathbb{N}}$ be a sequence of successive intervals of integers forming a partition of $\mathbb{N}$, i.e. $\forall k \in \mathbb{N}, \min I_{k+1} = \max I_k + 1$, and let $(\delta_k)_{k\in\mathbb{N}}$ be a positive decreasing sequence converging to 0. We shall say that (I_k), (δ_k) is a *rapidly converging system* if $\delta_1 \leqslant 1/2$ and for all $k \geqslant 1$:

(1) $|I_k|\delta_{k+1} \leqslant 1/4$;
(2) $|I_k|/2 > \sum_{j=1}^{k-1} |I_j|$.

For any $\alpha \in 2^\omega$, we define a sequence of positive numbers $(a_n(\alpha))_{n\in\mathbb{N}}$ by

$$a_n(\alpha) = \delta_{k+\alpha(k)}, \quad \forall k \in \mathbb{N},\ \forall n \in I_k.$$

Finally we define a map f from 2^ω into $bb_2(X \oplus Y)$ by

$$f(\alpha) = \big(a_i(\alpha)x_i + \big(1 - a_i(\alpha)\big)y_i\big)_{i\in\mathbb{N}}.$$

We shall say that f is the *map associated to* (I_k), (δ_k).

Lemma 2.1. *Assume X (respectively Y) is a Banach space with a normalized Schauder basis (x_n) (respectively (y_n)). Let (I_k), (δ_k) form a rapidly converging system and $f : 2^\omega \to bb_2(X \oplus Y)$ be the associated map. Then f Borel reduces the relation E_0 to permutative equivalence on $bb_2(X \oplus Y)$ or there exist $C \geqslant 1$, an infinite subset K of $\mathbb{N}$, and for each $k \in K$, a subset J_k of I_k with $|I_k \setminus J_k| \leqslant \sum_{j=0}^{k-1} |I_j|$, and distinct integers $(n_i)_{i\in J_k}$ such that*

$$(\delta_k x_i + y_i)_{i\in J_k} \sim^C (y_{n_i})_{i\in J_k}.$$

Proof. Without loss of generality we assume that (x_n) and (y_n) are bimonotone.

The map f is obviously Borel (even continuous) and whenever $\alpha E_0 \beta$, $f(\alpha)$ is equivalent, and thus permutatively equivalent to $f(\beta)$.

Assume f does not Borel reduce E_0 to permutative equivalence on $bb_2(X \oplus Y)$. We have $f(\alpha) \sim^{\text{perm}} f(\beta)$ for some α, β in 2^ω which are not E_0 related, and let C be the associated constant of equivalence. We may assume for arbitrarily large k that $\alpha(k)=0$ while $\beta(k)=1$. Let K be the infinite set of such integers, and let $k \in K$.

By the permutative equivalence between $f(\alpha)$ and $f(\beta)$, the sequence $(\delta_k x_i + (1-\delta_k)y_i)_{i\in I_k}$ satisfies

$$\big(\delta_k x_i + (1-\delta_k)y_i\big)_{i\in I_k} \sim^C \big(\delta_{k_i} x_{n_i} + (1-\delta_{k_i})y_{n_i}\big)_{i\in I_k},$$

where $(n_i)_{i\in I_k}$ is a sequence of distinct integers, and for all $i \in J_k$, k_i is equal to $m+\beta(m)$ if m is such that $n_i \in I_m$.

By condition (2), there exists $J_k \subset I_k$, of size at least $|I_k| - \sum_{j=1}^{k-1}|I_j| > 0$, for which we have

$$\big(\delta_k x_i + (1-\delta_k)y_i\big)_{i\in J_k} \sim^C \big(\delta_{k_i} x_{n_i} + (1-\delta_{k_i})y_{n_i}\big)_{i\in J_k},$$

where for $i \in J_k$, k_i is of the form $m+\beta(m)$ for some $m \geqslant k$. So if $m=k$, since $\beta(k)=1$, $k_i \geqslant k+1$, and if $m>k$, $k_i \geqslant k+1$ too. It follows that for all $i \in J_k$, $k_i \geqslant k+1$ and thus $\delta_{k_i} \leqslant \delta_{k+1}$.

Therefore, for any $(\lambda_i)_{i\in J_k}$,

$$\Big\|\sum_{J_k}\delta_{k_i}\lambda_i x_{n_i}\Big\| \leqslant \delta_{k+1}|J_k|\max_{i\in J_k}|\lambda_i|,$$

so as $\delta_{k+1}|J_k| \leqslant 1/4$, and by bimonotonicity,

$$\Big\|\sum_{J_k}\delta_{k_i}\lambda_i x_{n_i}\Big\| \leqslant \frac{1}{4}\Big\|\sum_{J_k}\lambda_i y_{n_i}\Big\|.$$

By the same type of estimate, we have that

$$\frac{3}{4}\Big\|\sum_{J_k}\lambda_i y_{n_i}\Big\| \leqslant \Big\|\sum_{J_k}(1-\delta_{k_i})\lambda_i y_{n_i}\Big\| \leqslant \frac{5}{4}\Big\|\sum_{J_k}\lambda_i y_{n_i}\Big\|.$$

Finally, $(\delta_{k_i}x_{n_i} + (1-\delta_{k_i})y_{n_i})_{i\in J_k} \sim^3 (y_{n_i})_{i\in J_k}$. Also,

$$\frac{1}{2}\Big\|\sum_{J_k}\lambda_i(\delta_k x_i + y_i)\Big\| \leqslant \Big\|\sum_{J_k}\lambda_i\big(\delta_k x_i + (1-\delta_k)y_i\big)\Big\| \leqslant \frac{3}{2}\Big\|\sum_{J_k}\lambda_i(\delta_k x_i + y_i)\Big\|,$$

since $\delta_k \leqslant 1/2$, so $(\delta_k x_i + (1-\delta_k)y_i)_{i\in J_k} \sim^3 (\delta_k x_i + y_i)_{i\in J_k}$, and it follows

$$(\delta_k x_i + y_i)_{i\in J_k} \sim^{9C} (y_{n_i})_{i\in J_k}. \qquad \square$$

Let $\preccurlyeq$ be a linear order on $\mathbb{N}$. When I is a finite subset of $\mathbb{N}$, we denote by $(I)_i^{\preccurlyeq}$ the ith element of I written in $\preccurlyeq$-increasing order.

Definition 2.2. Let (y_n) be a 1-subsymmetric 1-unconditional basic sequence. Let $\preccurlyeq$ be a linear order on $\mathbb{N}$. We define a norm $\|\cdot\|_{\preccurlyeq}$ on the linear span of $(y_n)_{n\in\mathbb{N}}$ by letting, for all $k\in\mathbb{N}$, for all $(\lambda_i)_{i=1}^k\in\mathbb{R}^k$,

$$\left\|\sum_{i=1}^k \lambda_i y_i\right\|_{\preccurlyeq} = \left\|\sum_{i=1}^k \lambda_i y_{\{1,\dots,k\}_i^{\preccurlyeq}}\right\|.$$

Note that the 1-subsymmetry of (y_n) is needed to ensure that this indeed defines a norm, and that (y_n) is a 1-unconditional basis of the completion of its span under this norm. If $\leqslant$ is the usual order relation on $\mathbb{N}$, then $((y_n),\|\cdot\|_{\leqslant})$ is obviously 1-equivalent to (y_n). When (y_n) is 1-symmetric (i.e. 1-equivalent to $(y_{\sigma(n)})$ for any permutation σ on $\mathbb{N}$), then the sequence $((y_n),\|\cdot\|_{\preccurlyeq})$ is always 1-equivalent to (y_n). We shall also be interested in $\|\cdot\|_{\geqslant}$, where $\geqslant$ is defined as usual on $\mathbb{N}$; note that $((y_n),\|\cdot\|_{\geqslant})$ is a 1-subsymmetric basic sequence, and that $((y_n),\|\cdot\|_{\geqslant\geqslant})$ is 1-equivalent to (y_n). We also note that whenever $(y_n)\leqslant(z_n)$, and $\preccurlyeq$ is a linear order on $\mathbb{N}$, it follows that $((y_n),\|\cdot\|_{\preccurlyeq})\leqslant((z_n),\|\cdot\|_{\preccurlyeq})$.

If (y_n) is a subsymmetric unconditional basis, then we define $\|\cdot\|_{\preccurlyeq}$ on $[y_n]$ by

$$\left\|\sum_{i\in\mathbb{N}} \lambda_i y_i\right\|_{\preccurlyeq} = \left\|\sum_{i\in\mathbb{N}} \lambda_i y'_i\right\|_{\preccurlyeq},$$

if (y'_n) is the canonical 1-subsymmetric 1-unconditional basis equivalent to (y_n). The previous observations are still valid up to some constant of equivalence.

Proposition 2.3. *Let X be a Banach space with a normalized unconditional basis (x_n) and Y be a Banach space with a normalized subsymmetric unconditional basis (y_n). The relation E_0 is Borel reducible to permutative equivalence on $bb_2(X\oplus Y)$ or there exists a linear order $\preccurlyeq$ on $\mathbb{N}$ such that $(y_n)\leqslant((y_n),\|\cdot\|_{\preccurlyeq})$ and $(x_n)\leqslant((y_n),\|\cdot\|_{\preccurlyeq})$.*

Proof. Without loss of generality we assume that (x_n) is bimonotone and that (y_n) is 1-unconditional and 1-subsymmetric. We consider the following.

Fact. *There exists $C\geqslant 1$ such that for all $n\in\mathbb{N}$, there exists a permutation σ_n of $\{1,\dots,n\}$ such that $(x_i)_{i=1}^n\leqslant^C(y_{\sigma_n(i)})_{i=1}^n$ and $(y_i)_{i=1}^n\leqslant^C(y_{\sigma_n(i)})_{i=1}^n$.*

We first assume Fact holds. This means that we may pick for each $n\in\mathbb{N}$ a linear order $\preccurlyeq_n$ on $\mathbb{N}$ such that

$$(x_1,\dots,x_n)\leqslant^C\big((y_1,\dots,y_n),\|\cdot\|_{\preccurlyeq_n}\big),$$

and

$$(y_1,\dots,y_n)\leqslant^C\big((y_1,\dots,y_n),\|\cdot\|_{\preccurlyeq_n}\big).$$

Let $\preccurlyeq$ be an accumulation point of the sequence $(\preccurlyeq_n)_{n\in\mathbb{N}}$ in the space of linear orders on $\mathbb{N}$, which is compact. For any $k \in \mathbb{N}$ we may find some $n \geqslant k$ such that $\preccurlyeq_n$ and $\preccurlyeq$ agree on $\{1, \dots, k\}$, so $\|\cdot\|_{\preccurlyeq_n}$ and $\|\cdot\|_{\preccurlyeq}$ agree on $[y_1, \dots, y_k]$. It follows that

$$(x_1, \dots, x_k) \leqslant^C \big((y_1, \dots, y_k), \|\cdot\|_{\preccurlyeq}\big),$$

therefore $((y_i), \|\cdot\|_{\preccurlyeq})$ dominates (x_i). Likewise, $((y_i), \|\cdot\|_{\preccurlyeq})$ dominates (y_i).

Assume now Fact does not hold. We may build by induction a rapidly converging system $(\delta_k), (I_k)$, so $\delta_1 \leqslant 1/2$ and for all $k \geqslant 1$:

(1) $|I_k|\delta_{k+1} \leqslant 1/4$;
(2) $|I_k|/2 > \sum_{j=1}^{k-1} |I_j|$,

and an increasing sequence of integers (K_k) so that for all $k \geqslant 1$,

(3) $K_k > \sum_{j=1}^{k-1} |I_j|$ and $K_k\delta_k \geqslant k$;
(4) for any permutation σ on $\{1, \dots, \max(I_k)\}$, there exists a sequence $(\mu_i)_{i\leqslant\max(I_k)}$ of non-negative numbers with

$$\Big\|\sum_{i\leqslant\max(I_k)} \mu_i y_{\sigma(i)}\Big\| \leqslant 1 \quad \text{and} \quad \Big\|\sum_{i\leqslant\max(I_k)} \mu_i x_i\Big\| + \Big\|\sum_{i\leqslant\max(I_k)} \mu_i y_i\Big\| \geqslant 5K_k.$$

We note that all μ_i's in (4) are smaller than 1. Also, any permutation on I_k may be extended to a permutation on $\{1, \dots, \max(I_k)\}$. Thus using (3) and the bimonotonicity of the basis, we deduce from (4):

(5) for any permutation τ on I_k, there exists a sequence $(\mu_i)_{i\in I_k}$ of non-negative numbers such that $\|\sum_{i\in I_k} \mu_i x_i\| + \|\sum_{i\in I_k} \mu_i y_i\| \geqslant 3K_k$ and such that $\|\sum_{i\in I_k} \mu_i y_{\tau(i)}\| \leqslant 1$.

Now we claim that the map associated to the system $(\delta_k), (I_k)$ Borel reduces E_0 to permutative equivalence on $bb_2(X \oplus Y)$. Otherwise, by Lemma 2.1, we find $C \geqslant 1$, an infinite subset K of $\mathbb{N}$, and for all $k \in K$, a subset J_k of I_k with $|I_k \setminus J_k| \leqslant \sum_{j=0}^{k-1} |I_j|$, and distinct integers $(n_i)_{i\in J_k}$ such that, for any $(\lambda_i)_{i\in J_k}$,

$$\delta_k\left(\Big\|\sum_{i\in J_k} \lambda_i x_i\Big\| + \Big\|\sum_{i\in J_k} \lambda_i y_i\Big\|\right) \leqslant \delta_k\Big\|\sum_{i\in J_k} \lambda_i x_i\Big\| + \Big\|\sum_{i\in J_k} \lambda_i y_i\Big\| \leqslant C\Big\|\sum_{i\in J_k} \lambda_i y_{n_i}\Big\|.$$

Now by 1-subsymmetry of (y_n), the sequence $(y_{n_i})_{i\in J_k}$ is 1-equivalent to some $(y_{\sigma(i)})_{i\in J_k}$ for some permutation σ of J_k. We may extend σ to a permutation $\tilde{\sigma}$ of I_k.

Applying the previous inequality to the coefficients μ_i given by (5) for $\tau = \tilde{\sigma}$, we obtain

$$\delta_k\big(3K_k - 2|I_k \setminus J_k|\big) \leqslant C\Big\|\sum_{i\in J_k} \mu_i y_{\sigma(i)}\Big\|,$$

so, by choice of J_k and by 1-unconditionality,

$$\delta_k\left(3K_k - 2\sum_{j=1}^{k-1}|I_j|\right) \leqslant C\left\|\sum_{i\in I_k}\mu_i y_{\tilde{\sigma}(i)}\right\|,$$

so by (3),

$$k \leqslant K_k\delta_k \leqslant C,$$

for arbitrary large $k \in K$, a contradiction. □

In the following we shall use the notation $(y_n)^{\preccurlyeq}$ to mean $((y_n), \|\cdot\|_{\preccurlyeq})$.

Proposition 2.4. *Let X (respectively Y) be a Banach space with a normalized subsymmetric unconditional basis $(x_n)_{n\in\mathbb{N}}$ (respectively $(y_n)_{n\in\mathbb{N}}$). Assume (x_n) and (y_n) are not equivalent. Then E_0 is Borel reducible to permutative equivalence on $bb_2(X \oplus Y)$.*

Proof. Assume (x_n) and (y_n) are 1-subsymmetric. We assume E_0 is not Borel reducible to permutative equivalence on $bb_2(X \oplus Y)$ and apply Proposition 2.3: let $\preccurlyeq$ be a linear order on $\mathbb{N}$ such that $(x_n) \leqslant (y_n)^{\preccurlyeq}$ and $(y_n) \leqslant (y_n)^{\preccurlyeq}$. By a standard application of Ramsey's theorem for sequences of length 2, we may find an infinite subset K of $\mathbb{N}$ on which either $\preccurlyeq$ coincides with $\leqslant$ or $\preccurlyeq$ coincides with $\geqslant$.

In the first case, by passing to a subsequence with indices in K, and by subsymmetry of (x_n) and (y_n), we obtain that $(x_n) \leqslant (y_n)$.

In the second case, we have $(x_n) \leqslant (y_n)^{\geqslant}$ and $(y_n) \leqslant (y_n)^{\geqslant}$. But this means that $(y_n)^{\geqslant} \leqslant (y_n)^{\geqslant\geqslant}$, and as $(y_n)^{\geqslant\geqslant}$ is equivalent to (y_n), that $(y_n) \geqslant (y_n)^{\geqslant}$. We deduce in that case that $(x_n) \leqslant (y_n)$ as well.

By symmetry we obtain that these two sequences are equivalent. □

An immediate consequence of this fact is that E_0 is Borel reducible to permutative equivalence between normalized block-sequences of $\ell_p \oplus \ell_q$, $1 \leqslant p < q < +\infty$, and of $c_0 \oplus \ell_p$, $1 \leqslant p < +\infty$.

We recall a conjecture by H.P. Rosenthal. A Schauder basis $(e_n)_{n\in\mathbb{N}}$ is said to be a *Rosenthal basis* if any normalized block-sequence of $(e_n)_{n\in\mathbb{N}}$ has a subsequence which is equivalent to $(e_n)_{n\in\mathbb{N}}$. A Banach space has the *Rosenthal property* if it admits a Rosenthal basis.

It is not difficult to see that a Rosenthal basis must be subsymmetric unconditional. Also, all spreading models generated by block-sequences are equivalent in a Banach space with a Rosenthal basis. Rosenthal conjectured that any Rosenthal basis must be equivalent to the canonical basis of c_0 or ℓ_p, $1 \leqslant p < +\infty$. For more details about this property, see [12].

Lemma 2.5. *Let X be a Banach space with an unconditional basis $(e_n)_{n\in\mathbb{N}}$. Assume E_0 is not Borel reducible to permutative equivalence on $bb(X)$. Then there is a subsequence $(f_n)_{n\in\mathbb{N}}$ of $(e_n)_{n\in\mathbb{N}}$ such that every normalized block-sequence in X has a subsequence which is equivalent to $(f_n)_{n\in\mathbb{N}}$. In particular $(f_n)_{n\in\mathbb{N}}$ is a Rosenthal basic sequence.*

Proof. Assume E_0 is not Borel reducible to permutative equivalence on $bb(X)$. Then E_0 is Borel reducible to permutative equivalence on the set of subsequences of $(x_n)_{n\in\mathbb{N}}$ for no $(x_n)_{n\in\mathbb{N}}$

in $bb(X)$. By [30, Proposition 22], it follows that every normalized block-sequence of X has a subsymmetric subsequence. It remains to show that any two subsymmetric block-sequences (x_n) and (y_n) in X are equivalent. We may assume, by passing to subsequences, that $x_k < y_k < x_{k+1}$ for all $k \in \mathbb{N}$. We then apply Proposition 2.4, since E_0 cannot be reduced to $\sim^{\text{perm}}$ on $bb_2([x_k]_{k\in\mathbb{N}} \oplus [y_k]_{k\in\mathbb{N}})$. $\square$

Let X have a Schauder basis $(e_n)_{n\in\mathbb{N}}$. For $1 \leqslant p \leqslant +\infty$, we say that ℓ_p *is block-finitely representable in* X if there exists $C \geqslant 1$ such that for all $n \in \mathbb{N}$, some length n block-sequence in X is C-equivalent to the canonical basis of ℓ_p^n. Note that this differs slightly from the usual definition where it is required that we may take $C = 1 + \epsilon$ for any $\epsilon > 0$. By Krivine's theorem [22], there always exists $p \in [1, +\infty]$ such that ℓ_p is block-finitely representable in X (with C arbitrarily close to 1 if you wish). We say that ℓ_p *is disjointly finitely representable in* X if there exists $C \geqslant 1$ such that $\forall n \in \mathbb{N}$, some length n sequence of disjointly supported blocks in X is C-equivalent to the canonical basis of ℓ_p^n.

Using the proof by Lemberg of Krivine's theorem [22], Odell et al. [27] proved that if X is a Banach space with a Schauder basis, $\bigoplus_{n\in\mathbb{N}} F_n$ is a decomposition of X in successive finite-dimensional subspaces of increasing dimension (where each F_n is equipped with the canonical basis which is a subsequence of the basis of X), (ϵ_n) is a sequence of positive reals, and $f : X \to \mathbb{R}$ is a Lipschitz function on X, then there exists a subsequence F_{k_n} of F_n, finite block-subspaces G_n of F_{k_n} of increasing dimension, and a map $\tilde{f}$ on $\mathbb{R}^{<\omega}$ such that, for all $k \in \mathbb{N}$, for all $k \leqslant n_1 < \cdots < n_k$, for all norm 1 vectors x_i in G_{n_i}, $i \leqslant k$, all coefficients $(\lambda_i)_{i\leqslant k}$, with $|\lambda_i| \leqslant 1$,

$$\left| f\left(\sum_{i=1}^{k} \lambda_i x_i\right) - \tilde{f}(\lambda_1, \ldots, \lambda_k)\right| \leqslant \epsilon_k.$$

We recall that a basic sequence $(x_n)_{n\in\mathbb{N}}$ generates a spreading model $(\tilde{x}_n)_{n\in\mathbb{N}}$ if for any $\epsilon > 0$, and $k \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that for all $N < n_1 < \cdots < n_k$, the sequences $(x_{n_i})_{i\leqslant k}$ and $(\tilde{x}_i)_{i\leqslant k}$ are $(1+\epsilon)$-equivalent. A spreading model is a basic sequence which is necessarily 1-subsymmetric.

The main application given in [27] for their result is about spreading models, and we derive from this the following lemma.

Lemma 2.6. *Let X be a Banach space with a Schauder* (*respectively unconditional*) *basis $(e_n)_{n\in\mathbb{N}}$. Let $p \in [1, +\infty]$ be such that ℓ_p is block* (*respectively disjointly*) *finitely representable in X. Then there exist a spreading model $(\tilde{y}_n)_{n\in\mathbb{N}}$ generated by a block-sequence in X, a normalized block-sequence* (*respectively sequence of disjointly supported blocks*) *(x_n) in X, successive intervals I_k forming a partition of $\mathbb{N}$ and some $C \geqslant 1$ such that*:

- *$|I_k| = k$ for all $k \in \mathbb{N}$;*
- *for all $k \in \mathbb{N}$, $(x_n)_{n\in I_k}$ is C-equivalent to the unit basis of ℓ_p^k;*
- *for any $k \in \mathbb{N}$, any $k < n_1 < \cdots < n_k$, any normalized sequence $(y_i)_{1\leqslant i\leqslant k}$ with $(y_i) \in [x_n]_{n\in I_{n_i}}$, $\forall i \leqslant k$, the sequence $(y_i)_{1\leqslant i\leqslant k}$ is 2-equivalent to $(\tilde{y}_i)_{1\leqslant i\leqslant k}$.*

Proof. Assume ℓ_p is block finitely representable in X. We construct a block-subspace of X of the form $\bigoplus_{n\in\mathbb{N}} F_n$, where each F_n is a block-subspace of dimension n whose basis is C-equivalent to the basis of ℓ_p^n and the F_n's are successive. We pick a sequence (ϵ_k) of positive real numbers smaller than 1 and decreasing to 0, and we apply the result of [27] to $\bigoplus_{n\in\mathbb{N}} F_n$ with the norm on X, which is a Lipschitz map on X.

We obtain finite block-subspaces G_k and a spreading model $\tilde{y}_n$ such that for any $k \in \mathbb{N}$, any $k < n_1 < \cdots < n_k$, any normalized sequence $(y_i)_{1\leqslant i\leqslant k}$ with $(y_i) \in G_i$ is $(1+\epsilon_k)$-equivalent to $\tilde{y}_n$. We let $(x_n)_{n\in I_k}$ be the canonical basis of G_k for all k and we pass to a subsequence to obtain the correct dimension k for each G_k: $(x_n)_{n\in I_k}$ is uniformly equivalent to the basis of l_p^k.

In the case when ℓ_p is disjointly finitely representable in X, we do the same construction with the difference that each F_n will have a basis C-equivalent to the basis of ℓ_p^n which is disjointly supported on X, instead of successive. □

Lemma 2.7. *Let X be a Banach space with an unconditional basis. Assume E_0 is not Borel reducible to permutative equivalence on $bb(X)$ (respectively on $dsb(X)$) and let $(f_n)_{n\in\mathbb{N}}$ be a Rosenthal basic sequence in X given by Lemma 2.5. Let $p \in [1,+\infty]$ be such that ℓ_p is block-finitely representable in $[f_n]_{n\in\mathbb{N}}$ (respectively disjointly finitely representable in X). Then $(f_n)_{n\in\mathbb{N}}$ is equivalent to the unit basis of ℓ_p (or c_0 if $p=+\infty$).*

Proof. Let (f_n) be a Rosenthal basic sequence in X. Let p be such that ℓ_p is block-finitely representable in $[f_n]_{n\in\mathbb{N}}$ (respectively disjointly finitely representable in X). Let (e_n) be the canonical basis of ℓ_p (or c_0 if $p=+\infty$). We need to prove that (f_n) is equivalent to (e_n).

We note that any spreading model $(\tilde{y}_n)$ generated by a block-sequence in X is equivalent to (f_n). Indeed, any block-sequence generating this spreading model has a subsequence equivalent to (f_n), so $(\tilde{y}_n)$ is equivalent to (f_n). So by Lemma 2.6, we find a block-sequence of $[f_n]_{n\in\mathbb{N}}$ (respectively sequence of disjointly supported blocks of X) $(x_n)_{n\in\mathbb{N}}$, a constant $C \geqslant 1$ and associated intervals (I_k) of length k so that

- for all $k\in\mathbb{N}$, $(x_n)_{n\in I_k}$ is C-equivalent to $(e_n)_{n\in I_k}$;
- for any $k\in\mathbb{N}$, any $k < n_1 < \cdots < n_k$, any normalized sequence $(y_i)_{1\leqslant i\leqslant k}$ with $(y_i) \in [x_n]_{n\in I_{n_i}}$, $\forall i \leqslant k$, the sequence $(y_i)_{1\leqslant i\leqslant k}$ is C-equivalent to $(f_i)_{1\leqslant i\leqslant k}$.

In the disjointly supported case, we may, by passing to a subsequence of (x_n), assume that for some subsequence (f'_n) of (f_n), x_n and f'_p are disjointly supported for all n, p in $\mathbb{N}$. In the $bb(X)$ case, we may, by replacing (x_n) by an appropriate subsequence of (x_{2n}), assume that for all $n\in\mathbb{N}$, $\min(\operatorname{supp}(x_{n+1})) \geqslant 2+\max(\operatorname{supp}(x_n))$, where the supports are with respect to (f_n). We may then find a subsequence (f'_n) of (f_n) such that $x_n < f'_n < x_{n+1}$ for all $n\in\mathbb{N}$ (recall that (x_n) is a block-sequence of $[f_n]$ in this case).

In both cases, we may therefore apply Proposition 2.3 to (x_n) and (f'_n), and using the fact that (f_n) is subsymmetric, we find a linear order $\preccurlyeq$ on $\mathbb{N}$ such that $(x_n) \leqslant^{C'} (f_n)^{\preccurlyeq}$, for some constant C'. In particular, for all $k\in\mathbb{N}$,

$$(x_n)_{n\in I_k} \leqslant^{C'} \left((f_n)^{\preccurlyeq}\right)_{n\in I_k}.$$

This implies that

$$(e_n)_{n\leqslant k} \leqslant^{cCC'} (f_{\sigma(n)})_{n\leqslant k},$$

where c is such that (f_n) is c-subsymmetric and σ is a permutation on $\{1,\dots,k\}$. By symmetry of the basis (e_n) and as k was arbitrary, we deduce that (f_n) cCC'-dominates (e_n).

We now prove that (f_n) is dominated by (e_n), and to simplify the notation, we assume $p < +\infty$; the case $p = +\infty$ is similar. Assume on the contrary that (f_n) is not dominated by (e_n). Then we may build by induction a rapidly converging system (δ_k), (I'_k) and some increasing sequence K_k such that for all $k \in \mathbb{N}$,

(6) $K_k > 2\sum_{j=1}^{k-1} |I'_j|$ and $K_k\delta_k \geqslant k$;
(7) there exists a sequence $(\mu_i)_{i\in I'_k}$ which satisfies $\|\sum_{i\in I'_k} \mu_i e_i\| \leqslant 1$ and $\|\sum_{i\in I'_k} \mu_i f_i\| \geqslant K_k$.

Note that $(|I'_k|)_{k\in\mathbb{N}}$ is increasing. We consider the previously defined sequence $(x_n)_{n\in\mathbb{N}}$ and, up to passing to the subsequence of $(x_n)_{n\in\mathbb{N}}$ corresponding to indices in $\bigcup_{k\in\mathbb{N}} I_{|I'_k|}$, we may assume that:

- for all $k \in \mathbb{N}$, $(x_n)_{n\in I'_k}$ is C-equivalent to $(e_n)_{n\in I'_k}$;
- for any $k \in \mathbb{N}$, any $k < n_1 < \cdots < n_k$, any normalized sequence $(y_i)_{1\leqslant i\leqslant k}$ with $(y_i) \in [x_n]_{n\in I'_{n_i}}$, the sequence $(y_i)_{1\leqslant i\leqslant k}$ is C-equivalent to $(f_i)_{1\leqslant i\leqslant k}$,

while we still have that for some subsequence (f'_n) of (f_n), $x_n < f'_n < x_{n+1}$ for all $n \in \mathbb{N}$ (respectively x_n and f'_p are disjointly supported for all n, p in $\mathbb{N}$).

By Lemma 2.1 applied to (f'_n) and (x_n) we may find $D \geqslant 1$, an infinite subset K of $\mathbb{N}$, and for all $k \in K$, a subset J_k of I'_k with $|I'_k \setminus J_k| \leqslant \sum_{j=0}^{k-1} |I'_j|$ and distinct integers $(n_i)_{i\in J_k}$ such that

$$\left(\delta_k f'_i + x_i\right)_{i\in J_k} \sim^D (x_{n_i})_{i\in J_k}.$$

The end of our proof now divides in two cases. For $k \in K$, let A_k be the set of n's such that $\{n_i,\ i \in J_k\} \cap I'_n \neq \emptyset$.

First case. We first assume that for any $m \in \mathbb{N}$, we may find $k \in K$ such that the set A_k is of cardinality at least m.

Let $m \in \mathbb{N}$. For infinitely many k's, we may find a set $L_k \subset J_k$ of cardinality m such that $\{n_i,\ i \in L_k\}$ meets I'_n for exactly m values of n which are strictly larger than m. Then $(x_{n_i})_{i\in L_k} \sim^C (f_{a_i})_{i\in L_k}$, where $(a_i)_{i\in L_k}$ is a reordering of $(1, \ldots, m)$.

We deduce that

$$(f_{a_i})_{i\in L_k} \leqslant^{CD} \left(\delta_k f'_i + x_i\right)_{i\in L_k}$$

so, as $L_k \subset I'_k$, for all $(\lambda_i)_{i\in L_k}$,

$$\left\|\sum_{i\in L_k} \lambda_i f_{a_i}\right\| \leqslant CD\left(\delta_k m \max_{i\in L_k} |\lambda_i| + C\left(\sum_{i\in L_k} |\lambda_i|^p\right)^{1/p}\right),$$

and by symmetry of the expression on the right-hand side, we deduce that for any sequence $(\lambda_i)_{1\leqslant i\leqslant m}$,

$$\left\|\sum_{i\leqslant m} \lambda_i f_i\right\| \leqslant CD\left(\delta_k m \max_{i\leqslant m} |\lambda_i| + C\left(\sum_{i\leqslant m} |\lambda_i|^p\right)^{1/p}\right).$$

Letting k tend to infinity and as m was arbitrary, we obtain that (f_n) is C^2D-dominated by $(e_n)_{n\in\mathbb{N}}$.

Second case. We now assume that there exists some $m \in \mathbb{N}$ such that for all $k \in K$, the set A_k contains at most m elements.

Then for any $k \in K$, all $(\lambda_i)_{i\in J_k}$,

$$\left\| \sum_{i\in J_k} \lambda_i x_{n_i} \right\| = \left\| \sum_{n\in A_k} \sum_{i\in J_k,\, n_i\in I'_n} \lambda_i x_{n_i} \right\| \leqslant Cm \left(\sum_{i\in J_k} |\lambda_i|^p \right)^{1/p}.$$

It follows that

$$\delta_k \left\| \sum_{i\in J_k} \lambda_i f'_i \right\| \leqslant CDm \left(\sum_{i\in J_k} |\lambda_i|^p \right)^{1/p}.$$

Applying this to the coefficients μ_i given by (7), we obtain

$$\delta_k \left(K_k - \sum_{j=1}^{k-1} |I'_j| \right) \leqslant cCDm,$$

where c is such that (f_n) is c-subsymmetric, so by (6),

$$k \leqslant \delta_k K_k \leqslant 2cCDm,$$

a contradiction. □

Theorem 2.8. *Let X be a Banach space with a Schauder basis (e_n). Assume E_0 is not Borel reducible to permutative equivalence on $bb(X)$. Then there exists $p \in [1, +\infty]$ such that every block-sequence of X has a block-sequence which is equivalent to the canonical basis of ℓ_p (or c_0 if $p = +\infty$).*

Proof. If (e_n) is unconditional, Lemma 2.5 applies, so there is a Rosenthal basic sequence (f_n) such that every normalized block basis in X has a subsequence equivalent to (f_n). Let p be such that ℓ_p is block finitely representable in $[f_n]_{n\in\mathbb{N}}$ (p exists by Krivine's theorem). By Lemma 2.7, (f_n) is equivalent to the basis of ℓ_p (or c_0 if $p = +\infty$).

In the general case, note that by [30, Theorem 16], every normalized block-sequence in X has a subsequence which is permutatively equivalent to its further subsequences. In particular, X contains no hereditarily indecomposable subspace (no subspace of a H.I. space is isomorphic to a proper subspace [16]), and by Gowers' dichotomy theorem [14], X is saturated with unconditional block-sequences.

By the unconditional case, we deduce that X is saturated with spaces isomorphic to c_0 or ℓ_p. Finally, if X contains ℓ_p and ℓ_q, for $p \neq q$, then as ℓ_p and ℓ_q are totally incomparable, X contains a direct sum $\ell_p \oplus \ell_q$, and we may assume that these copies are spanned by block-sequences (x_n) and (y_n) which alternate (i.e. $\forall n \in \mathbb{N}$, $x_n < y_n < x_{n+1}$). By Proposition 2.4, E_0 is Borel reducible to permutative equivalence on $bb_2(\ell_p \oplus \ell_q)$, so E_0 would be Borel reducible to $\sim^{\mathrm{perm}}$ on $bb(X)$, a contradiction. The same proof holds for c_0 and ℓ_p. We deduce that there is a unique p such that X contains copies of ℓ_p (or c_0 if $p = +\infty$). □

A Banach space X with an unconditional basis is said to satisfy an *upper p estimate* if there exists $C \geqslant 1$ such that for any disjointly supported vectors $x_1, \dots, x_n$,

$$\left\| \sum_{i=1}^{n} x_i \right\| \leqslant C \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{1/p} \quad \left(\text{or} \left\| \sum_{i=1} x_i \right\| \leqslant C \max_{i \leqslant n} \|x_i\| \text{ if } p = +\infty \right).$$

By a simple diagonalization argument, this is equivalent to saying that for any normalized disjointly supported sequence $(x_n)_{n\in\mathbb{N}}$ on X, (x_n) is dominated by the canonical basis of ℓ_p (or c_0 if $p = +\infty$).

Theorem 2.9. *Let X be a Banach space with an unconditional basis (e_n).*

- *Assume E_0 is not Borel reducible to permutative equivalence on $bb(X)$. Then there exists $p \in [1, +\infty]$ such that every normalized block-sequence of X has a subsequence which is equivalent to the canonical basis of ℓ_p (or c_0 if $p = +\infty$).*
- *Assume E_0 is not Borel reducible to permutative equivalence on $dsb(X)$. Then there is a unique $p \in [1, +\infty]$ such that ℓ_p is disjointly finitely representable in X. If $p = +\infty$ then (e_n) is equivalent to the unit vector basis of c_0. If $p < +\infty$ then X satisfies an upper p-estimate and every normalized block-sequence of X has a subsequence which is equivalent to the canonical basis of ℓ_p.*

Proof. The $bb(X)$ case is proved at the beginning of the proof of Theorem 2.8. Assume now that E_0 is not Borel reducible to permutative equivalence on $dsb(X)$. By Lemma 2.5, there is a Rosenthal basic sequence (f_n), necessarily unique up to equivalence, such that every normalized block basis in X has a subsequence equivalent to (f_n). Let p be such that ℓ_p is disjointly finitely representable in X. By Lemma 2.7, (f_n) is equivalent to the basis of ℓ_p (or c_0 if $p = +\infty$), so p is unique. It remains to show that (e_n) satisfies an upper p-estimate, which implies that (e_n) is equivalent to the basis of c_0 if $p = +\infty$.

For any $(x_n) \in dsb(X)$, we may find a normalized sequence $(v_n) \sim (f_n)$ which is disjointly supported from (x_{2n}). As E_0 is not Borel reducible to permutative equivalence on $bb_2([x_{2n}] \oplus [v_n])$, we deduce from Proposition 2.3 that $(x_{2n}) \leqslant (v_n)^{\preccurlyeq}$ for some linear order $\preccurlyeq$ on $\mathbb{N}$. As (v_n) is symmetric it follows that $(x_{2n}) \leqslant (v_n)$, that is for some C and any $(\lambda_n) \in c_{00}$,

$$\left\| \sum_{n\in\mathbb{N}} \lambda_{2n} x_{2n} \right\| \leqslant C \left(\sum_{n\in\mathbb{N}} |\lambda_{2n}|^p \right)^{1/p},$$

if $p < +\infty$, or

$$\left\| \sum_{n\in\mathbb{N}} \lambda_{2n} x_{2n} \right\| \leqslant C \max_{n\in\mathbb{N}} |\lambda_{2n}|,$$

if $p = +\infty$. We obtain a similar estimate for (x_{2n+1}) and deduce that (x_n) is dominated by the unit vector basis of ℓ_p (or c_0 if $p = +\infty$), and so finally (e_n) satisfies an upper p-estimate. □

Note that from this theorem, we may deduce that E_0 is Borel reducible to $\sim^{\text{perm}}$ on $bb(S)$, where S is Schlumprecht's space [1]. It is, however, still unknown if S is ergodic.

3. Permutative equivalence between unconditional basic sequences in X and in X^*

We obtain a complete dichotomy result by also looking at the disjointly supported sequences of the dual X^* of X, when X^* has a basis. Compare this theorem with Conjecture 1.4.

Theorem 3.1. *Let X be a Banach space with a shrinking normalized unconditional basis (e_n). Then either (e_n) is equivalent to the canonical basis of c_0 or some ℓ_p, $1 < p < +\infty$, or E_0 is Borel reducible to permutative equivalence on $dsb(X)$, or on $dsb(X^*)$.*

Proof. Assume E_0 is Borel reducible to permutative equivalence neither on $dsb(X)$ nor on $dsb(X^*)$. By Theorem 2.9, there exists $Y = c_0$ or ℓ_p for some $1 < p < +\infty$ such that every normalized block-sequence of X has a subsequence equivalent to the canonical basis of Y, and we may assume that $1 < p < +\infty$ and that X satisfies an upper p-estimate.

Some subsequence of (e_n) is equivalent to the basis of ℓ_p, so its dual basis identified with a subsequence of (e_n^*) is equivalent to the basis of $\ell_{p'}$ (where $1/p + 1/p' = 1$). Thus by Theorem 2.9 applied for X^*, X^* satisfies an upper p'-estimate. So $(e_n^*)_{n\in\mathbb{N}}$ is dominated by the unit vector basis of $\ell_{p'}$. It follows that $(e_n)_{n\in\mathbb{N}}$ dominates the unit vector basis of ℓ_p. Finally $(e_n)_{n\in\mathbb{N}}$ is equivalent to the unit vector basis of ℓ_p. □

We also deduce the following dichotomy result about the number of permutatively inequivalent sequences spanning subspaces, or quotients, of a Banach space with an unconditional basis which is not isomorphic to a Hilbert space. Note that by uniqueness of the unconditional basis of ℓ_2, any normalized unconditional basis of a subspace, or a quotient, of ℓ_2 must be (permutatively) equivalent to the canonical basis of ℓ_2.

Theorem 3.2. *Let X be a Banach space with an unconditional basis. Then either X is isomorphic to ℓ_2, or X contains 2^ω subspaces, or 2^ω quotients, spanned by normalized unconditional bases which are mutually permutatively inequivalent.*

Proof. Assume X is not isomorphic to ℓ_2. If X contains c_0 or ℓ_1, then we are done, since by [9], there is a Borel reduction of E_0 to permutative equivalence between the canonical unconditional bases of some subspaces of c_0 (respectively ℓ_1). So by the classical result of James (see [23]), we may assume X is reflexive.

We may assume the basis of X is normalized and we apply Theorem 3.1. If E_0 is Borel reducible to permutative equivalence on $dsb(X)$, then we obtain the desired result with subspaces of X. If E_0 is Borel reducible to permutative equivalence on $dsb(X^*)$, let $f: 2^\omega \to dsb(X^*)$ be the Borel reduction. We note that the bases $f(\alpha)$ and $f(\beta)$ are permutatively equivalent if and only if the dual bases $f(\alpha)^*$ and $f(\beta)^*$ are permutatively equivalent; and for $\alpha \in 2^\omega$, the dual basis $f(\alpha)^*$ is an unconditional basis of some quotient of X. We thus obtain continuum many permutatively inequivalent normalized unconditional bases of quotients of X in the family $f(\alpha)^*$, $\alpha \in 2^\omega$.

Finally if the basis of X is equivalent to the canonical basis of some ℓ_p, $1 < p < +\infty$, with $p < 2$, [9] gives an explicit construction of 2^ω subspaces of X with normalized unconditional bases which are mutually permutatively inequivalent (see the proof of Proposition 1.5; in fact we even obtain a reduction of E_0 to permutative equivalence between such unconditional bases in that case). If $p > 2$, then we use duality to deduce the existence of 2^ω quotients of X with normalized unconditional bases which are mutually permutatively inequivalent. □

The reader should compare this result with Conjecture 1.2, noting that the proof of Theorem 3.2 actually gives a reduction of E_0 to permutative equivalence on an appropriate space of basic sequences spanning subspaces or quotients of X, when X is not isomorphic to ℓ_2.

To conclude, let us mention two results with some similarity with Theorem 3.2, by the use their hypotheses make of both subspaces and duals (respectively quotients). By P. Mankiewicz and N. Tomczak-Jaegermann, if every subspace of every quotient of $\ell_2(X)$ has a Schauder basis, then the Banach space X must be isomorphic to ℓ_2 [24]. By V. Mascioni, if $\ell_2(X)$ is locally self-dual (i.e. finite-dimensional subspaces are uniformly isomorphic to their duals), then X must also be isomorphic to ℓ_2 [25].

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