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Uniform Hölder estimate for singularly perturbed parabolic systems of Bose–Einstein condensates and competing species ☆

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ABSTRACT

We prove the uniform Hölder continuity of solutions for two classes of singularly perturbed parabolic systems. These systems arise in Bose–Einstein condensates and in competing models in population dynamics. The proof relies upon the blow up technique and the monotonicity formulas by Almgren and Alt, Caffarelli, and Friedman.

1. Introduction

In this paper we establish the uniform C^α bound of the solutions for the following two parabolic systems for $\kappa \in (0, +\infty)$:

$$\begin{cases} \frac{\partial u_i}{\partial t} - d_i \Delta u_i = f_i(u_i) - \kappa u_i \sum_{j \neq i} b_{ij} u_j, & \text{in } \Omega \times (0, +\infty), \\ u_i = \varphi_i, & \text{on } \partial\Omega \times (0, +\infty), \\ u_i = \phi_i, & \text{on } \Omega \times \{0\}, \end{cases} \tag{1.1}$$

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and

$$\begin{cases} \frac{\partial u_i}{\partial t} - d_i \Delta u_i = f_i(u_i) - \kappa u_i \sum_{j \neq i} b_{ij} u_j^2, & \text{in } \Omega \times (0, +\infty), \\ u_i = \varphi_i, & \text{on } \partial\Omega \times (0, +\infty), \\ u_i = \phi_i, & \text{on } \Omega \times \{0\}. \end{cases} \tag{1.2}$$

Here $\Omega \subset \mathbb{R}^n$ ($n \geqslant 1$) is an open bounded domain with smooth boundary, $i, j = 1, 2, \dots, K$, $K \geqslant 2$; $d_i > 0$ and $b_{ij} > 0$ are constants (in (1.2), we also assume the symmetric condition $b_{ij} = b_{ji}$); φ_i are given nonnegative Lipschitz continuous functions on $\partial\Omega \times (0, +\infty)$; and ϕ_i are given nonnegative Lipschitz continuous functions on Ω, which satisfy $\phi_i(x) = \varphi_i(x, 0)$ for $x \in \partial\Omega$. They also satisfy the segregated property $\phi_i \phi_j = 0$ and $\varphi_i \varphi_j = 0$ for $i \neq j$. f_i are given Lipschitz functions, that is, $\exists C > 0$,

$$\left| f_i(u) - f_i(v) \right| \leqslant C|u - v|.$$

The first system (1.1) arises from population dynamics, known as the Volterra–Lotka competing system, and the second system (1.2) has its origin in Bose–Einstein condensation. K is the number of the species in (1.1) and is the number of hyperfine spin states in (1.2). For more background, see [7,8, 10,14] and references therein. Of course, in real applications, the most interesting cases are $n = 2, 3$. However, we do not assume this restriction on the dimension. This is possible by our conditions on f_i and the uniform bound on the solutions.

As $\kappa \to +\infty$, uniformly bounded solutions of (1.1) or (1.2) converge to a limiting configurations in some weak sense, $(u_1, u_2, \dots, u_K)$. The limit satisfies a separation condition (see [8]), that is, different components have separated supports:

$$u_i u_j \equiv 0, \quad \text{for } i \neq j.$$

The uniform Hölder regularity in related problems have been studied by many authors. In [8,14], Susanna Terracini and her coauthors proved the uniform Hölder regularity of solutions to the elliptic analogue of (1.1) and (1.2). Although they only state the result for dimension $n \leqslant 3$, it's essentially true in any dimension, as pointed out in their paper. In [2], Caffarelli, Karakhanyan and F. Lin also proved these estimate for (1.1), both in the elliptic case and the parabolic case (see also [6]). However, their result is a local one, only concerning the interior regularity. We will prove a global result and the proof is different from the one in [2]. In fact, our method mainly follows the blow up method, developed by Susanna Terracini and her coauthors in [8,14]. This method is a blow up analysis and needs us to prove some Liouville type theorems. This can be achieved by some monotonicity formulas of Alt–Caffarelli–Friedman type.

The original Alt–Caffarelli–Friedman monotonicity formula is only stated for the case of two phases, see [1] and [4, Chapter 12]. In the first section, we generalize this monotonicity formula to the case of multi phases. This can be seen as the analogue to the elliptic case in [8]. For (1.2), as in the elliptic case [14], we also need another monotonicity formula of Almgren type. This monotonicity formula was first introduced in [15], where it was used to prove a unique continuation property for parabolic equations, after the ideas of Garofalo and Lin [13].

Now let's give our settings. With minor assumptions on f_i (for example, if we take the classical logistic model $f_i(u) = a_i u - u^2$), for fixed κ, the existence of global solutions u_κ of both systems (1.1) and (1.2) can be guaranteed. Moreover, u_κ are nonnegative and Lipschitz continuous on $\overline{\Omega} \times [0, +\infty)$ (but the Lipschitz constants may depend on κ). We also assume that, $\exists C > 0$ independent of κ, such that $\sum_i u_{i,\kappa} \leqslant C$.

The main result of this paper is the following uniform regularity result:

Theorem 1.1. *For any $\alpha \in (0,1)$, there exists a constant C_α independent of κ, such that if u_κ is a solution of* (1.1) *or* (1.2)*, then*

$$\max_i \sup_{\Omega\times(0,+\infty)} \frac{|u_{i,\kappa}(x,t)-u_{i,\kappa}(y,s)|}{d^\alpha((x,t),(y,s))} \leqslant C_\alpha.$$

Here the parabolic distance is defined as

$$d\big((x,t),(y,s)\big) := \max\big\{|t-s|, |x-y|^2\big\}^{\frac{1}{2}}.$$

In this paper, we denote the parabolic dilating as follows: if $X=(x,t)$, then for $\lambda>0$

$$\lambda X = \big(\lambda x, \lambda^2 t\big).$$

We also denote $Q := \Omega \times (0,+\infty)$, $Q_R(x,t) = B_R(x) \times (t-R^2, t)$. The Gaussian measure on $\mathbb{R}^n$, $d\mu = e^{-\frac{|x|^2}{4}}\,dx$. With this measure we have the space $L^2(\mathbb{R}^n, d\mu)$ and the Sobolev space $H^1(\mathbb{R}^n, d\mu)$. $d(x,t) := d((x,t),(0,0))$ is the distance to the origin. In Sections 3 and 5, we sometimes denotes H as a half space of $\mathbb{R}^n$, for example, with the form $\{x_1 > t\}$ for some $t \in \mathbb{R}$.

In Section 2, we establish some monotonicity formulas and Liouville theorems. In Section 3, we perform the blow up procedure and prove Theorem 1.1 for the case of (1.1). In Section 4, we establish the Almgren monotonicity formula. In Section 5, by utilizing this monotonicity formula, we deal with the last case for (1.2) and finish the proof of Theorem 1.1.

2. The monotonicity formula

In this section, we prove some monotonicity formulas and use them to prove some Liouville type theorems, These results are generalizations of the corresponding results in the elliptic case (cf. [8, Section 7] and [14, Section 2]).

Define the kernel

$$G(x,t) = \frac{1}{(4\pi|t|)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4|t|}}$$

and

$$\beta(h) := 2 \inf_{v_i v_j = 0,\ \text{if}\ i\neq j} \sum_{i=1}^{h} \frac{\int_{\mathbb{R}^n} |\nabla v_i(y)|^2 G(y,1)\,dy}{\int_{\mathbb{R}^n} v_i^2(y) G(y,1)\,dy}.$$

By the isoperimetric inequality in Gaussian space, we have $\beta(2)=2$ (see [4, p. 232], the last part of the proof of Theorem 12.11) and $\beta(h)>2$ for $h>2$ (similar to the elliptic case, see [8, p. 557] and [9, Proposition 5.1]). This can also be compared to the corresponding quantity in the elliptic case defined in [8] and [14].

We also denote

$$G_i(x,t) = G\left(\frac{x}{\sqrt{d_i}}, t\right),$$

which satisfies, in $\{t>0\}$

$$\left(\frac{\partial}{\partial t} - d_i \Delta\right) G_i = 0.$$

Theorem 2.1. *Assume in $\mathbb{R}^n \times (-\infty, 0]$, u_i $(1 \leqslant i \leqslant h)$ are continuous functions satisfying*

$$\begin{cases} \frac{\partial u_i}{\partial t} - d_i \Delta u_i \leqslant 0, \\ u_i \geqslant 0, \\ u_i u_j = 0, \quad \text{if } i \neq j, \\ u_i(0,0) = 0. \end{cases} \tag{2.1}$$

Assume $\forall i$ and $t \in (-\infty, 0)$, $\int_{\mathbb{R}^n} |u_i(x,t)|^2 G_i(x,t)\,dx < +\infty$. Then

$$J(t) = \frac{1}{t^{\beta(h)}} \prod_{i=1}^{h} \int_{-t}^{0} \int_{\mathbb{R}^n} \left|\nabla u_i(x,s)\right|^2 G_i(x,s)\,dx\,ds$$

is a nondecreasing function on $(0, +\infty)$.

Proof. Denote $v(x,s) = u(x,-s)$ for $s \in (0,t)$, then

$$J(t) = \frac{1}{t^{\beta(h)}} \prod_{i=1}^{h} \int_{0}^{t} \int_{\mathbb{R}^n} |\nabla v_i|^2 G_i\,dx\,ds. \tag{2.2}$$

So

$$\frac{J'(t)}{J(t)} = -\frac{\beta(h)}{t} + \sum_{i=1}^{h} \frac{\int_{\mathbb{R}^n} |\nabla v_i(x,t)|^2 G_i(x,t)\,dx}{\int_0^t \int_{\mathbb{R}^n} |\nabla v_i(x,s)|^2 G_i(x,s)\,dx\,ds}. \tag{2.3}$$

By noting that

$$\frac{\partial v_i}{\partial s} + d_i \Delta v_i \geqslant 0, \tag{2.4}$$

we can integrate by parts to get

$$\begin{aligned}
&\int_{0}^{t} \int_{\mathbb{R}^n} \left|\nabla v_i(x,s)\right|^2 G_i(x,s)\,dx\,ds \\
&\quad = -\int_{0}^{t} \int_{\mathbb{R}^n} v_i(x,s) \Delta v_i(x,s) G_i(x,s) + v_i(x,s) \nabla v_i(x,s) \nabla G_i(x,s)\,dx\,ds \\
&\quad \leqslant \frac{1}{d_i} \int_{0}^{t} \int_{\mathbb{R}^n} \frac{\partial v_i}{\partial s} v_i(x,s) G_i(x,s)\,dx\,ds + \int_{0}^{t} \int_{R^n} \frac{v_i^2(x,s)}{2} \Delta G_i(x,s)\,dx\,ds \\
&\quad \leqslant \frac{1}{d_i} \int_{0}^{t} \int_{\mathbb{R}^n} \frac{\partial}{\partial s}\left(\frac{v_i^2(x,s)}{2}\right) G_i(x,s)\,dx\,ds + \frac{1}{d_i} \int_{0}^{t} \int_{R^n} \frac{v_i^2(x,s)}{2} \frac{\partial G_i}{\partial s}(x,s)\,dx\,ds
\end{aligned}$$

$$= \frac{1}{d_i} \int_0^t \int_{\mathbb{R}^n} \frac{\partial}{\partial s} \left(\frac{v_i^2(x,s)}{2} G_i(x,s) \right) dx\, ds$$

$$\leqslant \frac{1}{2d_i} \int_{\mathbb{R}^n} v_i^2(x,t) G_i(x,t)\, dx.$$

Substituting this into (2.3) we get

$$\frac{J'(t)}{J(t)} \geqslant -\frac{\beta(h)}{t} + 2 \sum_{i=1}^h d_i \frac{\int_{\mathbb{R}^n} |\nabla v_i(x,t)|^2 G_i(x,t)\, dx}{\int_{\mathbb{R}^n} v_i^2(x,t) G_i(x,t)\, dx}. \tag{2.5}$$

Define the rescaling

$$\bar{v}_i(x) = v_i(\sqrt{d_i t} x, t).$$

Since $\nabla \bar{v}_i(x) = \sqrt{d_i t} \nabla v_i(\sqrt{d_i t} x, t)$, by replacing $x = \sqrt{d_i t} y$, we get

$$d_i \frac{\int_{\mathbb{R}^n} |\nabla v_i(x,t)|^2 G_i(x,t)\, dx}{\int_{\mathbb{R}^n} v_i^2(x,t) G_i(x,t)\, dx} = \frac{1}{t} \frac{\int_{\mathbb{R}^n} |\nabla \bar{v}_i(y)|^2 G(y,1)\, dy}{\int_{\mathbb{R}^n} \bar{v}_i^2(y) G(y,1)\, dy}.$$

Because $\bar{v}_i$ have disjoint supports, we have

$$\sum_{i=1}^h \frac{\int_{\mathbb{R}^n} |\nabla \bar{v}_i(y)|^2 G(y,1)\, dy}{\int_{\mathbb{R}^n} \bar{v}_i^2(y) G(y,1)\, dy} \geqslant \frac{\beta(h)}{2}.$$

Substituting this into (2.5), we get $J'(t) \geqslant 0$. □

Corollary 2.2. *If (u_i) is a solution to (2.1), and there exist constants $b_{ij} > 0$ such that*

$$\left(\frac{\partial}{\partial t} - d_i \Delta \right) u_i - \sum_{j \neq i} \frac{b_{ij}}{b_{ji}} \left(\frac{\partial}{\partial t} - d_j \Delta \right) u_j \geqslant 0. \tag{2.6}$$

If each u_i has sublinear growth, that is, $\exists \alpha \in (0,1)$ and $C > 0$

$$u_i(x,t) \leqslant C\big(1 + d(x,t)\big)^\alpha,$$

then $\forall i$, $u_i \equiv 0$.

Proof. As in the proof of the monotonicity formula, we have

$$\int_{-t}^0 \int_{\mathbb{R}^n} |\nabla u_i|^2 G_i(x,s)\, dx\, ds \leqslant \frac{1}{2d_i} \int_{\mathbb{R}^n} u_i^2(x,-t) G_i(x,t)\, dx. \tag{2.7}$$

Take any two distinct i, j, we know

$$J(t) = \frac{1}{t^2} \left[\int_{-t}^0 \int_{R^n} |\nabla u_i|^2 G_i(x,s)\, dx\, ds \right] \left[\int_{-t}^0 \int_{R^n} |\nabla u_j|^2 G_j(x,s)\, dx\, ds \right]$$

is nondecreasing for $t > 0$. By (2.7) and the sublinear growth of u_i at infinity, we can bound this quantity by

$$\begin{aligned}
J(t) &\leqslant \frac{1}{t^2}\left[\int_{\mathbb{R}^n} \frac{u_i^2(x,-t)}{2d_i} G_i(x,t)\,dx\right]\left[\int_{\mathbb{R}^n} \frac{u_j^2(x,-t)}{2d_j} G_j(x,t)\,dx\right] \\
&\leqslant C\frac{1}{t^2}\left[\int_{\mathbb{R}^n} \left(|x|^2+t\right)^{\alpha} G_i(x,t)\,dx\right]\left[\int_{\mathbb{R}^n} \left(|x|^2+t\right)^{\alpha} G_j(x,t)\,dx\right] \\
&\leqslant C\frac{1}{t^2}\left[t^{\alpha}\int_{\mathbb{R}^n} \left(|y|^2+1\right)^{\alpha} G_i(y,1)\,dx\right]\left[t^{\alpha}\int_{\mathbb{R}^n} \left(|y|^2+1\right)^{\alpha} G_j(y,1)\,dx\right]^2 \\
&\leqslant Ct^{2\alpha-2}.
\end{aligned}$$

Since $2\alpha - 2 < 0$,

$$\lim_{t\to+\infty} J(t) = 0.$$

Because $J(t)$ is nondecreasing in t, we must have for $\forall t > 0$

$$J(t) = 0.$$

So there is one term in $J(t)$ vanishing. For example, if

$$\int_{-t}^{0}\int_{R^n} |\nabla u_i|^2 G_i(x,s)\,dx\,ds = 0,$$

then u_i is a function of t only. By the first inequality of (2.1) and (2.6), we see in the open set $\{u_i > 0\}$,

$$\left(\frac{\partial}{\partial t} - d_i\Delta\right)u_i = 0.$$

Noting that $u_i(0,0) = 0$, then we must have $u_i \equiv 0$. Since we can choose i, j arbitrarily, there is at most one component, assuming to be u_1, nonvanishing. Finally by (2.6), we get

$$\left(\frac{\partial}{\partial t} - d_1\Delta\right)u_1 = 0 \quad \text{in } \mathbb{R}^n \times (-\infty, 0]. \tag{2.8}$$

Then the standard Liouville theorem for the heat equation implies that u_1 is a constant function, too. $\square$

Lemma 2.3. *Assume in $R^n \times (-\infty, 0]$, u_i $(1 \leqslant i \leqslant M)$ are smooth and satisfy*

$$\begin{cases} \frac{\partial u_i}{\partial t} - d_i\Delta u_i = -u_i \sum_{j\neq i} u_j, \\ u_i \geqslant 0. \end{cases} \tag{2.9}$$

Then for any $2 \leqslant h \leqslant M$, $\forall \beta' < \beta(h)$ and t large enough

$$J(t) = \frac{1}{t^{\beta'}} \prod_{i=1}^{h} \int_{-t}^{0} \int_{R^n} \Big(d_i |\nabla u_i|^2 + u_i^2 \sum_{j \neq i} u_j \Big) G_i(x,s)\, dx\, ds$$

is a nondecreasing function of t.

Proof. Denote $v(x,s) = u(x,-s)$ for $s \in (-t,0)$, then

$$J(t) = \frac{1}{t^{\beta'}} \prod_{i=1}^{h} \int_{0}^{t} \int_{\mathbb{R}^n} \Big(d_i |\nabla v_i|^2 + v_i^2 \sum_{j \neq i} v_j \Big) G_i(x,s)\, dx\, ds. \tag{2.10}$$

We can calculate as in the previous lemma:

$$\frac{J'(t)}{J(t)} = -\frac{\beta'}{t} + \sum_{i=1}^{h} \frac{\int_{\mathbb{R}^n} (d_i |\nabla v_i|^2 + v_i^2 \sum_{j \neq i} v_j) G_i(x,t)\, dx}{\int_0^t \int_{\mathbb{R}^n} (d_i |\nabla v_i|^2 + v_i^2 \sum_{j \neq i} v_j) G_i(x,s)\, dx\, ds}. \tag{2.11}$$

By noting that

$$\frac{\partial v_i}{\partial s} + d_i \Delta v_i = v_i \sum_{j \neq i} v_j, \tag{2.12}$$

after integration by parts we have

$$\begin{aligned}
&\int_{0}^{t} \int_{\mathbb{R}^n} \Big(d_i |\nabla v_i|^2 + v_i^2 \sum_{j \neq i} v_j \Big) G_i(x,s)\, dx\, ds \\
&\quad = -\int_{0}^{t} \int_{\mathbb{R}^n} d_i \big[v_i(x,s) \Delta v_i(x,s) G_i(x,s) + v_i(x,s) \nabla v_i(x,s) \nabla G_i(x,s) \big]\, dx\, ds \\
&\qquad + \int_{0}^{t} \int_{\mathbb{R}^n} v_i^2 \sum_{j \neq i} v_j G_i(x,s)\, dx\, ds \\
&\quad = \int_{0}^{t} \int_{\mathbb{R}^n} \frac{\partial v_i}{\partial s} v_i(x,s) G_i(x,s)\, dx\, ds + d_i \int_{0}^{t} \int_{R^n} \frac{v_i^2(x,s)}{2} \Delta G_i(x,s)\, dx\, ds \\
&\quad = \int_{0}^{t} \int_{\mathbb{R}^n} \frac{\partial}{\partial s} \Big(\frac{v_i^2(x,s)}{2} \Big) G_i(x,s)\, dx\, ds + \int_{0}^{t} \int_{R^n} \frac{v_i^2(x,s)}{2} \frac{\partial G_i}{\partial s}(x,s)\, dx\, ds \\
&\quad = \int_{0}^{t} \int_{\mathbb{R}^n} \frac{\partial}{\partial s} \Big(\frac{v_i^2(x,s)}{2} G_i(x,s) \Big)\, dx\, ds
\end{aligned}$$

$$= \int_{\mathbb{R}^n} \frac{v_i^2(x,t)}{2} G_i(x,t)\,dx - \frac{v_i^2(0,0)}{2}$$

$$\leqslant \int_{\mathbb{R}^n} \frac{v_i^2(x,t)}{2} G_i(x,t)\,dx.$$

Substituting this into (2.11) we get

$$\frac{J'(t)}{J(t)} \geqslant -\frac{\beta'}{t} + 2\sum_{i=1}^{h} \frac{\int_{\mathbb{R}^n} (d_i|\nabla v_i|^2 + v_i^2 \sum_{j\neq i} v_j) G_i(x,t)\,dx}{\int_{\mathbb{R}^n} v_i^2(x,t) G_i(x,t)\,dx}. \tag{2.13}$$

Define the rescaling

$$\bar{v}_i(x) = v_i(\sqrt{d_i t}x, t).$$

Since $\nabla \bar{v}_i(x) = \sqrt{d_i t}\nabla v_i(\sqrt{d_i t}x, t)$,

$$\frac{\int_{\mathbb{R}^n} (d_i|\nabla v_i|^2 + v_i^2 \sum_{j\neq i} v_j) G_i(x,t)\,dx}{\int_{\mathbb{R}^n} v_i^2(x,t) G_i(x,t)\,dx} = \frac{1}{t} \frac{\int_{\mathbb{R}^n} (|\nabla \bar{v}_i(y)|^2 + t\bar{v}_i^2(y) \sum_{j\neq i} \bar{v}_j(y)) G(y,1)\,dy}{\int_{\mathbb{R}^n} \bar{v}_i^2(y) G(y,1)\,dy}.$$

Because $\bar{v}_i \geqslant 0$, we claim that

$$\lim_{t\to+\infty} \sum_{i=1}^{h} \frac{\int_{\mathbb{R}^n} (|\nabla \bar{v}_i(y)|^2 + t\bar{v}_i^2(y) \sum_{j\neq i} \bar{v}_j(y)) G(y,1)\,dy}{\int_{\mathbb{R}^n} \bar{v}_i^2(y) G(y,1)\,dy} \geqslant \frac{\beta(h)}{2}.$$

If this is true, then our lemma can be easily seen.

Assume this claim is wrong, then there exists a positive constant $\epsilon > 0$, a sequence $t_k \to +\infty$ and $v_{i,k} \in H^1(\mathbb{R}^n, d\mu)$ such that

$$\sum_{i=1}^{h} \frac{\int_{\mathbb{R}^n} (|\nabla v_{i,k}(y)|^2 + t_k v_{i,k}^2(y) \sum_{j\neq i} v_{j,k}(y)) G(y,1)\,dy}{\int_{\mathbb{R}^n} v_{i,k}^2(y) G(y,1)\,dy} \leqslant \frac{\beta(h)}{2} - \epsilon. \tag{2.14}$$

By renormalization, we can assume $\int_{\mathbb{R}^n} v_{i,k}^2(y) G(y,1)\,dy = 1$. Note that, by differentiation and the equation of v_i^2, we have

$$\int_{\mathbb{R}^n} \bar{v}_i^2(y) G(y,1)\,dy = \int_{\mathbb{R}^n} v_i^2(x,t) G_i(x,t)\,dx$$

is nondecreasing in t, so it has a uniform lower bound for $t \geqslant 1$. Then, $v_{i,k}$ are uniformly bounded in $H^1(\mathbb{R}^n, d\mu)$. So we can assume, after passing to a subsequence of k, $v_{i,k}$ converges to v_i weakly in $H^1(\mathbb{R}^n, d\mu)$. By the compactness embedding of $H^1(\mathbb{R}^n, d\mu)$ into $L^2(\mathbb{R}^n, d\mu)$, $v_{i,k}$ converges to w_i strongly in $L^2(\mathbb{R}^n, d\mu)$, so

$$\int_{\mathbb{R}^n} w_i^2(y) G(y,1)\,dy = 1.$$

Moreover

$$\int_{\mathbb{R}^n} \left|\nabla w_i(y)\right|^2 G(y,1)\,dy \leqslant \liminf_{k\to+\infty} \int_{\mathbb{R}^n} \left|\nabla v_{i,k}(y)\right|^2 G(y,1)\,dy. \tag{2.15}$$

(2.14) also implies

$$\int_{\mathbb{R}^n} v_{i,k}^2(y) \sum_{j\neq i} v_{j,k}(y) G(y,1)\,dy \to 0, \tag{2.16}$$

so $w_i w_j = 0$ for $i \neq j$. Then (2.14) and (2.15) contradict the definition of $\beta(h)$. $\square$

Lemma 2.4. *Assume in $R^n \times (-\infty, 0]$, u_i $(1 \leqslant i \leqslant M)$ satisfy*

$$\begin{cases} \frac{\partial u_i}{\partial t} - d_i \Delta u_i = -u_i \sum_{j\neq i} u_j^2, \\ u_i \geqslant 0. \end{cases} \tag{2.17}$$

Then for any $2 \leqslant h \leqslant M$, $\forall \beta' < \beta(h)$ and t large enough

$$J(t) = \frac{1}{t^{\beta'}} \prod_{i=1}^{h} \int_{-t}^{0} \int_{R^n} \left(d_i |\nabla u_i|^2 + u_i^2 \sum_{j\neq i} u_j^2\right) G_i(x,s)\,dx\,ds$$

is a nondecreasing function of t.

Proof. Denote $v(x,s) = u(x,-s)$ for $s \in (0,t)$, then

$$J(t) = \frac{1}{t^{\beta'}} \prod_{i=1}^{h} \int_{0}^{t} \int_{\mathbb{R}^n} \left(d_i |\nabla v_i|^2 + v_i^2 \sum_{j\neq i} v_j^2\right) G_i(x,s)\,dx\,ds. \tag{2.18}$$

We can calculate as in the previous lemma

$$\frac{J'(t)}{J(t)} = -\frac{\beta'}{t} + \sum_{i=1}^{h} \frac{\int_{\mathbb{R}^n} (d_i |\nabla v_i|^2 + v_i^2 \sum_{j\neq i} v_j^2) G_i(x,t)\,dx}{\int_0^t \int_{\mathbb{R}^n} (|\nabla v_i|^2 + v_i^2 \sum_{j\neq i} v_j^2) G_i(x,s)\,dx\,ds}. \tag{2.19}$$

By noting that

$$\frac{\partial v_i}{\partial s} + d_i \Delta v_i = v_i \sum_{j\neq i} v_j^2, \tag{2.20}$$

after integration by parts we have

$$\int_0^t\int_{\mathbb{R}^n}\Big(d_i|\nabla v_i|^2+v_i^2\sum_{j\neq i}v_j^2\Big)G_i(x,s)\,dx\,ds$$

$$=-\int_0^t\int_{\mathbb{R}^n}d_i\big[v_i(x,s)\Delta v_i(x,s)G_i(x,s)+v_i(x,s)\nabla v_i(x,s)\nabla G_i(x,s)\big]\,dx\,ds$$

$$+\int_0^t\int_{\mathbb{R}^n}v_i^2\sum_{j\neq i}v_j^2G_i(x,s)\,dx\,ds$$

$$=\int_0^t\int_{\mathbb{R}^n}\frac{\partial v_i}{\partial s}v_i(x,s)G_i(x,s)\,dx\,ds+\int_0^t\int_{R^n}d_i\frac{v_i^2(x,s)}{2}\Delta G(x,s)\,dx\,ds$$

$$=\int_0^t\int_{\mathbb{R}^n}\frac{\partial}{\partial s}\bigg(\frac{v_i^2(x,s)}{2}\bigg)G_i(x,s)\,dx\,ds+\int_0^t\int_{R^n}\frac{v_i^2(x,s)}{2}\frac{\partial G_i}{\partial s}(x,s)\,dx\,ds$$

$$=\int_0^t\int_{\mathbb{R}^n}\frac{\partial}{\partial s}\bigg(\frac{v_i^2(x,s)}{2}G_i(x,s)\bigg)\,dx\,ds$$

$$=\int_{\mathbb{R}^n}\frac{v_i^2(x,t)}{2}G_i(x,t)\,dx-\frac{v_i^2(0,0)}{2}$$

$$\leqslant\int_{\mathbb{R}^n}\frac{v_i^2(x,t)}{2}G_i(x,t)\,dx.$$

Thus

$$\frac{J'(t)}{J(t)}\geqslant-\frac{\beta'}{t}+2\sum_{i=1}^h\frac{\int_{\mathbb{R}^n}(d_i|\nabla v_i|^2+v_i^2\sum_{j\neq i}v_j^2)G_i(x,t)\,dx}{\int_{\mathbb{R}^n}v_i^2(x,t)G_i(x,t)\,dx}. \tag{2.21}$$

Define the rescaling

$$\bar v_i(x)=v_i(\sqrt{d_i t}x,t).$$

Since $\nabla\bar v_i(x)=\sqrt{d_i t}\nabla v_i(\sqrt{d_i t}x,t)$,

$$\frac{\int_{\mathbb{R}^n}(d_i|\nabla v_i|^2+v_i^2\sum_{j\neq i}v_j^2)G_i(x,t)\,dx}{\int_{\mathbb{R}^n}v_i^2(x,t)G_i(x,t)\,dx}=\frac{1}{t}\frac{\int_{\mathbb{R}^n}(|\nabla\bar v_i(y)|^2+t\bar v_i^2(y)\sum_{j\neq i}\bar v_j^2(y))G(y,1)\,dy}{\int_{\mathbb{R}^n}\bar v_i^2(y)G(y,1)\,dy}.$$

Because $\bar v_i\geqslant 0$, as in the previous lemma, we still have

$$\lim_{t\to+\infty}\sum_{i=1}^h\frac{\int_{\mathbb{R}^n}(|\nabla\bar v_i(y)|^2+t\bar v_i^2(y)\sum_{j\neq i}\bar v_j^2(y))G(y,1)\,dy}{\int_{\mathbb{R}^n}\bar v_i^2(y)G(y,1)\,dy}\geqslant\beta(h).$$

This is similar to the previous lemma. $\square$

Corollary 2.5. *If u_i is a solution to* (2.9) *or* (2.17), *and it has sublinear growth, that is,* $\exists\alpha\in(0,1)$

$$u_i(x,t)\leqslant C\bigl(1+d(x,t)\bigr)^{\alpha},$$

then each u_i is a constant function.

Proof. We only prove the first case and the second case is similar.

As in the proof of the monotonicity formula, we have

$$\int_{-t}^{0}\int_{\mathbb{R}^n}\Bigl(d_i|\nabla u_i|^2+u_i^2\sum_{j\neq i}u_j\Bigr)G_i(x,s)\,dx\,ds\leqslant\int_{\mathbb{R}^n}\frac{u_i^2(x,-t)}{2}G_i(x,t)\,dx. \tag{2.22}$$

Take any two distinct i, j, and $\epsilon>0$ small, we know

$$J(t)=\frac{1}{t^{2-2\epsilon}}\Biggl(\int_{-t}^{0}\int_{R^n}\Bigl(d_i|\nabla u_i|^2+u_i^2\sum_{j\neq i}u_j\Bigr)G_i\Biggr)\Biggl(\int_{-t}^{0}\int_{R^n}\Bigl(d_j|\nabla u_i|^2+u_i^2\sum_{j\neq i}u_j\Bigr)G_j\Biggr)$$

is nondecreasing for t large. Using (2.22), we can bound this quantity by

$$\begin{aligned}
&\frac{1}{t^{2-2\epsilon}}\Biggl(\int_{\mathbb{R}^n}\frac{u_i^2(x,-t)}{2}G_i(x,t)\,dx\Biggr)\Biggl(\int_{\mathbb{R}^n}\frac{u_i^2(x,-t)}{2}G_j(x,t)\,dx\Biggr)\\
&\leqslant C\frac{1}{t^{2-2\epsilon}}\Biggl[\int_{\mathbb{R}^n}\bigl(|x|^2+t\bigr)^{\alpha}G(x,t)\,dx\Biggr]\Biggl[\int_{\mathbb{R}^n}\bigl(|x|^2+t\bigr)^{\alpha}G_j(x,t)\,dx\Biggr]\\
&\leqslant C\frac{1}{t^{2-2\epsilon}}\Biggl[t^{\alpha}\int_{\mathbb{R}^n}\bigl(|y|^2+1\bigr)^{\alpha}G(y,1)\,dx\Biggr]\Biggl[t^{\alpha}\int_{\mathbb{R}^n}\bigl(|y|^2+1\bigr)^{\alpha}G_j(y,1)\,dx\Biggr]\\
&\leqslant Ct^{2\alpha-2+2\epsilon}.
\end{aligned}$$

So if we choose ϵ such that $2\alpha-2+2\epsilon<0$, then

$$\lim_{t\to+\infty}J(t)=0.$$

Since for t large, $J(t)$ is nondecreasing in t, we must have for t large

$$J(t)=0.$$

So there is one term in $J(t)$ vanishing. For example, if

$$\int_{-t}^{0}\int_{R^n}\Bigl(d_i|\nabla u_i|^2+u_i^2\sum_{j\neq i}u_j\Bigr)G_i(x,s)\,dx\,ds=0, \tag{2.23}$$

then u_i is a function of t only. Moreover, if $u_i\neq 0$, then by the maximum principle $u_i>0$ everywhere and thus for $\forall j\neq i$, $u_j\equiv 0$.

Since we can choose i, j arbitrarily, without loss of generality, we can assume $\forall i > 1$, $u_i \equiv 0$. Now u_1 satisfies the heat equation

$$\left(\frac{\partial}{\partial t} - d_1 \Delta\right) u_1 = 0 \quad \text{in } \mathbb{R}^n \times (-\infty, 0]. \tag{2.24}$$

Then the standard Liouville theorem for heat equation implies that u_1 is a constant function. $\square$

3. Blow up

Before going into the blow up procedure, we first give some preliminary results. Recall the definition of the heat kernel

$$G(x,t) = t^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}.$$

Lemma 3.1. *Assume u is a continuous nonnegative function on $\mathbb{R}^n \times [0, +\infty)$, satisfying $\frac{\partial u}{\partial t} - \Delta u \leqslant 0$ in the distributional sense and $u = 0$ on $\mathbb{R}^n \times \{0\}$. Moreover, assume $\forall t \geqslant 0$,*

$$\int_{\mathbb{R}^n} u^2(x,t) G(x,t)\, dx < +\infty.$$

Then $u \equiv 0$.

Proof. By convolution with a mollifier $\rho \in C_0^\infty(\mathbb{R}^n)$ and integration in the time t (changing u to $\int_t^{t+1} u(x,\tau)\, d\tau$), we can assume u is smooth enough. $\forall T > 0$, take $H(x,t) = G(x, T-t)$, then for $t \in (0, T)$, we can differentiate to get

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}^n} u^2(x,t) H(x,t)\, dx &= \int_{\mathbb{R}^n} 2uu_t H + u^2 H_t \\
&\leqslant \int_{\mathbb{R}^n} 2u \Delta u H - u^2 \Delta H \\
&\leqslant \int_{\mathbb{R}^n} 2u \Delta u H - \Delta u^2 H \\
&= -\int_{\mathbb{R}^n} 2|\nabla u|^2 H \\
&\leqslant 0.
\end{aligned}$$

So $\forall t \in (0, T)$, $\int_{\mathbb{R}^n} u(x,t)^2 H(x,t)\, dx = 0$. Thus $u \equiv 0$. $\square$

The following lemma is taken from [10, Lemma 5.7].

Lemma 3.2. *If in the parabolic cylinder Q_{2R}, u satisfies the following*

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u \leqslant -Mu, \\ u \geqslant 0, \\ u \leqslant A, \end{cases} \tag{3.1}$$

then

$$\sup_{Q_R} u \leqslant C_1(n) A e^{-C_2(n) R M^{\frac{1}{2}}},$$

where $C_1(n)$ and $C_2(n)$ depend on the dimension n only.

In order to prove the uniform Hölder bounds, we assume on the contrary, that there exists a sequence of solutions u_{κ_m} such that $\kappa_m \to +\infty$ (in the sequel, we will abbreviate this subindex m), but

$$L_\kappa := \max_i \sup_{X,Y\in\overline{\Omega}\times[0,+\infty),\, X\neq Y} \frac{|u_{i,\kappa}(X) - u_{i,\kappa}(Y)|}{d^\alpha(X,Y)} \to +\infty.$$

We can take $X_\kappa = (x_\kappa, t_\kappa), Y_\kappa = (y_\kappa, s_\kappa) \in \overline{\Omega} \times [0, +\infty)$ such that

$$\frac{L_\kappa}{2} = \frac{|u_{i,\kappa}(X_\kappa) - u_{i,\kappa}(Y_\kappa)|}{d^\alpha(X_\kappa, Y_\kappa)}. \tag{3.2}$$

Without loss of generality we can assume $s_\kappa \leqslant t_\kappa$. Since u_κ is uniformly bounded, we must have $d(X_\kappa, Y_\kappa) \to 0$.

Take a $r_\kappa > 0$, which will be determined later (but at least it is uniformly bounded). Define

$$\widetilde{u}_\kappa(X) := \frac{1}{L_\kappa r_\kappa^\alpha} u_\kappa(X_\kappa + r_\kappa X), \tag{3.3}$$

which is still defined on a cylindrical domain $\widetilde{Q}_\kappa := \frac{1}{r_\kappa}(Q - X_\kappa) = \widetilde{\Omega}_\kappa \times (-T_\kappa, +\infty)$. (Here $T_\kappa \geqslant 0$.)

Firstly, simple calculation shows

$$\max_i \sup_{X,Y\in\overline{\Omega}\times[0,+\infty),\, X\neq Y} \frac{|\widetilde{u}_{i,\kappa}(X) - \widetilde{u}_{i,\kappa}(Y)|}{d^\alpha(X,Y)} \leqslant 1. \tag{3.4}$$

If we denote $\widetilde{Y}_\kappa := \frac{1}{r_\kappa}(Y_\kappa - X_\kappa)$, we have

$$\frac{|\widetilde{u}_{i,\kappa}(\widetilde{Y}_\kappa) - \widetilde{u}_{i,\kappa}(0)|}{d^\alpha(\widetilde{Y}_\kappa, 0)} = \frac{1}{2}. \tag{3.5}$$

If u_κ is a solution to (1.1), then $\widetilde{u}_\kappa$ satisfies

$$\frac{\partial \widetilde{u}_{i,\kappa}}{\partial t}(X) - d_i \Delta \widetilde{u}_{i,\kappa}(X) = \frac{r_\kappa^{2-\alpha}}{L_\kappa} f_i\big(u_{i,\kappa}(X_\kappa + r_\kappa X)\big) - M_\kappa \widetilde{u}_{i,\kappa}(X) \sum_{j\neq i} b_{ij} \widetilde{u}_{j,\kappa}(X). \tag{3.6}$$

Here $M_\kappa = \kappa L_\kappa r_\kappa^{2+\alpha}$. If u_κ is a solution to (1.2), then $\widetilde{u}_\kappa$ satisfies

$$\frac{\partial \widetilde{u}_{i,\kappa}}{\partial t}(X) - d_i \Delta \widetilde{u}_{i,\kappa}(X) = \frac{r_\kappa^{2-\alpha}}{L_\kappa} f_i\big(u_{i,\kappa}(X_\kappa + r_\kappa X)\big) - M_\kappa \widetilde{u}_{i,\kappa}(X) \sum_{j\neq i} b_{ij} \widetilde{u}_{j,\kappa}^2(X). \tag{3.7}$$

Here $M_\kappa = \kappa L_\kappa^2 r_\kappa^{2+2\alpha}$.

Note that

$$\left|\frac{r_\kappa^{2-\alpha}}{L_\kappa} f_i\big(u_{i,\kappa}(X_\kappa + r_\kappa X)\big)\right| \leqslant C r_\kappa^2 |\widetilde{u}_{i,\kappa}|. \tag{3.8}$$

Here C is the Lipschitz constant of f_i.

Remark 3.3. Because the boundary values of $u_{i,\kappa}$ are fixed Lipschitz functions, for $X, Y \in \widetilde{\Omega}_\kappa \times \{-T_\kappa\}$ or $X, Y \in \partial\widetilde{\Omega}_\kappa \times (-T_\kappa, +\infty)$, we have a constant $C > 0$ such that

$$\left|\widetilde{u}_{i,\kappa}(X) - \widetilde{u}_{i,\kappa}(Y)\right| \leqslant \frac{1}{L_\kappa r_\kappa^\alpha} C r_\kappa d(X, Y). \tag{3.9}$$

Because $L_\kappa \to +\infty$ and r_κ is bounded, the boundary values of $\widetilde{u}_{i,\kappa}$ (minus a constant) will converge to 0 locally uniformly.

Remark 3.4. Consider the solution Φ_i of

$$\begin{cases} \frac{\partial \Phi_i}{\partial t} - d_i \Delta \Phi_i = f_i(\Phi_i), & \text{in } Q, \\ \Phi_i = u_{i,\kappa}, & \text{on } \partial_p Q. \end{cases} \tag{3.10}$$

By the comparison principle, we have $u_{i,\kappa} \leqslant \Phi_i$. We can define

$$\widetilde{\Phi}_{i,\kappa}(X) := \frac{1}{L_\kappa r_\kappa^\alpha} \Phi_i(X_\kappa + r_\kappa X).$$

Because Φ_i are Lipschitz continuous, $\widetilde{\Phi}_{i,\kappa}$ (minus a constant) will converge to 0 locally uniformly. This gives a control on $\widetilde{u}_{i,\kappa}$.

Lemma 3.5. *If $r_\kappa \to 0$, $\frac{d(X_\kappa, Y_\kappa)}{r_\kappa} \leqslant C$ and $M_\kappa \nrightarrow 0$, then $\widetilde{u}_\kappa(0)$ is bounded.*

Proof. Assume by the contrary, $N_\kappa := \widetilde{u}_{1,\kappa}(0) \to +\infty$. We will only treat the first case, i.e., u_κ satisfying (3.6). The second case is similar.

Claim. $\forall R > 0$, $\|M_\kappa \widetilde{u}_{1,\kappa} \sum_{i \neq 1} \widetilde{u}_{i,\kappa}\|_{L^\infty(Q_R(0) \cap \widetilde{Q}_\kappa)} \to 0$.

Because $\widetilde{u}_\kappa$ is uniformly Hölder continuous, we have

$$\widetilde{u}_{1,\kappa}(X) \geqslant N_\kappa - R^\alpha, \quad \text{in } Q_R(0) \cap \widetilde{Q}_\kappa. \tag{3.11}$$

In particular, on $Q_R(0) \cap \partial_p \widetilde{Q}_\kappa$, $\widetilde{u}_{1,\kappa} > 0$ and $\sum_{i>1} \widetilde{u}_{i,\kappa} \equiv 0$ (because they have disjoint supports on the boundary).

Take a standard cut-off function $\eta \in C_0^\infty(Q_R(0))$, such that $\eta \equiv 1$ on $Q_{R-1}(0)$ and $|\frac{\partial \eta}{\partial t}| + |\Delta \eta| \leqslant 4$. Multiplying the equation of $\widetilde{u}_{i,\kappa}$ ($i > 1$) by η and integrating by parts (noting the boundary condition of $\widetilde{u}_{i,\kappa}$), we get

$$\begin{aligned} &\iint\limits_{Q_R(0)\cap(\partial\widetilde{\Omega}_\kappa \times (0,+\infty))} \frac{\partial \widetilde{u}_{i,\kappa}}{\partial \nu} \eta + \iint\limits_{Q_R(0)\cap\widetilde{Q}_\kappa} \widetilde{u}_{i,\kappa}\left(-\frac{\partial \eta}{\partial t} - d_i \Delta \eta\right) + M_\kappa \widetilde{u}_{i,\kappa} \sum_{j \neq i} \widetilde{u}_{j,\kappa} \eta \\ &\quad \leqslant C r_\kappa^2 \iint\limits_{Q_R(0)\cap\widetilde{Q}_\kappa} \widetilde{u}_{i,\kappa} \eta. \end{aligned} \tag{3.12}$$

Here ν is the outward unit normal vector field to $\partial\widetilde{\Omega}_\kappa$. Since

$$\frac{\partial\widetilde{u}_{i,\kappa}}{\partial t}-d_i\Delta\widetilde{u}_{i,\kappa}\leqslant Cr_\kappa^2\widetilde{u}_{i,\kappa}, \tag{3.13}$$

and $\widetilde{u}_{i,\kappa}=0$ on $Q_R(0)\cap(\partial\widetilde{\Omega}_\kappa\times(0,+\infty))$, by comparing with the solution Φ_i of

$$\begin{cases}\frac{\partial\Phi_i}{\partial t}-d_i\Delta\Phi_i=Cr_\kappa^2\Phi_i, & \text{in } \widetilde{Q}_\kappa,\\ \Phi_i=\widetilde{u}_{i,\kappa}, & \text{on } \partial_p\widetilde{Q}_\kappa,\end{cases} \tag{3.14}$$

we get a constant $C>0$ independent of κ such that on $Q_R(0)\cap(\partial\widetilde{\Omega}_\kappa\times(0,+\infty))$

$$\left|\frac{\partial\widetilde{u}_{i,\kappa}}{\partial\nu}\right|\leqslant C. \tag{3.15}$$

By noting that $\sum_{j\neq i}\widetilde{u}_j\geqslant\widetilde{u}_1$ (because $i\neq 1$) and using (3.11),

$$\iint\limits_{Q_R(0)\cap\widetilde{Q}_\kappa} M_\kappa\big(N_\kappa-R^\alpha\big)\widetilde{u}_{i,\kappa}\eta\leqslant Cr_\kappa^2\iint\limits_{Q_R(0)\cap\widetilde{Q}_\kappa}\widetilde{u}_{i,\kappa}\eta+CR^{n+2}+4\iint\limits_{(Q_R(0)\setminus Q_{R-1}(0))\cap\widetilde{Q}_\kappa}\widetilde{u}_{i,\kappa}.$$

For κ large, we have $M_\kappa(N_\kappa-R^\alpha)-Cr_\kappa^2>0$, thus

$$\big[M_\kappa\big(N_\kappa-R^\alpha\big)-Cr_\kappa^2\big]\iint\limits_{Q_{R-1}(0)\cap\widetilde{Q}_\kappa}\widetilde{u}_{i,\kappa}\leqslant CR^{n+2}+4\iint\limits_{(Q_R(0)\setminus Q_{R-1}(0))\cap\widetilde{Q}_\kappa}\widetilde{u}_{i,\kappa}.$$

Since $\widetilde{u}_{i,\kappa}$ is Hölder continuous with constant 1, we get

$$\iint\limits_{(Q_R(0)\setminus Q_{R-1}(0))\cap\widetilde{Q}_\kappa}\widetilde{u}_{i,\kappa}\leqslant\iint\limits_{(Q_{R-1}(0)\setminus Q_{R-2}(0))\cap\widetilde{Q}_\kappa}\widetilde{u}_{i,\kappa}+CR^{n+1}.$$

Combing these, we get for κ and R large

$$\iint\limits_{Q_{R-1}(0)\cap\widetilde{Q}_\kappa}\widetilde{u}_{i,\kappa}\leqslant C\frac{R^{n+2}}{M_\kappa(N_\kappa-R^\alpha)}. \tag{3.16}$$

By (3.13) and the boundary condition, standard parabolic estimate shows

$$\sup_{Q_{R-2}(0)\cap\widetilde{Q}_\kappa}\widetilde{u}_{i,\kappa}\leqslant C\frac{R^{n+2}}{M_\kappa(N_\kappa-R^\alpha)}. \tag{3.17}$$

In $Q_{R-2}(0)$, we can substitute (3.11) into (3.6) to get

$$\frac{\partial\widetilde{u}_{i,\kappa}}{\partial t}-\Delta\widetilde{u}_{i,\kappa}\leqslant-\big[M_\kappa\big(N_\kappa-R^\alpha\big)-Cr_\kappa^{2-\alpha}\big]\widetilde{u}_{i,\kappa}. \tag{3.18}$$

Now we can use Lemma 3.2 to get

$$\sup_{Q_{\frac{R}{2}}(0)\cap\widetilde{Q}_\kappa} \widetilde{u}_{i,\kappa} \leqslant C_1(n)\frac{R^{n+2}}{M_\kappa(N_\kappa - R^\alpha)} e^{-C_2(n)RM_\kappa^{\frac{1}{2}}(N_\kappa - R^\alpha)^{\frac{1}{2}}}, \tag{3.19}$$

and the claim can be easily seen.

Define

$$\widehat{u}_{1,\kappa}(X) := \widetilde{u}_{1,\kappa}(X) - \widetilde{u}_{1,\kappa}(0).$$

We have $\widehat{u}_{1,\kappa}(0) = 0$, and it is Hölder continuous with constant 1, and

$$\frac{|\widehat{u}_{i,\kappa}(\widetilde{Y}_\kappa) - \widehat{u}_{i,\kappa}(0)|}{d^\alpha(\widetilde{Y}_\kappa, 0)} = \frac{1}{2}. \tag{3.20}$$

Moreover, by the claim, it satisfies

$$\frac{\partial \widehat{u}_{1,\kappa}}{\partial t} - d_1 \Delta \widehat{u}_{1,\kappa} = \varepsilon_\kappa,$$

where $\varepsilon_\kappa \to 0$ uniformly on any $Q_R(0) \cap \widetilde{Q}_\kappa$.

From this equation, we can defer from standard parabolic estimate that $\widehat{u}_{1,\kappa}$ are uniformly Lipschitz continuous. This fact, combing with (3.20), implies that $\exists C > 0$,

$$d(\widetilde{Y}_\kappa, 0) \geqslant C. \tag{3.21}$$

On the other hand, the assumption of the lemma says

$$d(\widetilde{Y}_\kappa, 0) \leqslant C. \tag{3.22}$$

After passing to a subsequence, $\widehat{u}_{1,\kappa}$ converges to a continuous function $\widetilde{u}$ uniformly on compact sets. $\widehat{u}$ satisfies the equation

$$\frac{\partial \widehat{u}}{\partial t} - d_1 \Delta \widehat{u} = 0.$$

By (3.21) and (3.22), we can assume $\lim_\kappa \widetilde{Y}_\kappa = \widetilde{Y}$. (3.20) can be passed to the limit, which is

$$\frac{|\widehat{u}(\widetilde{Y}) - \widehat{u}(0)|}{d^\alpha(\widetilde{Y}, 0)} = \frac{1}{2}. \tag{3.23}$$

In particular, $\widehat{u}$ is not a constant function.

$\widehat{u}$ is defined on the domain Q_∞, which could be of four types.

1. $Q_\infty = \mathbb{R}^n \times (-\infty, +\infty)$.
   In this case, since $\widehat{u}$ is a positive solution to the heat equation, it must be a constant function by the Liouville theorem. This contradicts (3.23).

2. $Q_\infty = \mathbb{R}^n \times [-t_0, +\infty)$.
   In this case, by Remark 3.3 we have the initial value condition

$$\widehat{u} \equiv \text{const.} \quad \text{on } \mathbb{R}^n \times \{-t_0\}.$$

   We claim that $t_0 > 0$. In fact, if $t_0 = 0$, then because $\widetilde{Y} := (\widetilde{y}, s) \in Q_\infty$ and $s \leqslant 0$ (from our construction), we must have $s = 0$. Now the initial value condition and (3.23) contradicts each other, and the claim is proven.
   By this claim and the growth rate of $\widehat{u}$ at infinity, we can use the uniqueness of initial value problem for the heat equation to conclude that $\widehat{u} \equiv \text{const}$. This contradicts (3.23), too.
3. $Q_\infty = H \times (-\infty, +\infty)$.
   Here H is a half space, for example $\{x_1 \geqslant t\}$ for some $t \in \mathbb{R}$. In this case, by Remark 3.3 we have the boundary condition

$$\widehat{u} \equiv \text{const.} \quad \text{on } \partial H \times (-\infty, +\infty).$$

   After an odd extension of $\widehat{u} - c$ to $\mathbb{R}^n \times (-\infty, +\infty)$, we can use Liouville theorem again to obtain a contradiction.
4. $Q_\infty = H \times [-t_0, +\infty)$.
   This case can be treated similarly. Since we have the initial-boundary value condition, we can use the uniqueness result, too. □

Lemma 3.6. *If u_κ satisfy* (3.6)*, then $\kappa L_\kappa d^{2+\alpha}(X_\kappa, Y_\kappa) \to +\infty$. If u_κ satisfy* (3.7)*, then $\kappa L_\kappa^2 d^{2+\alpha}(X_\kappa, Y_\kappa) \to +\infty$.*

Proof. We will only treat the first case, i.e., u_κ satisfying (3.6). The second case is exactly the same. Assume by the contrary, $\exists C > 0$ such that $\kappa L_\kappa d^{2+\alpha}(X_\kappa, Y_\kappa) \leqslant C$.

Take r_κ such that

$$M_\kappa = \kappa L_\kappa r_\kappa^{2+\alpha} = 1.$$

Then $\lim_{\kappa \to +\infty} r_\kappa = 0$ and $(\frac{d(X_\kappa, Y_\kappa)}{r_\kappa})^{2+\alpha} \leqslant C$. So we can use Lemma 3.5 to conclude that $\widetilde{u}_\kappa(0)$ is uniformly bounded.

Our choice of r_κ implies the equation of $\widetilde{u}_\kappa$ is

$$\frac{\partial \widetilde{u}_{i,\kappa}}{\partial t} - d_i \Delta \widetilde{u}_{i,\kappa} = \frac{r_\kappa^{2-\alpha}}{L_\kappa} f_i\big(u_{i,\kappa}(X_\kappa + r_\kappa X)\big) - \widetilde{u}_{i,\kappa} \sum_{j \neq i} b_{ij} \widetilde{u}_{j,\kappa}. \tag{3.24}$$

First, from this equation we can derive a uniform Lipschitz estimate of $\widetilde{u}_{i,\kappa}$. Thus, as in the previous lemma, (3.21) is still valid. By our choice of r_κ, (3.22) holds too.

After passing to a subsequence, $\widetilde{u}_{i,\kappa}$ converges to a continuous function $\widehat{u}_i$ uniformly on compact sets. $\widehat{u}_i$ satisfies the equation

$$\frac{\partial \widehat{u}_i}{\partial t} - d_i \Delta \widehat{u}_i = -\widehat{u}_i \sum_{j \neq i} b_{ij} \widehat{u}_j. \tag{3.25}$$

By (3.21) and (3.22), as in the previous lemma we can assume $\lim_\kappa \widetilde{Y}_\kappa = \widetilde{Y}$. (3.20) can be passed to the limit, which is

$$\frac{|\widehat{u}(\widetilde{Y})-\widehat{u}(0)|}{d^{\alpha}(\widetilde{Y},0)}=\frac{1}{2}. \tag{3.26}$$

In particular, $\widehat{u}$ is not a constant function.

$\widehat{u}$ is defined on the domain Q_∞, which could be of four types.

1. $Q_\infty=\mathbb{R}^n\times(-\infty,+\infty)$.
   In this case, since $\widehat{u}$ is a positive solution to (3.25). By Corollary 2.5 it must be a constant function. This contradicts (3.26).
2. $Q_\infty=\mathbb{R}^n\times[-t_0,+\infty)$.
   In this case, by Remark 3.3 we have the initial value condition
   $$\widehat{u}\equiv\text{const.}\quad\text{on } \mathbb{R}^n\times\{-t_0\}.$$
   Similar to the previous lemma we still have $t_0>0$. Now we can use the uniqueness of initial value problem for the equation of $\widehat{u}$ to conclude that $\widehat{u}\equiv$ const., which contradicts (3.26), too.
3. $Q_\infty=H\times(-\infty,+\infty)$ for some half space H.
   In this case, by Remark 3.3 we have the boundary condition: for constants $c_i\geqslant 0$,
   $$\widehat{u}_i\equiv c_i\quad\text{on } \partial H\times(-\infty,+\infty).$$
   By the separation condition on the boundary, for $i\neq j$, $c_ic_j=0$, so there is only one c_i which is nonzero. By Remark 3.4 (after passing to the limit), there is only one nonvanishing component of $\widehat{u}$. In particular, the right-hand side of (3.25) is 0. Thus we are reduced to Case 3 in the previous lemma, and we can use Liouville theorem again to obtain a contradiction.
4. $Q_\infty=H\times[-t_0,+\infty)$ for some half space H.
   This case can be treated similarly. Since we have the initial-boundary value condition, we can use the uniqueness result, too. □

Now we come to the proof of our main result. Firstly, we prove the case (1.1).

Proof. From the previous lemma, we must have $\kappa L_\kappa d^{2+\alpha}(X_\kappa,Y_\kappa)\to+\infty$. We take $r_\kappa=d(X_\kappa,Y_\kappa)$. Thus after the blow up, we have

$$d(\widetilde{Y}_\kappa,0)=1. \tag{3.27}$$

We also have $\lim_{\kappa\to+\infty}r_\kappa=0$ and $\lim_{\kappa\to+\infty}M_\kappa=+\infty$. Then by Lemma 3.5 we know $\widetilde{u}_\kappa(0)$ are uniformly bounded.

Simple calculation shows that $\widetilde{u}_\kappa$ satisfies the following parabolic inequalities.

$$\begin{cases}\frac{\partial\widetilde{u}_{i,\kappa}}{\partial t}-d_i\Delta\widetilde{u}_{i,\kappa}\leqslant Cr_\kappa^2\widetilde{u}_{i,\kappa},\\ \left(\frac{\partial}{\partial t}-d_i\Delta\right)\widetilde{u}_{i,\kappa}-\sum_{j\neq i}\frac{b_{ij}}{b_{ji}}\left(\frac{\partial}{\partial t}-d_j\Delta\right)\widetilde{u}_{j,\kappa}\geqslant Cr_\kappa^2\left(\widetilde{u}_{i,\kappa}-\sum_{j\neq i}\frac{b_{ij}}{b_{ji}}\widetilde{u}_{j,\kappa}\right).\end{cases} \tag{3.28}$$

After passing to a subsequence, $\widetilde{u}_{i,\kappa}$ converges to a continuous function $\widehat{u}_i$ uniformly on compact sets, which is defined on the domain Q_∞.

By (3.27), we can assume (after passing to a subsequence again) that $\widetilde{Y}_\kappa\to\widetilde{Y}$ which satisfies $d(\widetilde{Y},0)=1$. So after passing to the limit in (3.5), we get

$$\frac{|\widehat{u}(\widetilde{Y})-\widehat{u}(0)|}{d^{\alpha}(\widetilde{Y},0)}=\frac{1}{2}. \tag{3.29}$$

In particular, $\widehat{u}$ is not a constant function.

$\forall K \Subset Q_\infty$, we know if κ is large, $K \Subset \widetilde{Q}_\kappa$. Then we can take a smooth function $\eta \in C_0^\infty(\widetilde{Q}_\kappa)$ and $\eta \equiv 1$ on K. Multiplying the equation of $\widetilde{u}_{i,\kappa}$, (3.6), by η and integrating by parts, we get

$$\iint\limits_{\widetilde{Q}_\kappa} \widetilde{u}_{i,\kappa}\left(-\frac{\partial \eta}{\partial t} - d_i \Delta \eta\right) + M_\kappa \widetilde{u}_{i,\kappa} \sum_{j \neq i} \widetilde{u}_{j,\kappa}\eta \leqslant C r_\kappa^2 \iint\limits_{\widetilde{Q}_\kappa} \widetilde{u}_{i,\kappa}\eta. \tag{3.30}$$

Because $\widetilde{u}_{i,\kappa}$ are uniformly bounded on any compact set (by the boundedness of $\widetilde{u}_\kappa(0)$ and the uniform Hölder continuity) and $M_\kappa \to +\infty$, we get

$$\lim_{\kappa \to +\infty} \iint\limits_{K} \widetilde{u}_{i,\kappa} \sum_{j \neq i} \widetilde{u}_{j,\kappa} = 0. \tag{3.31}$$

So the limit $\widehat{u}$ satisfies

$$\widehat{u}_i \widehat{u}_j = 0, \quad \text{if } i \neq j. \tag{3.32}$$

(3.28) can be passed to the limit, so in the distributional sense $\widehat{u}_i$ satisfies

$$\begin{cases} \frac{\partial \widehat{u}_i}{\partial t} - d_i \Delta \widehat{u}_i \leqslant 0, \\ \left(\frac{\partial}{\partial t} - d_i \Delta\right)\widehat{u}_i - \sum_{j \neq i} \frac{b_{ij}}{b_{ji}}\left(\frac{\partial}{\partial t} - d_j \Delta\right)\widehat{u}_j \geqslant 0. \end{cases} \tag{3.33}$$

Q_∞ could be of four types.

1. $Q_\infty = \mathbb{R}^n \times (-\infty, +\infty)$.
   In this case, since $\widehat{u}$ is a positive continuous solution to (3.33), by Corollary 2.2 it must be a constant function. This is a contradiction.
2. $Q_\infty = \mathbb{R}^n \times [-t_0, +\infty)$.
   In this case, by Remark 3.3 we have the initial value condition

   $$\widehat{u}_i \equiv c_i \quad \text{on } \mathbb{R}^n \times \{-t_0\},$$

   for some constants $c_i \geqslant 0$. From the segregation condition (3.32) and the continuity of $\widehat{u}_i$, we have

   $$c_i c_j = 0, \quad \text{if } i \neq j.$$

   Similar to Lemma 3.5 we still have $t_0 > 0$. If there is one $c_i > 0$, then for $j \neq i$, $c_j = 0$. Since each $\widehat{u}_j$ is globally Hölder continuous, Lemma 3.1 implies $\widehat{u}_j \equiv 0$ for $j \neq i$. Combining both inequalities in (3.33) we know

   $$\frac{\partial \widehat{u}_i}{\partial t} - \Delta \widehat{u}_i = 0.$$

   Now we can use the uniqueness of initial value problem for the heat equation to conclude that $\widehat{u}_i \equiv \text{const}$. If all of $c_i = 0$, then by Lemma 3.1, $\forall i$, $\widehat{u}_i \equiv 0$. This is again a contradiction.
3. $Q_\infty = H \times (-\infty, +\infty)$ for some half space H.
   In this case, by Remark 3.3 we have the boundary condition

   $$\widehat{u}_i \equiv c_i, \quad \text{on } \partial H \times (-\infty, +\infty).$$

From the segregation condition (3.32) and the continuity of $\widehat{u}_i$, we have

$$c_i c_j = 0, \quad \text{if } i \neq j.$$

If there is one $c_i > 0$, then for $j \neq i$, $c_j = 0$. Similar to the previous lemma, by Remark 3.4, we see $\widehat{u}_j \equiv 0$ for $j \neq i$. Then by combining both inequalities in (3.33) we have

$$\frac{\partial \widehat{u}_i}{\partial t} - \Delta \widehat{u}_i = 0.$$

Then we can use Liouville theorem (after odd extension to $\mathbb{R}^n \times (-\infty, +\infty)$) again to obtain that $\widehat{u}_i \equiv \text{const}$. This is a contradiction. If all of $c_i = 0$, we can get a contradiction directly from Remark 3.4.

4. $Q_\infty = H \times [-t_0, +\infty)$ for some half space H.
This case can be treated similarly. Since we have the initial-boundary value condition, we can use results similar to Lemma 3.1 (we can replace the Gaussian kernel with the heat kernel on $H \times [-t_0, +\infty)$) and the uniqueness result of the initial-boundary value problem for the heat equation in $H \times [-t_0, +\infty)$, too. $\square$

Remark 3.7. If we consider the case of (1.2), in Cases 2 and 4 above, we can still prove that $\widehat{u}_i$ is a constant function, although we do not know whether $\widehat{u}_i$ satisfies the heat equation a priori. For example, in Case 2, without loss of generality, we can assume $\widehat{u}_i = c_i > 0$ on the boundary. Since $\widehat{u}_i$ is globally Hölder continuous, $\exists \epsilon > 0$ such that $\widehat{u}_i > 0$ in the time interval $[-t_0, -t_0 + \epsilon)$. Because $\widehat{u}_i$ satisfies the heat equation when $\widehat{u}_i > 0$, we can apply the uniqueness of initial-boundary value problem for the heat equation to conclude that $\widehat{u}_i \equiv c_i$ in the time interval $[-t_0, -t_0 + \epsilon)$. Then we can extend further to get that $\widehat{u}_i \equiv c_i$ in the whole time interval $[-t_0, +\infty)$.

4. Almgren monotonicity formula

In this section, we establish a monotonicity formula of Almgren type. This monotonicity formula is standard and it has been indicated in [3]. But here we mainly consider local solutions (also intended for the applications in [11]), so the calculation is a little involved. We will present the calculation in full details.

4.1. Definitions

Note that (1.2) and (3.7) has a gradient structure. For example, under suitable boundary conditions, (3.7) is the gradient flow of the following functional.

$$\int_{\mathbb{R}^n} \frac{1}{2} \sum_i d_i |\nabla \widetilde{u}_{i,\kappa}|^2 + M_\kappa \sum_{i \neq j} \widetilde{u}_{i,\kappa}^2 \widetilde{u}_{j,\kappa}^2 - \sum_i F_{i,\kappa}(\widetilde{u}_{i,\kappa}). \tag{4.1}$$

Here $F_{i,\kappa}(s) = \int_0^s \frac{r_\kappa^{2-\alpha}}{L_\kappa} f_i(L_\kappa r_\kappa^\alpha t)\, dt$.

In the following of this section, we assume $u_\kappa = (u_{i,\kappa})$ are a sequence of uniformly Hölder continuous solutions of (1.2) in $Q_1(0) := B_1(0) \times (-1, 0)$, and as $\kappa \to +\infty$, u_κ converges to u uniformly in $Q_1(0)$. For simplicity, we assume in (1.2), $d_i = 1$. The general case can be easily changed if we change the heat kernel as in Section 2. Note that for any fixed $\kappa < +\infty$, u_κ is smooth. The limit functions u satisfies (see [3, (3.14)])

$$\begin{cases} \frac{\partial u_i}{\partial t} - \Delta u_i = f_i(u_i) - \mu_i, & \text{in } Q_1(0), \\ u_i \geqslant 0, & \text{in } Q_1(0), \\ u_i u_j = 0, & \text{in } Q_1(0). \end{cases} \tag{4.2}$$

Here μ_i is a Radon measure supported on $\partial\{u_i > 0\}$. This implies

$$\left(\frac{\partial u_i}{\partial t} - \Delta u_i\right)u_i = f_i(u_i)u_i. \tag{4.3}$$

Define the backward heat kernel for $t < 0$:

$$G(x,t) = \left(4\pi|t|\right)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4|t|}}. \tag{4.4}$$

Take a $\varphi \in C_0^\infty(B_{\frac{1}{2}}(0))$. For any fixed $(x_0, t_0) \in Q_{\frac{1}{2}}(0)$ and $t \in (0, \frac{1}{2})$, define

$$\begin{aligned} D_\kappa(t) = \int\limits_{\mathbb{R}^n} \Bigg[\frac{1}{2}\sum_i \left|\nabla u_{i,\kappa}(x_0 - x, t_0 - t)\right|^2 + \sum_i F_i\left(u_{i,\kappa}(x_0 - x, t_0 - t)\right) \\ + \frac{\kappa}{4}\sum_{i \neq j} u_{i,\kappa}^2(x_0 - x, t_0 - t) u_{j,\kappa}^2(x_0 - x, t_0 - t)\Bigg]\varphi^2(x_0 - x)G(x,t)\,dx, \end{aligned} \tag{4.5}$$

$$H_\kappa(t) := \int\limits_{\mathbb{R}^n} \frac{1}{2}\sum_i u_{\kappa,i}^2(x_0 - x, t_0 - t)\varphi^2(x_0 - x)G(x,t)\,dx. \tag{4.6}$$

Concerning the limit function u, we have a similar definition,

$$D(t) = \int\limits_{\mathbb{R}^n} \left[\frac{1}{2}\sum_i \left|\nabla u_i(x_0 - x, t_0 - t)\right|^2 + \sum_i F_i\left(u_{i,\kappa}(x_0 - x, t_0 - t)\right)\right]\varphi^2(x_0 - x)G(x,t)\,dx, \tag{4.7}$$

$$H(t) := \int\limits_{\mathbb{R}^n} \frac{1}{2}\sum_i u_i^2(x_0 - x, t_0 - t)\varphi^2(x_0 - x)G(x,t)\,dx. \tag{4.8}$$

Then we have the following monotonicity formula.

Theorem 4.1. *For $t \in (0, \frac{1}{2})$, $\exists C > 0$ independent of $(x_0, t_0) \in Q_{\frac{1}{2}}$, such that*

$$e^{Ct}\frac{tD(t)}{H(t)} + Ct$$

is nondecreasing in t.

4.2. Calculations

Now let's compute $D_\kappa'(t)$. In the calculation of monotonicity formula we will simply take (x_0, t_0) as the origin $(0, 0)$.

Denote

$$D_\kappa(t) = \int\limits_{\mathbb{R}^n} \left[\frac{1}{2}\left|\nabla u_\kappa(x, -t)\right|^2 + F\left(u_\kappa(x, -t)\right) + H_\kappa\left(u_\kappa(x, -t)\right)\right]\varphi^2(x)G(x,t)\,dx. \tag{4.9}$$

Here we abbreviate the index i, and $H_\kappa(u_\kappa)=\frac{\kappa}{4}\sum_{i\neq j}u_{i,\kappa}^2u_{j,\kappa}^2$. Concerning u, we denote $D(t)$ as

$$D(t)=\int\limits_{\mathbb{R}^n}\left[\frac{1}{2}\left|\nabla u(x,-t)\right|^2+F\big(u(x,-t)\big)\right]\varphi^2(x)G(x,t)\,dx. \tag{4.10}$$

Firstly, by changing the coordinates through

$$x=t^{\frac{1}{2}}y,$$

we get

$$D_\kappa(t)=\int\limits_{\mathbb{R}^n}\left[\frac{1}{2}\left|\nabla u_\kappa\big(t^{\frac{1}{2}}y,-t\big)\right|^2+F\big(u_\kappa\big(t^{\frac{1}{2}}y,-t\big)\big)+H_\kappa\big(u_\kappa\big(t^{\frac{1}{2}}y,-t\big)\big)\right]\varphi^2\big(t^{\frac{1}{2}}y\big)G(y,1)\,dy.$$

So, if we denote

$$u_\kappa^t(y)=u_\kappa\big(t^{\frac{1}{2}}y,-t\big),$$

we have

$$D_\kappa(t)=\int\limits_{\mathbb{R}^n}\left[\frac{1}{2}t^{-1}\left|\nabla u_\kappa^t(y)\right|^2+F\big(u_\kappa^t(y)\big)+H_\kappa\big(u_\kappa^t(y)\big)\right]\varphi^2\big(t^{\frac{1}{2}}y\big)G(y,1)\,dy.$$

Now we can compute the derivative.

$$\begin{aligned}D_\kappa'(t)=\int\limits_{\mathbb{R}^n}&t^{-1}\nabla u_\kappa^t(y)\cdot\nabla\frac{\partial u_\kappa^t}{\partial t}(y)\varphi^2\big(t^{\frac{1}{2}}y\big)G(y,1)-\frac{1}{2}t^{-2}\left|\nabla u_\kappa^t(y)\right|^2\varphi^2\big(t^{\frac{1}{2}}y\big)G(y,1)\\&+\frac{\partial H_\kappa}{\partial u_\kappa}\frac{\partial u_\kappa^t}{\partial t}(y)\varphi^2\big(t^{\frac{1}{2}}y\big)G(y,1)+\frac{\partial F}{\partial u_\kappa}\frac{\partial u_\kappa^t}{\partial t}(y)\varphi^2\big(t^{\frac{1}{2}}y\big)G(y,1)\\&+\left[\frac{1}{2}t^{-1}\left|\nabla u_\kappa^t(y)\right|^2+F\big(u_\kappa^t(y)\big)+H_\kappa\big(u_\kappa^t(y)\big)\right]t^{-\frac{1}{2}}\varphi\big(t^{\frac{1}{2}}y\big)\nabla\varphi\big(t^{\frac{1}{2}}y\big)\cdot yG(y,1).\end{aligned} \tag{4.11}$$

Through integration by parts, the term involving $\nabla u_\kappa^t(y)\cdot\nabla\frac{\partial u_\kappa^t}{\partial t}(y)$ can be transformed into

$$\begin{aligned}-\int\limits_{\mathbb{R}^n}&\Delta u_\kappa^t(y)\frac{\partial u_\kappa^t}{\partial t}(y)\varphi^2\big(t^{\frac{1}{2}}y\big)G(y,1)+\frac{\partial u_\kappa^t}{\partial t}(y)\nabla u_\kappa^t(y)\cdot\nabla G(y,1)\varphi^2\big(t^{\frac{1}{2}}y\big)\\&+\frac{\partial u_\kappa^t}{\partial t}(y)\nabla u_\kappa^t(y)\cdot\nabla\big(\varphi^2\big(t^{\frac{1}{2}}y\big)\big)G(y,1).\end{aligned}$$

Note that

$$\nabla G(y,1) = -\frac{y}{2} G(y,1),$$
$$\frac{\partial}{\partial t} u_\kappa^t(y) = \frac{1}{2} t^{-\frac{1}{2}} y \cdot \nabla u_\kappa\big(t^{\frac{1}{2}} y, -t\big) - \frac{\partial u_\kappa}{\partial t}\big(t^{\frac{1}{2}} y, -t\big),$$
$$\Delta u_\kappa^t(y) = t \Delta u_\kappa\big(t^{\frac{1}{2}} y, -t\big),$$

and the equation for u_κ is

$$\frac{\partial u_\kappa}{\partial t} - \Delta u_\kappa + \frac{\partial H_\kappa}{\partial u_\kappa} = \frac{\partial F}{\partial u_\kappa}.$$

Substituting these into (4.11), we have

$$\begin{aligned}
D_\kappa'(t) = \int_{\mathbb{R}^n} & -t^{-1} \Delta u_\kappa^t(y) \frac{\partial u_\kappa^t}{\partial t}(y) \varphi^2\big(t^{\frac{1}{2}} y\big) G(y,1) \\
& - t^{-1} \frac{\partial u_\kappa^t}{\partial t}(y) \nabla u_\kappa^t(y) \cdot \nabla G(y,1) \varphi^2\big(t^{\frac{1}{2}} y\big) \\
& - t^{-1} \frac{\partial u_\kappa^t}{\partial t}(y) \nabla u_\kappa^t(y) \cdot \nabla\big(\varphi^2\big(t^{\frac{1}{2}} y\big)\big) G(y,1) - \frac{1}{2} t^{-2} \big|\nabla u_\kappa^t(y)\big|^2 \varphi^2\big(t^{\frac{1}{2}} y\big) G(y,1) \\
& + \frac{\partial H_\kappa}{\partial u} \frac{\partial u_\kappa^t}{\partial t}(y) \varphi^2\big(r^{\frac{1}{2}} y\big) G(y,1) + \frac{\partial F}{\partial u} \frac{\partial u_\kappa^t}{\partial t}(y) \varphi^2\big(r^{\frac{1}{2}} y\big) G(y,1) \\
& + \left[\frac{1}{2} \big|\nabla u_\kappa\big(t^{\frac{1}{2}} y, -t\big)\big|^2 + F\big(u_\kappa^t(y)\big) + H_\kappa\big(u_\kappa\big(t^{\frac{1}{2}} y, -t\big)\big)\right] t^{-\frac{1}{2}} \varphi\big(t^{\frac{1}{2}} y\big) \nabla\varphi\big(t^{\frac{1}{2}} y\big) \cdot y G(y,1) \\
= \int_{\mathbb{R}^n} & -\Delta u_\kappa(x,-t) \left[\frac{1}{2} t^{-1} x \cdot \nabla u_\kappa(x,-t) - \frac{\partial u_\kappa}{\partial t}(x,-t)\right] \varphi^2(x) G(x,t) \\
& + \frac{1}{2} t^{-1} x \cdot \nabla u_\kappa(x,t) \left[\frac{1}{2} t^{-1} x \cdot \nabla u_\kappa(x,-t) - \frac{\partial u_\kappa}{\partial t}(x,-t)\right] \varphi^2(x) G(x,t) \\
& - 2\left[\frac{1}{2} t^{-1} x \cdot \nabla u_\kappa(x,-t) - \frac{\partial u_\kappa}{\partial t}(x,-t)\right] \nabla u_\kappa(x,-t) \cdot \nabla\varphi(x) \varphi(x) G(x,t) \\
& - \frac{1}{2} t^{-1} \big|\nabla u_\kappa(x,t)\big|^2 \varphi^2(x) G(x,t) \\
& + \left(\frac{\partial H_\kappa}{\partial u_\kappa}(x,-t) + \frac{\partial F}{\partial u}\right) \left[\frac{1}{2} t^{-1} x \cdot \nabla u_\kappa(x,-t) - \frac{\partial u_\kappa}{\partial t}(x,-t)\right] \varphi^2(x) G(x,t) \\
& + \left[\frac{1}{2} \big|\nabla u_\kappa(x,-t)\big|^2 + H_\kappa\big(u_\kappa(x,-t)\big)\right] t^{-1} \varphi(x) x \cdot \nabla\varphi(x) G(x,t) \\
= \int_{\mathbb{R}^n} & \left[\frac{1}{2} t^{-1} x \cdot \nabla u_\kappa(x,-t) - \frac{\partial u_\kappa}{\partial t}(x,-t)\right]^2 \varphi^2(x) G(x,t) \\
& - 2\left[\frac{1}{2} t^{-1} x \cdot \nabla u_\kappa(x,-t) - \frac{\partial u_\kappa}{\partial t}(x,-t)\right] \nabla u_\kappa(x,-t) \cdot \nabla\varphi(x) \varphi(x) G(x,t)
\end{aligned}$$

$$+\left[\frac{1}{2}\left|\nabla u_\kappa(x,-t)\right|^2+F\big(u_\kappa(x,-t)\big)+H_\kappa\big(u_\kappa(x,-t)\big)\right]t^{-1}\varphi(x)x\cdot\nabla\varphi(x)G(x,t)$$
$$-\int_{\mathbb{R}^n}\frac{1}{2}t^{-1}\left|\nabla u_\kappa(x,t)\right|^2\varphi^2(x)G(x,t). \tag{4.12}$$

As $\kappa\to+\infty$, since we have the weak convergence of ∇u_κ and $\frac{\partial u_\kappa}{\partial t}$, after passing to the limit in the above formula (similar to the local energy inequality, cf. [16, Section 3]), we get

$$\begin{aligned}D'(t)\geqslant{}&\int_{\mathbb{R}^n}\left[\frac{1}{2}t^{-1}x\cdot\nabla u(x,-t)-\frac{\partial u}{\partial t}(x,-t)\right]^2\varphi^2(x)G(x,t)\\&-2\left[\frac{1}{2}t^{-1}x\cdot\nabla u(x,-t)-\frac{\partial u}{\partial t}(x,-t)\right]\nabla u(x,-t)\cdot\nabla\varphi(x)\varphi(x)G(x,t)\\&+\left[\frac{1}{2}\left|\nabla u(x,-t)\right|^2+F\big(u(x,-t)\big)\right]t^{-1}\varphi(x)x\cdot\nabla\varphi(x)G(x,t)\\&-\int_{\mathbb{R}^n}\frac{1}{2}t^{-1}\left|\nabla u(x,t)\right|^2\varphi^2(x)G(x,t).\end{aligned}\tag{4.13}$$

In particular

$$\begin{aligned}\frac{d}{dt}\big(tD(t)\big)\geqslant{}&\int_{\mathbb{R}^n}t\left[\frac{1}{2}t^{-1}x\cdot\nabla u(x,-t)-\frac{\partial u}{\partial t}(x,-t)\right]^2\varphi^2(x)G(x,t)\\&-2t\left[\frac{1}{2}t^{-1}x\cdot\nabla u(x,-t)-\frac{\partial u}{\partial t}(x,-t)\right]\nabla u(x,-t)\cdot\nabla\varphi(x)\varphi(x)G(x,t)\\&+\left[\frac{1}{2}\left|\nabla u(x,-t)\right|^2+F\big(u(x,-t)\big)\right]\varphi(x)x\cdot\nabla\varphi(x)G(x,t)\\&+F(u)\varphi^2(x)G(x,t).\end{aligned}\tag{4.14}$$

These inequalities are understood in the integral sense.

Remark 4.2. Because f is Lipschitz, $\exists C>0$, $|F(u)|\leqslant C|u|^2$. So

$$\int_{\mathbb{R}^n}|\nabla u|^2\varphi^2(x)G(x,t)-F\big(u(x)\big)\varphi^2(x)G(x,t)\geqslant -C\int_{\mathbb{R}^n}|u|^2\varphi^2(x)G(x,t). \tag{4.15}$$

That is, $D(t)\geqslant -CH(t)$.

4.3. Take (x_0,t_0) as the origin $(0,0)$, and denote

$$H(t):=\int_{\mathbb{R}^n}\frac{1}{2}\sum_i u_i^2(x,-t)\varphi^2(x)G(x,t)\,dx. \tag{4.16}$$

We compute its derivative. By (4.2),

$$\begin{aligned}
H'(t) &= \int_{\mathbb{R}^n} -u(x,-t)\frac{\partial u}{\partial t}(x,-t)\varphi^2(x)G(x,t) + \frac{1}{2}\left|u(x,-t)\right|^2\varphi^2(x)\frac{\partial G}{\partial t}(x,t) \\
&= \int_{\mathbb{R}^n} -u(x,-t)\Delta u(x,-t)\varphi^2(x)G(x,t) + \frac{1}{2}\left|u(x,-t)\right|^2\varphi^2(x)\Delta G(x,t) \\
&\quad - \int_{\mathbb{R}^n} f\big(u(x,-t)\big)u(x,-t)\varphi^2(x)G(x,t) \\
&= \int_{\mathbb{R}^n} -u(x,-t)\Delta u(x,-t)\varphi^2(x)G(x,t) \\
&\quad + \int_{\mathbb{R}^n} u(x,-t)\Delta u(x,-t)\varphi^2(x)G(x,t) + \left|\nabla u(x,-t)\right|^2\varphi^2(x)G(x,t) \\
&\quad + 2\int_{\mathbb{R}^n} \varphi(x)u(x,-t)\nabla u(x,-t)\nabla\varphi(x)G(x,t) \\
&\quad + \int_{\mathbb{R}^n} \left|u(x,-t)\right|^2\big(\varphi(x)\Delta\varphi(x) + \left|\nabla\varphi(x)\right|^2\big)G(x,t) \\
&\quad - \int_{\mathbb{R}^n} f\big(u(x,-t)\big)u(x,-t)\varphi^2(x)G(x,t).
\end{aligned}$$

By integrating by parts, the third term can be transformed into:

$$\begin{aligned}
&2\int_{\mathbb{R}^n} \varphi(x)u(x,-t)\nabla u(x,-t)\nabla\varphi(x)G(x,t) \\
&\quad = \frac{1}{2}\int_{\mathbb{R}^n} \nabla\left|u(x,-t)\right|^2\nabla\varphi^2(x)G(x,t) \\
&\quad = -\frac{1}{2}\int_{\mathbb{R}^n} \left|u(x,-t)\right|^2\left[\Delta\varphi^2(x)G(x,t) + \nabla\varphi^2(x)\nabla G(x,t)\right] \\
&\quad = -\int_{\mathbb{R}^n} \left|u(x,-t)\right|^2\left[\left|\nabla\varphi(x)\right|^2 + \varphi(x)\Delta\varphi(x) - \varphi(x)\frac{x\cdot\nabla\varphi(x)}{2t}\right]G(x,t).
\end{aligned}$$

Thus

$$\begin{aligned}
H'(t) &= \int_{\mathbb{R}^n} \left|\nabla u(x,-t)\right|^2\varphi^2(x)G(x,t) + \int_{\mathbb{R}^n} \left|u(x,-t)\right|^2\varphi(x)\frac{x\cdot\nabla\varphi(x)}{2t}G(x,t) \\
&\quad - \int_{\mathbb{R}^n} f\big(u(x,-t)\big)u(x,-t)\varphi^2(x)G(x,t). \qquad (4.17)
\end{aligned}$$

The following lemma follows the proof of Lemma 1 in [12].

Lemma 4.3. *$\forall t \in (-\frac{1}{2}, 0)$, $\int_{B_1(0)} |u(x,t)|^2\,dx$ has a uniform lower bound.*

Proof. By the Cauchy–Schwarz inequality and the Lipschitz property of f, we have

$$H'(t) \geqslant -\int_{\mathbb{R}^n} \left|u(x,-t)\right|^2 \varphi(x) \left|\frac{x\cdot\nabla\varphi(x)}{2t}\right| G(x,t) - CH(t). \tag{4.18}$$

Because u is bounded, and $\varphi \in C_0^\infty(B_1)$, $\varphi \equiv 1$ on $B_{\frac{1}{2}}$, we get a constant $C > 0$ such that

$$\frac{d}{dt}\left(e^{Ct}H(t)\right) \geqslant -Ce^{-\frac{1}{Ct}}. \tag{4.19}$$

Now writing the dependence on (x_0, t_0) explicitly, and integrating in $t \in (0, s)$, we get

$$C\int_{\mathbb{R}^n} \left|u(x, t_0 - t)\right|^2 \varphi^2(x) G(x - x_0, t)\,dx \geqslant \left|u(x_0, t_0)\right|^2 - Ce^{-\frac{1}{Ct}}.$$

Integrating this inequality over $x_0 \in B_{\frac{1}{2}}$ and recalling that $\int_{\mathbb{R}^n} G(x - x_0, t)\,dx_0 = 1$, we get

$$C\int_{B_1} \left|u(x, t_0 - t)\right|^2 dx \geqslant \int_{B_{\frac{1}{2}}} \left|u(x_0, t_0)\right|^2 dx_0 - Ce^{-\frac{1}{Ct}}.$$

Thus if we assume $\int_{B_{\frac{1}{2}}} |u(x,0)|^2\,dx > 0$, then the claim follows. $\square$

4.4. For simplicity, denote $v(x,t) := u(x_0 + x, t_0 - t)$. Define $N(t) := \frac{tD(t)}{H(t)}$. We have

$$N'(t) = \frac{(tD(t))'H(t) - tD(t)H'(t)}{H(t)^2},$$

and

$$\begin{aligned}
&\left(tD(t)\right)'H(t) - tD(t)H'(t) \\
&\quad \geqslant \left(\int_{\mathbb{R}^n} t\left[\frac{x\cdot\nabla v}{2t} + v_t\right]^2 \varphi^2 G + I_1\right)\left(\int_{\mathbb{R}^n} \frac{1}{2}|v|^2\varphi^2 G\right) \\
&\qquad - t\left(\int_{\mathbb{R}^n} \frac{1}{2}|\nabla v|^2\varphi^2 G + F(v)\varphi^2 G\right)\left(\int_{\mathbb{R}^n} |\nabla v|^2 \varphi^2 G + I_2\right) \\
&\quad = \frac{t}{2}\left(\int_{\mathbb{R}^n} \left[\frac{x\cdot\nabla v}{2t} + v_t\right]^2 \varphi^2 G\right)\left(\int_{\mathbb{R}^n} |v|^2\varphi^2 G\right) - \frac{t}{2}\left(\int_{\mathbb{R}^n} |\nabla v|^2\varphi^2 G\right)^2 \\
&\qquad + I_1\left(\int_{\mathbb{R}^n} \frac{1}{2}|v|^2\varphi^2 G\right) - tI_2\left(\int_{\mathbb{R}^n} \frac{1}{2}|\nabla v|^2\varphi^2 G\right).
\end{aligned} \tag{4.20}$$

Here

$$I_1 = \int_{\mathbb{R}^n} -2t\left[\frac{x \cdot \nabla v}{2t} + v_t\right]\nabla v \cdot \nabla\varphi\varphi G + \left[\frac{1}{2}|\nabla v|^2 + F(v)\right]\varphi x \cdot \nabla\varphi G + F(v)\varphi^2 G, \tag{4.21}$$

$$I_2 = \int_{\mathbb{R}^n} |v|^2\varphi\frac{x \cdot \nabla\varphi}{2t}G - \int_{\mathbb{R}^n} f(v)v\varphi^2 G. \tag{4.22}$$

With the equation of v in mind, after integration by parts, we have

$$\begin{aligned}\int_{\mathbb{R}^n} vv_t\varphi^2 G &= \int_{\mathbb{R}^n} v\Delta v\varphi^2 G + f(v)v\varphi^2 G \\ &= -\int_{\mathbb{R}^n} |\nabla v|^2\varphi^2 G + v\nabla v\nabla G\varphi^2 + v\nabla v\nabla\varphi^2 G + \int_{\mathbb{R}^n} f(v)v\varphi^2 G.\end{aligned} \tag{4.23}$$

Because $\nabla G = \frac{x}{2t}G$, we get

$$\int_{\mathbb{R}^n} |\nabla v|^2\varphi^2 G = -\int_{\mathbb{R}^n} v\left(v_t + \frac{x \cdot \nabla v}{2t}\right)\varphi^2 G - \int_{\mathbb{R}^n} 2v\varphi\nabla v\nabla\varphi G + \int_{\mathbb{R}^n} f(v)v\varphi^2 G. \tag{4.24}$$

Substituting this into (4.20), we get

$$\begin{aligned}\big(tD(t)\big)'H(t) - tD(t)H'(t) \geqslant{}& -\frac{t}{2}I_3 + I_1\left(\int_{\mathbb{R}^n} \frac{1}{2}|v|^2\varphi^2 G\right) - tI_2\left(\int_{\mathbb{R}^n} \frac{1}{2}|\nabla v|^2\varphi^2 G\right) \\ & - t\left(\int F(v)\varphi^2 G\right)\left(\int |\nabla v|^2\varphi^2 G + I_2\right),\end{aligned}$$

where

$$\begin{aligned}I_3 ={}& \left(\int_{\mathbb{R}^n} 2v\varphi\nabla v\nabla\varphi G\right)^2 + \left(\int_{\mathbb{R}^n} f(v)v\varphi^2 G\right)^2 \\ & + 2\left(\int_{\mathbb{R}^n} v\left(v_t + \frac{x \cdot \nabla v}{2t}\right)\varphi^2 G\right)\left(\int_{\mathbb{R}^n} 2v\varphi\nabla v\nabla\varphi G\right) \\ & - 2\left(\int_{\mathbb{R}^n} v\left(v_t + \frac{x \cdot \nabla v}{2t}\right)\varphi^2 G\right)\left(\int_{\mathbb{R}^n} f(v)v\varphi^2 G\right) \\ & - 2\left(\int_{\mathbb{R}^n} 2v\varphi\nabla v\nabla\varphi G\right)\left(\int_{\mathbb{R}^n} f(v)v\varphi^2 G\right).\end{aligned} \tag{4.25}$$

Substituting (4.24) into the above formula, we get

$$
\begin{aligned}
I_3 = {} & \left(\int_{\mathbb{R}^n} 2v\varphi\nabla v\nabla\varphi G\right)^2 + \left(\int_{\mathbb{R}^n} f(v)v\varphi^2 G\right)^2 \\
& - 2\left(\int_{\mathbb{R}^n} |\nabla v|^2\varphi^2 G + 2v\varphi\nabla v\nabla\varphi G - f(v)v\varphi^2 G\right)\left(\int_{\mathbb{R}^n} 2v\varphi\nabla v\nabla\varphi G\right) \\
& - 2\left(\int_{\mathbb{R}^n} |\nabla v|^2\varphi^2 G + 2v\varphi\nabla v\nabla\varphi G - f(v)v\varphi^2 G\right)\left(\int_{\mathbb{R}^n} f(v)v\varphi^2 G\right) \\
& - 2\left(\int_{\mathbb{R}^n} 2v\varphi\nabla v\nabla\varphi G\right)\left(\int_{\mathbb{R}^n} f(v)v\varphi^2 G\right).
\end{aligned} \tag{4.26}
$$

By the Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
|I_3| \leqslant {} & \left(\int_{\mathbb{R}^n} |\nabla v|^2|\nabla\varphi|^2 G\right)\left(\int_{\mathbb{R}^n} v^2\varphi^2 G\right) + C\left(\int_{\mathbb{R}^n} v^2\varphi^2 G\right)^2 \\
& + C\left(\int_{\mathbb{R}^n} |\nabla v|^2\varphi^2 G + v^2|\nabla\varphi|^2 G + v^2\varphi^2 G\right)\left(\int_{\mathbb{R}^n} v^2\varphi^2 G + |\nabla v|^2|\nabla\varphi|^2 G\right) \\
& + C\left(\int_{\mathbb{R}^n} |\nabla v|^2\varphi^2 G + v^2|\nabla\varphi|^2 G + v^2\varphi^2 G\right)\left(\int_{\mathbb{R}^n} v^2\varphi^2 G\right) \\
& + C\left(\int_{\mathbb{R}^n} v^2|\nabla\varphi|^2 G + |\nabla v|^2\varphi^2 G\right)\left(\int_{\mathbb{R}^n} v^2\varphi^2 G\right).
\end{aligned} \tag{4.27}
$$

Because $\forall t$, $\int_{B_1} |v|^2 \leqslant C$, we get

$$
\int_{\mathbb{R}^n} v^2|\nabla\varphi|^2 G \leqslant C \int_{B_{\frac{1}{2}}\setminus B_{\frac{1}{4}}} v^2 t^{-\frac{n}{2}} e^{-\frac{1}{64t}} \leqslant Ct^{-\frac{n}{2}} e^{-\frac{1}{64t}}. \tag{4.28}
$$

Because $\forall t$, $\int_{B_1} |\nabla v|^2 \leqslant C$, we also have

$$
\int_{\mathbb{R}^n} |\nabla v|^2|\nabla\varphi|^2 G \leqslant C \int_{B_{\frac{1}{2}}\setminus B_{\frac{1}{4}}} |\nabla v|^2 t^{-\frac{n}{2}} e^{-\frac{1}{64t}} \leqslant Ct^{-\frac{n}{2}} e^{-\frac{1}{64t}}. \tag{4.29}
$$

Moreover, recalling the definition of $D(t)$ and $H(t)$, we get

$$
|I_3| \leqslant Ct^{-\frac{n}{2}} e^{-\frac{1}{64t}}\big[H(t) + D(t)\big] + CH(t)^2 + CH(t)D(t). \tag{4.30}
$$

Concerning I_1, we have

$$|I_1| = \int\limits_{\mathbb{R}^n} -2t\left[\frac{x\cdot\nabla v}{2t}+v_t\right]\nabla v\cdot\nabla\varphi\varphi G+\left[\frac{1}{2}|\nabla v|^2+F(v)\right]\varphi x\cdot\nabla\varphi G+F(v)\varphi^2 G$$
$$\leqslant C\int\limits_{\mathbb{R}^n}\left|\frac{x\cdot\nabla v}{2}+tv_t\right||\nabla v||\nabla\varphi|\varphi G+\big(|\nabla v|^2+|v|^2\big)\varphi|\nabla\varphi|G+|v|^2\varphi^2 G$$
$$\leqslant C\left[t^{-\frac{n}{2}}e^{-\frac{1}{64t}}+H(t)+t^{-\frac{n}{2}+1}e^{-\frac{1}{64t}}\int\limits_{\mathbb{R}^n}|v_t|^2\varphi\right]. \tag{4.31}$$

While I_2 can be controlled by

$$|I_2| = \int\limits_{\mathbb{R}^n}\varphi v\nabla v\nabla\varphi G+\int\limits_{\mathbb{R}^n}|v|^2\big(\varphi\Delta\varphi+|\nabla\varphi|^2\big)G-\int\limits_{\mathbb{R}^n}f(v)v\varphi^2 G$$
$$\leqslant Ct^{-\frac{n}{2}}e^{-\frac{1}{64t}}+CH(t). \tag{4.32}$$

Combing all of these together, we get

$$N'(t)\geqslant -Ct\frac{t^{-\frac{n}{2}}e^{-\frac{1}{64t}}[H(t)+D(t)]+H(t)^2+H(t)D(t)}{H^2(t)}$$
$$-C\frac{t^{-\frac{n}{2}}e^{-\frac{1}{64t}}+H(t)+t^{-\frac{n}{2}+1}e^{-\frac{1}{64t}}\int_{\mathbb{R}^n}|v_t|^2\varphi}{H(t)}-Ct\frac{(t^{-\frac{n}{2}}e^{-\frac{1}{64t}}+H(t))D(t)}{H^2(t)}$$
$$-\frac{CtH(t)[D(t)+CH(t)+Ct^{-\frac{n}{2}}e^{-\frac{t}{64}}]}{H^2(t)}. \tag{4.33}$$

Noting that

$$H(t)\geqslant t^{-\frac{n}{2}}e^{-\frac{1}{64t}}\int\limits_{B_{\frac{1}{2}}}|v|^2\geqslant\frac{1}{C}t^{-\frac{n}{2}}e^{-\frac{1}{64t}},$$

thus

$$N'(t)\geqslant -C-CN(t)-Ct\int\limits_{\mathbb{R}^n}|v_t|^2\varphi^2 G. \tag{4.34}$$

For $t\in(0,1)$,

$$\frac{d}{dt}\big(e^{Ct}N(t)\big)\geqslant -Ce^{Ct}-Ce^{Ct}t\int\limits_{\mathbb{R}^n}|v_t|^2\varphi\geqslant -C-Ct\int\limits_{\mathbb{R}^n}|v_t|^2\varphi.$$

Thus

$$e^{Ct}N(t)+Ct+C\int\limits_0^t s\int\limits_{\mathbb{R}^n}\left|\frac{\partial v}{\partial t}(x,s)\right|^2\varphi(x)\,dx\,ds$$

is a nondecreasing function of t.

Remark 4.4. Because $|\frac{\partial v}{\partial t}|^2$ is locally integrable on $\mathbb{R} \times (0, +\infty)$, the last term can be controlled by t:

$$\int_0^t s \int_{\mathbb{R}^n} \left| \frac{\partial v}{\partial t}(x,s) \right|^2 \varphi(x)\, dx\, ds \leqslant Ct.$$

Remark 4.5. Because $|f(v)| \leqslant C|v|$, we have $|F(v)| \leqslant C|v|^2$. Thus

$$\left| \int_{\mathbb{R}^n} F(v)\varphi^2 G \right| \leqslant CH(t).$$

For $t \in (0, 1)$, if we replace the constant C in the second term by a larger $\widetilde{C}$

$$e^{Ct}N(t) + \widetilde{C}t + C\int_0^t s \int_{\mathbb{R}^n} \left| \frac{\partial v}{\partial t}(x,s) \right|^2 \varphi(x)\, dx\, ds$$

is still a nondecreasing function of t and is nonnegative.

Now for $X = (x, t) \in Q_1$, we can define

$$\Theta(X; u) := \lim_{t \to 0^+} N(t; X, u).$$

In the following, if no ambiguity appears, we often omit u or X in $\Theta(X; u)$ and $N(t; X, u)$, writing as $\Theta(X)$ and $N(t)$.

5. Completion of the proof

In this section we consider the second case, (1.2). The main difference is, now we do not have the second inequalities of (3.33). As pointed out in Remark 3.7, this only affects the proof when $Q_\infty = \mathbb{R}^n \times (-\infty, +\infty)$ or $Q_\infty = H \times (-\infty, +\infty)$ for some half space H.

If $Q_\infty = \mathbb{R}^n \times (-\infty, +\infty)$, now we can't apply Corollary 2.2. However, if we check the proof of Corollary 2.2 carefully, we still have the partial result that there is at most one component, say $\widehat{u}_1$, which may not be a constant function. By Lemma 3.2, in the open set $\{\widehat{u}_1 > 0\}$,

$$\frac{\partial \widehat{u}_1}{\partial t} - \Delta \widehat{u}_1 = 0.$$

If $\{\widehat{u}_1 > 0\} = \mathbb{R}^n \times (-\infty, +\infty)$, because $\widehat{u}_1$ is a globally Hölder continuous function, $\widehat{u}_1$ must be a constant function. This is a contradiction. Therefore, $\{\widehat{u}_1 = 0\}$ is not empty. Without loss of generality, we can assume

$$\widehat{u}_1(0, 0) = 0.$$

We will use the monotonicity formula of Almgren type to exclude this possibility, because this monotonicity formula will give the growth rate at infinity.

If $Q_\infty = H \times (-\infty, +\infty)$, similar to the treatment of Case 3 in the case of (1.1), we can still get that there is only one nonvanishing component of $\widehat{u}$, say $\widehat{u}_1$. After odd extension to the whole space, we can still apply the Almgren monotonicity formula to get a contradiction. In the following, we only treat the case of $Q_\infty = \mathbb{R}^n \times (-\infty, +\infty)$.

Similar to the calculation in the previous section, we have an Almgren type monotonicity formula for the limit $\widehat{u}$. That is, for $t > 0$, if we define

$$D(t; x_0, t_0) := \int_{\mathbb{R}^n} \sum_i d_i \left|\nabla \widehat{u}_i(x - x_0, t_0 - t)\right|^2 G_i(x, t)\, dx,$$

$$H(t; x_0, t_0) := \int_{\mathbb{R}^n} \sum_i \left|\widehat{u}_i(x - x_0, t_0 - t)\right|^2 G_i(x, t)\, dx,$$

and

$$N(t; x_0, t_0) = \frac{tD(t)}{H(t)}.$$

Here $G_i(x, t)$ is defined in Section 2. Then $N(t; x_0, t_0)$ is a nondecreasing function of t. In particular,

$$\Theta(x_0, t_0) = \lim_{t \to 0^+} N(t; x_0, t_0)$$

is well defined. Note that here we need not localize as in the previous section, thus these formulas have clean forms.

Firstly, due to the global Hölder continuity of $\widehat{u}_1$, we have the following result.

Lemma 5.1. $\forall (x_0, t_0) \in \{\widehat{u}_1 = 0\}$, $N(t; x_0, t_0) \equiv \frac{\alpha}{2}$.

Proof. By (4.17), direct calculation shows

$$t \frac{d}{dt} \log H(t; x_0, t_0) = 2N(t; x_0, t_0). \tag{5.1}$$

First assume $\exists \tau > 0$ such that $N(\tau; x_0, t_0) \leqslant \frac{\alpha}{2} - \delta$ for a constant $\delta > 0$. By the monotonicity of $N(t; x_0, t_0)$, $\forall t \leqslant \tau$,

$$N(t; x_0, t_0) \leqslant \frac{\alpha}{2} - \delta.$$

Substituting this into (5.1), we get

$$\frac{H(t; x_0, t_0)}{t^{2\alpha - 2\delta}}$$

is nonincreasing in $t \in (0, \tau)$. In particular, $\exists C_1 > 0$ such that

$$H(t; x_0, t_0) \geqslant C_1 t^{2\alpha - 2\delta}. \tag{5.2}$$

However, since $\widehat{u}$ is C^α continuous, and $\widehat{u}(x_0, t_0) = 0$, we have

$$\left|\widehat{u}(x + x_0, t_0 - t)\right| \leqslant \left(|x| + |t|^{\frac{1}{2}}\right)^\alpha.$$

Substituting this into the definition of $H(t; x_0, t_0)$, we get for some constant $C > 0$,

$$H(t; x_0, t_0) \leqslant Ct^{2\alpha}. \tag{5.3}$$

This contradicts (5.4) for $t > 0$ small.

Next assume $\exists\tau > 0$ such that $N(\tau; x_0, t_0) \geqslant \frac{\alpha}{2} + \delta$ for a constant $\delta > 0$. By the monotonicity of $N(t; x_0, t_0)$, $\forall t \geqslant \tau$,

$$N(t; x_0, t_0) \geqslant \frac{\alpha}{2} + \delta.$$

Substituting this into (5.1), we get

$$\frac{H(t; x_0, t_0)}{t^{2\alpha+2\delta}}$$

is nondecreasing in $t \in (\tau, +\infty)$. In particular, $\exists C_2 > 0$ such that

$$H(t; x_0, t_0) \geqslant C_2 t^{2\alpha+2\delta}. \tag{5.4}$$

However, since $\widehat{u}$ is global C^α continuous, and $\widehat{u}(x_0, t_0) = 0$, we have

$$\left|\widehat{u}(x + x_0, t_0 - t)\right| \leqslant \left(|x| + |t|^{\frac{1}{2}}\right)^\alpha.$$

Substituting this into the definition of $H(t; x_0, t_0)$, we get for some constant $C > 0$,

$$H(t; x_0, t_0) \leqslant C t^{2\alpha}. \tag{5.5}$$

This contradicts (5.4) for $t > 0$ large. $\square$

By the method of Section 6.1 in [11] (see (6.24) therein), this lemma implies, $\forall(x_0, t_0)$, if $\widehat{u}_1(x_0, t_0) = 0$, then $\forall\lambda > 0$,

$$\widehat{u}\left(\lambda x + x_0, t_0 - \lambda^2 t\right) = \lambda^\alpha \widehat{u}(x + x_0, t_0 - t). \tag{5.6}$$

Proposition 6.3 in [11] also implies $\{\widehat{u}_1 = 0\}$ forms a self-similar linear subspace of $\mathbb{R}^n \times (-\infty, +\infty)$ (for the notation, see Definition 8.4 in [5]). By [5], either $\{\widehat{u}_1 = 0\} = \mathbb{R}^d \times \mathbb{R}$ or $\{\widehat{u}_1 = 0\} = \mathbb{R}^d \times \{0\}$.

By [11, Proposition 6.3], $\widehat{u} = (\widehat{u}_1, 0, \ldots, 0)$ is homogeneous of degree α with respect to $(0, 0)$. That is, $\forall\lambda > 0$,

$$\widehat{u}_1\left(\lambda x, \lambda^2 t\right) = \lambda^\alpha \widehat{u}_1(x, t).$$

If we denote $w(x) = \widehat{u}_1(x, -1)$, then it satisfies

$$\begin{cases} \Delta w - \frac{x}{2} \cdot \nabla w + \frac{\alpha}{2} w = \nu, & \text{in } \mathbb{R}^n, \\ w \geqslant 0, & \text{in } \mathbb{R}^n, \\ w(\Delta w - \frac{x}{2} \cdot \nabla w + \frac{\alpha}{2} w) = 0, & \text{in } \mathbb{R}^n. \end{cases} \tag{5.7}$$

Here ν is a positive Radon measure supported on $\partial\{w > 0\}$. In particular, by integration by parts (using the third equation of (5.7)) we have

$$\frac{\alpha}{2} = \frac{\int_{\mathbb{R}^n} |\nabla w(x)|^2 e^{-\frac{|x|^2}{4}}\, dx}{\int_{\mathbb{R}^n} w(x)^2 e^{-\frac{|x|^2}{4}}\, dx}. \tag{5.8}$$

In other words, $\frac{\alpha}{2}$ is the first eigenvalue of the above quadric form in $H_0^1(\{w > 0\}, e^{-\frac{|x|^2}{4}}\,dx)$. We also need to note that w is global Hölder continuous and it is not a constant function. We show this is impossible.

If $\{\widehat{u}_1 = 0\} = \mathbb{R}^d \times \{0\}$, then $w > 0$ strictly on $\mathbb{R}^n$. Then $\nu = 0$. Take a test function $\eta \in C_0^\infty(\mathbb{R}^n)$ such that, $\eta \equiv 1$ in $B_R(0)$, $\eta \equiv 0$ outside of $B_{R+1}(0)$, and

$$|\Delta\eta| + |\nabla\eta| \leqslant 16.$$

Multiplying the equation of w with $\eta e^{-\frac{|x|^2}{4}}$ and integrating by parts, we get

$$\frac{\alpha}{2}\int\limits_{\mathbb{R}^n} w(x)\eta(x)e^{-\frac{|x|^2}{4}}\,dx = \int\limits_{\mathbb{R}^n} w(x)\left(\Delta\eta - \frac{x}{2}\cdot\nabla\eta\right)e^{-\frac{|x|^2}{4}}\,dx. \tag{5.9}$$

If we take R large enough, the right-hand side can be arbitrarily small. So we must have $w \equiv 0$. This is a contradiction.

If $\{\widehat{u}_1 = 0\} = \mathbb{R}^d \times \mathbb{R}$, then $\{w = 0\} = \mathbb{R}^d$. If $d \leqslant n - 2$, then the set $\{w = 0\}$, which is the support of the measure ν, has capacity 0. So $\nu = 0$ again, and we can get a contradiction as above. If $d = n - 1$, without loss of generality, we can assume $\{w = 0\} = \{x_1 = 0\}$. Then $\{w > 0\}$ has exactly two components, $\{x_1 > 0\}$ and $\{x_1 < 0\}$. However, direct calculation shows that $f(x) = |x_1|$ is the first eigenfunction on $\{w > 0\}$, with eigenvalue $\frac{1}{2} > \frac{\alpha}{2}$. This contradicts the uniqueness of the first eigenfunction and the proof is finished.

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