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The Euler Characteristic of Vector Fields on Banach Manifolds and a Globalization of Leray–Schauder Degree

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In our search [14] to find a unified approach to characteristics of vector fields on finite- and infinite-dimensional manifolds we studied the notion of a Fredholm vector field on Banach manifolds M with special differentiable structures. If X is a Fredholm vector field on M with a finite number of zeros we defined the Euler characteristic $\chi(X)$. For bounded Hilbert submanifolds M of finite codimension of a Hilbert space H and a special class of Fredholm vector fields the Euler characteristic was shown to be independent of X and equal to $\chi(M)$ the Euler characteristic of M.

The attempt in [14] to apply the theory to the geodesic problem failed. In this paper we abandon special differentiable structures and speak only of Fredholm vector fields with respect to a connection on M.

The index theory of such vector fields applies to the study of many intrinsically nonlinear problems [e.g., harmonic mappings between finite-dimensional Riemannian manifolds, geodesics (see Section 6), and to simply connected minimal surfaces spanning a fixed curve $\Gamma \subset \mathbb{R}^3$ [17]].

In addition this theory is related to the theory of the degree of Fredholm maps on Banach manifolds as developed in [3]. The degree of Fredholm maps, which includes the Leray–Schauder degree, also applies to intrinsically nonlinear problems in analysis. By intrinsically nonlinear we mean the study of solutions to systems of nonlinear partial differential or integral equations on spaces of mappings which are not linear or affine.

1. PRELIMINARIES ON FUNCTIONAL ANALYSIS

Most of the results in this section are well known and unless otherwise stated can be found in Palais [8].

Let E, F be Banach spaces and $\mathscr{L}(E, F)$ denote the linear continuous maps from E to F. If $F = E$ we denote $\mathscr{L}(E, F)$ by $\mathscr{L}(E)$.

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DEFINITION. $C \in \mathscr{L}(E, F)$ is compact if for any bounded set $B \subset E$, $\overline{C(B)}$ is compact in F. Denote the linear space of compact linear maps by $C(E, F)$ or $C(E)$ if $F = E$.

PROPOSITION 1. *$C(E, F)$ is a closed subspace of $\mathscr{L}(E, F)$.*

DEFINITION. $T \in \mathscr{L}(E, F)$ is linear *Fredholm* if

(i) $\dim \operatorname{Ker} T < \infty$,

(ii) $\dim \operatorname{Coker} T = \dim F/T(E) < \infty$,

(iii) Range T is closed.

Actually (i) + (ii) $\Rightarrow$ (iii) but most authors assume (iii) in order to avoid proving it. Let $\mathscr{F}(E, F) \subset \mathscr{L}(E, F)$ denote the subset of Fredholm maps.

By the *index* of a linear Fredholm map T we mean

$$\operatorname{ind} T = \dim \operatorname{Ker} T - \dim \operatorname{Coker} T.$$

PROPOSITION 2. *$\mathscr{F}(E, F)$ is open in $\mathscr{L}(E, F)$ and* $\operatorname{ind}: \mathscr{F}(E, F) \to Z$ *is continuous and therefore constant on components of $\mathscr{F}(E, F)$.*

DEFINITION. Let $\mathscr{L}_c(E) = \{T \mid T = I + C,\ I \text{ the identity},\ C \in C(E)\}$. If $T \in \mathscr{L}_c(E)$ then T is Fredholm of index zero.

Let $G\mathscr{L}(E)$ denote the general linear group of E; i.e., the set of invertible linear maps in $\mathscr{L}(E)$. When E is a Hilbert space Kuiper showed in [5] that $G\mathscr{L}(E)$ is contractible as a topological space. Since that time most of the function spaces in analysis (e.g., Hölder spaces, Sobelev spaces) have been shown to have contractible general linear groups. This of course implies that tangent bundles of Banach manifolds are trivial which means, in some sense, $G\mathscr{L}(E)$ is too big a group for much of finite-dimensional topology to go through in ∞ dimensions. Thus we introduce a Lie subgroup $G\mathscr{L}_c(E)$ of $G\mathscr{L}(E)$, the so-called *Fredholm group*.

DEFINITION. Let $G\mathscr{L}_c(E) = \mathscr{L}_c(E) \cap G\mathscr{L}(E)$.

PROPOSITION 3. *$G\mathscr{L}_c(E)$ is a Lie subgroup of $G\mathscr{L}(E)$ which, if E has a Schauder basis, has the homotopy type of $G\mathscr{L}(\infty) =$* lim inf *$G\mathscr{L}(n)$, the direct limit of the general linear group of $\mathbb{R}^n$ (for details see* [3, 10]*). In particular, $\pi_0(G\mathscr{L}_c(E)) \approx Z_2$, the integers* mod 2. *Thus $G\mathscr{L}_c(E)$ has two components, which remains true if E does not have a Schauder basis.* ∎

Let $G\mathscr{L}_c^+(E)$ be the component of the identity in $G\mathscr{L}_c(E)$ and $G\mathscr{L}_c^-(E)$ the other component. We would like to give an example of an element in

$G\mathscr{L}_c^-(E)$. Write $E = E_0 \times R$ where R is a one-dimensional subspace of E. Define $J \in G\mathscr{L}_c(E)$ by

$$J(x, y) = (x, -y), \qquad (x, y) \in E_0 \times R.$$

Then $J \in G\mathscr{L}_c^-(E)$.

It is easy to see that $G\mathscr{L}_c(E)$ is not open on $G\mathscr{L}(E)$. For the development of vector field theory on Banach manifolds we need a slightly larger set than $G\mathscr{L}_c(E)$, one which will have two components, which contains $G\mathscr{L}_c(E)$, and yet is open in $G\mathscr{L}(E)$. Let $\mathscr{S}(E) \subset G\mathscr{L}(E)$ be the maximal starred neighborhood of the identity in $G\mathscr{L}(E)$. Formally

$$\mathscr{S}(E) = \{T \in G\mathscr{L}(E) \mid (tT + (1-t)I) \in G\mathscr{L}(E)\ \forall t \in [0, 1]\}.$$

PROPOSITION 4. *$\mathscr{S}(E)$ is open in $G\mathscr{L}(E)$ and $T \in \mathscr{S}(E)$ iff $T^{-1} \in \mathscr{S}(E)$.*

Proof. Since the unit interval is compact we can find an $\epsilon > 0$ so that if for any $t \in [0, 1]$

$$\| A - (tT + (1-t)I)\| < \epsilon$$

then $A \in G\mathscr{L}(E)$. Pick $\tilde{T} \in G\mathscr{L}(E)$ with $\| \tilde{T} - T\| < \epsilon$. Then $\tilde{T} \in \mathscr{S}(E)$ which proves openness.

Suppose $(tT + (1-t)I) \in G\mathscr{L}(E)$ for all $t \in [0, 1]$. Then $T\{tI + (1-t)T^{-1}\} \in G\mathscr{L}(E)$. Setting $s = 1 - t$, we see that $(1-s)I + sT^{-1} \in G\mathscr{L}(E)$ which implies that $T^{-1} \in \mathscr{S}(E)$. ∎

PROPOSITION 5. *$\mathscr{S}(E)$ is contractible onto the identity.*

DEFINITION. Let $\mathscr{R}(E) = \{A \mid A = T + C,\ T \in \mathscr{S}(E) \text{ and } C \in C(E)\}$ and let $G\mathscr{R}(E) = \mathscr{R}(E) \cap G\mathscr{L}(E)$.

It follows at once that $\mathscr{L}_c(E) \subset \mathscr{R}(E)$ and $G\mathscr{L}_c(E) \subset G\mathscr{R}(E)$.

We shall call $\mathscr{R}(E)$ the Rothe set of E and $G\mathscr{R}(E)$ the invertible members of the Rothe set.

PROPOSITION 6. *$\mathscr{R}(E)$ is open in $\mathscr{L}(E)$ and hence $G\mathscr{R}(E)$ is open in $G\mathscr{L}(E)$.*

Proof. Let $A \in \mathscr{R}(E)$. We must show that there is an $\epsilon > 0$ so that whenever $\| S - A\| < \epsilon$, $S \in \mathscr{R}(E)$. Now $A = T + C$, $T \in \mathscr{S}(E)$, $C \in C(E)$. Since $\mathscr{S}(E)$ is open in $G\mathscr{L}(E)$ there is an $\epsilon > 0$ so that whenever $\| \tilde{T} - T\| < \epsilon$, $\tilde{T} \in \mathscr{S}(E)$. Let $S \in \mathscr{L}(E)$ with $\| S - A\| < \epsilon$. Set $\tilde{T} = (S - A) + T$. Then $S = (S - A) + A = \tilde{T} + C$ where $\| \tilde{T} - T\| < \epsilon$. Thus $\tilde{T} \in \mathscr{S}(E)$ and $S \in \mathscr{R}(E)$. ∎

We have already observed that $G\mathscr{R}(E) \supset G\mathscr{L}_c(E)$ and that $G\mathscr{L}_c(E)$ has two components, and we would like to show this also holds for $G\mathscr{R}(E)$. To do this we shall show that $G\mathscr{R}(E)$ is homotopically equivalent to $G\mathscr{L}_c(E)$.

PROPOSITION 7. *The quotient space $G\mathscr{R}(E)/C(E)$ is a contractible paracompact space.*

Proof. It is paracompact because it is an open subset of the quotient of Banach spaces which is a Banach space and hence it is metrizable and therefore paracompact. To see that it is contractible consider the canonical projection map $\pi\colon G\mathscr{R}(E) \to G\mathscr{R}(E)/C(E)$. Obviously π is surjective. Let $A \in G\mathscr{R}(E)$. Then

$$A = T + C, \qquad T \in \mathscr{S}(E), \quad C \in C(E).$$

The homotopy $(t, A) \to (tT + (1 - t)I)(I + T^{-1}C)$, $0 \leqslant t \leqslant 1$ is not well defined on $G\mathscr{R}(E)$ but the projection of this homotopy onto $G\mathscr{R}(E)/C(E)$ is well defined, continuous, and contracts $G\mathscr{R}(E)/C(E)$ to a point. ▌

PROPOSITION 8. *$\pi\colon \mathscr{L}(E) \to \mathscr{L}(E)/C(E)$ is a trivial vector bundle over the space $\mathscr{L}(E)/C(E)$.*

Proof. The result follows from a theorem of Bartle and Graves which can be found in [6, Proposition 7.2]. ▌

Consider the map $\pi\colon G\mathscr{R}(E) \to G\mathscr{R}(E)/C(E)$. Since $G\mathscr{R}(E)/C(E)$ is open in $\mathscr{L}(E)/C(E)$, by Proposition 9 the triple $(\pi, G\mathscr{R}(E), G\mathscr{R}(E)/C(E))$ admits local sections. $\pi^{-1}[T]$ can be identified with $G\mathscr{L}_c(E)$ which acts transitively on $\pi^{-1}[T]$. Thus the existence of local sections implies that $(\pi, G\mathscr{R}(E), G\mathscr{R}(E)/C(E))$ is in fact a principle bundle. Since $G\mathscr{R}(E)/C(E)$ is a contractible paracompact space this implies that this bundle is trivial. Thus we have

PROPOSITION 9. *The triple $(\pi, G\mathscr{R}(E), G\mathscr{R}(E)/C(E))$ is a trivial principle bundle with fiber $G\mathscr{L}_c(E)$ over the contractible base space $G\mathscr{R}(E)/C(E)$. Therefore*

$$G\mathscr{R}(E) \approx (G\mathscr{R}(E)/C(E)) \times G\mathscr{L}_c(E)$$

where $\approx$ denotes bundle equivalence. ▌

As immediate corollaries we have

COROLLARY 1. *$G\mathscr{R}(E)$ is homotopically equivalent to $G\mathscr{L}_c(E)$.*

COROLLARY 2. *$G\mathscr{R}(E)$ has two components which we shall denote by $G\mathscr{R}^+(E)$ and $G\mathscr{R}^-(E)$.*

2. A Review of the Brouwer Degree of a Fredholm Map

In discussing the degree of a Fredholm map we shall be following [3, 4]. This degree is an extension of the Leray–Schauder degree using some simple techniques of differential topology. An elementary discussion of the finite-dimensional theory can be found in Milnor's book [7].

DEFINITION. A "Φ-structure" or "Fredholm structure," M_φ modeled on E on a manifold M consists of a maximal atlas $\{(\mathscr{U}_i, \varphi_i)\}$ for M, $\varphi_i: \mathscr{U}_i \to E$ such that when defined, the derivative, $D(\varphi_i \circ \varphi_j^{-1})(x) \in G\mathscr{L}_c(E)$. The structure M_φ is said to be "orientable" if there is a subatlas of $\{(\mathscr{U}_i, \varphi_i)\}$ with $D(\varphi_i \circ \varphi_j^{-1})(x) \in G\mathscr{L}_c^+(E)$. A maximal subatlas will be called an "orientation" of M_c.

If M is connected, any orientable Φ-structure on M will admit precisely two orientations. M will be called "completely orientable" if it admits a Φ-structure and if every such structure that it admits is orientable. Each Φ-structure M_φ on M gives rise to a class $\nu(M_\varphi)$ in the first singular cohomology group $H^1(M, Z_2)$ of M with the property that M_φ is orientable iff $\nu(M_\varphi) = 0$. It can be shown that if M admits C^r partitions of unity, $r \geqslant 3$ and $G\mathscr{L}(E)$ is contractible then M is completely orientable iff $H^1(M; Z_2) = 0$, however, we shall not need this fact for our theory.

DEFINITION. Let M and N be smooth Banach manifolds. A C^1 map $f: M \to N$ is *Fredholm* if $Df_x: T_xM \to T_{f(x)}N$ is linear Fredholm for each $x \in M$. By the index of f we mean the index of Df_x. If M is connected this does not depend on x. If M is not connected, we shall assume the index to be the same for all components. A Fredholm map of index n will be called a Φ_n map.

DEFINITION. A map $f: M \to N$ is a "proper map" if the inverse image of any compact set in N is compact; it is σ *proper* if M can be written as the countable union of closed subsets $M = \bigcup_j M_j$ with $f \mid M_j$ a proper map. Recall from [12] that a Fredholm map is locally proper; thus a Fredholm map with separable domain is necessarily σ proper. We shall rely strongly on Smale's infinite-dimensional version of Sard's theorem. The following generalization is due to Quinn:

THEOREM 1 (Smale–Sard). *Suppose that $f: M \to N$ is a C^r, σ-proper Fredholm map between Banach manifolds where $r > \max(\operatorname{ind} f, 0)$. Then the set $\mathscr{C}$ of regular values of f is a Baire subset of N. If f is proper $\mathscr{C}$ is open and dense.* ∎

Suppose that M, N are manifolds with Φ-structures M_φ, N_φ modeled in E. A C^r map $f: M \to N$ will be called a *$\Phi(I)$-map* from M_φ to N_φ if

$$D(\psi_i \circ f \circ \varphi_j^{-1})(\varphi_j(x)) \in \mathscr{L}_c(E)$$

for all $x \in M$ and charts ψ_i, φ_j of M_φ, N_φ for which it is defined. A $\Phi(I)$ map is necessarily a Φ_0-map. If $r \geqslant 2$ and M_φ, N_φ have orientations and if f is proper we may apply Smale's theorem to obtain an oriented degree for f just as in the finite-dimensional case; namely, take a regular value y of f in N and let $\deg f$ be the algebraic number of points in $f^{-1}(y)$

$$\deg f = \sum_{x \in f^{-1}(y)} \operatorname{sgn} Df_x \tag{1}$$

where $\operatorname{sgn} Df_x = \pm 1$ depending on whether $D(\psi_i \circ f \circ \varphi_j^{-1})(\varphi_j(x))$ lies in $G\mathscr{L}_c^+(E)$ or $G\mathscr{L}_c^-(E)$ for oriented charts ψ_i, φ_j at $f(x)$, x. If $f^{-1}(y) = \phi$, $\deg f = 0$. This Brouwer degree gives an invariant of f under proper C^r homotopies through proper $\Phi(I)$-maps of M_φ into N_φ.

The following theorem shows that Φ_0 maps are often $\Phi(I)$ maps for some Φ-structure. Its proof can be found in [3, 4].

THEOREM 2 (Pull Back Theorem). (i) *A Φ_0-map f: $M \to E$ induces a unique Φ-structure $\{M, f\}_\varphi$ on M modeled on E, with respect to which f is a $\Phi(I)$-map into E with its trivial structure.*

(ii) *If M_φ is a Φ-structure on M modeled on E and if M admits C^r partitions of unity there is a C^r Φ_0-map f: $M \to E$ with $\{M, f\}_\varphi = M_\varphi$.* ▮

Theorem 2 has several extensions, in particular the range space E of f in (i) could be replaced by any manifold with a Φ-structure. However, for most applications to analysis it suffices to consider the case of maps whose range is a Banach space. In fact, from the analysis point of view the Brouwer degree is, in effect, also a local theory, a theory which gives the local degree of a Palais–Smale vector field (cf. Section 3).

It follows from Theorem 2 that one may obtain an oriented degree for a proper C^r Φ_0-map f: $M \to E$, $r \geqslant 2$ by considering it as a $\Phi(I)$-map on $\{M, f\}_\varphi$, provided the latter is orientable. In general one would have to use some theorem like the vanishing of $H^1(M; Z_2)$ to guarantee that $\{M, f\}_\varphi$ is orientable. This is a major point of departure between our vector field theory and the general Fredholm theorem. As we shall see further on, the maps f we will be considering will naturally induce an "orientation" on M.

This degree for Φ_0-maps f: $M \to E$ will not, however, be an invariant of proper homotopies through Φ_0-maps. For example, suppose E is an infinite-dimensional Hilbert space and $T \in G\mathscr{L}_c^-(E)$. Then $\deg T = -1$, although $G\mathscr{L}(E)$ is connected and so T is homotopic in $G\mathscr{L}(E)$ to the identity map, which as degree $+1$. However, as mentioned earlier, the degree is a proper homotopy invariant through $\Phi(I)$-maps and we shall use this fact.

A more direct extension of the Leray–Schauder degree occurs when one has a closed domain B of E (or M) whose boundary will be denoted by ∂B, together with a point y of E and a proper C^r Φ_0-map f: B, $\partial B \to E$, $E - \{y\}$, $r \geqslant 2$. In the same way as before, given orientability, we can define an integer $\deg(f, \partial B, y)$ by looking at the inverse image of a regular value of f lying on the component of y in $E - f(\partial B)$. If f is an identity plus compact field and B is bounded in E, this reduces to the Leray–Schauder degree.

This degree has been used by many people to obtain existence and uniqueness theorems in partial differential and integral equations. But for intrinsically nonlinear problems the Leray–Schauder theory is insufficient. We are interested in a globalization of this theory which can aid us in obtaining theorems on the number of solutions to intrinsically nonlinear partial differential equations (e.g., the Plateau problem) and integral equations.

We shall now introduce the notion of a Rothe map and the degree of such a map. A C^2 function $f: B \to F$, B a domain in E, is *Rothe* if for each $x \in B \subset E$ the Frechet derivative $Df(x) \in \mathscr{R}(E)$. Rothe maps are clearly Fredholm of index zero, and moreover we have

THEOREM 3. *A Rothe map f induces in a natural way a unique oriented Φ-structure B_φ on B^0, the interior of B, with respect to which f is a $\Phi(I)$-map.*

Proof. Let $x_0 \in B^0$. Then $Df(x_0) = T + C$ where $T \in \mathscr{S}(E)$ and C is compact linear. Consequently we can find a linear operator $S \in \mathscr{L}(E)$ with finite-dimensional range so that $Df(x_0) + S \in G\mathscr{R}^+(E)$. Consider the map $\varphi: \mathscr{U} \to E$ defined by $\varphi(x) = f(x) + S(x)$. Then $D\varphi(x_0)$ is an isomorphism and by the inverse function theorem φ restricted to some open ball $\mathscr{U}$ is a diffeomorphism onto $W \subset E$ with $D\varphi(x) \in G\mathscr{R}^+(E)$ and inverse $\varphi^{-1}: W \to E$. Thus φ is a chart for B^0 about x_0. Do this for each x_0 of B^0 and we obtain an atlas for B^0 consisting of such charts. Then

$$\begin{aligned} f \circ \varphi^{-1}(w) &= \varphi \circ \varphi^{-1}(w) + S \circ \varphi^{-1}(w) \\ &= w + S \circ \varphi^{-1}(w) \end{aligned}$$

which shows that f is a $\Phi(I)$-map with respect to the atlas B_φ. Next we show that B_φ is an orientation for B^0. Suppose now that at two points $x_0, x_1 \in B^0$ there are neighborhoods V_0, V_1 about x_0, x_1 and charts $\varphi_0, \varphi_1: V_0, V_1 \to E$ in B_φ chosen as above. Then

$$f \circ \varphi_0^{-1} = I + S_0 \circ \varphi_0^{-1} = I + k_0,$$

$$f \circ \varphi_1^{-1} = I + S_1 \circ \varphi_1^{-1} = I + k_1$$

and thus

$$(I + k_0)\varphi_0 = (I + k_1)\varphi_1$$

giving

$$(I + k_0)\, \varphi_0 \circ \varphi_1^{-1} = I + k_1,$$

thus $\psi = \varphi_0 \circ \varphi_1^{-1}$ is of the form identity plus finite dimensional and hence for all $w \in \varphi_1(V_0 \cap V_1)$, $D\psi(w) = D(\varphi_0 \circ \varphi_1^{-1})(w) \in G\mathscr{L}_c(E)$, which implies that this atlas is a C structure for B^0. The following lemma is all that is required to finish the proof of Theorem 3.

LEMMA. *For all* $w \in \varphi_1(V_0 \cap V_1)$ *we have that*

$$D\psi(w) \in G\mathscr{L}_c^+(E).$$

Proof. $D\psi(w) = D\varphi_0(x) \circ D\varphi_1^{-1}(w) = [Df(x) + S_0][Df(x) + S_1] = [T + C_0][T + C_1]^{-1} = [I + C_0T^{-1}][I + C_1T^{-1}]^{-1}$ where $T \in \mathscr{S}(E)$ and C_0, C_1 are compact linear with $(T + C_0) \in G\mathscr{R}^+(E)$ and $[T + C_1] \in G\mathscr{R}^+(E)$ which implies that $(I + C_0T^{-1}) \in G\mathscr{L}_c^+(E)$ and $(I + C_1T^{-1}) \in G\mathscr{L}_c^+(E)$. Since $G\mathscr{L}_c^+(E)$ is a group it follows that

$$D\psi(w) \in G\mathscr{L}_c^+(E). \quad \blacksquare$$

Let f: B, $\partial B \to E$, $E - \{y\}$ be a proper Rothe map. Since f is proper $f(\partial B)$ is closed and $y \in \mathscr{O}$, $\mathscr{O}$ the open component of y in $E - f(\partial B)$. Let B_φ be the orientable C structure on B^0 given by Theorem 3. $\mathscr{O}$ as an open submanifold of E inherits a natural orientation. Let $M = f^{-1}(\mathscr{O})$ and let M_φ be the oriented Φ-structure on M induced by B_φ. Then f: $M \to \mathscr{O}$ is a proper $\Phi(I)$-map from M_φ to $\mathscr{O}_\varphi$ and therefore has a Brouwer degree, $\deg f$ given by formula 1 of this section. We shall denote this degree by

$$\deg(f, B, y)$$

which is the degree of a Rothe map.

From what we have stated in this section and from the standard properties of degree it follows that $\deg(f, B, y)$ has the properties:

(i) If $z \in \mathscr{O}$ then $\deg(f, B, z) = \deg(f, B, y)$.

(ii) If $\deg(f, B, y) \neq 0$ then there exists an $x \in B^0$ with $f(x) = y$.

(iii) If $\gamma^{-1}(y) = \bigcup_{i=1}^n C_i$ where the C_i are disjoint compact sets and if $C_i \subset \mathscr{U}_i^0$, $\mathscr{U}_i \cap \mathscr{U}_j = \phi$, $i \neq j$, then

$$\deg(f, B, y) = \sum_i \deg(f, \mathscr{U}_i, y).$$

This is the additivity property of degree.

(iv) (Invariance under homotopy.) If f_t: B, $\partial B \to E$, $E - \{y\}$, $0 \leqslant t \leqslant 1$ is a homotopy of Rothe maps then

$$\deg(f_0, B, y) = \deg(f_1, B, y).$$

When y is a regular value a very nice interpretation can be given of this degree, an interpretation which is already implicit in what we have done.

If y is regular, let $x_1, \dots, x_k$ be the finite number of points in Φ^0 in $f^{-1}(y)$. Then

$$\deg(f, B, y) = \sum_{x_i \in f^{-1}(y)} \operatorname{sgn} Df(x_i)$$

where $\operatorname{sgn} Df(x_i) = +1$ if $Df(x_i) \in G\mathscr{R}^+(E)$ and $\operatorname{sgn} Df(x_i) = -1$ if $Df(x_i) \in G\mathscr{R}^-(E)$.

3. Fredholm and Palais–Smale Vector Fields

Let $X\colon M \to TM$ be a C^r, $r \geqslant 1$ vector field on a C^{r+1} Banach manifold, modeled on a space E. How can we define X to be Fredholm? If we fix $p \in M$ and look at $DX_{(p)}\colon T_pM \to T_{X(p)}(TM)$ we see that, since T_pM is linearly isomorphic to E and $T_{X(p)}(TM)$ is isomorphic to $E \times E$, $DX(p)$ could not possibly be linear Fredholm if $\dim E = \infty$. Thus this definition does not work.

Now let $\varphi\colon \mathscr{U} \to \mathscr{O} \subset E$ be a coordinate patch about $p \in M$. Consider the principle part $X^\varphi\colon \mathscr{O} \to E$ of X in this coordinate system. One can ask that this be nonlinear Fredholm. But it is not hard to see that this notion depends on the coordinate mapping φ and thus is not intrinsic. To see this suppose $X(p) \neq 0$, then by the flow box theorem [1] there is a coordinate patch $\varphi\colon \mathscr{U} \to \mathscr{O}$ about p such that for $x \in \mathscr{O}$, $X^\varphi(x) = v$ where v is some fixed vector. Thus $DX^\varphi(x) \equiv 0$ which is not Fredholm if $\dim E = \infty$.

In [14] the author defined a notion of Fredholm vector fields on manifolds with special structures and then unsuccessfully tried to apply this notion to intrinsically nonlinear problems in analysis.

In the paragraphs below we define the notions of Fredholm and Palais–Smale vector fields on M with respect to a connection K on M. It is our contention that the latter class of vector fields arise naturally in analysis and they have as their zeros long studied objects of geometry and analysis, as, for example, geodesics, harmonic maps between finite-dimensional Riemannian manifolds, and simply connected minimal surfaces. We begin our study with the notion of a connection.

For the exposition of connections on bunles we follow Eliasson [2] and the reader is referred to that source. Let $\pi\colon \xi \to M$ be a Banach bundle of class C^r with fiber E over a paracompact Banach manifold M (possibly with boundary). Let $\lambda\colon T\xi \to \xi$ denote the tangent bundle to the bundle ξ with canonical projection map λ.

A local trivialization of π is given by a bundle equivalence

$$\begin{array}{ccc} \pi^{-1}(\mathscr{U}) & \xrightarrow{\ \Phi\ } & \phi(\mathscr{U}) \times E \\ \downarrow & & \downarrow \\ \mathscr{U} & \xrightarrow{\ \phi\ } & \phi(\mathscr{U}) \end{array}$$

where ϕ is a chart for M. $\Phi_p = \Phi \mid \pi^{-1}(p)$ is a linear isomorphism of $E_p = \pi^{-1}(p)$ onto $\phi(p) \times E \approxeq E$.

DEFINITION. A connection map K for the bundle ξ is a bundle map K: $T\xi \to \xi$ over the map π. Thus we have the commutative diagram

$$\begin{array}{ccc} T\xi & \xrightarrow{K} & \xi \\ \lambda \downarrow & & \downarrow \pi \\ \xi & \xrightarrow{\pi} & M \end{array}$$

which implies that K maps the fiber $T_z\xi$ linearly into $E_{\pi\lambda(z)}$.

In addition we assume that for any local trivialization $(\mathscr{U}, \phi, \Phi)$ of π: $\xi \to M$ there is a C^l map Γ_ϕ: $\phi(\mathscr{U}) \to \mathscr{L}(E \times F; E)$, the E valued bilinear maps on $E \times F$, which gives the local representative of K, $K_\phi = \Phi \circ K \circ T\Phi^{-1}$ by the formula

$$K_\phi(x, \rho, y, \eta) = (x, \eta + \Gamma_\phi(x) \cdot (y, \rho)).$$

It follows that K is of class C^l. The map Γ_ϕ is the local connector for K with respect to the chart ϕ. When M is C^{r+1}, $\xi = TM$ the local connector Γ_ϕ corresponds to the classical Christoffel symbols and in finite dimension we have $[\Gamma_\phi(x)(y, z)]^i = \Sigma \Gamma^i_{jk}(x)\, y^j z^k$.

If $\xi = TM$ we say that K is *symmetric* if locally there is a chart ϕ so that $\Gamma_\phi(x)$: $E \times E \to E$ is a symmetric bilinear map. It is easy to check that this property is independent of the choice of coordinate chart ϕ. It follows easily that if M admits a partition of unity of class C^{r-1} then there exists a C^{r-1} connection map for π: $\xi \to M$.

A connection for a manifold M is defined to be a connection for its tangent bundle TM. The important thing about bundle connections from our point of view is that they occur naturally in the context of manifolds of maps. This was proved in great generality by Eliasson [2, Theorem 5.4] and by the author at a later date. It is this fact that motivated the present theory.

"Roughly speaking" if X and Y are two finite-dimensional manifolds of class C^s with Y admitting a connection K of class C^{s-2} and $\mathscr{G}$ is a manifold of maps functor (e.g., $\mathscr{L}_k{}^p$, $k > (\dim X)/p$, C^k, $C^{k,\alpha}$) then K induces a C^{s-4} connection $\mathscr{G}(K)$ on the manifold of maps $\mathscr{G}(X, Y)$. If K is symmetric then $\mathscr{G}(K)$ is also symmetric.

Again let M be a differentiable Banach manifold with tangent bundle τ: $TM \to M$. In addition let π: $\xi \to M$ be a Banach vector bundle with a connection map K on $T\xi$. If s: $M \to \xi$ is a smooth section of ξ, we define the covariant derivative of s to be the bundle map ∇s: $TM \to \xi$ defined by $\nabla s(p) = K \circ Ds(p)$. $Ds(p)$, the Frechet derivative of s at p, takes the fiber T_pM linearly into $T_{s(p)}\xi$ and K maps $T_{s(p)}\xi$ into E_p. Therefore $\nabla s(p)$ is a linear map from T_pM to E_p.

DEFINITION. A C^k, $k \leqslant r$, section s of a C^r bundle $\pi: \xi \to M$ is said to be *Fredholm* with respect to a C^l connection K on ξ if for each $p \in M$, $\nabla s(p)$: $T_pM \to E_p$ is a linear Fredholm map. The Fredholm index of s at p is the Fredholm index of $\nabla s(p)$. Thus

$$\operatorname{ind} \nabla s(p) = \dim \operatorname{Ker} \nabla s(p) - \dim \operatorname{Coker} \nabla s(p).$$

if M is connected this is independent of p and we call this common integer the index of the Fredholm section s. If M is not connected we shall require the index to agree on all components of M.

Remark 1. If p is a zero of a section s, $s(p) = 0$, then $\nabla s(p)$: $T_pM \to E_p$ does not depend on the connection K. To see this note that in a local coordinate chart ϕ with local representative s^ϕ of s

$$\nabla s^\phi(p)[h] = Ds^\phi(p)[h] + \Gamma_\phi(p)(h, s^\phi(p)).$$

Consequently if $s(p) = 0$

$$\nabla s(p)[h] = Ds(p)[h].$$

Remark 2. The notion of Fredholm section depends strongly on the choice of connection K. A section may be Fredholm with respect to a connection K_1 and yet not Fredholm with respect to another connection K_2. However, we can define an equivalence relation on the space of connections in such a way that the property of being Fredholm for a section s is independent of the choice of representative connection in any equivalence class. To be more precise let K_1, K_2: $T\xi \to \xi$ be two connections for the bundle ξ. Define the torsion $\mathscr{T}(K_1, K_2)$: $T\xi \to \xi$ by $\mathscr{T}(K_1, K_2) = K_1 - K_2$. Let $\pi_\#: \mathscr{L}_2(\xi) \to M$ be the bundle of maps over M with fiber over p, $\pi_\#^{-1}(p)$ consisting of bilinear maps on $E_p = \pi^{-1}(p)$ with values in E_p.

The torsion tensor $\mathscr{T}(K_1, K_2)$ can be interpreted as a section of $\mathscr{L}_2(\xi)$. In local coordinates ϕ: $\mathscr{U} \to E$, $x \in \phi(\mathscr{U})$, $(h, k) \in E \times E$

$$\mathscr{T}_\phi(K_1, K_2)(x)(h, k) = \Gamma_\phi^1(x)(h, k) - \Gamma_\phi^2(x)(h, k)$$

where Γ_ϕ^1 and Γ_ϕ^2 are the local connectors of K_1 and K_2. We say that $\mathscr{T}(K_1, K_2)$ is left completely continuous (or left compact) if for every $p \in M$ and every fixed $k \in E_p$ the linear correspondence

$$h \mapsto \mathscr{T}(K_1, K_2)(p)[h, k] \in \mathscr{L}(E_p)$$

is in addition a compact linear map.

K_1 is equivalent to $K_2 (K_1 \sim K_2)$ if $\mathscr{T}(K_1, K_2)$ is left compact. If $\xi = TM$ and K_1 and K_2 are symmetric then $\mathscr{T}(K_1, K_2)$ is a symmetric tensor and there is no distinction between left compact and right compact. The following proposition follows immediately from the invariance of Fredholm linear maps under additive translations by linear compact maps.

PROPOSITION. *If* $K_1 \sim K_2$ *then a section* $s\colon M \to \xi$ *is Fredholm with respect to* K_1 *iff it is Fredholm with respect to* K_2.

We conclude this section with the definitions of Palais–Smale and Rothe vector fields.

DEFINITION. A C^1 vector field X on a C^r manifold M with a C^l connection K is *Palais–Smale* or simply *PS* with respect to K if for each $p \in M$ the covariant derivative $\nabla X(p)$ of X at p is an element of $G\mathscr{L}_c(T_pM)$. Thus Palais–Smale vector fields are Fredholm vector fields of index zero.

DEFINITION. A *Rothe* vector field X is one for which $\nabla X(p) \in \mathscr{R}(T_pM)$ for every $p \in M$. These are also Fredholm of index zero.

DEFINITION. A C^1 vector field X on a Banach manifold M is *ZPS* if whenever $X(p) = 0$ the Frechet derivative $X_*(p) \in \mathscr{L}_c(T_pM)$.

Thus a *ZPS* vector field is one which is Palais–Smale on its zero set. It is clear that a *PS* vector field is a *ZPS* field. The author is unaware of a vector field arising naturally in an intrinsically nonlinear problem which is *ZPS* but not *PS* with respect to some connection.

A C^1 vector field is *Z-Fredholm* of index n (resp. *Z-Rothe*, or *ZR*) if whenever $X(p) = 0$, $X_*(p)$ is linear Fredholm of index n (resp. $X_*(p) \in \mathscr{R}(T_pM)$).

We are now prepared to introduce the Euler characteristic of *ZPS* vector fields.

4. THE EULER CHARACTERISTIC OF *ZPS* VECTOR FIELDS

Let M be a C^{r+1}, $r \geqslant 2$ paracompact manifold (perhaps with boundary) modeled on a space E which admits an equivalent C^2 norm. Thus M admits C^2 partitions of unity.

Let $X\colon M \to TM$ be a C^2 *ZPS* vector field with a *finite* number of isolated zeros $p_1 \cdots p_k$ in the interior of M.

Let $\varphi_i\colon \mathscr{U}_i \to \mathscr{O}$ with $\varphi_i(p_i) = 0 \in E$. Then X^{φ_i}, the principal part of X in the coordinate system φ_i, is a map $X^{\varphi_i}\colon \mathscr{O} \to E$ with derivative $DX^{\varphi_i}(0) \in G\mathscr{L}_c(E)$. For a sufficiently small closed ball B_i about 0, $X^{\varphi_i}\colon B_i \to E$ is a proper Rothe map (this follows from the fact that $\mathscr{R}(E)$ is open in $\mathscr{L}(E)$), and hence has a local degree $\deg(X^{\varphi_i}, B_i, 0)$. As in finite dimensions one checks, using the homotopy property of degree, that this does not depend on the coordinate patch φ_i. We define the Euler characteristic $\chi(X)$ by the formula

$$\chi(X) = \sum_i \deg(X^{\varphi_i}, B_i, 0) \tag{1}$$

and if X has no zeros we set $\chi(X) = 0$.

If X is a ZR field with finitely many zeros then formula (1) also defines the Euler characteristic for X.

DEFINITION. A vector field $X\colon M \to TM$ is *proper* if the set of zeros of X form a compact subset of M.

If the zeros of a proper vector field are isolated then there are only finitely many of them. If in addition the vector field is ZPS (or ZR) and the zeros are in the interior of M the Euler characteristic $\chi(X)$ is defined.

DEFINITION. A zero p of a C^1 vector field X is *nondegenerate* if $X_*(p) = DX(p)\colon T_pM\circlearrowleft$ is an isomorphism.

It is easy to see that nondegenerate zeros are isolated. In many applications to analysis one is given a family of PS vector fields $\{X_a\}_{a\in\mathcal{A}}$ depending on a parameter space $\mathcal{A}$ (e.g., $\mathcal{A}$ could be a space of boundary conditions, a space of connections, and so on). In [16] sufficient conditions are given when, for an open dense set (or Baire subset) of $\mathcal{A}$, X_a has nondegenerate zeros.

When a proper ZPS (ZR) vector field has only nondegenerate zeros one can give a particularly simple interpretation of the Euler characteristic. Let p be a nondegenerate zero of X. Then for $E = T_pM$, $DX(p) \in G\mathscr{R}(E)$. Define

$$\operatorname{sgn} DX(p) = \begin{cases} +1 & \text{if } DX(p) \in G\mathscr{R}^+(E), \\ -1 & \text{if } DX(p) \in G\mathscr{R}^-(E). \end{cases}$$

The Euler characteristic is then given by the formula

$$\chi(X) = \sum_{p\in \mathrm{Zeros}(X)} \operatorname{sgn} DX(p). \tag{2}$$

We would now like to show that given a proper ZPS field X with isolated zeros in the interior of M we can perturb it slightly to produce a proper ZR field Y which

(a) has nondegenerate zeros,

(b) equals X outside an arbitrarily small neighborhood of the zeros and

(c) $\chi(Y) = \chi(X)$.

To see this it clearly suffices to consider the case X has only one isolated zero p of X. Let $\varphi\colon \mathscr{U} \to E$ be a coordinate neighborhood of p containing no other zero of X with $\varphi(p) = 0$, $\varphi(\mathscr{U}) = B_r$ a ball of radius r centered at 0 in E and such that $X^\varphi: B \to E$ is a proper Rothe map (cf. Section 2). Thus $X^\varphi(0) = 0$ and $\inf_{x \in B_r - B_{r/2}} \| X^\varphi(x) \| = \eta > 0$.

Let $\gamma: E \to R$ be a C^2 function, $0 \leqslant \gamma \leqslant 1$, which equals one on $B_{r/2}$ and is zero outside B_r. By the Smale–Sard theorem [1] of Section 2 given ϵ, $\eta/2 > \epsilon > 0$, we can find a regular value $y \in E$ for X^φ with $\| y \| < \epsilon$. Define $Y^\varphi: B_r \to E$ by $Y^\varphi(x) = X^\varphi(x) - \gamma(x) y$ and the C^2 vector field Y on M by

$$Y(q) = \begin{cases} X(q), & q \in M - \mathscr{U}, \\ Y^\varphi(x), & x = \varphi(q), \quad q \in \mathscr{U}. \end{cases} \tag{3}$$

If ϵ is sufficiently small $DY^\varphi(x) \in \mathscr{R}(E)$ which implies that Y is Z-Rothe. Note also that $Y(q) \neq 0$ if $q \notin \mathscr{U}$ and $Y(q) = 0$ in $\mathscr{U}$ iff for $x = \varphi(q)$, $X^\varphi(x) = y$. Since y is a regular value and $Y^\varphi: B_r \to E$ is still nonlinear Fredholm of index zero (Y^φ is a finite-dimensional perturbation of X^φ) this implies that all zeros of Y^φ in B_r (and hence all zeros of Y in M) are nondegenerate.

Now $(t, x) \to tX^\varphi(x) + (1 - t) Y^\varphi(x)$ is a homotopy between X^φ and Y^φ which has no zeros on ∂B_r. Thus by the homotopy property of degree $\deg(X^\varphi, B_r, 0) = \deg(Y^\varphi, B_r, 0)$ which immediately implies that $\chi(X) = \chi(Y)$. Let us state this result formally as

THEOREM 1. *Let $X: M \to TM$ be a C^2 ZPS vector field with a finite number of isolated zeros in the interior of M. Then given an arbitrarily small neighborhood $\mathscr{U}$ of the zeros of X there exists a proper C^2 Rothe field Y with the properties*

(a) *Y has a finite number of isolated nondegenerate zeros,*

(b) *Y equals X outside $\mathscr{U}$,*

(c) $\chi(X) = \chi(Y)$. ▮

Let $\mathscr{V}_{ZPS}(M)$, $\mathscr{R}(M)$, $\mathscr{R}^*(M)$, $\mathscr{F}_0(M)$, $\mathscr{F}_0^*(M)$ be the spaces of C^2 proper ZPS fields, proper ZR fields, proper ZR fields with nondegenerate zeros, proper Z-Fredholm fields of index 0, and those with nondegenerate zeros, respectively. Using the techniques of Theorem 1 we can prove a somewhat stronger version of the result.

THEOREM 2. *If $X \in \mathscr{V}_{ZPS}(M)$ with zeros in the interior of M then given any neighborhood $\mathscr{U}$ of the zeros of X there exists a $Y \in \mathscr{R}^*(M)$ with $Y = X$ on $M - \mathscr{U}$. If $\mathscr{U}$ is sufficiently small $\chi(Y)$ is independent of all choices made. We can take this to be the definition of $\chi(X)$.* ▮

Along the same lines but less interesting from the point of view of degree theory one can also show that if

THEOREM 3. *If $X \in \mathscr{F}_0(M)$, with zeros in the interior of M, then given any neighborhood $\mathscr{U}$ of the zeros of X there exists a $Y \in \mathscr{F}_0^*(M)$ with $Y = X$ on $M - \mathscr{U}$.* ▮

Let us now move on to an investigation of how the Euler characteristic behaves under homotopy.

DEFINITION. Let $\pi_0: M \to [0, 1]$ be a C^3 smooth fiber bundle over the unit interval with $\pi_0^{-1}(t) = M_t$ a C^2 Banach manifold.

Two C^2 proper *ZPS* vector fields $X_0: M_0 \to TM_0$ and $X_1: M_1 \to TM_1$ are equivalent ($X_0 \sim X_1$) if there is a C^2 proper vector field $X: M \to TM$ such that

(i) $X_t = X \mid M_t: M_t \to TM_t$.

Thus X_t is a "vertical" family of vector fields (e.g., see [16]). Geometrically this means that X is a section of the vertical subbundle $v(TM)$ of TM, where the fiber of $v(TM)$ over $m \in TM$ is the kernel of the Frechet derivative $D\pi_0(m)$ of π_0 at the point m. This can be naturally identified with $T_m M_{\pi_0(m)}$.

(ii) For each t, X_t is *ZPS*.

(iii) The zeros of X are in the interior of M.

The following is the globalization of the property of invariance of degree under homotopy.

THEOREM 4 (Invariance of the Euler characteristic under homotopy.) *Suppose $X_0 \sim X_1$. Then $\chi(X_0)$ and $\chi(X_1)$ are both defined and equal.*

Proof. That both $\chi(X_0)$ and $\chi(X_1)$ are defined follows from Theorem 2. By the same theorem we can assume that both X_0 and X_1 have nondegenerate zeros. Since X is proper we can apply the Smale–Sard theorem and simple transversality arguments to find a proper vertical *ZR* vector field Y on M and on $\epsilon > 0$ with

(a) $Y \mid M_t = X \mid M_t, \quad 0 \leqslant t \leqslant \epsilon, \quad 1 - \epsilon \leqslant t \leqslant 1,$

(b) $Y(M) \pitchfork \mathscr{Z}(v(TM))$;

that is, Y, viewed as a section of the vertical subbundle $v(TM)$ of TM, is transverse to the zero section of this bundle; and

(c) the zeros of Y are in the interior of M.

Identifying $\mathscr{Z}(v(TM))$ with M we see that these conditions imply that $Y(M) \cap M$ is a compact one-dimensional submanifold $\mathscr{P}$ of M^0. We are interested only in those components of $\mathscr{P}$ with boundary. Let $\{p_1, ..., p_k\}$ be the zeros of X_0 and $\{q_1, ..., q_m\}$ be the zeros of X_1. Each p_j and q_i is a boundary point of some component of $\mathscr{P}$.

Since the zeros of X_0 and X_1 are nondegenerate it follows that

$$\chi(X_0) = \sum \operatorname{sgn} DX_0(p_j)$$

and

$$\chi(X_1) = \sum_i \operatorname{sgn} DX_1(q_i).$$

Applying the techniques developed in [3] based on geometric ideas due to Pontrjagin we see that

(1) if p_{j_1} and p_{j_2} are boundary points of the same component $\mathscr{P}_{j_1 j_2}$ of $\mathscr{P}$ then $\operatorname{sgn} DX_0(p_{j_1}) = -\operatorname{sgn} DX_0(p_{j_2})$;

(2) if q_{i_1} and q_{i_2} are boundary points of the same component $\mathscr{P}_{i_1 i_2}$ of $\mathscr{P}$ then $\operatorname{sgn} DX_1(q_{i_1}) = -\operatorname{sgn} DX_1(q_{i_2})$;

(3) if p_j and q_i are boundary points of the same component of $\mathscr{P}$ then $\operatorname{sgn} DX_0(p_j) = \operatorname{sgn} DX(q_i)$.

Putting (1), (2), and (3) together we can immediately conclude that

$$\chi(X_0) = \chi(X_1)$$

which concludes the proof of Theorem 4. ▮

5. The Euler–Hopf Theorem

In this section we state conditions under which the Euler characteristic of a proper ZPS vector field on a manifold M is equal to the Euler characteristic of M. This of course would imply that the cohomology of M, $H^i(M, Q)$, with rational coefficients, vanishes for sufficiently large i and hence the Euler characteristic would be defined. We shall have the blanket assumption that our manifold M is a C^3 complete Finsler manifold (possibly with boundary) modeled on a real Banach space E with an equivalent C^2 norm.

Definition. A set $S \subset M$ is bounded if for all $p, q \in S$ $\sup_{p,q\in S} \rho(p, q) < \infty$ where ρ is the distance function induced by the Finsler on M (for relevant definitions see [11]).

Definition. A vector field X: $M \to TM$ satisfies condition (CV) if whenever $\{p_i\}$ is a bounded sequence in M and $\| X(p_i)\| \to 0$ then there is a subsequence $\{p_{ij}\}$ which converges.

We have an immediate consequence of this definition, namely,

Proposition. *Let X be a vector field on M satisfying condition (CV), and $S \subset M$ any bounded set. Then the set of zeros of X in $\bar{S}$ is compact. Hence if the zeros of X in any closed set A are isolated then A contains at most finitely many of these zeros.* ▮

We wish now to define what it means for a vector field to behave like a gradient with respect to some scalar function. Let $t \to \sigma_p(t)$ denote the trajectory of X with initial condition p. Further let $f\colon M \to R$ be a C^2 function.

DEFINITION. We say that a C^1 vector field X is *gradient like* for f if:

(G0) X satisfies (CV).

(G1) $X_p(f) = Df(p)(X_p) \geqslant 0$ and equals zero only if p is simultaneously a critical point of f and a zero of X.

This condition implies that f increases along the trajectories of X.

(G2) Let $p \in M$. The trajectory σ_p of X through p has a maximal domain $(\alpha, \beta) \subset \mathbb{R}$. Then as $t \to \beta$ either

(i) $f(\sigma_p(t)) \to +\infty$ or

(ii) $\| X(\sigma_p(t))\| \to 0$ and $\sigma_p[0, \beta)$ is bounded.

Similarly as $t \to \alpha$ either

(iii) $\gamma(\sigma_p(t)) \to -\infty$ or

(iv) $\| X(\sigma_p(t))\| \to 0$ and $\sigma_p(\alpha, 0]$ is bounded.

(G3) (Regularity condition.) Let $K(a, b)$ denote the zeros of X in $f^{-1}[a, b]$, $-\infty < a - b < \infty$. Then $K(a, b)$ is bounded. From condition (G0) and Proposition 1 it follows that $K(a, b)$ is also compact.

PROPOSITION 1. *In axiom* (G2), *if, as* $t \to \beta$, $\| X(\sigma_p(t))\| \to 0$ *and* $\sigma_p[0, \beta)$ *is bounded then* $\beta = +\infty$ *and* $\sigma_p(t)$ *has a critical point as a limit point as* $t \to \infty$.

Similarly if, as $t \to \alpha$, $\| X(\sigma_p(t))\| \to 0$ and $\sigma_p(\alpha, 0]$ is bounded, then $\alpha = -\infty$ and $\sigma_p(t)$ has a critical point as a limit point as $t \to -\infty$.

Proof. Condition (G0) implies that if, as $t \to \beta$, $\| X(\sigma_p(t))\| \to 0$ with $\sigma_p[0, \beta)$ bounded, then $\sigma_p(t)$ has a limit point in M as $t \to \beta$. By Theorem 3.9 of [11] this is impossible unless $\beta = \infty$. Since $\| X(\sigma_p(t))\| \to 0$ as $t \to \beta$, this limit point must be a zero of X and hence a critical point of f.

The proof for $t \to \alpha$ is exactly the same. ∎

DEFINITION. Let $f\colon M \to R$ be C^2 with X a C^1 gradient-like vector field for f. A critical point p of f is *B-nondegenerate* with respect to X if:

(a) $DX(p)\colon T_pM \to T_pM$, the Frechet derivative of X at p is symmetric with respect to the Hessian $H_p(f) = D^2f(p)\colon T_pM \times T_pM \to R$.

(b) $DX(p)$ is an isomorphism with spectrum off the imaginary axis.

(c) $H_p(f)(DX(p)u, u) > 0$ if $u \neq 0$.

DEFINITION. Let $p \in M$ be the critical point of a C^2 function $f: M \to R$. By the index of f at p we mean the dimension of the maximal subspace on which the Hessian $H_p(f)$ is a negative definite bilinear form (for details see [9, 15]).

The following theorem connects the local vector field index and the index of the zero of a ZPS vector field.

THEOREM 1. *Let X be a C^1 ZPS vector field which is gradient like with respect to $f: M \to R$. Let p be a B-nondegenerate critical point of f in M^0 with respect to X and set β = index of f at p. Then $\beta < \infty$ and*

$$\operatorname{sgn} DX(p) = (-1)^\beta.$$

Proof. Let $A = DX(p): T_pM \circlearrowleft$. We shall show that $T_pM = T_pM_+ \oplus T_pM_-$ with $H_p(f)$ positive on T_pM_+ and negative on T_pM_- with $\dim T_pM_- = \beta < \infty$. Moreover T_pM_+ is $H_p(f)$ orthogonal to T_pM_-, each of these subspaces is invariant under A, and $A \mid T_pM_+$ has real spectrum to the right of the imaginary axis and $A \mid T_pM_-$ has real compact discrete spectrum to the left of the imaginary axis.

Since X is ZPS, A is invertible and of the form $I + C$. Therefore -1 is not an eigenvalue of C (recall all points in the spectrum of a completely continuous operator except 0 are eigenvalues). Let $\lambda_1, ..., \lambda_m$ be the real eigenvalues of C less than -1 with multiplicities $\beta_1, ..., \beta_m$. Recall (e.g., see Taylor [13]) that there is a least integer $n_i < \infty$ with $F_i = \text{Kernel}(\lambda_i I + C)^{n_i + r} = \text{Kernel}(\lambda_i I + C)^{n_i}$ for all positive integers r. By definition $\beta_i = \dim F_i < \infty$. Set $\beta_* = \Sigma\beta_i$, $T_pM_- = \oplus F_i$. Then $\beta_* = \dim T_pM_- < \infty$. Clearly T_pM_- is invariant under A and since T_pM_- is also finite dimensional it has an $H_p(f)$ orthogonal complement $T_pM_- = T_pM_+$ so that $T_pM = T_pM_- \oplus T_pM_+$.

Since A is symmetric with respect to $H_p(f)$ it follows that T_pM_+ is also invariant under A.

LEMMA 1. *Under the above circumstances the spectrum of A is real.*

Proof. Let $E = T_pM$. By passing to the complexification of E and A we may view E as a Banach space over the complex field $\mathbb{C}$. Suppose $\lambda \in \mathbb{C}$ belongs to the spectrum of A. Then $(\lambda - 1)$ belongs to the spectrum of C, but all members of the spectrum of C (except 0) are eigenvalues. Thus if $\lambda \neq 1$ belongs to the spectrum of A then $\lambda - 1$ is an eigenvalue of C and hence λ is an eigenvalue of A. Since $H_p(f)(Au, u) > 0$ for $u \neq 0$ (this still holds on the complexification space) it follows as in the Hilbert space case that λ must be real. Thus the entire spectrum of A is real.

LEMMA 2. *If λ belongs to the spectrum of $A \mid T_pM^+$ then $\lambda > 0$.*

Proof. If $\lambda < 0$ ($\lambda \neq 0$) then $\lambda - 1$ is an eigenvalue of C which is less than -1, but this is impossible since C has no eigenvalue less than -1 on T_pM_+.

From Lemmas 1 and 2 we can conclude that the spectrum of $A \mid T_pM_- = A_-$ is real negative and the spectrum of $A \mid T_pM_+ = A_+$ is real positive.

LEMMA 3. *$H_p(f)$ is positive on T_pM_+ and negative definite on T_pM_-.*

Proof. Using the operational calculus we can define square roots S_+ and S_- of A_+ and $-A_-$ which can be expressed as power series in A_+ and A_-, respectively. Thus $S_+^2 = A_+$, $S_-^2 = -A_-$, and S_+ and S_- are symmetric with respect to $H_p(f)$ and take T_pM_+ and T_pM_- isomorphically onto themselves. Let $u \in T_pM_+$, then $u = S_+v$ and

$$\begin{aligned} H_p(f)(u, u) &= H_p(f)(S_+v, S_+v) \\ &= H_p(f)(S_+^2v, v) = H_p(f)(Av, v) > 0 \end{aligned}$$

whence $H_p(f)$ is positive on T_pM_+. In a similar way we see that $H_p(f)$ is negative on T_pM_- but $\dim T_pM_- = \beta_* < \infty$ and so $H_p(f)$ is negative definite on T_pM_-.

From Lemma 3 it follows easily that $\beta_* = \beta$. The proof of Theorem 1 will be completed once we show

LEMMA 4. $\operatorname{sgn} \nabla X(p) = (-1)^{\beta_*}$.

Proof. Consider the homotopy A_τ, $0 \leqslant \tau \leqslant 1$ defined by $A_\tau \mid T_pM_+ = (I + \tau C) \mid T_pM_+$ and $A_\tau \mid T_pM_- = A$. Then $\tau \to A_\tau$ is a path in $G\mathscr{L}_c(T_pM)$ connecting A with a linear map which is the identity on T_pM_+. Thus, whether or not $A \in G\mathscr{L}_c^+(T_pM)$, depends only on $A \mid T_pM_-$. From the Jordan canonical form it follows that $\operatorname{sgn} \det(I + C \mid T_pM_-) = (-1)^{\beta_*}$. Hence if $(-1)^{\beta_*} < 0$, $(I + C) \in G\mathscr{L}_c^-(T_pM)$ and if $(-1)^{\beta_*} > 0$, $(I + C) \in G\mathscr{L}_c^+(T_pM)$. This proves the lemma and concludes the proof of the theorem. ▌

THEOREM 2. *Let $f: M \to R$ and $X: M \to TM$ be a proper C^1 ZPS gradient-like vector field with f having B-nondegenerate critical points in M^0 with respect to X. Then*

$$\chi(X) = \sum_i (-1)^{\beta_i} C_{\beta_i} \tag{1}$$

where C_{β_i} is the number of critical points of index β_i. If M has no boundary

$$\chi(X) = \chi(M). \tag{2}$$

If M has a boundary and X points outward along ∂M equality (2) *still holds. Thus M has a well-defined Euler characteristic in either case.*

Proof. Equality (1) follows directly from Theorem 1. Equality (2) is a consequence of the Morse theory as developed in [15]. ∎

In conclusion we apply Theorem 4 of Section 4 to obtain

THEOREM 3 (Euler–Hopf). *Let M be a complete C^3 Banach–Finsler manifold modeled on a Banach space E with an equivalent C^2 norm. Suppose $\partial M = \phi$ and Y is a proper C^2 ZPS vector field which is equivalent to a proper C^2 ZPS vector field X which is gradient like for $f: M \to R$ where f has B-nondegenerate critical points with respect to X. Then*

$$\chi(y) = \chi(X) = \chi(M). \tag{3}$$

If $\partial M \neq \phi$ and Y points outward along ∂M and is equivalent to a gradient-like X as above we again have equality (3). ∎

Remark. In finite dimensions if M is compact with $\partial M = \phi$ Eq. (3) holds for all vector fields. But even in finite dimensions if the manifold M is not compact there is no guarantee that the Euler characteristic of a proper vector field will be the Euler characteristic of the underlying space without some further assumptions. For example, let $M = \mathbb{R}^2$, and $X = \operatorname{grad} f$, where $f(x, y) = \frac{1}{2}(x^2 + y^2)$. Then $X(x, y) = (x, y)$ and X is clearly a proper vector field which is gradient like for f and has one nondegenerate zero (which is a B-nondegenerate critical point for f). Thus $\chi(X) = \chi(\mathbb{R}^2) = 1$.

Consider the constant vector field Y on $\mathbb{R}^2$ given by $Y(x, y) = (1, 0)$. Y has no zeros and so $\chi(Y) = 0 \neq \chi(M)$. One readily checks that Y is not equivalent to X.

6. Applications to Geodesics on Finite-Dimensional Riemannian Manifolds

Let V be a C^{r+k} Riemannian manifold. By the Nash imbedding theorem we may assume that V is a Riemannian submanifold of $\mathbb{R}^N$ for some N.

Denote by $\mathscr{L}_k{}^p$ ($k \geqslant 1$) the Sobelev space of maps functor (these are maps whose kth derivatives are in $\mathscr{L}_p$). Let P, Q be two points in V and $\Lambda_k{}^p(P, Q)$ be the space of $\mathscr{L}_k{}^p$ paths which join P and Q. Specifically

$$\Lambda_k{}^p(P, Q) = \{\sigma: I \to V \mid \sigma \in \mathscr{L}_k{}^p(I, \mathbb{R}^N), \sigma(0) = P, \sigma(0) = Q\}.$$

Then $M = \Lambda_k{}^p(P, Q)$ is a C^r Banach submanifold of $\mathscr{L}_k{}^p(I, \mathbb{R}^N)$ (see [19]), the $\mathscr{L}_k{}^p$ maps of the unit interval to $\mathbb{R}^N$. If $p = 2$ then $\Lambda_k{}^2(P, Q)$ is a Hilbert submanifold of $\mathscr{L}_k{}^2(I, \mathbb{R}^N)$. The tangent space to $\Lambda_k{}^p(P, Q)$ at a path σ, say

$\Lambda_k{}^p(P, Q)_\sigma$, can be identified with the $\mathscr{L}_k{}^p$ paths $h: I \to \mathbb{R}^N$ with $h(t) \in T_{\sigma(t)}V$ and $h(0) = 0 = h(1)$. Define the energy functional $E: \Lambda_k{}^p(P, Q) \to R$ by

$$E(\sigma) = \tfrac{1}{2} \int_0^1 \| \sigma'(t)\|^2 \, dt.$$

It is well known that E is C^r smooth [9], and that the critical points of E are the geodesics joining P and Q parameterized by arc length.

In [9] the functional E was studied in the case $p = 2$, $k = 1$. Define the vector field $\lambda: M \to TM$ by the elliptic differential equation

$$\frac{D^2\lambda(\sigma)}{\partial t^2} = \frac{D\sigma'}{\partial t} \tag{1}$$

with $\lambda(\sigma)(0) = 0$, $\lambda(\sigma)(1) = 0$ and where $D/\partial t$ denotes covariant differentiation with respect to the unique symmetric connection induced by the Riemannian structure on V. If $\sigma \in \Lambda_k{}^p(P, Q) = M$ then (1) uniquely defines a vector field $\lambda(\sigma)$ over σ and hence an element of $\Lambda_k{}^p(P, Q)_\sigma = T_\sigma M$.

In [15] it was shown in the case $p = 2$, $k = 2$ and in [9] for $p = 2$, $k = 1$ that λ is a C^{r-1} vector field whose zeros are the critical points of E and that λ is gradient like for E. Thus, in particular, $\lambda_\sigma(E) = DE(\sigma)(\lambda(\sigma)) \geqslant 0$ as the reader may easily verify. Using the same techniques as in [15] or by appealing to standard regularity results in linear elliptic systems one has in general that for any p, $k \geqslant 1$ Eq. (1) defines a C^{r-1} vector field on $\Lambda_k{}^p(P, Q)$ whose zeros are geodesics parameterized by arc length and which is gradient like for $E: \Lambda_k{}^p(P, Q) \to R$.

The Riemannian connection on V induces a connection K on $\Lambda_k{}^p(P, Q)$ (see remarks in Section 3). If $\beta: M \to TM$ is a vector field the covariant derivative $\nabla\beta(\sigma): T_\sigma M \to T_\sigma M$ with respect to K at a point σ is characterized as follows.

Let $h \in T_\sigma M$ be arbitrary and let $s \to \sigma_s$ be a smooth path in M with $\sigma_0 = \sigma$ and $(d/ds)\sigma_s |_{s=0} = h$; i.e., $(d/ds)\sigma_s(t) |_{s=0} = h(t)$ for all $t \in [0, 1]$. Then

$$\nabla\beta(\sigma)[h] = \frac{D}{\partial s} \beta(\sigma_s) \Big|_{s=0}. \tag{2}$$

The aim of this section is to show that if $\lambda: M \to TM$ is the gradient-like vector field defined by (1) then λ is PS with respect to the induced connection K on M.

We shall show that λ is PS on $\Lambda_k{}^p(P, Q)$ in the case $k \geqslant 2$. With the introduction of the more exotic $\mathscr{L}^p_{-k}$ sections of vector bundles our proof can be pushed through for $k = 1$ but we shall not do this. Using the methods of [15] a proof can be given in the case $k = 1$ which avoids the use of the $\mathscr{L}^p_{-k}$ spaces.

Consider a parameterized surface $\varphi: R^2 \to V$ and let β be any vector field along φ. One can apply two covariant operators $D/\partial x$ and $D/\partial y$ to β, to obtain the relation on mixed partials

$$\frac{D}{\partial x}\frac{D}{\partial y}\beta - \frac{D}{\partial y}\frac{D}{\partial x}\beta = R\left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}\right)\beta \tag{3}$$

where R: $TV \times TV \times TV \to TV$ is the Riemann curvature tensor.

PROPOSITION 1. *The covariant derivative* $\nabla\lambda(\sigma)$: $T_\sigma M\circlearrowleft$ *of the vector field* λ *at a path* $\sigma \in M$ *satisfies the second-order equation*

$$\begin{aligned}\frac{D^2}{\partial t^2}\{\nabla\lambda(\sigma)[h] - h\} &= R\left(\frac{d\sigma}{dt}, h\right)\left(\frac{d\sigma}{dt} - \frac{D\lambda}{\partial t}\right) - \frac{D}{\partial t}\left\{R\left(\frac{d\sigma}{dt}, h\right)\lambda\right\} \\ &= R\left(\frac{d\sigma}{dt}, h\right)\left(\frac{d\sigma}{dt}\right) - (\tilde{\nabla}_{d\sigma/dt}R)\left(\frac{d\sigma}{dt}, h, \lambda\right) \\ &\quad - R\left(\frac{D\sigma'}{\partial t}, h\right)\lambda - R\left(\frac{d\sigma}{dt}, \frac{Dh}{\partial t}\right)\lambda - 2R\left(\frac{d\sigma}{dt}, h\right)\frac{D\lambda}{\partial t}\end{aligned} \tag{4}$$

where $\tilde{\nabla}_{d\sigma/dt}$ *denotes the covariant derivative of the Riemann curvature tensor in the direction* $d\sigma/dt$.

Thus, in particular, if $D\sigma'/\partial t = 0$ [*thus* σ *is a zero of* λ]

$$\frac{D^2}{\partial t^2}\{\nabla\lambda(\sigma)[h] - h\} = \frac{D^2}{\partial t^2}\{\lambda_*(\sigma)[h] - h\} = R\left(\frac{d\sigma}{dt}, h\right)\frac{d\sigma}{dt}$$

where $\lambda_*(\sigma)$: $T_\sigma M\circlearrowleft$ *denotes the Frechet derivative of* λ *at* σ.

Proof. By (2)

$$\nabla\lambda(\sigma)[h] = \frac{D}{\partial s}\lambda(\sigma_s)\Big|_{s=0}.$$

But

$$\frac{D^2}{\partial t^2}\lambda(\sigma) = \frac{D\sigma'}{\partial t}.$$

Using (3) we get

$$\begin{aligned}\frac{D}{\partial s}\frac{D^2\lambda}{\partial t^2}(\sigma_s) &= \frac{D}{\partial s}\frac{D}{\partial t}\frac{d}{\partial t}\sigma_s = \frac{D}{\partial t}\frac{D}{\partial s}\frac{d}{\partial t}\sigma_s + R\left(\frac{d\sigma_s}{dt}, \frac{d\sigma_s}{ds}\right)\frac{d\sigma_s}{dt} \\ &= \frac{D^2}{\partial t^2}\frac{d}{d\sigma}\sigma_s + R\left(\frac{d\sigma_s}{dt}, \frac{d\sigma_s}{ds}\right)\frac{d\sigma_s}{dt}.\end{aligned}$$

Setting $s = 0$ we get

$$\frac{D}{\partial s}\left[\frac{D^2}{\partial t^2}\lambda(\sigma_s)\right]_{s=0} = \frac{D^2h}{\partial t^2} + R\left(\frac{d\sigma}{dt}, h\right)\frac{d\sigma}{dt}. \tag{5}$$

Moreover

$$\frac{D}{\partial s}\frac{D^2}{\partial t^2}\lambda(\sigma_s) = \frac{D}{\partial t}\frac{D}{\partial s}\frac{D}{\partial t}\lambda(\sigma_s) + R\left(\frac{d\sigma_s}{dt}, \frac{d\sigma_s}{ds}\right)\frac{D}{\partial t}\lambda(\sigma_s)$$

$$= \frac{D^2}{\partial t^2}\frac{D}{\partial s}\lambda(\sigma_s) + \frac{D}{\partial t}\left\{R\left(\frac{d\sigma_s}{dt}, \frac{d\sigma_s}{ds}\right)\lambda(\sigma_s)\right\}$$

$$+ R\left(\frac{d\sigma_s}{dt}, \frac{d\sigma_s}{ds}\right)\frac{D}{\partial t}\lambda(\sigma_s).$$

Letting $s = 0$ we obtain

$$\frac{D}{\partial s}\frac{D^2}{\partial t^2}\lambda(\sigma_s)\Big|_{s=0} = \frac{D^2}{\partial t^2}\{\nabla\lambda(\sigma)[h]\} + \frac{D}{\partial t}\left\{R\left(\frac{d\sigma}{dt}, h\right)\lambda\right\} + R\left(\frac{d\sigma}{dt}, h\right)\frac{D\lambda}{\partial t}. \quad (6)$$

But applying a little elementary tensor calculus we get

$$\frac{D}{\partial t}\left\{R\left(\frac{d\sigma}{dt}, h\right)\lambda\right\} = (\tilde{\nabla}_{d\sigma/dt}R)\left(\frac{d\sigma}{dt}, h, \lambda\right) + R\left(\frac{D\sigma'}{\partial t}, h\right)\lambda$$

$$+ R\left(\frac{d\sigma}{dt}, \frac{Dh}{\partial t}\right)\lambda + R\left(\frac{d\sigma}{dt}, h\right)\frac{D\lambda}{\partial t}.$$

Making this substitution and equating (5) and (6) we obtain (4) concluding the proof of the proposition. ▮

Let $\hat{\Lambda}_k^p(P, Q)_\sigma = \{h: I \to TV \mid h(t) \in T_{\sigma(t)}V, \sigma \in \Lambda_k^p(P, Q), h \in L_k^p\}$. Thus

$$\hat{\Lambda}_k^p(P, Q)_\sigma \supset \Lambda_k^p(P, Q)_\sigma.$$

PROPOSITION 2. *Suppose* $\sigma \in \Lambda_k^p(P, Q)$, $k \geqslant 2$, *and* $h \in \Lambda_k^p(P, Q)_\sigma$. *The linear map* $T: \Lambda_k^p(P, Q)_\sigma \to \hat{\Lambda}_{k-2}^p(P, Q)_\sigma$ *defined by*

$$h \mapsto R\left(\frac{d\sigma}{dt}, h\right)\frac{d\sigma}{dt} - (\tilde{\nabla}_{d\sigma/dt}R)\left(\frac{d\sigma}{dt}, h, \lambda\right)$$

$$- R\left(\frac{D\sigma'}{\partial t}, h\right)\lambda - R\left(\frac{d\sigma}{dt}, \frac{Dh}{\partial t}\right)\lambda - 2R\left(\frac{d\sigma}{dt}, h\right)\frac{D\lambda}{\partial t}$$

is a compact map.

Proof. We must show that if $\{h_n\}$ is a bounded sequence in $\Lambda_k^p(P, Q)_\sigma$ then Th_n has a convergent subsequence in $\hat{\Lambda}_{k-2}^p(P, Q)_\sigma$. The basic reason why this is true is that T is a linear map into Λ_{k-2}^p but only the first derivatives of h appear in the expression for T.

Rather then prove this for all the terms above we shall consider only one. The proof for the rest is more or less the same. First let us look at the case

$k = 2$. Consider $h_n \to R(D\sigma'/\partial t, h_n)\lambda$ for fixed σ. Since $\{h_n\}$ is bounded in $\Lambda_2{}^p(P, Q)$ there is a subsequence (say $\{h_n\}$, again) which converges to $h \in \Lambda_2{}^p(P, Q)$ in supremum norm. Thus $\sup_t \| h_n(t) - h(t)\| \to 0$ as $n \to \infty$. Here $\| h(t)\| = \langle h(t), h(t)\rangle^{1/2}$ where $\langle\ ,\ \rangle_p$ is the Riemannian inner product on T_pV. Now if $\sigma \in \Lambda_2{}^p(P, Q)$, $R(D\sigma'/\partial t, h_n)\lambda \in \hat{\Lambda}^p(P, Q)_\sigma$. All that remains to prove is that $R(D\sigma'/\partial t, h_n)\lambda \to R(D\sigma'/\partial t, h)\lambda$ in the norm of $\hat{\Lambda}^p(P, Q)_\sigma$. Let the C_i's below denote constants:

$$\left\| R\left(\frac{D\sigma'}{\partial t}, h_n(t)\right)\lambda(t) - R\left(\frac{D\sigma'}{\partial t}, h(t)\right)\lambda(t)\right\|$$

$$= \left\| R\left(\frac{D\sigma'}{\partial t}, h_n(t) - h(t)\right)\lambda(t)\right\| \leqslant C_0 \left\| \frac{D\sigma'}{\partial t}\right\| \cdot \| h_n(t) - h(t)\| \cdot \| \lambda(t)\|. \quad (7)$$

But $\sup_t \| \lambda(t)\| \leqslant C_1 \| \lambda \|_{\mathscr{L}_2{}^p}$, the $\mathscr{L}_2{}^p$ norm of λ which in turn is bounded by some constant C_2 which depends on σ (recall $D^2\lambda/\partial t^2 = D\sigma'/\partial t$). Therefore (7) is bounded by

$$C_0C_2 \left\| \frac{D\sigma'}{\partial t}\right\| \| h_n(t) - h(t)\|.$$

Setting $C_0C_2 = C_3$, raising (7) to the pth power and integrating we get

$$\int_0^1 \left\| R\left(\frac{D\sigma'}{\partial t}, h_n(t) - h(t)\right)\lambda(t)\right\|^p dt \leqslant C_3(\sup_t \| h_n(t) - h(t)\|)^p \int_0^1 \left\| \frac{D\sigma'}{\partial t}\right\|^p dt.$$

But the right-hand side clearly goes to zero as $n \to \infty$, and the proposition is established for this term when $k = 2$.

For $k > 2$ we proceed inductively. So, for example, if $k = 3$ we must show not only that $R(D\sigma'/\partial t, h_n)\lambda \to R(D\sigma'/\partial t, h)\lambda$ in L_p but the covariant derivative $D/\partial t$ of the right term tends in $\mathscr{L}_p$ to the covariant derivative to the left term in $\mathscr{L}_p$. However, the same sort of calculation we have already done also establishes this fact. ▮

THEOREM 1. *Let $M = \Lambda_k{}^p(P, Q)$. Then the vector field λ: $M \to TM$ defined by Eq.* (1) *is PS with respect to the connection K on M induced by the Riemannian connection on V.*

Proof. We must show that

$$\nabla\lambda(\sigma)[h] = h + S(h)$$

where S: $T_\sigma M \to T_\sigma M$ is compact. By Proposition 2 we have shown that

$$\frac{D^2}{\partial t^2}\lambda(\sigma)[h] = \frac{D^2 h}{\partial t^2} + Th$$

where

$$T: \Lambda_k^p(P, Q)_\sigma \to \hat{\Lambda}_{k-2}^p(P, Q)_\sigma$$

is compact. Note that $L = D^2/\partial t^2$ is an isomorphism

$$\Lambda_k^p(P, Q)_\sigma \xrightarrow{L} \hat{\Lambda}_{k-2}^p(P, Q).$$

Set $S = L^{-1}T$, and since T is compact so is S which completes the proof of the theorem. ∎

In [16] it is shown that for fixed P and "for almost all Q" the vector field λ has nondegenerate zeros and therefore $E: \Lambda_k^p(P, Q) \to R$ has B-nondegenerate critical points with respect to λ. In [18] it is shown that the regular values of E form a residual Baire subset of R. The following corollary follows immediately from the previous theorem, Theorem 2, Section 5, and the fact that λ is gradient like for E which is proven in [15].

COROLLARY. *Suppose $E: \Lambda_k^p(P, Q) \to R$ has only B-nondegenerate critical points with respect to which the vector field λ is defined by the elliptic equation* (1). *Let $a \in R$ be a regular value for E and $M^a = \{\sigma \in \Lambda_k^p(P, Q) \mid E(\sigma) \leqslant a\}$. Then λ points outward along $\partial M^a = E^{-1}(a)$ and $\chi(\lambda) = \chi(M^a)$.*

One can eliminate the need to assume that E has only B-nondegenerate critical points. We conclude this paper with the following.

THEOREM 2. *Let $E: \Lambda_k^p(P, Q) \to R$ be the energy functional, λ the gradient-like field for E defined by Eq.* (1), *and $a \in R$ a regular value for E. Suppose further that the underlying Riemannian manifold V is C^{r+k}, $r \geqslant 3$. Then $\chi(\lambda) = \chi(M^a)$.*

Proof. First observe that λ points outward along ∂M^a and therefore has no zeros on ∂M^a. Since λ is gradient like for E the zeros of λ are a compact subset of the interior of M^a which by Theorem 2 of Section 4 implies that $\chi(\lambda)$ is defined. Let $W \subset V$ be a path connected neighborhood of Q so that for all $Q^* \in W$ "a" is a regular value for $E: \Lambda_k^p(P, Q^*) \to R$. We can do this since the vector field λ^* on $\Lambda_k^p(P, Q^*)$ defined by (1) satisfies (CV) and λ^* varies continuously with Q^*. Let $Q_1 \in W$ be sufficiently close to Q so that $E: \Lambda_k^p(P, Q_1)$ has only nondegenerate zeros. Let $t \to Q_t$, $0 \leqslant t \leqslant 1$ be a smooth imbedding of the unit interval into W joining $Q_0 = Q$ and Q_1.

Set $M_t = \Lambda_k^p(P, Q_t)$ and λ_t the PS vector field on M_t defined by (1). Let $\xi = \bigcup_t M_t$ with $\pi: \xi \to I$ the obvious fiber bundle over $I = [0, 1]$. Then $t \to \lambda_t$ is an equivalence between $\lambda = \lambda_0$ and λ_1 and therefore $\chi(\lambda) = \chi(\lambda_1)$.

Consider E as a map on ξ. Then $\lambda: \xi \to T\xi$ defined by $\lambda \mid M_t = \lambda_t$ is nonzero on $E^{-1}(a)$. From this it follows that $E^{-1}(a)$ is transverse to the fibers of ξ. To see this note that $E^{-1}(a)$ is a codimension one submanifold of ξ with $\lambda(\sigma)$ "orthogonal" to $E^{-1}(a)$ at any point $\sigma \in E^{-1}(a)$. M_t is also a codimension one submanifold of ξ. Consequently if $E^{-1}(a)$ were tangent to M_t for any t this would imply that $\lambda_t \notin TM_t$ which is impossible. Therefore $\xi^a = \{\sigma \in \xi \mid E(\sigma) \leqslant a\}$ is a smooth fiber bundle over I with boundary $E^{-1}(a)$ which implies that $M^a = M_0{}^a$ is diffeomorphic to $M_1{}^a$. Thus $\chi(M^a) = \chi(M_1{}^a) = \chi(\lambda_1) = \chi(\lambda)$ which concludes the proof of the theorem and this paper. ∎

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