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Stability results for uniquely determined sets from two directions in discrete tomography

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ABSTRACT

In this paper we prove several new stability results for the reconstruction of binary images from two projections. We consider an original image that is uniquely determined by its projections and possible reconstructions from slightly different projections. We show that for a given difference in the projections, the reconstruction can only be disjoint from the original image if the size of the image is not too large. We also prove an upper bound for the size of the image given the error in the projections and the size of the intersection between the image and the reconstruction.

1. Introduction

Discrete tomography is concerned with problems such as reconstructing binary images on a lattice from given projections in lattice directions [6]. Each point of a binary image has a value equal to zero or one. The line sum of a line through the image is the sum of the values of the points on this line. The projection of the image in a certain lattice direction consists of all the line sums of the lines through the image in this direction.

Several problems related to the reconstruction of binary images from two or more projections have been described in the literature [6,7]. Already in 1957, Ryser gave an algorithm to reconstruct binary images from their horizontal and vertical projections and characterised the set of projections that correspond to a unique binary image [11]. For any set of directions, it is possible to construct images that are not uniquely determined by their projections in those directions [6, Theorem 4.3.1]. The problem of deciding whether an image is uniquely determined by its projections and the problem of reconstructing it are NP-hard for any set of more than two directions [4].

Aside from various interesting theoretical problems, discrete tomography also has applications in a wide range of fields. The most important are electron microscopy [8] and medical imaging [5,13], but there are also applications in nuclear science [9,10] and various other fields [12,15].

An interesting problem in discrete tomography is the stability of reconstructions. Even if an image is uniquely determined by its projections, a very small error in the projections may lead to a completely different reconstruction [1,3]. Alpers et al. [1,2] showed that in the case of two directions a total error of at most 2 in the projections can only cause a small difference in the reconstruction. They also proved a lower bound on the error if the reconstruction is disjoint from the original image.

In this paper we improve this bound, and we resolve the open problem of stability with a projection error greater than 2.

2. Notation and statement of the problems

Let F_1 and F_2 be two finite subsets of $\mathbb{Z}^2$ with characteristic functions χ_1 and χ_2. (That is, $\chi_h(x,y) = 1$ if and only if $(x,y) \in F_h$, $h \in \{1,2\}$.) For $i \in \mathbb{Z}$, we define *row* i as the set $\{(x,y) \in \mathbb{Z}^2 : x = i\}$. We call i the index of the row. For $j \in \mathbb{Z}$,

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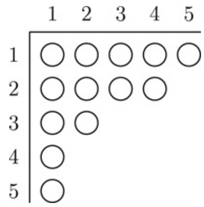

Fig. 1. A uniquely determined set with the assumed row and column ordering.

we define *column* j as the set $\{(x, y) \in \mathbb{Z}^2 : y = j\}$. We call j the index of the column. Following matrix notation, we use row numbers that increase when going downwards and column numbers that increase when going to the right.

The *row sum* $r_i^{(h)}$ is the number of elements of F_h in row i, that is $r_i^{(h)} = \sum_{j\in\mathbb{Z}} \chi_h(i, j)$. The *column sum* $c_j^{(h)}$ of F_h is the number of elements of F_h in column j, that is $c_j^{(h)} = \sum_{i\in\mathbb{Z}} \chi_h(i, j)$. We refer to both row and column sums as the *line sums* of F_h.

Throughout this paper, we assume that F_1 is uniquely determined by its row and column sums. Such sets were studied by, among others, Ryser [11] and Wang [14]. Let a be the number of rows and b the number of columns that contain elements of F_1. We renumber the rows and columns such that we have

$$r_1^{(1)} \geq r_2^{(1)} \geq \cdots \geq r_a^{(1)} > 0,$$
$$c_1^{(1)} \geq c_2^{(1)} \geq \cdots \geq c_b^{(1)} > 0,$$

and such that all elements of F_2 are contained in rows and columns with positive indices. By [14, Theorem 2.3] we have the following property of F_1 (see Fig. 1):

- in row i the elements of F_1 are precisely the points $(i, 1), (i, 2), \ldots, (i, r_i^{(1)})$,
- in column j the elements of F_1 are precisely the points $(1, j), (2, j), \ldots, (c_j^{(1)}, j)$.

We will refer to this property as the *triangular shape* of F_1.

Everywhere except in Section 6 we assume that $|F_1| = |F_2|$. Note that we do not assume F_2 to be uniquely determined.

As F_1 and F_2 are different and F_1 is uniquely determined by its line sums, F_2 cannot have exactly the same line sums as F_1. Define the *difference* or *error in the line sums* as

$$\sum_{j\geq 1} |c_j^{(1)} - c_j^{(2)}| + \sum_{i\geq 1} |r_i^{(1)} - r_i^{(2)}|.$$

As in general $|t - s| \equiv t + s \bmod 2$, the above expression is congruent to

$$\sum_{j\geq 1} \left(c_j^{(1)} + c_j^{(2)}\right) + \sum_{i\geq 1} \left(r_i^{(1)} + r_i^{(2)}\right) \equiv 2|F_1| + 2|F_2| \equiv 0 \bmod 2,$$

hence the error in the line sums is always even. We will denote it by 2α, where α is a positive integer.

For notational convenience, we will often write p for $|F_1 \cap F_2|$.

We consider two problems concerning stability.

Problem 1. *Suppose $F_1 \cap F_2 = \emptyset$. How large can $|F_1|$ be in terms of α?*

Alpers et al. [2, Theorem 29] proved that $|F_1| \leq \alpha^2$. They also showed that there is no constant c such that $|F_1| \leq c\alpha$ for all F_1 and F_2. In Section 4 of this paper we will prove the new bound $|F_1| \leq \alpha(1 + \log\alpha)$ and show that this bound is asymptotically sharp.

Problem 2. *How small can $|F_1 \cap F_2|$ be in terms of $|F_1|$ and α, or, equivalently, how large can $|F_1|$ be in terms of $|F_1 \cap F_2|$ and α?*

Alpers ([1, Theorem 5.1.18]) showed in the case $\alpha = 1$ that

$$|F_1 \cap F_2| \geq |F_1| + \frac{1}{2} - \sqrt{2|F_1| + \frac{1}{4}}.$$

This bound is sharp: if $|F_1| = \frac{1}{2}n(n + 1)$ for some positive integer n, then there exists an example for which equality holds. A similar result is stated in [2, Theorem 19].

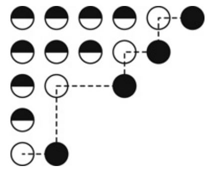

Fig. 2. A staircase. The set F_1 consists of the white and the black-and-white points, while F_2 consists of the black and the black-and-white points. The staircase is indicated by the dashed line segments.

While [1,2] only deal with the case $\alpha = 1$, we will give stability results for general α. In Section 5 we will give two different upper bounds for $|F_1|$. The bounds have different asymptotic behaviour. Writing p for $|F_1 \cap F_2|$, the second bound reduces to

$$|F_1| \leq p + 1 + \sqrt{2p+1}$$

in case $\alpha = 1$, which is equivalent to

$$p \geq |F_1| - \sqrt{2|F_1|}.$$

Hence the second new bound can be viewed as a generalisation of Alpers' bound. The first new bound is different and better in the case that α is very large.

In Section 6 we will generalise the results to the case $|F_1| \neq |F_2|$.

3. Staircases

Alpers introduced the notion of a staircase to characterise $F_1 \triangle F_2$ in the case $\alpha = 1$. We will use a slightly different definition and then show that for general α the symmetric difference $F_1 \triangle F_2$ consists of α staircases.

Definition 3. A set of points $(p_1, p_2, \ldots, p_n)$ in $\mathbb{Z}^2$ is called a *staircase* if the following two conditions are satisfied:

- for each i with $1 \leq i \leq n-1$ one of the points p_i and p_{i+1} is an element of $F_1 \setminus F_2$ and the other is an element of $F_2 \setminus F_1$;
- either for all i the points p_{2i} and p_{2i+1} are in the same column and the points p_{2i+1} and p_{2i+2} are in the same row, or for all i the points p_{2i} and p_{2i+1} are in the same row and the points p_{2i+1} and p_{2i+2} are in the same column.

This definition is different from [1,2] in the following way. Firstly, the number of points does not need to be even. Secondly, the points p_1 and p_n can both be either in $F_1 \setminus F_2$ or in $F_2 \setminus F_1$. So this definition is slightly more general than the one used in [1,2] for the case $\alpha = 1$.

Consider a point $p_i \in F_1 \setminus F_2$ of a staircase $(p_1, p_2, \ldots, p_n)$. Assume that p_{i-1} is in the same column as p_i and that p_{i+1} is in the same row as p_i. Owing to the triangular shape of F_1, the row index of p_{i-1} must be larger than the row index of p_i, and the column index of p_{i+1} must be larger than the column index of p_i. Therefore, the staircase looks like a real-world staircase (see Fig. 2). From now on, we assume for all staircases that p_1 is the point with the largest row index and the smallest column index, while p_n is the point with the smallest row index and the largest column index. We say that the staircase *begins* with p_1 and *ends* with p_n.

Lemma 4. *Let F_1 and F_2 be finite subsets of $\mathbb{Z}^2$ such that*

- *F_1 is uniquely determined by its row and column sums, and*
- *$|F_1| = |F_2|$.*

Let α be defined as in Section 2. Then the set $F_1 \triangle F_2$ is the disjoint union of α staircases.

Proof. We will construct the staircases one by one and delete them from $F_1 \triangle F_2$. For a subset A of $F_1 \triangle F_2$, define

$$\rho_i(A) = |\{j \in \mathbb{Z} : (i,j) \in A \cap F_1\}| - |\{j \in \mathbb{Z} : (i,j) \in A \cap F_2\}|, \quad i \in \mathbb{Z},$$
$$\sigma_j(A) = |\{i \in \mathbb{Z} : (i,j) \in A \cap F_1\}| - |\{i \in \mathbb{Z} : (i,j) \in A \cap F_2\}|, \quad j \in \mathbb{Z},$$
$$\tau(A) = \sum_i |\rho_i(A)| + \sum_j |\sigma_j(A)|.$$

We have $2\alpha = \tau(F_1 \triangle F_2)$.

Assume that the rows and columns are ordered as in Section 2. Owing to the triangular shape of F_1, for any point $(i,j) \in F_1 \setminus F_2$ and any point $(k,l) \in F_2 \setminus F_1$ we then have $k > i$ or $l > j$.

Suppose we have deleted some staircases and are now left with a non-empty subset A of $F_1 \triangle F_2$. Let $(p_1, p_2, \ldots, p_n)$ be a staircase of maximal length that is contained in A. Let (x_1, y_1) and (x_n, y_n) be the coordinates of the points p_1 and p_n respectively. Each of those two points can be either in $A \cap F_1$ or in $A \cap F_2$, so there are four different cases. (If $n = 1$, so p_1 and p_n are the same point, then there are only two cases.) We consider two cases; the other two are similar.

First suppose $p_1 \in A \cap F_1$ and $p_n \in A \cap F_2$. If (x, y_1) is a point of $A \cap F_2$ in the same column as p_1, then $x > x_1$, so we can extend the staircase by adding this point. That contradicts the maximal length of the staircase. So there are no points of $A \cap F_2$ in column y_1. Therefore $\sigma_{y_1}(A) > 0$.

Similarly, since $p_n \in A \cap F_2$, there are no points of $A \cap F_1$ in the same column as p_n. Therefore $\sigma_{y_n}(A) < 0$.

All rows and all columns that contain points of the staircase, except columns y_1 and y_n, contain exactly two points of the staircase, one in $A \cap F_1$ and one in $A \cap F_2$. Let $A' = A \setminus \{p_1, p_2, \ldots, p_n\}$. Then $\rho_i(A') = \rho_i(A)$ for all i, and $\sigma_j(A') = \sigma_j(A)$ for all $j \neq y_1, y_n$. Furthermore, $\sigma_{y_1}(A') = \sigma_{y_1}(A) - 1$ and $\sigma_{y_n}(A') = \sigma_{y_n}(A) + 1$. Since $\sigma_{y_1}(A) > 0$ and $\sigma_{y_n}(A) < 0$, this gives $\tau(A') = \tau(A) - 2$.

Now consider the case $p_1 \in A \cap F_1$ and $p_n \in A \cap F_1$. As above, we have $\sigma_{y_1}(A) > 0$. Suppose (x_n, y) is a point of $A \cap F_2$ in the same row as p_n. Then $y > y_n$, so we can extend the staircase by adding this point. That contradicts the maximal length of the staircase. So there are no points of $A \cap F_2$ in row x_n. Therefore $\rho_{x_n}(A) > 0$.

All rows and all columns that contain points of the staircase, except column y_1 and row x_n, contain exactly two points of the staircase, one in $A \cap F_1$ and one in $A \cap F_2$. Let $A' = A \setminus \{p_1, p_2, \ldots, p_n\}$. Then $\rho_i(A') = \rho_i(A)$ for all $i \neq x_n$, and $\sigma_j(A') = \sigma_j(A)$ for all $j \neq y_1$. Furthermore, $\sigma_{y_1}(A') = \sigma_{y_1}(A) - 1$ and $\rho_{x_n}(A') = \rho_{x_n}(A) - 1$. Since $\sigma_{y_1}(A) > 0$ and $\rho_{x_n}(A) > 0$, this gives $\tau(A') = \tau(A) - 2$.

We can continue deleting staircases in this way until all points of $F_1 \triangle F_2$ have been deleted. Since $\tau(A) \geq 0$ for all subsets $A \subset F_1 \triangle F_2$, this must happen after deleting exactly α staircases. □

Remark 5. Some remarks about the above lemma and its proof.

(i) The α staircases from the previous lemma have 2α endpoints in total (where we count the same point twice in case of a staircase consisting of one point). Each endpoint contributes a difference of 1 to the line sums in one row or column. Since all these differences must add up to 2α, they cannot cancel each other.

(ii) A staircase consisting of more than one point can be split into two or more staircases. So it may be possible to write $F_1 \triangle F_2$ as the disjoint union of more than α staircases. However, in that case some of the contributions of the endpoints of staircases to the difference in the line sums cancel each other. On the other hand, it is impossible to decompose $F_1 \triangle F_2$ into fewer than α staircases.

(iii) The endpoints of a staircase can be in $F_1 \setminus F_2$ or $F_2 \setminus F_1$. For a staircase T of which the two endpoints are in different sets, we have $|T \cap F_1| = |T \cap F_2|$. For a staircase T of which the two endpoints are in the same set, we have $|T \cap F_1| = 1 + |T \cap F_2|$ or $|T \cap F_2| = 1 + |T \cap F_1|$. Since $|F_1 \setminus F_2| = |F_2 \setminus F_1|$, the number of staircases with two endpoints in $F_1 \setminus F_2$ must be equal to the number of staircases with two endpoints in $F_2 \setminus F_1$. This implies that of the 2α endpoints, exactly α are in the set $F_1 \setminus F_2$ and α are in the set $F_2 \setminus F_1$.

Consider a decomposition of $F_1 \triangle F_2$ as in the proof of Lemma 4. We will now show that for our purposes we may assume that all these staircases begin with a point $p_1 \in F_1 \setminus F_2$ and end with a point $p_n \in F_2 \setminus F_1$.

Suppose there is a staircase beginning with a point $(x, y) \in F_2 \setminus F_1$. Then there also exists a staircase ending with a point $(x', y') \in F_1 \setminus F_2$: otherwise more than half of the 2α endpoints would be in $F_2 \setminus F_1$, which is a contradiction to Remark 5(iii). Owing to the Remark 5(i) we must have $r_x^{(1)} < r_x^{(2)}$ and $r_{x'}^{(1)} > r_{x'}^{(2)}$.

Let y'' be such that $(x', y'') \notin F_1 \cup F_2$. Delete the point (x, y) from F_2 and add the point (x', y'') to F_2. Then $r_x^{(2)}$ decreases by 1 and $r_{x'}^{(2)}$ increases by 1, so the difference in the row sums decreases by 2. Meanwhile, the difference in the column sums increases by at most 2. So α does not increase, while F_1, $|F_2|$ and $|F_1 \triangle F_2|$ do not change. So the new situation is just as good or better than the old one. The staircase that began with (x, y) in the old situation now begins with a point of $F_1 \setminus F_2$. The point that we added becomes the new endpoint of the staircase that previously ended with (x', y').

Therefore, in our investigations we may assume that all staircases begin with a point of $F_1 \setminus F_2$ and end with a point of $F_2 \setminus F_1$. This is an important assumption that we will use in the proofs throughout the paper. An immediate consequence of the assumption is that $r_i^{(1)} = r_i^{(2)}$ for all i. The only difference between corresponding line sums occurs in the columns.

4. A new bound for the disjoint case

Using the concept of staircases, we can prove a new bound for Problem 1.

Theorem 6. *Let F_1 and F_2 be finite subsets of $\mathbb{Z}^2$ such that*

- *F_1 is uniquely determined by its row and column sums,*
- *$|F_1| = |F_2|$, and*
- *$F_1 \cap F_2 = \emptyset$.*

Let α be defined as in Section 2. Then

$$|F_1| \leq \sum_{i=1}^{\alpha} \left\lfloor \frac{\alpha}{i} \right\rfloor .$$

Proof. Assume that the rows and columns are ordered as in Section 2. Let a be the number of rows and b the number of columns that contain elements of F_1. Let $(k, l) \in F_1$. Then all the points in the rectangle $\{(i, j) : 1 \leq i \leq k, 1 \leq j \leq l\}$ are elements of F_1. Since F_1 and F_2 are disjoint, none of the points in this rectangle is an element of F_2, and all the points belong to $F_1 \triangle F_2$. So all of the kl points must belong to different staircases, which implies $\alpha \geq kl$. For all i with $1 \leq i \leq a$ we have

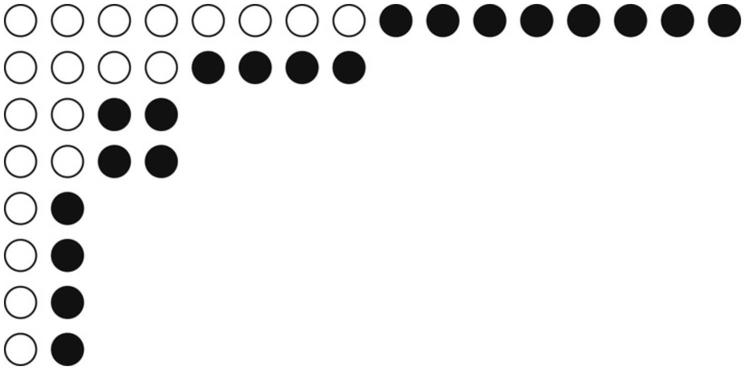

Fig. 3. The construction from Example 8 with $m = 3$.

$(i, r_i^{(1)}) \in F_1$, hence $r_i^{(1)} \le \frac{\alpha}{i}$. Since $r_i^{(1)}$ must be an integer, we have

$$|F_1| = \sum_{i=1}^{a} r_i^{(1)} \le \sum_{i=1}^{a} \left\lfloor \frac{\alpha}{i} \right\rfloor .$$

Since $(a, 1) \in F_1$, we have $a \le \alpha$, so

$$|F_1| \le \sum_{i=1}^{\alpha} \left\lfloor \frac{\alpha}{i} \right\rfloor . \quad \square$$

Corollary 7. *Let F_1, F_2 and α be defined as in Theorem 6. Then*

$$|F_1| \le \alpha(1 + \log \alpha).$$

Proof. We have

$$|F_1| \le \sum_{i=1}^{\alpha} \left\lfloor \frac{\alpha}{i} \right\rfloor \le \alpha \sum_{i=1}^{\alpha} \frac{1}{i} \le \alpha \left(1 + \int_1^{\alpha} \frac{1}{x} \, \mathrm{d}x \right) = \alpha \left(1 + \log \alpha \right) . \quad \square$$

The following example shows that the upper bound cannot even be improved by a factor $\frac{1}{2 \log 2} \approx 0.72$.

Example 8 (*Taken from [1]*). Let $m \ge 1$ be an integer. We construct sets F_1 and F_2 as follows (see also Fig. 3).

- Row 1:
  - $(1, j) \in F_1$ for $1 \le j \le 2^m$,
  - $(1, j) \in F_2$ for $2^m + 1 \le j \le 2^{m+1}$.
- Let $0 \le l \le m - 1$. Row i, where $2^l + 1 \le i \le 2^{l+1}$:
  - $(i, j) \in F_1$ for $1 \le j \le 2^{m-l-1}$,
  - $(i, j) \in F_2$ for $2^{m-l-1} + 1 \le j \le 2^{m-l}$.

The construction is almost completely symmetrical: if $(i, j) \in F_1$, then $(j, i) \in F_1$; and if $(i, j) \in F_2$ with $i > 1$, then $(j, i) \in F_2$. Since it is clear from the construction that each row contains exactly as many points of F_1 as points of F_2, we conclude that each column j with $2 \le j \le 2^m$ contains exactly as many points of F_1 as points of F_2 as well. The only difference in the line sums occurs in the first column (which has 2^m points of F_1 and none of F_2) and in columns $2^m + 1$ up to 2^{m+1} (each of which contains one point of F_2 and none of F_1). So we have

$$\alpha = 2^m.$$

Furthermore,

$$|F_1| = 2^m + \sum_{l=0}^{m-1} 2^l 2^{m-l-1} = 2^m + m 2^{m-1}.$$

Hence for this family of examples it holds that

$$|F_1| = \alpha + \frac{1}{2} \alpha \log_2 \alpha,$$

which is very close to the bound we proved in Corollary 7.

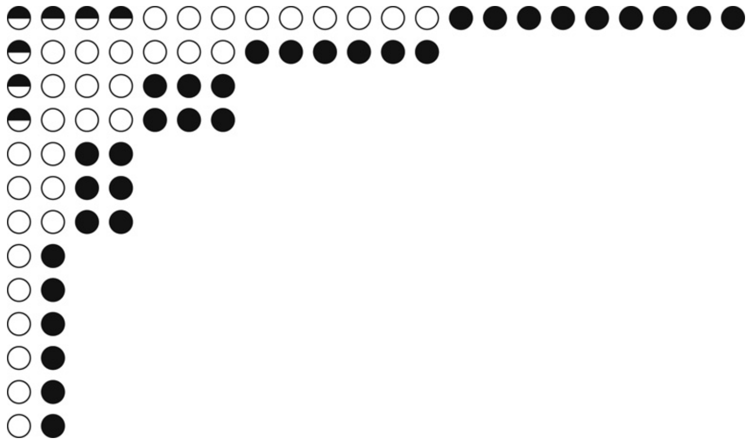

Fig. 4. The construction from Example 11 with $k = 3$ and $m = 4$.

5. Two bounds for general α

In case F_1 and F_2 are not disjoint, we can use an approach very similar to Section 4 in order to derive a bound for Problem 2.

Theorem 9. *Let F_1 and F_2 be finite subsets of $\mathbb{Z}^2$ such that*

- *F_1 is uniquely determined by its row and column sums, and*
- *$|F_1| = |F_2|$.*

Let α be defined as in Section 2, and let $p = |F_1 \cap F_2|$. Then

$$|F_1| \le \sum_{i=1}^{\alpha+p} \left\lfloor \frac{\alpha + p}{i} \right\rfloor.$$

Proof. Assume that the rows and columns are ordered as in Section 2. Let $(k, l) \in F_1$. Then all the points in the rectangle $\{(i, j) : 1 \le i \le k, 1 \le j \le l\}$ are elements of F_1. At most p of the points in this rectangle are elements of F_2, so at least $kl - p$ points belong to $F_1 \triangle F_2$. None of the points in the rectangle is an element of $F_2 \setminus F_1$, so all of the $kl - p$ points of $F_1 \triangle F_2$ in the rectangle must belong to different staircases, which implies $\alpha + p \ge kl$. For all i with $1 \le i \le a$ we have $(i, r_i^{(1)}) \in F_1$, hence $r_i^{(1)} \le \frac{\alpha+p}{i}$. Since $r_i^{(1)}$ must be an integer, we have

$$|F_1| = \sum_{i=1}^{a} r_i^{(1)} \le \sum_{i=1}^{a} \left\lfloor \frac{\alpha + p}{i} \right\rfloor.$$

Since $(a, 1) \in F_1$, we have $a \le \alpha + p$, so

$$|F_1| \le \sum_{i=1}^{\alpha+p} \left\lfloor \frac{\alpha + p}{i} \right\rfloor. \quad \square$$

Corollary 10. *Let F_1, F_2, α and p be defined as in Theorem 9. Then*

$$|F_1| \le (\alpha + p)(1 + \log(\alpha + p)).$$

Proof. Analogous to the proof of Corollary 7. $\square$

The following example shows that the upper bound cannot even be improved by a factor $\frac{1}{2\log 2} \approx 0.72$, provided that $\alpha > \frac{p+1}{2\log 2 - 1} \log(p + 1)$.

Example 11. Let k and m be integers satisfying $k \ge 2$ and $m \ge 2k - 2$. We construct sets F_1 and F_2 as follows (see also Figs. 4 and 5).

- Row 1:
  - $(1, j) \in F_1 \cap F_2$ for $1 \le j \le 2^{k-1}$,
  - $(1, j) \in F_1$ for $2^{k-1} + 1 \le j \le 2^m - 2^{k-1} + 1$,
  - $(1, j) \in F_2$ for $2^m - 2^{k-1} + 2 \le j \le 2^{m+1} - 2^k - 2^{k-1} + 2$.

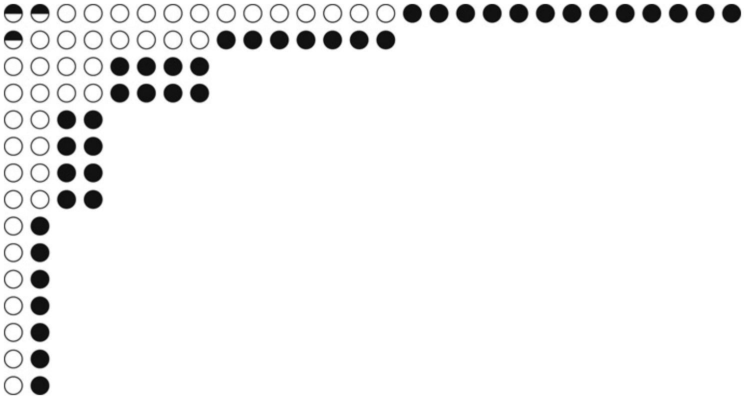

Fig. 5. The construction from Example 11 with $k = 2$ and $m = 4$.

- Let $0 \leq l \leq k-2$. Row i, where $2^l + 1 \leq i \leq 2^{l+1}$:
  - $(i, 1) \in F_1 \cap F_2$,
  - $(i, j) \in F_1$ for $2 \leq j \leq 2^{m-l-1} - 2^{k-l-2} + 1$,
  - $(i, j) \in F_2$ for $2^{m-l-1} - 2^{k-l-2} + 2 \leq j \leq 2^{m-l} - 2^{k-l-1} + 1$.
- Let $k-1 \leq l \leq m-k$. Row i, where $2^l + 1 \leq i \leq 2^{l+1}$:
  - $(i, j) \in F_1$ for $1 \leq j \leq 2^{m-l-1}$,
  - $(i, j) \in F_2$ for $2^{m-l-1} + 1 \leq j \leq 2^{m-l}$.
- Let $m-k+1 \leq l \leq m-1$. Row i, where $2^l - 2^{l-m+k-1} + 2 \leq i \leq 2^{l+1} - 2^{l-m+k} + 1$:
  - $(i, j) \in F_1$ for $1 \leq j \leq 2^{m-l-1}$,
  - $(i, j) \in F_2$ for $2^{m-l-1} + 1 \leq j \leq 2^{m-l}$.

The construction is almost symmetrical: if $(i, j) \in F_1$, then $(j, i) \in F_1$; if $(i, j) \in F_1 \cap F_2$, then $(j, i) \in F_1 \cap F_2$; and if $(i, j) \in F_2$ with $i > 1$, then $(j, i) \in F_2$. Since it is clear from the construction that each row contains exactly as many points of F_1 as points of F_2, we conclude that each column j with $2 \leq j \leq 2^m - 2^{k-1} + 1$ contains exactly as many points of F_1 as points of F_2 as well. The only difference in the line sums occurs in the first column (which has $2^m - 2^{k-1} + 1$ points of F_1 and only 2^{k-1} of F_2) and in columns $2^m - 2^{k-1} + 2$ up to $2^{m+1} - 2^k - 2^{k-1} + 2$ (each of which contains one point of F_2 and none of F_1). So we have

$$\begin{aligned}\alpha &= \frac{1}{2}\left((2^m - 2^{k-1} + 1) - 2^{k-1} + (2^{m+1} - 2^k - 2^{k-1} + 2) - (2^m - 2^{k-1} + 1)\right)\\ &= 2^m - 2^k + 1.\end{aligned}$$

It is easy to see that

$$p = |F_1 \cap F_2| = 2^k - 1.$$

Now we count the number of elements of F_1.

- Row 1 contains $2^m - 2^{k-1} + 1$ elements of F_1.
- Let $0 \leq l \leq k-2$. Rows $2^l + 1$ up to 2^{l+1} together contain $2^l(2^{m-l-1} - 2^{k-l-2} + 1) = 2^{m-1} - 2^{k-2} + 2^l$ elements of F_1.
- Let $k-1 \leq l \leq m-k$. Rows $2^l + 1$ up to 2^{l+1} together contain $2^l \cdot 2^{m-l-1} = 2^{m-1}$ elements of F_1.
- Let $m-k+1 \leq l \leq m-1$. Rows $2^l - 2^{l-m+k-1} + 2$ up to $2^{l+1} - 2^{l-m+k} + 1$ together contain $(2^l - 2^{l-m+k-1})(2^{m-l-1}) = 2^{m-1} - 2^{k-2}$ elements of F_1.

Hence the number of elements of F_1 is

$$\begin{aligned}|F_1| &= 2^m - 2^{k-1} + 1 + (k-1)(2^{m-1} - 2^{k-2}) + \sum_{l=0}^{k-2} 2^l + (m - 2k + 2)2^{m-1} + (k-1)(2^{m-1} - 2^{k-2})\\ &= 2^m + m2^{m-1} + 2^{k-1} - k2^{k-1}.\end{aligned}$$

For this family of examples we now have

$$|F_1| = \alpha + p + \frac{\alpha + p}{2}\log_2(\alpha + p) + \frac{p+1}{2} - \frac{p+1}{2}\log_2(p+1).$$

We will now prove another bound, which is better if $p = |F_1 \cap F_2|$ is large compared to α. Let u be an integer such that $2u = |F_1 \triangle F_2|$. We will first derive an upper bound on u in terms of a, b and α. Then we will derive a lower bound on $|F_1|$ in terms of a, b and α. By combining these two, we find an upper bound on u in terms of α and p.

Lemma 12. *Let F_1 and F_2 be finite subsets of $\mathbb{Z}^2$ such that*

- *F_1 is uniquely determined by its row and column sums, and*
- *$|F_1| = |F_2|$.*

Let α, a and b be defined as in Section 2. Define u as $2u = |F_1 \triangle F_2|$. Then we have

$$u^2 \leq \frac{\alpha}{4}(a+b)(a+b+\alpha-1).$$

Proof. Decompose $F_1 \triangle F_2$ into α staircases as in Lemma 4, and let $\mathcal{T}$ be the set consisting of these staircases. Let $T \in \mathcal{T}$ be a staircase and $i \leq a+1$ a positive integer. Consider the elements of $T \cap F_2$ in rows $i, i+1, \ldots, a$. If such elements exist, then let $w_i(T)$ be the largest column index that occurs among these elements. If there are no elements of $T \cap F_2$ in those rows, then let $w_i(T)$ be equal to the smallest column index of an element of $T \cap F_1$ (no longer restricted to rows $i, \ldots, a$). We have $w_i(T) \geq 1$. Define $W_i = \sum_{T \in \mathcal{T}} w_i(T)$.

Let d_i be the number of elements of $F_1 \setminus F_2$ in row i. Let $y_1 < \cdots < y_{d_i}$ be the column indices of the elements of $F_1 \setminus F_2$ in row i, and let $y'_1 < \cdots < y'_{d_i}$ be the column indices of the elements of $F_2 \setminus F_1$ in row i. Let $\mathcal{T}_i \subset \mathcal{T}$ be the set of staircases with elements in row i. The elements in $F_2 \setminus F_1$ of these staircases are in columns $y'_1, y'_2, \ldots, y'_{d_i}$, hence the set $\{w_i(T) : T \in \mathcal{T}_i\}$ is equal to the set $\{y'_1, y'_2, \ldots, y'_{d_i}\}$. The elements in $F_1 \setminus F_2$ are in columns $y_1, y_2, \ldots, y_d$ and are either the first element of a staircase or correspond to an element of $F_2 \setminus F_1$ in the same column but in a row with index at least $i+1$. In either case, for a staircase $T \in \mathcal{T}_i$ we have $w_{i+1}(T) = y_j$ for some j. Hence the set $\{w_{i+1}(T) : T \in \mathcal{T}_i\}$ is equal to the set $\{y_1, y_2, \ldots, y_{d_i}\}$. We have

$$\sum_{T \in \mathcal{T}_i} w_{i+1}(T) = \sum_{j=1}^{d_i} y_j \leq \sum_{j=1}^{d_i} (y_{d_i} - j + 1) = d_i y_{d_i} - \frac{1}{2}(d_i - 1)d_i,$$

and

$$\sum_{T \in \mathcal{T}_i} w_i(T) = \sum_{j=1}^{d_i} y'_j \geq \sum_{j=1}^{d_i} (y_{d_i} + j) = d_i y_{d_i} + \frac{1}{2}(d_i + 1)d_i.$$

Hence

$$\begin{aligned} W_i &= W_{i+1} + \sum_{T \in \mathcal{T}_i} (w_i(T) - w_{i+1}(T)) \\ &\geq W_{i+1} + \frac{1}{2}(d_i+1)d_i + \frac{1}{2}(d_i-1)d_i \\ &= W_{i+1} + d_i^2. \end{aligned}$$

Since $W_{a+1} \geq \alpha$, we find

$$W_1 \geq \alpha + d_1^2 + \cdots + d_a^2.$$

We may assume that if (x, y) is the endpoint of a staircase, then (x, y') is an element of $F_1 \cup F_2$ for $1 \leq y' < y$ (i.e. there are no gaps between the endpoints and other elements of $F_1 \cup F_2$ on the same row). After all, by moving the endpoint of a staircase to another empty position on the same row, the error in the columns can only become smaller (if the new position of the endpoint happens to be in the same column as the first point of another staircase, in which case the two staircases fuse together to one) but not larger, and u, a and b do not change.

So on the other hand, as W_1 is the sum of the column indices of the endpoints of the staircases, we have

$$W_1 \leq (b+1) + (b+2) + \cdots + (b+\alpha) = \alpha b + \frac{1}{2}\alpha(\alpha+1).$$

We conclude

$$\alpha + \sum_{i=1}^{a} d_i^2 \leq \alpha b + \frac{1}{2}\alpha(\alpha+1).$$

Note that $\sum_{i=1}^{a} d_i = u$. By the Cauchy–Schwarz inequality, we have

$$\left(\sum_{i=1}^{a} d_i^2\right)\left(\sum_{i=1}^{a} 1\right) \geq \left(\sum_{i=1}^{a} d_i\right)^2 = u^2,$$

so

$$\sum_{i=1}^{a} d_i^2 \geq \frac{u^2}{a}.$$

From this it follows that

$$\alpha b + \frac{1}{2}\alpha(\alpha+1) \geq \alpha + \frac{u^2}{a},$$

or, equivalently,

$$u^2 \leq \alpha ab + \frac{1}{2}\alpha(\alpha-1)a.$$

By symmetry we also have

$$u^2 \leq \alpha ab + \frac{1}{2}\alpha(\alpha-1)b.$$

Hence

$$u^2 \leq \alpha ab + \frac{1}{4}\alpha(\alpha-1)(a+b).$$

Using that $\sqrt{ab} \leq \frac{a+b}{2}$, we find

$$u^2 \leq \alpha\left(\frac{(a+b)^2}{4} + \frac{(\alpha-1)(a+b)}{4}\right) = \frac{\alpha}{4}(a+b)(a+b+\alpha-1). \quad \square$$

Lemma 13. *Let F_1 and F_2 be finite subsets of $\mathbb{Z}^2$ such that*
- *F_1 is uniquely determined by its row and column sums, and*
- *$|F_1| = |F_2|$.*

Let α, a and b be defined as in Section 2. Then we have

$$|F_1| \geq \frac{(a+b)^2}{4(\alpha+1)}.$$

Proof. Without loss of generality, we may assume that all rows and columns that contain elements of F_1 also contain at least one point $F_1 \triangle F_2$: if a row or column does not contain any points of $F_1 \triangle F_2$, we may delete it. By doing so, $F_1 \triangle F_2$ does not change, while $|F_1|$ becomes smaller, so the situation becomes better.

First consider the case $r_{i+1}^{(1)} < r_i^{(1)} - \alpha$ for some i. We will show that this is impossible. If a column does not contain an element of $F_2 \setminus F_1$, then by the assumption above it contains an element of $F_1 \setminus F_2$, which must then be the first point of a staircase. Consider all points of $F_2 \setminus F_1$ and all first points of staircases in columns $r_{i+1}+1, r_{i+1}+2, \ldots, r_i$. Since these are more than α columns, at least two of those points must belong to the same staircase. On the other hand, if $(x, y) \in F_1 \setminus F_2$ is the first point of a staircase with $r_{i+1} < y \leq r_i$, then we have $x \leq i$, so the second point (x', y') in the staircase, which is in $F_2 \setminus F_1$, must satisfy $x' \leq i$ and therefore $y' > r_i$. So the second point cannot also be in one of the columns $r_{i+1}+1$, $r_{i+1}+2, \ldots, r_i$. If two points of $F_2 \setminus F_1$ in columns $r_{i+1}+1, r_{i+1}+2, \ldots, r_i$ belong to the same staircase, then they must be connected by a point of $F_1 \setminus F_2$ in the same columns. However, by a similar argument this forces the next point to be outside the mentioned columns, while we assumed that it was in those columns. We conclude that it is impossible for row sums of two consecutive rows to differ by more than α.

By the same argument, column sums of two consecutive columns cannot differ by more than α. Hence we have $r_{i+1}^{(1)} \geq r_i^{(1)} - \alpha$ for all i, and $c_{j+1}^{(1)} \geq c_j^{(1)} - \alpha$ for all j.

We now have $r_2^{(1)} \geq b - \alpha$, $r_3^{(1)} \geq b - 2\alpha$, and so on. Also, $c_2^{(1)} \geq a - \alpha$, $c_3^{(1)} \geq a - 2\alpha$, and so on. Using this, we can derive a lower bound on $|F_1|$ for fixed a and b. Consider Fig. 6. The points of F_1 are indicated by black dots. The number of points is equal to the grey area in the picture, which consists of all 1×1-squares with a point of F_1 in the upper left corner. We can estimate this area from below by drawing a line with slope α through the point $(a+1, 1)$ and a line with slope $\frac{1}{\alpha}$ through the point $(b+1, 1)$; the area closed in by these two lines and the two axes is less than or equal to the number of points of F_1.

For $\alpha = 1$ those lines do not have a point of intersection. Under the assumption we made at the beginning of this proof, we must in this case have $a = b$ and the number of points of F_1 is equal to

$$\frac{a(a+1)}{2} \geq \frac{a^2}{\alpha+1} = \frac{(a+b)^2}{4(\alpha+1)},$$

so in this case we are done.

In order to compute the area for $\alpha \geq 2$ we switch to the usual coordinates in $\mathbb{R}^2$, see Fig. 7. The equation of the first line is $y = \alpha x - a$, and the equation of the second line is $y = \frac{1}{\alpha}x - \frac{1}{\alpha}b$. We find that the point of intersection is given by

$$(x, y) = \left(\frac{a\alpha - b}{\alpha^2 - 1}, \frac{-b\alpha + a}{\alpha^2 - 1}\right).$$

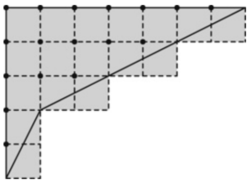

Fig. 6. The number of points of F_1 (indicated by small black dots) is equal to the grey area.

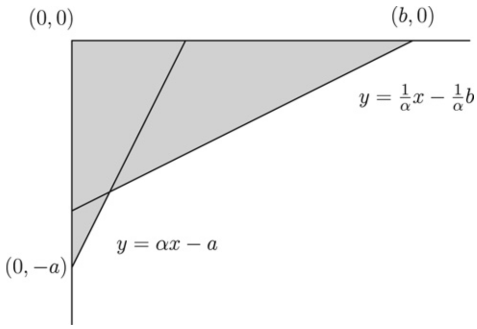

Fig. 7. Computing the area bounded by the two lines and the two axes.

The area of the grey part of Fig. 7 is equal to

$$\frac{1}{2}a \cdot \frac{a\alpha - b}{\alpha^2 - 1} + \frac{1}{2}b \cdot \frac{b\alpha - a}{\alpha^2 - 1} = \frac{a^2\alpha + b^2\alpha - 2ab}{2(\alpha^2 - 1)}.$$

We now have

$$|F_1| \geq \frac{\alpha(a^2 + b^2) - 2ab}{2(\alpha^2 - 1)} \geq \frac{\alpha\frac{(a+b)^2}{2} - \frac{(a+b)^2}{2}}{2(\alpha^2 - 1)} = \frac{(a+b)^2}{4(\alpha + 1)}. \quad \square$$

Theorem 14. *Let F_1 and F_2 be finite subsets of $\mathbb{Z}^2$ such that*

- *F_1 is uniquely determined by its row and column sums, and*
- *$|F_1| = |F_2|$.*

Let α be defined as in Section 2, and let $p = |F_1 \cap F_2|$. Write $\beta = \sqrt{\alpha}(\alpha + 1)$. Then

$$|F_1| \leq p + \sqrt{\frac{\alpha}{4}\left(\beta + \sqrt{\beta(\alpha - 1) + 4(\alpha + 1)p + \beta^2} + \frac{\alpha - 1}{2}\right)^2 - \frac{(\alpha - 1)^2\alpha}{16}}.$$

Proof. Write $s = a + b$ for convenience of notation. From Lemma 12 we derive

$$u \leq \frac{\sqrt{\alpha}}{2}\left(s + \frac{\alpha - 1}{2}\right).$$

We substitute $|F_1| = u + p$ in Lemma 13 and use the above bound for u:

$$\frac{\sqrt{\alpha}}{2}\left(s + \frac{\alpha - 1}{2}\right) + p \geq |F_1| \geq \frac{s^2}{4(\alpha + 1)}.$$

Solving for s, we find

$$\begin{aligned} s &\leq \sqrt{\alpha}(\alpha + 1) + \sqrt{\sqrt{\alpha}(\alpha^2 - 1) + 4(\alpha + 1)p + \alpha(\alpha + 1)^2} \\ &= \beta + \sqrt{\beta(\alpha + 1) + 4(\alpha + 1)p + \beta^2}. \end{aligned}$$

Finally we substitute this in Lemma 12:

$$u \leq \sqrt{\frac{\alpha}{4}\left(\beta + \sqrt{\beta(\alpha - 1) + 4(\alpha + 1)p + \beta^2} + \frac{\alpha - 1}{2}\right)^2 - \frac{(\alpha - 1)^2\alpha}{16}}.$$

This, together with $|F_1| = u + p$, yields the claimed result. $\square$

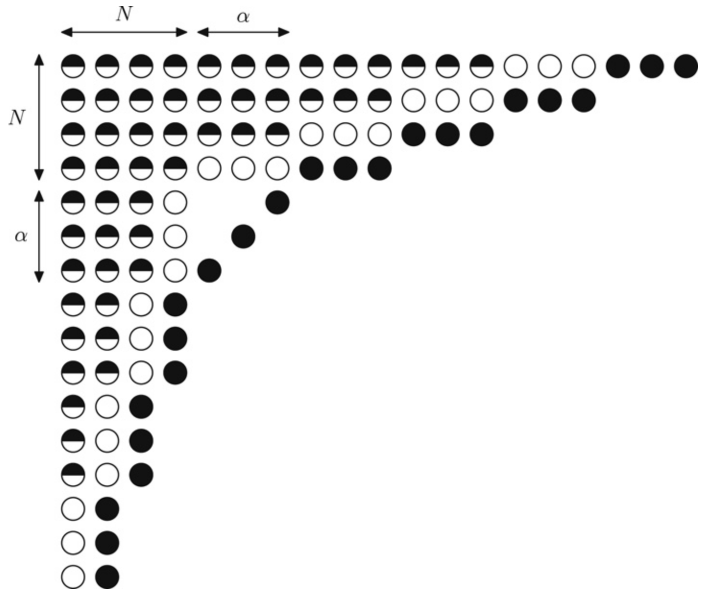

Fig. 8. The construction from Example 16 with $N = 4$ and $\alpha = 3$.

Remark 15. By a straightforward generalisation of [2, Proposition 13 and Lemma 16], we find a bound very similar to the one in Theorem 14:

$$|F_1| \le p + (\alpha + 1)\left(\alpha - \frac{1}{2}\right) + (\alpha + 1)\sqrt{2p + \frac{(2\alpha - 1)^2}{4}}.$$

Theorem 14 says that $|F_1|$ is asymptotically bounded by $p + \alpha\sqrt{p} + \alpha^2$. The next example shows that $|F_1|$ can be asymptotically as large as $p + 2\sqrt{\alpha p} + \alpha$.

Example 16. Let N be a positive integer. We construct F_1 and F_2 with total difference in the line sums equal to 2α as follows (see also Fig. 8). Let $(i, j) \in F_1 \cap F_2$ for $1 \le i \le N$, $1 \le j \le N$. Furthermore, for $1 \le i \le N$:

- Let $(i, j), (j, i) \in F_1 \cap F_2$ for $N + 1 \le j \le N + (N - i)\alpha$.
- Let $(i, j), (j, i) \in F_1$ for $N + (N - i)\alpha + 1 \le j \le N + (N - i + 1)\alpha$.
- Let $(i, j), (j, i) \in F_2$ for $N + (N - i + 1)\alpha + 1 \le j \le N + (N - i + 2)\alpha$.

Finally, for $1 \le t \le \alpha$, let $(i, j) \in F_2$ with $i = N + t$ and $j = N + \alpha + 1 - t$.

The only differences in the line sums occur in the first column (a difference of α) and in columns $N + N\alpha + 1$ up to $N + N\alpha + \alpha$ (a difference of 1 in each column). We have

$$p = N^2 + 2 \cdot \frac{1}{2}N(N - 1)\alpha = N^2 + N^2\alpha - N\alpha,$$

and

$$|F_1| = N^2 + 2 \cdot \frac{1}{2}N(N + 1)\alpha = N^2 + N^2\alpha + N\alpha.$$

From the first equality we derive

$$N = \frac{\alpha}{2(\alpha + 1)} + \sqrt{\frac{p}{\alpha + 1} + \frac{\alpha^2}{4(\alpha + 1)^2}}.$$

Hence

$$|F_1| = p + 2N\alpha = p + \frac{\alpha^2}{\alpha + 1} + \sqrt{\frac{4\alpha^2 p}{\alpha + 1} + \frac{\alpha^4}{(\alpha + 1)^2}}.$$

6. Generalisation to unequal sizes

Until now, we have assumed that $|F_1| = |F_2|$. However, we can easily generalise all the results to the case $|F_1| \neq |F_2|$.

Suppose $|F_1| > |F_2|$. Then there must be a row i with $r_i^{(1)} > r_i^{(2)}$. Let $j > b$ be such that $(i, j) \notin F_2$ and define $F_3 = F_2 \cup \{(i, j)\}$. We have $r_i^{(3)} = r_i^{(2)} + 1$, so the error in row i has decreased by one, while the error in column j has increased by one. In this way, we can keep adding points until F_2 together with the extra points is just as large as F_1, while the total difference in the line sums is still 2α. Note that $p = |F_1 \cap F_2|$ and $|F_1|$ have not changed during this process, so the results from Theorem 14 and Corollary 10 are still valid in exactly the same form.

Suppose on the other hand that $|F_1| < |F_2|$. Then there must be a row with $r_i^{(1)} < r_i^{(2)}$. Let j be such that $(i, j) \in F_2 \setminus F_1$ and define $F_3 = F_2 \setminus \{(i, j)\}$. The error in row i has now decreased by one, while the error in column j has at most increased by one, so the total error in the line sums has not increased. We can keep deleting points of F_2 until there are exactly $|F_1|$ points left, while the total difference in the line sums is at most 2α.

By using $|F_1 \triangle F_2| = 2(|F_1| - p)$, we can state the results from Theorem 14 and Corollary 10 in a more symmetric way, not depending on the size of F_1.

Theorem 17. *Let F_1 and F_2 be finite subsets of $\mathbb{Z}^2$ such that F_1 is uniquely determined by its row and column sums. Let α be defined as in Section 2, and let $p = |F_1 \cap F_2|$. Write $\beta = \sqrt{\alpha}(\alpha + 1)$. Then*

(1) $|F_1 \triangle F_2| \leq 2\alpha + 2(\alpha + p) \log(\alpha + p)$.

(2) $|F_1 \triangle F_2| \leq \sqrt{\alpha \left(\beta + \sqrt{\beta(\alpha - 1) + 4(\alpha + 1)p + \beta^2} + \frac{\alpha-1}{2}\right)^2 - \frac{(\alpha-1)^2\alpha}{4}}$.

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