Periodic solutions of second order superquadratic Hamiltonian systems with potential changing sign (II)

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Abstract
We consider the periodic solutions of the second order Hamiltonian system

\[-\ddot{x} - \dot{x} = h(t)V'(x)\]

with $V$ being positive and superquadratic at infinity, $h$ being $C^1$, $2\pi$-periodic, sign changing and all zeros being simple. Some existence and multiplicity results of periodic solutions are given.
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1. Introduction

This paper is sequel to [17]. We continue the study of periodic solutions of the second order Hamiltonian system

\[-\ddot{x} - \dot{x} = h(t)V'(x)\]  \hfill (1.1)
with $h$ changing sign via the Morse theory, $x \in \mathbb{R}^m$. This problem has been considered by some authors (see [1,3,6,9,12,13,15,18,19,21]). In the superquadratic case, in these papers it is assumed that the potential $V$ is either positive homogeneous of degree $p$ or asymptotic to $|x|^p$ at infinity for some $p > 2$. We assumed in [17] that $V$ is $C^2$ and satisfies

$$V'(x) \cdot x \geq \theta V(x) > 0 \quad |x| \geq r$$  \hspace{1cm} (V1)

for some constants $\theta > 2$ and $r > 0$, and $h$ is a continuous, $2\pi$-periodic function and satisfies the thick zero condition

$$\{t \in [0, 2\pi] | h(t) < 0\} \cap \{t \in [0, 2\pi] | h(t) > 0\} = \emptyset,$$  \hspace{1cm} (h0)

and the sets $S_+ = \{t \in [0, 2\pi] | h(t) > 0\}$, $S_- = \{t \in [0, 2\pi] | h(t) < 0\}$, $S_0 = S^1 \setminus (S_+ \cup S_-)$, are nonempty, consist of finite intervals. Following the idea in [5], we showed that the functional $I$ associated with (1.1) satisfies the Palais–Smale condition and all critical groups of $I$ at infinity are zero. Some existence and multiplicity results of periodic solutions of (1.1) then follow from this fact by the Morse theory easily. However, the verification of (P.S) condition and the proof that all critical groups of $I$ at infinity are zero in [17] depend on a decomposition lemma of $H^1(S^1, \mathbb{R}^m)$, which in turn relies on (h0) in a crucial way.

In this paper, we consider the case that the function $h$ changes sign and satisfies the thin zero condition

$$h \in C^1, \quad h'(t) \neq 0 \quad \text{whenever} \quad h(t) = 0.$$  \hspace{1cm} (h1)

We will prove the following result in this paper.

**Theorem 1.** Let $V_1$ and $V_2$ be $C^2$ functions on $\mathbb{R}^m$ satisfying

$$\Lambda(V)(x) = \text{the lowest eigenvalue of} \quad V''(x) \to +\infty \quad \text{as} \quad |x| \to +\infty$$  \hspace{1cm} (V1)'

and let $h$ be a $C^1$ function on $S^1$ satisfying (h1). Set $h_-(t) = \min\{0, h(t)\}$, $h_+(t) = \max\{0, h(t)\}$. Then

$$-\ddot{x} - \lambda x = h_-(t)V_1'(x) + h_+(t)V_2'(x)$$  \hspace{1cm} (1.2)

has a nonzero $2\pi$-periodic solution if either $\lambda \notin \sigma(S^1)$ and $V_1$ and $V_2$ satisfy

$$|V_1'(x)| = |V_2'(x)| = o(|x|), \quad |x| \to 0,$$  \hspace{1cm} (V2)
or there is a symmetric neighborhood $U$ of 0 in $\mathbb{R}^m$ such that

$$V_1(-x) = V_1(x), \quad V_2(-x) = V_2(x), \quad x \in U.$$  \hspace{1cm} (V3)

If $V_1$ and $V_2$ are even in $x$, then (1.2) has an unbounded sequence of $2\pi$-periodic solutions.

As in [17] the variational approach is used to prove this theorem. That is, we look for the critical points of the functional

$$I(x) = \frac{1}{2} \int_0^{2\pi} (|\dot{x}|^2 - \lambda |x|^2) \, dt - \int_0^{2\pi} h(t) V_1(x) \, dt - \int_0^{2\pi} h(t) V_2(x) \, dt$$

which is defined on

$$H^1(S^1) = \left\{ x : [0, 2\pi] \to \mathbb{R}^m, x(0) = x(2\pi), \int_0^{2\pi} (|\dot{x}|^2 + |x|^2) \, dt < \infty \right\}.$$

The idea of the proof is as follows. We take a sequence of $C^1$ bounded functions $\{h_n\}$, each $h_n$ satisfies the condition (h_0) and $\|h_n - h\|_{C^0} \to 0$ as $n \to 0$. Consider

$$-\ddot{x} - \lambda x = h_{n,-}(t) V_1'(x) + h_{n,+}(t) V_2'(x),$$

by Theorem 1 in [17], we have a $2\pi$-periodic solution $x_n$ for (1.3). Moreover, the Morse index of $x_n$ has a uniform bound. Using this fact, we can get an $L^\infty$ estimate of $x_n$, which enables us to take the limit and $x = \lim_{n \to \infty} x_n$ is a solution of (1.2). Thus, the key point is to get the Morse index and $L^\infty$ estimates of the solutions of perturbed equation (1.3). The idea using the Morse index to get the $L^\infty$ estimate of the solutions has been used in [2, 16, 20] for the semilinear elliptic BVP. A similar argument has been used in [7, 8, 10] for the periodic solutions of Hamiltonian systems.

2. $L^\infty$ estimate in terms of Morse index

For simplicity, we assume that the function $h$ has only two zeros $0 \leq t_1 < t_2 < 2\pi$ with $h'(t_1) > 0$ and $h'(t_2) < 0$. Then $S^1 = S_+ \cup S_-$, where $S_+ = \{t \in S^1, h(t) \geq 0\} = [t_1, t_2]$ and $S_- = \{t \in S^1, h(t) \leq 0\} = [t_2, t_1 + 2\pi]$. Set $S_{n,+} = [t_1 + \frac{1}{n}, t_2 - \frac{1}{n}], S_{n,-} = [t_2 + \frac{1}{n}, 2\pi + t_1 - \frac{1}{n}], S_{n,0} = [t_1 - \frac{1}{n}, t_1 + \frac{1}{n}], S_{n,0}^2 = [t_2 - \frac{1}{n}, t_2 + \frac{1}{n}], S_{n,0} = S_{n,0}^1 \cup S_{n,0}^2$. Now we pick up a sequence of $C^1$ bounded $2\pi$-periodic function $h_n$ satisfying:

(i) $h_n(t) = h(t)$ if $|t - t_1| < |t - t_2| \geq \frac{2}{n}$;
(ii) $h_n(t) = 0$ if $|t - t_1| \leq \frac{1}{n}$ or $|t - t_2| \leq \frac{1}{n}$,
(iii) $h_n'(t) \geq 0$ if $|t - t_1| \leq \delta, h_n'(t) \leq 0$ if $|t - t_2| \leq \delta$ for some $\delta > 0$. 

Then \( h_n \to h \) in \( C^0 \) as \( n \to \infty \). Consider \( 2\pi \)-periodic solutions of

\[
-\ddot{x} - \lambda x = h_{n,-}(t)V'_1(x) + h_{n,+}(t)V'_2(x),
\]

which are the critical points of the functional

\[
I^n(x) = \frac{1}{2} \int_0^{2\pi} (|\dot{x}|^2 - \lambda |x|^2) \, dt - \int_0^{2\pi} h_{n,-}(t)V_1(x) \, dt - \int_0^{2\pi} h_{n,+}(t)V_2(x) \, dt.
\]

Since \( h_n \) satisfies the condition \((h_0)\), \( I^n \) satisfies (P.S) condition on \( X \) if \( V_1 \) and \( V_2 \) satisfy \((V1)\) and \( n \) is large enough (see [17]).

The main result of this section is the following proposition, which plays an important role in our approximation argument.

**Proposition 2.** Let the functions \( h, V_1 \) and \( V_2 \) satisfy the conditions \((h_1)\) and \((V1)'\), respectively, and let \( M \) be a constant. If \( x_n \) is a critical point of \( I^n \) with the Morse index \( i(x_n) \) satisfying

\[
i(x_n) \leq M.
\]

Then there is a constant \( C \) independent of \( n \) such that \( \|x_n\|_{C^2} \leq C \).

The proof is divided into several lemmas. Let \( S_{+,\delta} = [t_1 + \delta, t_2 - \delta] \) and \( B^r_n = \{ t \in S_{+,\delta} | |x_n(t)| \leq r \} \). The following lemma is taken from [10]. We denote a constant independent of \( n \) by \( C \).

**Lemma 3.** If \( \lim_{n \to \infty} \|x_n\|_{C(S_{+,\delta})} = \infty \), then for any \( r > 0 \),

\[
\lim_{n \to \infty} \mu(B^r_n) = 0,
\]

where \( \mu \) is the Lebesgue measure in \( \mathbb{R} \).

**Proof.** Given \( \varepsilon > 0 \), we choose \( R > 2r \) such that

\[
\frac{2r}{(R - r)} < \varepsilon.
\]

The set \( B^R_n = \{ t \in S_{+,\delta} | |x_n(t)| < R \} \) is a countable union of mutually disjoint open intervals \( B^R_n = \cup_k (s_k, t_k) \).

**Claim.** If \( n \gg 1 \), then

\[
\mu((s_k, t_k) \cap B^r_n) \leq \varepsilon(t_k - s_k).
\]

This implies \( \mu(B^r_n) \leq \mu(S_{+,\delta}) \varepsilon \). Hence (2.3) holds because \( B^r_n \subset B^R_n \).
For $t \in [t_1 + \delta, t_2 - \delta]$, we have $h_n(t) = h_+(t) \geq \eta_0 > 0$. The condition $(V1)'$ implies
\[
\frac{V_2(x)}{|x|^2} \to \infty \quad \text{as } |x| \to \infty.
\]
From this we get
\[
-\frac{1}{2} \lambda |x|^2 + h(t)V_2(x) \geq \frac{\eta_0}{2} V_2(x) - C. \tag{2.6}
\]
Let
\[
H_n(t) = \frac{1}{2} (|\dot{x}_n(t)|^2 - \lambda |x_n(t)|^2) + h_+(t)V_2(x_n(t)),
\]
then
\[
|H'_n(t)| = |h'_+ V_2(x_n(t))| \leq C(H_n(t) + 1). \tag{2.7}
\]
Since $\|x_n\|_{C(S_+)} \to \infty$, there exists $\tau_n \in S_+$ such that
\[
H_n(\tau_n) \geq \frac{1}{2} |\dot{x}_n(\tau_n)|^2 + \frac{\eta_0}{2} V_2(x(\tau_n)) - C \to +\infty \tag{2.8}
\]
by (2.6) and $(V1)'$. Set
\[
M_n = \max_{t \in S_+} H_n(t), \quad m_n = \min_{t \in S_+} H_n(t),
\]
(2.7) and (2.8) then conclude
\[
\frac{M_n}{m_n} \leq C, \tag{2.9}
\]
\[
|\dot{x}_n(t)|^2 \geq 2(m_n - E_R) \to +\infty, \quad t \in B^R_n, \tag{2.10}
\]
and
\[
|\dot{x}_n(t)|^2 \leq 2(M_n + E_R), \quad t \in B^R_n, \tag{2.11}
\]
where $E_R = \max_{|x| \leq R, t \in S_+} | - \frac{1}{2} \lambda |x|^2 + h(t)V_2(x) |$. Now we estimate $t_k - s_k$. For simplicity of notation, we denote $(s_k, t_k) = (s, t)$. Then $|x_n(s)| = |x_n(t)| = R$ and
|x(τ)| < R for τ ∈ (s, t). By the mean value theorem and $x_n$ solving (2.1), we have
\[
|\dot{x}_n(τ) - \dot{x}_n(s)| ≤ 2\pi \hat{E}_R, \quad τ ∈ (s, t),
\]
(2.12)
where $\hat{E}_R = \max_{|x| ≤ R, t ∈ S_{+,δ}} |\dot{x} - h(t) V_2(x)|$. Integrating (2.12) w.r.t. τ from s to t we get
\[
|x_n(t) - x_n(s) - \dot{x}_n(s)(t - s)| ≤ 2\pi \hat{E}_R(t - s),
\]
hence
\[
(|\dot{x}_n(s)| - 2\pi \hat{E}_R)(t - s) ≤ |x_n(t) - x_n(s)| ≤ 2R.
\]
(2.13)
Then from (2.10) and (2.13) we have
\[
t - s ≤ \frac{2R}{\sqrt{2(m_n - E_R) - 2\pi \hat{E}_R}}.
\]
(2.14)
(2.10) and (2.12) yield for $n \gg 1$,
\[
\dot{x}_n(τ) · \dot{x}_n(s) ≥ \frac{1}{2} |\dot{x}_n(s)|^2, \quad τ ∈ (s, t).
\]
(2.15)
Suppose $(s, t) \cap B^r_n \neq ∅$, then we can define
\[
s^* = \min\{τ ∈ [s, t], |x_n(τ)| ≤ r\}, \quad t^* = \max\{τ ∈ [s, t], |x_n(τ)| ≤ r\},
\]
and they satisfy $|x_n(t^*)| = |x_n(s^*)| = r$ and
\[
r < |x_n(τ)| < R \quad \text{for} \quad τ ∈ (s, s^*) ∪ (t^*, t).
\]
In view of (2.15) we have
\[
\frac{d}{dτ} (x_n(τ) · \dot{x}_n(s)) ≥ \frac{1}{2} |\dot{x}_n(s)|^2 \quad \text{for} \quad τ ∈ (s^*, t^*).
\]
This concludes
\[
2r |\dot{x}_n(s)| ≥ (x_n(t^*) - x_n(s^*)) · \dot{x}_n(s) ≥ \frac{1}{2} |\dot{x}_n(s)|^2 (t^* - s^*).
\]
(2.16)
Consequently by (2.10) and (2.16) we get
\[
\mu((s, t) \cap B^r_n) ≤ t^* - s^* ≤ \frac{4r}{|\dot{x}_n(s)|} ≤ \frac{4r}{\sqrt{2(m_n - E_R)}}.
\]
(2.17)
From (2.11) and the mean value theorem we have

\[
R - r \leq |x_n(s^*) - x_n(s)| \leq \sqrt{2(M_n + E_R)}(s^* - s),
\]

\[
R - r \leq |x_n(t) - x_n(t^*)| \leq \sqrt{2(M_n + E_R)}(t - t^*).
\]

Hence

\[
(t - s) \geq \frac{2(R - r)}{\sqrt{2(M_n + E_R)}}.
\] (2.18)

Thus, from (2.4), (2.17) and (2.18), we finally get

\[
\mu((s, t) \cap B^r_n) \leq \frac{2r}{(R - r)} \frac{\sqrt{2(M_n + E_R)}}{\sqrt{2(m_n + E_R)}} (t - s) \leq \varepsilon(t - s)
\] (2.19)

if \( n \gg 1 \). This completes the proof of the claim and the lemma. \( \square \)

**Lemma 4.** If \( i(x_n) \leq M \), then there is a constant \( C \) such that

\[
\|x_n\|_{C^2(S_{+},\delta)} \leq C.
\] (2.20)

**Proof.** From (2.1), it suffices to show \( \|x_n\|_{C(S_{+},\delta)} \leq C \). Take an integer \( k \geq M + 1 \), let \( \{e_1, \ldots, e_k\} \) be an orthogonormal set such that \( \text{supp}(e_i) \subset S_{+},\delta \). Set \( X_0 = \text{span}\{e_1, \ldots, e_k\} \). For \( \xi = \sum c_i e_i \) with \( \|\xi\|_{L^2} = 1 \) we have

\[
\int_0^{2\pi} (|\xi|^2 - \lambda|\xi|^2) \, dt \leq C(k)\|\xi\|^2_{L^2}
\] (2.21)

for some constant \( C(k) \). Now using (V1)', we can take an \( r \) such that

\[
(V_2''(x)\xi, \xi) \geq \frac{(C(k) + 1)}{\eta_0} |\xi|^2 \quad \text{for } |x| \geq r.
\]

If \( \|x_n\|_{C(S_{+},\delta)} \to \infty \), then by Lemma 3 we have

\[
\mu(B^r_n) \to 0 \quad \text{as } n \to \infty,
\]
where $B^*_{n} = \{ t \in S_{+}\delta||x_n(t)| \leq r}$. This implies, for $n \geq 1$,

$$
\int_{0}^{2\pi} (V''_2(x_n)\xi, \xi) dt = \int_{S_{+}\delta} (V''_2(x_n)\xi, \xi) dt
$$

$$
= \int_{S_{+}\delta \setminus B^*_{n}} (V''_2(x_n)\xi, \xi) dt + \int_{B^*_{n}} (V''_2(x_n)\xi, \xi) dt
$$

$$
\geq \left( \frac{C(k) + \frac{1}{2}}{\eta_0} \right) \int_{0}^{2\pi} |\xi|^2 dt
$$

(2.22)

because supp(\xi) \subset S_{+}\delta and (V1)'. From (2.21) and (2.22), using $h_n(t) = h(t) \geq \eta_0$ for $t \in S_{+}\delta$, we obtain

$$
(I''(x_n)\xi, \xi) = \int_{0}^{2\pi} (|\dot{x}_n(t)|^2 - \lambda |x_n(t)|^2) dt - \int_{0}^{2\pi} h_n(t)(V''_2(x_n)\xi, \xi) dt
$$

$$
\leq C(k)\|\dot{x}\|_{L^2}^2 - \left( C(k) + \frac{1}{2} \right) \|\xi\|_{L^2}^2 < 0.
$$

(2.23)

This concludes

$$
i(x_n) \geq k \geq M,
$$

which contradicts with the assumption. Hence $\|x_n\|_{C(S_{+}\delta)} \leq C$ for some constant C. $\square$

**Lemma 5.** Let $S_0^1 = [t_1, t_1 + \delta]$ and $S_0^2 = [t_2 - \delta, t_2]$. There is a constant C such that

$$
\|x_n\|_{C^2(S_0^1)}, \|x_n\|_{C^2(S_0^2)} \leq C.
$$

(2.24)

**Proof.** From (V1)' we have

$$
H'_n(t) = h'_n(t)V_2(x_n(t)) \geq -C, \quad t \in S_0^1.
$$

This implies

$$
H_n(t) = \frac{1}{2} (|\dot{x}_n(t)|^2 - \lambda |x_n(t)|^2) + h_n(t)V_2(x_n(t))
$$

$$
\leq H_n(t_1 + \delta) + C \leq C, \quad t \in S_0^1
$$

(2.25)
by Lemma 4. Then by (2.25) we have
\[
\frac{1}{2} |\dot{x}_n(t)|^2 \leq \frac{1}{2} \lambda |x_n(t)|^2 - h_{n,+}(t)V_2(x_n(t)) + C \leq \frac{1}{2} \lambda |x_n(t)|^2 + C.
\]
Hence
\[
|\dot{x}_n(t)| \leq C (|x_n(t)| + 1), \quad t \in [t_1, t_1 + \delta].
\]
Since $|x_n(t_1 + \delta)| \leq C$ by Lemma 4, we have
\[
|x_n(t)| \leq C, \quad t \in [t_1, t_1 + \delta] \tag{2.26}
\]
by the Gronwall inequality. Then (2.1) implies $\|x_n\|_{C^2(S_{1,0}^1)} \leq C$. Similarly, we can show $\|x_n\|_{C^2(S_{0,\delta}^2)} \leq C$. \hfill $\Box$

**Lemma 6.**
\[
\|x_n\|_{C^2(S_-)} \leq C. \tag{2.27}
\]

**Proof.** We note that $x_n$ satisfies
\[
\ddot{x} - \lambda x = h_n(t)V_1'(x), \quad t \in S_- \tag{2.28}
\]
We have shown that $|\dot{x}_n(t_2)|, |x_n(t_2)| \leq C$ and $|\dot{x}_n(t_1 + 2\pi)|, |x_n(t_1 + 2\pi)| \leq C$. (2.28) and integration by parts show
\[
\int_{S_-} (|\dot{x}_n(t)|^2 - \lambda |x_n(t)|^2 - h_n(t)V_1'(x_n(t)) \cdot x_n(t)) dt \\
= x_n(t_2) \cdot \dot{x}_n(t_2) - x_n(t_1 + 2\pi) \cdot \dot{x}_n(t_1 + 2\pi) \leq C. \tag{2.29}
\]
From this we can obtain
\[
\int_{S_-} |x_n|^2 \, dt \leq C \tag{2.30}
\]
by the condition $h_n \leq 0$ on $S_-$ and (V1)'. Hence
\[
\int_{S_-} |\dot{x}_n|^2 \, dt \leq C \tag{2.31}
\]
by (2.29) since \( h_n \leq 0 \). Then \( \| x_n \|_{C(S_-)} \leq C \), and \( \| x_n \|_{C^2(S_-)} \leq C \) by Eq. (2.1). Now we prove (2.30). Let us first consider the function \( F(t) = \frac{V'(tx) \cdot (tx)}{|x|^2} \) with \( |x| = 1 \), then

\[
\lim_{t \to +\infty} F(t) = \lim_{t \to +\infty} \frac{(V''(tx)x, x)}{|x|^2} = +\infty
\]

uniformly in \( |x| = 1 \) by (V1). Therefore

\[
\lim_{|x| \to \infty} \frac{V'(x) \cdot x}{|x|^2} = +\infty.
\]

This shows that for any \( M > 0 \), there is a constant \( C > 0 \) such that

\[
V'(x) \cdot x \geq M|x|^2 - C. \tag{2.32}
\]

Substituting (2.32) into (2.29) we have

\[
- \int_{S_-} h_n(t)(M|x_n(t)|^2 - C) \, dt \leq C - \int_{S_-} (|\dot{x}_n(t)|^2 - \lambda|x_n(t)|^2) \, dt. \tag{2.33}
\]

If \( \int_{S_-} |x_n|^2 \, dt \to \infty \), we set \( \tilde{x}_n = \frac{x_n}{\sqrt{\int_{S_-} |x_n|^2 \, dt}} \). Then (2.29) implies

\[
\int_{S_-} |\dot{\tilde{x}}_n|^2 \, dt \leq C
\]

since \( h_n \leq 0 \) on \( S_- \). So there is a subsequence such that \( \tilde{x}_n \to \tilde{x}_0 \) weakly in \( H^1_0(S_-) \) and \( \int_{S_-} |\tilde{x}_0|^2 \, dt = 1 \). Divided by \( \int_{S_-} |x_n|^2 \, dt \) both sides in (2.33), using \( \int_{S_-} |x_n|^2 \, dt \to \infty \) we get

\[
-M \int_{S_-} h(t)|\tilde{x}_0|^2 \, dt \leq \dot{\lambda}. \tag{2.34}
\]

Since \( M \) is arbitrary, we obtain \( \tilde{x}_0 = 0 \). This contradicts with \( \int_{S_-} |\tilde{x}_0|^2 \, dt = 1 \). Thus, we have proved \( \int_{S_-} |x_n|^2 \, dt \leq C \). \( \square \)

Combining Lemmas 4–6, we get Proposition 2.

We end this section with a remark. In verifying the (P.S) condition, condition (V1) plays an important role. (V1) and (V1)' overlap in some cases, in general they are not comparable. We need both (V1) and (V1)' later. But in Theorem 1, we only assume
(V1)′. This can be done by an approximating argument. Let \( \theta > 2 \) and \( V \) be a \( C^2 \) function such that (V1)′ holds. For \( R > 0 \), let \( \chi \) be a function such that

\[
\chi_R(s) = 1 \quad \text{if} \quad s \leq R, \quad \chi_R(s) = 0 \quad \text{if} \quad s \geq R + 1, 
\]

and let

\[
V_R(x) = \chi_R(|x|)V(x) + (1 - \chi_R(|x|))|x|^\theta. 
\]

Clearly, \( V_R \) satisfies (V1) and (V1)′. From the proof of Proposition 2, we can see that the estimate is uniformly in \( R \geq 1 \), that is, if \( x_{n,R} \) is a solution of

\[
-\ddot{x} - \dot{x} = h_{n,-}(t)V_{1,R}(x) + h_{n,+}(t)V_{2,R}(x) 
\]

with \( i(x_{n,R}) \leq M \), then \( \|x_{n,R}\| \leq C \) for some constant \( C \), which is independent of \( n \) and \( R \). With this fact, in order to get Theorem 1, we may assume that \( V \) satisfies (V1) and (V1)′.

3. Topology of level sets

Let \( h_n \) be the function introduced in the last section and

\[
I^n(x) = \frac{1}{2} \int_0^{2\pi} (|\dot{x}|^2 - \lambda |x|^2) \, dt - \int_0^{2\pi} h_{n,-}(t)V_1(x) \, dt - \int_0^{2\pi} h_{n,+}V_2(x) \, dt. 
\]

Then \( I^n \) is \( C^2 \) on \( X = H^1(S^1) \). We have shown in [17] that if (V1) holds, for fixed \( \lambda \), \( I^n \) satisfies (P.S) condition if \( n \gg 1 \) and there is a number \( A_n \) such that there is no critical point \( x \) of \( I^n \) with \( I^n(x) \leq A_n \), and the critical groups of \( I^n \) at infinity satisfy

\[
C_\ast(I^n, \infty) = H_\ast(X, I^n_{A_n}) = 0, \quad \ast = 0, 1, 2, \ldots, 
\]

where the coefficient group of the homology is \( G \) and \( I^n_c = \{ x \in X | I^n(x) \leq c \} \).

Let \( I \) be a \( C^2 \) functional on a Hilbert space \( X \) satisfying (P.S) condition, and let \( x \) be an isolated critical point of \( I \). The critical group of \( I \) at \( x \) is

\[
C_\ast(I, x) = H_\ast(U \cap I_c, (U \setminus \{ x \}) \cap I_c),
\]

where \( U \) is a neighborhood of \( x \) such that \( I \) has no critical point other than \( x \) in \( U \) and \( c = I(x) \). Set \( m_\ast(I, x) = \text{rank}C_\ast(I, x) \) and

\[
P(x, t) = \sum_{i \geq 0} m_i(I, x)t^i. 
\]
In our case \( I = I^n \), \( P(x, t) \) is finite for any isolated critical point \( x \) by the Shifting Lemma and Morse Lemma. Let \( a < b \) be regular values, \( \beta_i = \text{rank}H_i(I_b, I_a) \), and let \( \{x_1, x_2, \cdots\} \) be the critical points of \( I \) with \( a < I(x_i) < b \). Assume each \( x_i \) is isolated. Then the following Morse inequality holds:

\[
\sum P(x_i, t) = \sum_{j \geq 0} \beta_j t^j + (1 + t)Q(t)
\]

(3.3)

where \( Q(t) \) is a polynomial with nonnegative integer as its coefficients. We refer to [4] for the detail.

**Proposition 7.** Let \( \{h_n\} \) be the functions defined in the last section, \( V_1 \) and \( V_2 \) be \( C^{2m+2} \) functions satisfying the conditions (V1) and (V1)’. Then for any positive integer \( N \), there exists a constant \( a = a(N) \) such that

\[
H_\ast(I^n, I^n) = 0, \quad \ast = 0, 1, \ldots, N.
\]

(3.4)

Before the proof, we recall a lemma from [14], which is a corollary of the Sard’s theorem in \( \mathbb{R}^{2m} \). We repeat their proof for the completeness.

**Lemma 8.** Let \( V_1 \) and \( V_2 \) be in \( C^{2m+2} \) satisfying (V1) and (V1)’. Then the set of the critical values of \( I^n \) is of measure zero in \( \mathbb{R} \).

**Proof.** Clearly, it is enough to show that the set of critical values of \( I^n \) located in a fixed interval \([a, b]\) is of measure zero. We fix an interval \([a, b]\). Consider the solution \( x(t, x_0, x_1) \) of

\[
-\ddot{x} - \lambda x = h_{n,-}(t)V_1'(x) + h_{n,+}(t)V_2'(x)
\]

(3.5)

with the initial value \( x(0) = x_0, \dot{x}(0) = x_1, (x_0, x_1) \in \mathbb{R}^{2m} \). For each \( t \) let \( \phi^t \) be the map

\[
\phi^t : (x_0, x_1) \to (x(t, x_0, x_1), \dot{x}(t, x_0, x_1)).
\]

Let

\[
S = \{(x_0, x_1)|x = x(t, x_0, x_1) \text{ is } 2\pi\text{-periodic and } I^n(x) \in [a, b]\}.
\]

Then \( S \) is compact due to the (P.S) condition. For \((x_0, x_1) \in S\), the solution \( x(t, x_0, x_1) \) exists globally, hence there is a small neighborhood \( U \) of \( S \) in \( \mathbb{R}^{2m} \) such that for any \((x_0, x_1) \in U\), the solution \( x(t, x_0, x_1) \) of (3.5) with initial value \((x_0, x_1)\) exists for
Let $t \in [-1, 2]$. Let $f$ be a smooth function which is equal to $1$ near $0$ and equal to $0$ near $1$, and set
\[
\psi(t, x_0, x_1) = f(t) \phi^j(x_0, x_1) + (1 - f(t)) \phi^j((\phi^1)^{-1}(x_0, x_1)).
\]
Let $\psi_1(t, x_0, x_1) \in \mathbb{R}^m$ be the first factor of $\psi(t, x_0, x_1)$, then for every $(x_0, x_1) \in U$, $t \rightarrow \psi_1(t, x_0, x_1)$ is a loop belonging to $X$. We define a map $\Psi : U \rightarrow X$ by setting $\Psi(x_0, x_1)(t) = \psi_1(t, x_0, x_1)$. Consider the function $I^n(\Psi) : U \rightarrow \mathbb{R}$, which is $C^{2m+1}$ and has the property that every $(x_0, x_1) \in S$ is a critical point of $I^n(\Psi)$. This implies that the critical values of $I^n$ are contained in those of $I^n(\Psi)$, which is of measure $0$ by Sard’s theorem. The proof of the lemma is finished. □

Lemma 9. Let $I$ be a $C^2$ functional on a Hilbert space $X$ satisfying (P.S) condition, and let $I'(x)$ be Fredholm with index $0$, $a < b$ be regular values of $I$. If $H_k(I_b, I_a) \neq 0$ for some integer $k$, then there is a critical point $x$ of $I$ such that the Morse index $i(x)$ satisfies $i(x) \leq k$.

This is a corollary of the well known Marino-Prodi perturbation and the Morse inequality (3.3).

Proof. We argue by contradiction. Let $K = \{x | I'(x) = 0, a < I(x) < b\}$. Then $K$ is compact by (P.S) condition. For $\varepsilon > 0$, by Marino-Prodi perturbation (see [11]), there is a $C^2$ functional $g_\varepsilon$ with $\|g_\varepsilon\|_{C^2} \leq \varepsilon$ such that all critical points of $I_\varepsilon = I + g_\varepsilon$ are nondegenerate and $g_\varepsilon \equiv 0$ outside a small neighborhood of $K$. Let $K_\varepsilon = \{x | (I + g_\varepsilon)'(x) = 0, a < I(x) + g_\varepsilon(x) < b\}$, then $K_\varepsilon$ is a finite set. Suppose for all $x \in K$ we have $i(x) \geq k + 1$. Then for small $\varepsilon$,

\[
i(x) \geq k + 1 \quad \forall x \in K_\varepsilon.
\]

Hence for each $x \in K_\varepsilon$,

\[
C_\ast(I_\varepsilon, x) = 0, \quad \ast = 0, 1, \ldots, k.
\]  

(3.6)

Applying the Morse inequality (3.3) to $I_\varepsilon$ we get

\[
\sum_{I_\varepsilon' = 0, a < I_\varepsilon(x) < b} P(x, t) = \sum_{j \geq 0} \beta_j t^j + (1 + t) Q(t),
\]  

(3.7)

where $\beta_\ast = \text{rank } H_\ast(I_\varepsilon, b, I_\varepsilon, a) = \text{rank } H_\ast(I_b, I_a)$ if $\varepsilon$ is small. Therefore, we have

\[
C_k(I_\varepsilon, x) \neq 0
\]  

(3.8)

for some critical point $x$ of $I_\varepsilon$ with $a < I_\varepsilon(x) < b$ since the RHS of (3.7) contains the term $\beta_k t^k \neq 0$. This is impossible because of (3.6). □
Remark. Similarly if we denote \( n(x) = \dim(\ker(I''(x))) \), then

\[
k - n(x) \leq i(x). \tag{3.9}
\]

Proof of Proposition 7. Let \( x \) be a critical point of \( I^n \) with Morse index \( i(x) \leq N + 1 \). Then by Proposition 2, there exists a constant \( c(N) \) independent of \( n \) such that

\[
\|x\|_{C^2} \leq c(N). \tag{3.10}
\]

This implies, for some constant \( a = a(N) \) such that

\[
I^n(x) \leq a(N) - 1. \tag{3.11}
\]

Now we show

\[
H_*(X, I^n_{a(N)}) = 0, \quad *= 0, 1, \ldots, N + 1. \tag{3.12}
\]

We fix an integer \( k \leq N + 1 \). If \( H_k(X, I^n_a) \neq 0 \), then for some constant \( b > a \) such that \( H_k(I^n_b, I^n_a) \neq 0 \). We may assume that \( a \) and \( b \) are regular values for all \( I^n \) by Lemma 8. From Lemma 9, there is a critical point \( x \) of \( I^n \) with

\[
a < I^n(x) < b \quad \text{and} \quad i(x) \leq k.
\]

This is impossible since \( i(x) \leq k \), we have \( I^n(x) \leq a - 1 \) by (3.11). Hence (3.12) holds. Using the homology exact sequences of the triple \( I^n_{A_n} \subset I^n_a \subset X \)

\[
\cdots \to H_{q+1}(X, I^n_{a}) \to H_q(I^n_{a}, I^n_{A_n}) \to H_q(X, I^n_{A_n}) \to H_q(X, I^n_{a}) \to \cdots
\]

and (3.12), we have

\[
H_q(I^n_{a}, I^n_{A_n}) \cong H_q(X, I^n_{A_n}), \quad *= 0, 1, \ldots, N. \tag{3.13}
\]

Then (3.1) concludes (3.4). \( \square \)

4. Critical points with the prescribed Morse index

Let \( I^n \) be the functional in the last section. In this section we study the existence of critical points of \( I^n \) with the prescribed Morse index.

The following is the main result of this section.
Proposition 10. Let $V_1$ and $V_2$ be $C^{2m+2}$ functions on $\mathbb{R}^m$ satisfying the conditions (V1) and (V1'). If $\lambda \notin \sigma(S^1)$, $V_1$ and $V_2$ satisfy

$$|V'_1(x)| = |V'_2(x)| = o(|x|), \quad |x| \to 0,$$

or there is a symmetric neighborhood $U$ of 0 in $\mathbb{R}^m$ such that

$$V_1(-x) = V_1(x), \quad V_2(-x) = V_2(x), \quad x \in U$$

and $x = 0$ is an isolated solution of

$$-\ddot{x} - \lambda x = h_-(t)V'_1(x) + h_+(t)V'_2(x). \quad (4.1)$$

Then there is a constant $N_0$ independent of $n$ and a nonzero critical point $x^n$ of $I^n$ such that

$$i(x^n) \leq N_0 + 1 \quad \text{for } n \geq 1. \quad (4.2)$$

If $V_1$ and $V_2$ are even in $x$, then there is a constant $N_0$ such that for each integer $k > N_0 + 2m$, $I^n$ has a critical point $x_k^n$ with

$$k - 2m \leq i(x_k^n) \leq k. \quad (4.3)$$

The proof is similar to that of Proposition 7. It consists of several steps. By our assumption, 0 is an isolated critical point of $I$, but it may not be isolated for $I^n$. In order to apply the Morse inequality for isolated critical points, we need to modify $I^n$ is a neighborhood of 0 first.

Lemma 11. Let 0 be an isolated critical point of $I$. There is a small ball $B_r(0)$ in $X$ such that for $n \geq 1$, there is a functional $\hat{I}^n$ which satisfies

$$\hat{I}^n(x) = I(x) \text{ for } x \in B_r(0),$$

$$\hat{I}^n(x) = I^n(x) \text{ for } x \in X \setminus B_{2r}(0)$$

and all nonzero critical points of $\hat{I}^n$ are the critical points of $I^n$.

Proof. Let $r > 0$ be a number such that 0 is the only critical point of $I$ in $B_{2r}(0)$. Then there is an $\varepsilon_0$ such that $\|I'(x)\| \geq \varepsilon_0$ for $x \in B_{2r}(0) \setminus B_r(0)$ by the (P.S) condition. Take a smooth function $\chi$ such that

$$\chi(s) = 1 \text{ if } s \leq r, \quad \chi(s) = 0 \text{ if } s \geq 2r$$
and set

\[ \hat{I}^n(x) = \chi(\|x\|)I(x) + (1 - \chi(\|x\|))I^n(x). \]

The lemma follows from the fact that for \( n \geq 1 \), \( \hat{I}^n \) has no critical point in \( B_{2r}(0) \setminus B_r(0) \). Indeed, we have

\[ (\hat{I}^n)'(x) = I'(x) + (1 - \chi(\|x\|))((\hat{I}^n)'(x) - I'(x)) - (\chi(\|x\|))'(I^n(x) - I(x)). \quad (4.4) \]

Using \( \|I'(x)\| \geq \varepsilon_0 \) for \( x \in B_{2r}(0) \setminus B_r(0) \) and \( \|I^n - I\|_{C^2(B_{2r}(0))} \to 0 \), from (4.4), for \( n \geq 1 \) we have \( \|(\hat{I}^n)'(x)\| \geq \varepsilon_0 / 2 \). \( \Box \)

For the functional \( \hat{I}^n \), 0 is an isolated critical point. Moreover, the critical groups of \( \hat{I}^n \) at 0 are same as those of \( I \), and the critical groups at infinity of \( \hat{I}^n \) are same as those of \( I^n \):

\[ C_*(\hat{I}^n, 0) = C_*(I, 0), \quad * = 0, 1, \ldots, \quad (4.5) \]

\[ C_*(\hat{I}^n, \infty) = C_*(I^n, \infty) = 0, \quad * = 0, 1, \ldots. \quad (4.6) \]

**Remark.** In Lemma 11, without the assumption that 0 is an isolated solution of (4.1), we can modify the functional \( \hat{I}^n \) in such a way that 0 is the only critical point of \( \hat{I}^n \) in \( B_r(0) \) which is nondegenerate and \( \hat{I}^n(x) = I^n(x) \) if \( x \in X \setminus B_{2r}(0) \), all critical points of \( \hat{I}^n \) in \( B_{2r}(0) \) satisfying \( i(x) \leq N_0 \), where \( N_0 \) is a constant independent of \( n \). If \( V_1 \) and \( V_2 \) are even, then \( \hat{I}^n \) can be chosen even too.

**Proof of Proposition 10.** Suppose 0 is an isolated critical point of \( I \). Consider the functional \( \hat{I}^n \) given by Lemma 11, 0 is an isolated critical point of \( \hat{I}^n \). Let \( C_*(\hat{I}^n, 0) \) be the critical groups. It is independent of \( n \) by (4.5). We have shown in [17] that \( P(0, t) = \sum_i \text{rank} C_i(\hat{I}^n, 0)t^i \) is a nonzero polynomial and \( P(0, -1) \) is odd if \( (V2) \) or \( (V3) \) holds. Let \( N_0 \) be the degree of \( P(0, t) \). For simplicity of notation, we denote \( \hat{I}^n \) by \( I \). By Proposition 7, for integer \( k \geq N_0 + 1 \), we can take a regular value \( a \geq 1 \) of \( I \) such that

\[ H_*(I_a, I_{A_n}) = 0, \quad * = 0, 1, \ldots, k + 2. \quad (4.7) \]

Similar to Lemma 9, using the Marino-Prodi perturbation argument, we may assume that all nonzero critical points of \( I \) are nondegenerate. Let \( m_i \) be the number of critical points \( x \neq 0 \) of \( I \) with Morse index \( i \) such that \( A_n < I(x) < a \) and \( \beta_i = \text{rank} H_i(I_a, I_{A_n}). \)
Applying the Morse inequality we have

\[ P(0, t) + \sum_i m_i t^i = \sum_{i \geq 0} \beta_i t^i + (1 + t)Q(t) = t^{k+2}P_1(t) + (1 + t)Q(t), \quad (4.8) \]

since \( \beta_i = 0 \) for \( i \leq k + 2 \), where \( P_1 \) and \( Q \) are polynomials with nonnegative integral coefficients. Let \( Q(t) = q_0 + q_1 t + \cdots q_k t^k + \cdots \). From (4.8) we conclude that there must be an integer \( i \) with \( 0 \leq i \leq N_0 + 1 \) such that

\[ m_i \neq 0. \quad (4.9) \]

Hence, there is a critical point \( x \neq 0 \) of \( I \), which is a critical point of \( I^n \) by Lemma 11, and satisfies

\[ i(x) \leq N_0 + 1. \quad (4.10) \]

Indeed, if \( m_i = 0 \) for \( 0 \leq i \leq N_0 \), comparing the coefficients of the term \( t^{N_0} \) in the LHS and the RHS of (4.8), we have

\[ q_{N_0-1} + q_{N_0} \neq 0. \]

If \( q_{N_0} = 0 \), then from (4.8) we have

\[ P(0, t) = (1 + t) \left( q_0 + q_1 t + \cdots + q_{N_0-1} t^{N_0-1} \right). \]

This contradicts with \( P(0, -1) \) is odd. Therefore

\[ q_{N_0} \neq 0. \]

Substituting this into (4.8), we obtain

\[ m_{N_0+1} = q_{N_0} + q_{N_0+1} \geq q_{N_0} > 0. \quad (4.11) \]

This proves (4.9) if (V2) or (V3) holds and 0 is in isolated solution of (4.1).

Now we do not assume that 0 is an isolated solution of (4.1) but \( V_1 \) and \( V_2 \) are even in \( x \). Let \( N_0 \) be the constant given by the remark at the end of the proof of Lemma 11. We will show

**Claim.** If all critical points of \( I \) are nondegenerate, then for \( k \geq N_0 + 1 \), there is a critical point \( x_k \) of \( I \) such that

\[ i(x_k) = k. \quad (4.12) \]
Using (4.12) and the Marino-Prodi perturbation, without the assumption of nondegeneracy of critical points we have a critical point $x_k$ such that

$$k - 2m \leq i(x_k) \leq k.$$  \hspace{1cm} (4.13)

Such $x_k$ is a critical point of $I^n$ if $k > N_0 + 2m$ by the remark. (4.12) also follows from (4.8) and each $m_i$ is even. Since $I$ is even, the nonzero critical points of $I$ appear in pair. If $m_k = 0$, then comparing the coefficients in (4.8) we get

$$q_{k-1} + q_k = 0,$$

hence $q_{k-1} = q_k = 0$. Consider the LHS and the RHS of (4.8) of degree equal to or less than $k - 1$ we have

$$P(0, t) + \sum_{i=1}^{k-1} m_i t^i = (1 + t) \left( q_0 + q_1 t + \cdots + q_{k-2} t^{k-2} \right).$$  \hspace{1cm} (4.14)

Setting $t = -1$ in (4.14), we obtain

$$P(0, -1) + \sum_{i=1}^{k-1} m_i (-1)^i = 0.$$  \hspace{1cm} (4.15)

But $P(0, -1)$ is odd and each $m_i$ is even, so the LHS of (4.15) is odd. Thus we get a contradiction. The proof of the claim, hence that of the proposition is finished. \hspace{1cm} □

5. Proof of Theorem 1

In this section we prove Theorem 1. We first assume that $V_1$ and $V_2$ are $C^{2m+2}$. The case that $V_1$ and $V_2$ are $C^2$ can be obtained by an approximate argument. For the existence of one nonzero $2\pi$-periodic solution, we may assume $0$ is isolated.

Proof of Theorem 1. Let $\{h_n\}$ be the functions in Section 2. Consider the $2\pi$-periodic solutions of the equation

$$-\ddot{x} - \lambda x = h_{n,-}(t) V'_1(x) + h_{n,+}(t) V'_2(x),$$  \hspace{1cm} (5.1)

which are the critical points of

$$I^n(x) = \frac{1}{2} \int_0^{2\pi} (|\dot{x}|^2 - \lambda |x|^2) \, dt - \int_0^{2\pi} h_{n,-}(t) V_1(x) \, dt - \int_0^{2\pi} h_{n,+}(t) V_2(x) \, dt.$$
By Proposition 10, for each $I^n$, there is a critical point $x^n$ such that
\[
\|x^n\| \geq r, \quad i(x^n) \leq N_0 + 1. \tag{5.2}
\]

From Proposition 2, there is a constant $C$ such that
\[
\|x^n\|_{L^\infty} \leq C.
\]

Using Eq. (5.1), this yields
\[
\|x^n\|_{C^2} \leq C.
\]
So we can assume for $n \to \infty$, $x^n \to x$ in $C^1$. Then $x \in C^2$ and solves
\[
-\ddot{x} - \lambda x = h_-(t)V_1'(x) + h_+(t)V_2'(x). \tag{5.3}
\]

$x \neq 0$ because $\|x\| \geq r > 0$.

If in addition we assume that $V_1$ and $V_2$ are even in $x$. Then by Proposition 10, for each integer $k \geq N_0$ we have a critical point $x^n_k$ with
\[
k - 2m \leq i(x^n_k) \leq k.
\]
As before we can take limit
\[
x_k = \lim_{n \to \infty} x^n_k
\]
and each $x_k$ is a $2\pi$-periodic solution of (5.3). Moreover the Morse index $i(x)$ of $x$ satisfies
\[
i(x) \leq \lim_{n \to \infty} i(x^n_k) \leq k
\]
and
\[
i(x) \geq \lim_{n \to \infty} i(x^n_k) - 2m \geq k - 4m. \tag{5.5}
\]
This is because $\dim \ker I''(x) \leq 2m$. Thus, we obtain a sequence of $2\pi$-periodic solutions $\{x_k\}_1^\infty$ of (5.3) such that $i(x_k) \to \infty$. For such a sequence, we have $\|x_k\|_{L^\infty} \to \infty$ as $k \to \infty$. 
If $V_1$ and $V_2$ are $C^2$, we can take two sequences of $C^{2m+2}$ functions $V_{n,1}$ and $V_{n,2}$ such that

$$V_{n,1} \to V_1, \quad V_{n,2} \to V_2 \quad C^2_{loc}(\mathbb{R}^{2m}).$$

Then for the equation

$$-\ddot{x} - \lambda x = h_-(t)V_{n,1}'(x) + h_+(t)V_{n,2}'(x), \quad (5.6)$$

we have a solution $x^n$ with $\|x^n\|_{L^\infty} \leq C$ by Propositions 2 and 10. Then we can take the limit and $x = \lim_{n \to \infty} x^n = x$ is a solution of (5.3). If $V_1$ and $V_2$ are even, we can get an unbounded sequence of $2\pi$-periodic solutions. □

References

[13] M. Girardi, M. Matzeu, On periodic solutions of the system $\ddot{x}(t) + b(t) \left( V_1'(x(t)) + V_2'(x(t)) \right) = 0$ where $b(\cdot)$ changes sign and $V_1, V_2$ have different superquadratic growths, in: Proceedings of the Local and Variational Methods on Hamiltonian Systems, World Scientific, Singapore, 1995, pp. 65–76.