Overgroups of primitive groups, II

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Abstract

We continue our study of the overgroup lattices of subgroups of finite alternating and symmetric groups, with applications to the question of Palfy and Pudlak as to whether each finite lattice is an interval in the lattice of subgroups of some finite group.

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Assume $\Omega$ is a set of finite order $n$, $S = \text{Sym}(\Omega)$ is the symmetric group on $\Omega$, $G$ is $S$ or the alternating group on $\Omega$, and $H$ is a subgroup of $G$ primitive on $\Omega$. We continue our study (begun in [A4]) of the set $\mathcal{O}_G(H)$ of overgroups of $H$ in $G$. See also work of Cheryl Praeger in [P], for a different approach to the subject.

In this paper we concentrate on the lattice structure on $\mathcal{O}_G(H)$, particularly in the case where $H$ is the intersection of many pairs of maximal subgroups of $G$. The overgroup lattice is described in terms of generalized Fitting subgroups of overgroups and certain natural structures on $\Omega$ associated to the overgroups, most particularly regular product structures on $\Omega$. The machinery describing the relationship among regular product structures on $\Omega$, which is developed in Section 5, may be of interest in its own right.

One motivation for many of our results is the theorem of Palfy and Pudlak in [PP], published in 1980, which focused attention on the question of whether each nonempty finite lattice is isomorphic to an interval in the lattice of subgroups of some finite group. That question remains open to this day.

John Shareshian and the author have begun a program (cf. [A2] and [A3]) to show that a certain class of lattices are not of the form $\mathcal{O}_G(H)$ for any finite group $G$ and subgroup $H$ of $G$. We now define that class of lattices.

Let $\Lambda$ be a finite lattice. Then $\Lambda$ has a greatest member $\infty$ and a least member 0. Set $\Lambda' = \Lambda - \{0, \infty\}$. Regard $\Lambda$ as a graph with adjacency relation the comparability relation, and define $\Lambda$ to be disconnected if the graph $\Lambda'$ is disconnected. The lattice $\Lambda$ is an $M_m$-lattice if $|\Lambda'| = m$ and the graph

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$\Lambda'$ has no edges. Write $\Delta(m)$ for the lattice of subsets of an $m$-set, partially ordered by inclusion. Define a $D\Delta$-lattice to be a lattice $\Lambda$ such that $\Lambda'$ has $r > 1$ connected components $\Lambda'_i$, $1 \leq i \leq r$, and for each $i$, $\Lambda'_i \cong \Delta(m_i)$ for some $m_i > 2$.

Aschbacher and Shreshian conjecture that no $D\Delta$-lattice is an interval in the subgroup lattice of a finite group. Further [A2] and [A3] reduce the verification of this conjecture to two problems about the overgroup lattices in almost simple groups. The first problem is to prove that if $X$ is almost simple and $Y \leq X$, then $O_X(Y)$ is not a $D\Delta$-lattice. This is accomplished, when $X$ is alternating or symmetric and $Y$ is primitive, in Theorem E below.

We now state our main theorems. See [FGT] for the notation and terminology involving finite groups used in the paper. See Section 2 for a discussion of the various types of primitive subgroups of $S$. Recall that a point stabilizer in a primitive group $H$ is a maximal subgroup of $H$, that the socle of $H$ is equal to its generalized Fitting subgroup $F^*(H)$, and that $F^*(H)$ is either an elementary abelian $p$-group or the direct product of isomorphic nonabelian simple groups. The primitive groups fall into five or six classes, depending on the structure of the socle and the embedding of the point stabilizer. See Section 1 for our notation for lattices.

A finite lattice $\Lambda$ is an $l$-lattice if $\Lambda' \neq \emptyset$ and for each maximal member $x$ of $\Lambda'$, there exists a maximal member $y$ of $\Lambda'$ with $x \land y = 0$. Observe that disconnected lattices are $l$-lattices. Thus the class of $l$-lattices provides common ground for simultaneously studying disconnected lattices and related classes, and seems to be the right class in which to carry out the arguments which are available. Several of our theorems are about $l$-lattices.

Recall $G$ is the alternating or symmetric group on a set $\Omega$ of finite order $n$, and $H \leq G$. Write

$$O_G(H)^{''} = \{ M \in O_G(H) : F^*(G) \nleq M \}$$

for the set of “proper” overgroups of $H$ in $G$, and $\mathcal{M}(H) = M_G(H)$ for the set of maximal members of $O_G(H)^{''}$. Recall a finite group $X$ is almost simple if $F^*(X)$ is a nonabelian simple group.

Suppose $H$ is almost simple. We say $H$ is octal if $H \cong L_3(2)$ and $n = 8$. Further $H$ is product indecomposable unless it satisfies one of the three conditions appearing at the end of Section 2.

Our first result follows from the list of maximal containments of almost simple primitive groups in [LPS2], together with some hard work.

**Theorem A.** Assume $\Omega$ is a set of finite order $n$ and $H$ is an almost simple primitive subgroup of $S = \text{Sym}(\Omega)$ which is product indecomposable and not octal. Then all members of $O_S(H)$ are almost simple, product indecomposable, and not octal, and setting $U = F^*(H)$, one of the following holds:

1. $|\mathcal{M}_S(H)| = 1$.
2. $U = H$, $|\mathcal{M}_S(H)| = 3$, $\text{Aut}(U) \cong N_S(U) \in \mathcal{M}_S(U)$, $N_S(U)$ is transitive on $\mathcal{M}_S(H) - \{N_S(U)\}$, and $U$ is maximal in $V$, where $K \in \mathcal{M}_S(H) - \{N_S(U)\}$ and $V = F^*(K)$. Further $(U, V, n)$ is one of the following:
   a. $(HS, A_m, 15400)$, where $m = 176$ and $n = \binom{m}{2}$.
   b. $(G_2(3), \Omega_7(3), 3159)$.
   c. $(L_2(q), M_n, n)$, where $q \in \{11, 23\}$, $n = q + 1$, and $M_n$ is the Mathieu group of degree $n$.
   d. $(L_2(17), Sp_8(2), 136)$.
3. $U \cong L_3(4)$, $n = 280$, $|\mathcal{M}_S(U)| = 4$, $\text{Aut}(U) \cong N_S(U) \in \mathcal{M}_S(U)$, $N_S(U)$ is transitive on $\mathcal{M}_S(U) - \{N_S(U)\}$, and $K \in \mathcal{M}_S(H) - \{N_S(U)\}$ is isomorphic to $\text{Aut}(U_4(3))$.
4. $U \cong S_2(q)$, $q = 2^k$, $n = q^2(q^2 + 1)/2$, $\mathcal{M}_S(U) = \{K_1, K_2\}$ where $K_1 = N_S(V_1) \cong \text{Aut}(V_1)$, $V_1 = A_{q^2+1}$, $V_2 = Sp_{4k}(2)$, and $N_S(U) \cong \text{Aut}(U)$ is maximal in $V_1$.
5. $H \cong L_2(11)$, $n = 55$, $\text{PGL}_2(11) \cong N_S(H)$, and $\mathcal{M}_S(H) = \{N_S(H), K, K'\}$, $t \in N_S(H) - H$, is of order 3, where $K \cong S_11$ and $O_K(H) = \{H < L < V < K\}$, with $L \cong M_{11}$ and $V \cong A_{11}$.

Theorem A is proved in Section 8. Recall [LPS2] shows that for almost all nonabelian simple subgroups $L$ of $G$ such that $O_G(L)$ is primitive on $\Omega$, $O_G(L)$ is the unique maximal member of $O_G(H)^{''}$ for each primitive subgroup $H$ of $G$ with $F^*(H) = L$. Further [LPS2] lists (up to some notion of equivalence) the pairs $H \leq K$ such that $H$ and $K$ are proper almost simple primitive subgroups of $G,$
$F^*(H)$ is proper in $F^*(K)$, and $H$ is maximal in $K$. Various results in this paper give a qualitative description of $O_C(H)$ for $H$ an almost simple primitive subgroup of $G$ such that $N_G(F^*(H))$ is not the unique maximal member of $O_C(H)$". The proofs of these results use the lists in [LPS2]. In particular Theorem A says that, with very rare exceptions, for each such $H$, there is a unique maximal member of $O_C(H)$". Then Theorems A and B and lemmas in Sections 7 and 8 supplying more detailed information, give a precise description of $O_C(H)$ when $O_C(H)$ does not have a unique maximal member.

**Theorem B.** Assume $G$ is the alternating or symmetric group on a set of finite order, and $H$ is an almost simple primitive subgroup of $G$ which is product indecomposable and not octal, and such that $\Lambda = O_G(H)$ is an I-lattice. Then $\Lambda = M_2, T_{2,2}, T_{1,2,2}, M_{1,3}, T_1 \ast H_7$, or $T_{-1} \circ \Gamma(k)$ for some integer $k > 1$.

Theorem B is proved in Section 9. The definitions of the lattices $M_m$, $T_{r,s}$, etc., appear in Section 1.

**Theorem C.** Assume $G$ is the alternating or symmetric group on a set $\Omega$ of finite nonprime order $n$, and $H$ is a primitive subgroup of $G$ such that $\Lambda = O_G(H)$ is an I-lattice. Let $\mathcal{M}$ be the set of maximal members of $\Lambda - [G]$ and $D = F^*(H)$. Then one of the following holds:

1. All members of $\Lambda$ are almost simple, product indecomposable, and not octal.
2. $H$ is semisimple, $\mathcal{F}(H) = \{F, F\}$, $\mathcal{M} = \{N_C(F), N_C(\bar{F})\}$, and $\Lambda = M_2, T_{1,3}, T_{1,4},$ or $M_{1,4}$.
3. $n$ is a prime power, $H$ is affine, $\mathcal{M} = \{N_C(D), N_C(F(D)) : D \in D(H)\}$, and one of the following holds:
   1. $\Lambda = M_2$.
   2. $n = 25$, $G$ is the alternating group, for $\omega \in \Omega$, $H_\omega \cong Z_4 * Q_8$, and $\Lambda \cong T_{2,1,1,1}$.
   3. $n$ is $5^2, 7^2, 11^2, 3^4$, or $5^4$ and $\Lambda$ is $T_{1,2}$.
   4. $n$ is a power of $2$ and $\Lambda$ is $T_{1,3}, T_{1,4},$ or $M_{1,4}$.
4. $n = 8$, $G$ is alternating, $H \cong L_3(2)$ is octal, $\mathcal{M}$ consists of the stabilizers of the two $H$-invariant affine structures on $\Omega$, and $\Lambda = M_2$.
5. $n = 8$, $G$ is symmetric, $H \cong L_3(2)$ is octal, $N_G(H) \cong PGL_2(7)$, $\mathcal{M} = \{F^*(G), N_G(H)\}$, $\Lambda = \mathcal{M} \cup \{H, K_1, K_2, G\}$, where $K_1$ and $K_2$ are the stabilizers of the two $H$-invariant affine structures on $\Omega$, and $\Lambda = M_{1,3}$.
6. $G$ is symmetric, $N_G(H)$ is the stabilizer of an affine structure, regular product structure, or diagonal structure on $\Omega$, $H$ is the stabilizer in $F^*(G)$ of that structure, $O_C(H) = [H, F^*(G), N_G(H), G]$, and $\Lambda = M_2$.

Theorem C is proved in Section 6. All the lattices listed do indeed occur, as can be seen by tracing through the proof in Section 6 and [A4]. Definitions of the various structures on $\Omega$, definitions of the notation $\mathcal{F}(H), \mathcal{F}(D), \text{ and } D(H)$, and definitions of affine and semisimple primitive groups appear in Section 3.

Combining Theorems B and C, and recalling from Section 1 that $T_{1,m} \cong T_{-1} \circ \Gamma(p^{m-1})$ for $m \geq 2$, and $M_{1,4} \cong T_{-1} \circ \Gamma(pq)$ for distinct primes $p, q$, we obtain

**Theorem D.** Assume $G$ is the alternating or symmetric group on a set of finite nonprime order $n$, and $H$ is a primitive subgroup of $G$ such that $\Lambda = O_G(H)$ is an I-lattice. Then $\Lambda = M_2, T_{2,2}, T_{1,2,2}, T_{2,1,1,1}, M_{1,3}, T_1 \ast H_7$, or $T_{-1} \circ \Gamma(k)$ for some integer $k > 1$.

Again all the lattices listed in Theorem D do actually occur as intervals. Observe that if the lattice $O_C(H)$ is disconnected, then $O_C(H)$ is an I-lattice. Moreover $D\Delta$-lattices and $M_m$-lattices for $m > 1$ are disconnected, so we obtain the following immediate corollaries to Theorem D:

**Theorem E.** Assume $G$ is the alternating or symmetric group on a set of finite nonprime order $n$, and $H$ is a primitive subgroup of $G$. Then $O_C(H)$ is not a $D\Delta$-lattice.

**Theorem F.** Assume $G$ is the alternating or symmetric group on a set of finite nonprime order, and $H$ is a primitive subgroup of $G$ such that $O_C(H) \cong M_m$ for some $m$. Then $m \leq 2$. 
Theorem F is a special case of a result of A. Basile in [Be], which determines the possible integers \( m \) for which \( OC(H) \) is isomorphic to \( M_m \) in an alternating or symmetric group \( G \). Such results are relevant to one of the cases left open after the reduction by Baddeley and Lucchini in [BL], aimed at showing that "most" \( M \)-lattices are not intervals in the subgroup lattice of any finite group.

Observe that in several theorems, the case where \( n \) is prime is not addressed. That case is treated by P. Perepelitsky in [Pe].

Our last theorem is a restatement of Theorem F. Define a finite lattice \( \Lambda \) to be of depth \( d \) if \( d \) is the maximal length of a chain in \( \Lambda \). Define a subgroup \( H \) of a group \( G \) to be of depth \( d \) in \( G \) if \( OC(H) \) is of depth \( d \). As the \( M \)-lattices are the lattices of depth 2, we can restate Theorem F as follows:

**Theorem G.** Assume \( G \) is a finite alternating or symmetric group of nonprime degree, and \( H \) is a primitive subgroup of \( G \) of depth 2 in \( G \). Then \( H \) is contained in at most two maximal subgroups of \( G \).

1. Lattices

In this section we assume that \( \Lambda \) is a nonempty finite lattice. Then \( \Lambda \) has a greatest element \( \infty \) and least element 0.

Regard \( \Lambda \) as an undirected graph with adjacency relation the comparability relation on \( \Lambda \). We say that \( \Lambda \) is disconnected if the subgraph \( \Lambda' = \Lambda - \{0, \infty \} \) is disconnected as a graph.

If \( \Delta \) is another finite lattice, write \( \Lambda \star \Delta \) for the lattice \( L' \) such that the poset \( L' \) is the disjoint union of \( \Lambda' \) and \( \Delta' \). In particular the connected components of \( L' \) are the union of the connected components of \( \Lambda' \) and \( \Delta' \). Observe that \( \Lambda \star \Delta \) is the coproduct of \( \Lambda \) and \( \Delta \) in the category of lattices.

Write \( \Lambda \sqcap \Delta \) for the lattice \( K \) such that \( K' \) is the disjoint union of \( \Lambda - \{0\} \) and \( \Delta - \{0\} \). Write \( \Lambda \circ \Delta \) for the lattice \( J \) such that \( J' \) is the disjoint union of \( \Lambda \) and \( \Delta \).

For \( m \equiv -1 \) an integer, write \( T_m \) for the tower of height \( m + 2 \). That is \( T_m \) is the poset which is a chain with \( m + 2 \) elements. Given positive integers \( m_1, \ldots, m_r \), set \( T_{m_1, \ldots, m_r} = T_{m_1} \star \cdots \star T_{m_r} \).

Write \( M_r \) for \( T_{m_1, \ldots, m_r} \) with \( m_i = 1 \) for \( 1 \leq i \leq r \). The lattice \( \Lambda \) is an \( M \)-lattice if \( \Lambda \cong M_r \) for some positive integer \( r \).

Set \( M_{1,3} = T_0 \sqcap M_2 \) and \( M_{1,4} = T_{-1} \circ M_2 \).

Write \( \Delta(m) \) for the poset of all subsets of \( m \). A \( D \Delta \)-lattice is a lattice of the form \( L_1 \star \cdots \star L_r \) for some \( r > 1 \) and \( L_i \cong \Delta(m_i) \) for all \( 1 \leq i \leq r \) and some \( m_i > 2 \).

Given a positive integer \( k \), write \( \Gamma(k) \) for the set of all positive integers dividing \( k \), partially ordered by \( d \leq e \) if \( d \) divides \( e \). For example if \( p \) is a prime and \( e \) a positive integer then \( \Gamma(p^e) \cong \Gamma(e-1) \).

Define \( H_7 \) to be the lattice \( L \) of order 9 such that \( L' \) has three maximal members \( m_i, 0 \leq i \leq 2 \), for \( i = 1, 2 \) we have \( [0, m_i] = \{0 < a_i < b_i < m_i\} \) and \( m_0 \land m_i = b_i \), and \( m_1 \land m_2 = 0 \).

The proof of the following lemma is straightforward:

\[(1.1) \text{ Assume } \Lambda \text{ is an } I \text{ lattice such that } \Lambda' \text{ has exactly two maximal members } x \text{ and } y. \text{ Then } \Lambda \cong \{0, x\} \sqcap \{0, y\}, \text{ where } [0, x] = \{z \in \Lambda: z \leq x\}.\]

2. Primitive groups

In this section we assume that \( \Omega \) is a finite set of order \( n \) and let \( S = \text{Sym}(\Omega) \) be the symmetric group on \( \Omega \). Recall that [FGT] is our reference for notation, terminology, and concepts from finite group theory. For example if \( p \) is a prime and \( e \) is a positive integer, then \( E_{pe} \) denotes the direct product of \( e \) copies of the group of order \( p \).

We first recall some structures on \( \Omega \) defined in Section 1 of [A4]. The notion of an affine structure on \( \Omega \) is defined in Definition 2.2 of [A4]. For our purposes it suffices to recall from Lemma 2.4 in [A4] that if \( p \) is a prime and \( E_{pe} \cong D \) is a subgroup of \( S \) regular on \( \Omega \), then \( D \) defines an affine structure \( R = R(D) \) on \( \Omega \), the stabilizer \( N_S(R) \) of \( R \) is \( N_S(D) \), and \( N_S(D) \) is the split extension of \( D \) by \( \text{Aut}(D) \cong GL_e(p) \).
Next we recall the definition of a regular product structure on $\Omega$ from Definition 2.5 in [A4]. Let $m,k$ be integers with $m \geq 5$ and $k > 1$. Informally, a regular $(m,k)$-product structure on $\Omega$ is a bijection $f : \Omega \to \Gamma^I$, where $I = \{1, \ldots, k\}$ and $\Gamma$ is an $m$-set. The function $f$ may be thought of as a family of functions $(f_i : \Omega \to \Gamma^i : i \in I)$ via $f(\omega) = (f_1(\omega), \ldots, f_k(\omega))$ for $\omega \in \Omega$.

Formally a product structure is a set $\mathcal{F} = \{\Omega_i : i \in I\}$ of partitions $\Omega_i$ of $\Omega$ into $m$ blocks of size $m^{k-1}$, such that $\mathcal{F}$ is injective: For each pair of distinct points $\omega, \omega' \in \Omega$, $\mathcal{F}(\omega) \neq \mathcal{F}(\omega')$, where $\mathcal{F}(\omega)$ is the family $((\omega)_i : i \in I)$ of blocks defined by $\omega \in [\omega]_i \in \Omega_i$.

The set $\mathcal{F} = \mathcal{F}(f)$ of partitions defined by $f$ has its partition $\Omega_i = \{f_i^{-1}(\gamma) : \gamma \in \Gamma^i\}$, the fibers of $f_i$. An indexing of $\mathcal{F}$ is an indexing $\Omega_i = \{\Omega_i, \gamma : \gamma \in \Gamma^i\}$ of the blocks of the various partitions $\Omega_i$. The function $f$ defines the indexing $\Omega_i, \gamma = f_i^{-1}(\gamma)$, while an indexing of $\mathcal{F}$ defines a function $f$ via $\omega \in [\omega]_i \in \Omega_i$. As $\mathcal{F}$ is injective, the function $f$ defined by the indexing is injective, so as $|\Omega| = |\Gamma|^I$, $f : \Omega \to \Gamma^I$ is a bijection. In short the formal definition is a “coordinate free” definition of product structure.

The formal product structure $\mathcal{F}$ can also be regarded as a chamber system in the sense of Tits [T].

The stabilizer $N_S(\mathcal{F})$ in $S$ of $\mathcal{F}$ is the subgroup consisting of those $g \in S$ such that $\mathcal{F}g = \mathcal{F}$. From 2.8 in [A4], $N_S(\mathcal{F})$ is isomorphic to the wreath product of $S_m$ by $S_k$.

Next Lemma 3.2 in [A4] describes the possible structures of subgroups $H$ of $S$ primitive on $\Omega$. There are five types, but in this paper we almost always deal with only two of these types. Namely let $D = F^*(H)$ and $\omega \in \Omega$. Then $H$ is affine if $D \cong E_{pr}$ for some prime $p$, $D$ is regular on $\Omega$, and $H_\omega$ is a complement to $D$ in $H$ which is irreducible on $D$ regarded as an $F_pH_\omega$-module. If $H$ is affine then $n = p^6$ is a prime power, and by an earlier remark, $N_S(D)$ is the stabilizer of the affine structure $R(D)$.

Next $H$ is semisimple if $D$ is the direct product of the set $L$ of components of $H$, $H$ is transitive on $L$, $D_\omega$ is the direct product of the groups $L_\omega$, $L \in L$, $L_\omega \neq 1$, and $Aut_{H_\omega}(L)$ is maximal in $Aut_H(L)$. In this case $n = m^k$, where $m = |L : L_\omega|$ and $k = |L|$. Moreover if $k > 1$ then from Notation 2.6 in [A4], there is a regular $(m,k)$-product structure $\mathcal{F}(H) = \mathcal{F}(L)$ on $\Omega$ with $H \subseteq N_S(\mathcal{F}(H))$ and $D \leq F^*(N_S(\mathcal{F}(H)))$. The partitions in $\mathcal{F}(H)$ are indexed by $L$, and the partition determined by $L \in L$ consists of the orbits of $(\langle \mathcal{L} - L \rangle)$ on $\Omega$.

In either case, as in Definition 3.6 in [A4], write $\mathcal{F}(H)$ for the set of $H$-invariant regular product structures on $\Omega$. For example if $H$ is semisimple and product indecomposable, $\mathcal{F}(H)$ is the greatest member of $\mathcal{F}(H)$ under the partial order defined in Section 5. Further if $H$ is affine write $\mathcal{D}(H)$ for the set of systems $\mathcal{D} = \{D_1, \ldots, D_k\}$ of imprimitivity for $H$ on $D$. That is $k > 1$, $D = D_1 \times \cdots \times D_k$, and $H$ permutes $D$ transitively via conjugation. From 2.6 in [A4], for $\mathcal{D} \in \mathcal{D}(H)$, $\mathcal{F}(\mathcal{D})$ is an $H$-invariant regular $(d,k)$-product structure on $\Omega$, where $d = |D_1|$.

If $H$ is semisimple and $k = 1$ then $H$ is almost simple. Conversely it turns out that each almost simple primitive subgroup $H$ of $S$ is semisimple, and of course with $k = 1$. We say a semisimple group $H$ is octal if the components $L$ of $H$ are isomorphic to $L_3(2)$ and the orbits of $L$ are of length 8. The semisimple group $H$ is product decomposable if one of the three cases in 5.8.4 in [A4] holds:

(i) $L \cong A_6$ and $c = 6^2$.
(ii) $L \cong M_{12}$ and $c = 12^2$.
(iii) $L \cong Sp_q(q)$ for some $q > 2$ even, and $c = (q^2(q^2 - 1)/2)^2$.

Here $L \in L$ and $c$ is the length of the orbits of $L$ on $\Omega$. Finally $H$ is product indecomposable if it is not product decomposable.

3. Partitions and chamber systems on $\Omega$

In this section we assume that $\Omega$ is a finite set and let $S = Sym(\Omega)$ be the symmetric group on $\Omega$.

Write $\mathcal{P} = \mathcal{P}(\Omega)$ for the set of partitions of $\Omega$. Each $P \in \mathcal{P}$ determines an equivalence relation $\sim_P$ on $\Omega$, whose equivalence classes are the blocks of $P$. Of course in the other direction, $P$ is also determined by $\sim_P$.

Define a partial order on $\mathcal{P}$ by $P \leq Q$ if $Q$ is a refinement of $P$. Equivalently, if $\alpha, \beta \in \Omega$ and $\alpha \sim_Q \beta$ then also $\alpha \sim_P \beta$.
Write 0 for the member of \( \mathcal{P} \) with a unique block \( \{ \Omega \} \), and set \[ \infty = \{ \{ \omega \}: \omega \in \Omega \} \in \mathcal{P}. \]

Thus 0 is the least element and \( \infty \) the greatest element of the poset \( \mathcal{P} \).

Indeed \( \mathcal{P} \) is a lattice: For \( P, Q \in \mathcal{P} \),
\[
P \lor Q = \{ A \cap B: A \in P, B \in Q, \text{ and } A \cap B \neq \emptyset \},
\]
while \( P \land Q \) is the partition such that \( \sim_{P \land Q} \) is the equivalence relation generated by \( \sim_P \) and \( \sim_Q \).

The stabilizer \( N_S(P) \) of \( P \) in \( S \) is the subgroup of all \( g \in S \) such that \( P g = P \).

A partition \( P \) is a \( (m, k) \)-partition if \( P \) has \( k \) blocks, each of size \( m \).

If \( Q \subseteq P \) and \( B \in \Omega \), set \( P_B = \{ A \in P: A \subseteq B \} \) and observe that \( P_B \in \mathcal{P}(B) \) and \( Q/P = \{ P_B: B \in \Omega \} \in \mathcal{P}(P) \).

**Definition 3.1.** A rank 2 chamber system on \( \Omega \) is a pair \( \rho = (P, Q) \) of partitions of \( \Omega \). The stabilizer of \( \rho \) is the subgroup \( N_S(\rho) = N_S(P) \cap N_S(Q) \).

If we view \( \rho \) as the pair \( \sim_P, \sim_Q \) of equivalence relations on \( \Omega \), then \( \rho \) is a rank 2 chamber system in the sense of Tits (cf. Section 2.1 in [T]); i.e. \( \rho \) is a family of two equivalence relations on \( \Omega \), with \( \Omega \) the set of chambers of \( \rho \). In the terminology of [T], \( \rho \) is connected if \( P \land Q = 0 \). Define \( \rho \) to be injective if \( P \lor Q = \infty \). If one forms the complex \( \Delta(\rho) \) of the chamber system \( \rho \) as in Section 2.2 of [T], then \( \rho \) is injective iff the map \( \{ A, B \} \mapsto A \cap B \) is a bijection of the set of chambers of \( \Delta(\rho) \) with \( \Omega \).

Define \( \rho \) to be regular if \( P \) and \( Q \) are regular partitions.

### 4. Systems of imprimitivity on \( F_p \)-spaces

In this section \( p \) is a prime, \( V \) is a finite dimensional \( F_p \)-space, and \( H \) is an irreducible subgroup of \( GL(V) \).

**Definition 4.1.** Write \( \mathcal{D}(H) = \mathcal{D}(H, V) \) for the set of systems of imprimitivity for \( H \) on \( V \); that is the members of \( \mathcal{D}(H) \) are \( H \)-invariant sets \( \mathcal{D} = \{ V_1, \ldots, V_k \} \) of nonzero proper subspaces of \( V \) such that \( V = V_1 \oplus \cdots \oplus V_k \). Partially order \( \mathcal{D}(H) \) by \( \mathcal{D} \subseteq \mathcal{D}' \) if each member of \( \mathcal{D}' \) is contained in a member of \( \mathcal{D} \).

**4.2.** Let \( \mathcal{D} = \{ V_1, \ldots, V_k \} \in \mathcal{D}(H) \). Then

1. \( H \) is transitive on \( \mathcal{D} \) and \( N_H(V_i) \) is irreducible on \( V_i \) for each \( i \).
2. \( \dim(V_i) = d \) is independent of \( i \), and \( \dim(V) = kd \).

**Proof.** Part (1) follows as \( \mathcal{D} \) is \( H \)-invariant and \( H \) is irreducible on \( V \). Then (1) implies (2). \( \square \)

**4.3.** Let \( \mathcal{D}, \mathcal{D}' \in \mathcal{D}(H) \) such that \( U \cap U' \neq 0 \) for some \( U \in \mathcal{D} \) and \( U' \in \mathcal{D}' \). Set
\[
\mathcal{E} = \{ W \cap W': W \in \mathcal{D}, \ W' \in \mathcal{D}', \text{ and } W \cap W' \neq 0 \}.
\]

Then \( \mathcal{E} = \mathcal{D} \lor \mathcal{D}' \in \mathcal{D}(H) \).

**Proof.** As \( \mathcal{D} \) and \( \mathcal{D}' \) are \( H \)-invariant, so is \( \mathcal{E} \). Let \( |\mathcal{D}| = k, |\mathcal{D}'| = k' \), \( I = \{ 1, \ldots, k \} \), and \( I' = \{ 1, \ldots, k' \} \). For \( i \in I \), let
\[
I'(i) = \{ j \in I': V_i \cap V'_j \neq 0 \}.
\]
By 4.2.1, \( H \) is transitive on \( D \), so as \( U \cap U' \neq \emptyset \) for some \( U \in D \) and \( U' \in D' \), it follows that \( |I'(i)| = s > 0 \) is independent of \( i \in I \). Let

\[
D_i = \{V_i \cap V'_j : j \in I'(i)\} \quad \text{and} \quad V(i) = (D_i).
\]

Then \( H_i = N_H(V_i) \) acts on \( D_i \) and by 4.2.1, \( H_i \) is irreducible on \( V_i \), so \( V_i = V(i) \). Then as \( D' \in D(H) \), \( V_i \) is the direct sum of the members of \( D_i \), so \( D_i \in D(H_i, V_i) \). It follows that the union \( E \) of the sets \( D_i, i \in I \), is in \( D(H) \), and that \( D' \leq E \). By symmetry, \( D' \leq E \).

On the other hand suppose \( F = \{U_1, \ldots, U_m\} \in D(H) \) with \( D, D' \leq F \). Then for each \( I, U_l \leq V_i(l) \cap V_j(0) = X_i \) for some \( i(l) \in I \) and \( j(l) \in I' \). By definition, \( X_i \in D(l(0)) \), so each member of \( F \) is contained in a member of \( E \). That is \( E \leq F \), so \( E = D \cup D' \).

\[\square\]

**Notation 4.4.** For \( D \in D(H) \) and \( G \leq GL(V) \), the stabilizer in \( G \) of \( D \) is the group \( N_G(D) \) of all \( g \in G \) with \( Dg = D \). Write \( K(D) \) for the kernel of the action of \( N_{GL(V)}(D) \) on \( D \).

**Hypothesis 4.5.** Assume \( D = \{V_1, \ldots, V_k\} \in D(H) \) such that \( c = |V_i| \geq 5 \), and \( P \) is a normal subgroup of \( H \) and \( K(D) \) such that \( K(D)/P \) is a group of exponent at most 2. Set \( I = \{1, \ldots, k\} \), \( d = \dim(V_i) \), and \( K = K(D) \). For \( i \in I \), let \( V^i = \{V_i : i \in I - \{i\}\} \), \( K_i = C_K(V^i) \), \( P_i = P \cap K_i \), and \( S_i = K_i \cap SL(V) \). Pick a basis \( \{x_i, 1, \ldots, xi_d\} \) for \( V_i \), and let \( T_i \) be the set of \( t \in K_i \) permuting the elements of \( F \times xi_1 \) and centralizing \( x_i \) for \( i > 1 \). Let \( T \) be the set of transvections and pseudo-reflections (semisimple elements with centralizer of codimension 1) in \( P \), and \( T_i = P_i \cap T \). Thus \( T_i \cap P_i \subseteq T_i \).

Most of the result in this section assume Hypothesis 4.5. At the end of the section, those results are used to prove Lemma 4.12, which is used in turn in Section 6 to prove Theorem C. Lemma 4.12 essentially gives the overgroups in an alternating or symmetric group of a primitive intersection of the stabilizers of an affine structure and a regular product structure.

**(4.6). Assume Hypothesis 4.5. Then**

1. \( K = K_1 \times \cdots \times K_k \) with \( K_i \cong GL(V_i) \cong GL_d(p) \).
2. \( K_i = S_i T_i \).
3. For each \( i \in I \), \( S_i \subseteq P_i \) and \( |T_i : T_i \cap P_i| \leq 2 \), so \( |K_i : P_i| \leq 2 \).
4. \( S_i = (S_i \cap T) \) and \( T_i \cap P_i \subseteq T_i \), so \( P_i = \langle T_i \rangle \).

**Proof.** Part (1) is trivial. Part (2) follows as \( S_i \) is the kernel of the determinant map \( \text{det} : K_i \to F_p^d \), \( T_i \cong F_p^d \), and \( T_i \cap S_i = 1 \). This also shows that \( K_i/S_i \) is cyclic and \( S_i \cong SL_d(p) \). Then as \( c \geq 5 \), \( S_i = [K_i, K_i] \). Hence as \( K/P \) is of exponent at most 2, (3) follows. Finally \( S_i \) is generated by transvections, and \( T_i \) consists of pseudo-reflections, so (4) follows from (2) and (3).

**\(\square\)**

**((4.7). Assume Hypothesis 4.5, and assume further that \( E = \{U_1, \ldots, U_m\} \in D(H) \) with \( E \not\subseteq D \). Then**

1. For \( U \in E \), \( c = |U| \), so \( P = 5 \) and \( \dim(U) = 1 \).
2. For \( i \in I \), \( |P_i| = 2 \) and the image \( P_i^E \) of \( P_i \) in \( \text{Sym}(E) \) is generated by a transposition.
3. \( k = 2r = m \) is even and \( B = D \cap E \) is of order \( r \).

**Proof.** Let \( K' = K(E) \) and for \( j \in J = \{1, \ldots, m\} \), set \( e = \dim(U_j) \). Applying 4.6.1 to \( H' = N_{GL(V)}(E) \) and \( P' = K' \), we obtain a decomposition \( K' = K'_1 \times \cdots \times K'_m \) with \( K'_j \cong GL_e(p) \). We first prove:

1. \( K_i = K'_l \) if \( d > 1 \), while \( P_i \cap K'_l = 1 \) if \( d = 1 \).

Assume (a) fails for some \( i \). From parts (3) and (4) of 4.6, and as \( c \geq 5 \), there is \( t \in T_i \cap K'_l \), with \( t \in S_i \) if \( d > 1 \). Let \( J_i = \{j \in J : t\pi_j \neq 1\} \), where \( \pi_j : K' \to K'_j \) is the projection map. Then \( 0 \neq [U_j, t] \).
for \( j \in J_i \), so as \( \dim((V, t)) = 1 \), it follows that \( J_i = \{ j \} \) is of order 1, and \( [V, t] = [U_j, t] \preceq U_j \). Hence if \( d > 1 \) then \( t \in S_i \preceq K' \) by assumption, so \([V, t] \preceq V_i\); then as \( S_i \) is irreducible on \( V_i \), it follows that \( V_i = [V, t] \preceq U_j \). On the other hand if \( d = 1 \) then \( V_i = [V, t] \preceq U_j \). Therefore as \( H \) is transitive on \( D \), for each \( i \in I \), \( V_i \preceq U_{j(t)} \) for some \( j(t) \in J \), contradicting \( E \not= D \). This completes the proof of (a).

Next let \( t \in T \). Suppose first that \( e > 1 \). Then as \( \dim(V/C_V(t)) = 1 \), for each \( j \in J \), \( 0 \not= C_U(t) \). Thus \( t \in K' \), so for \( i \in I \), \( S_i \preceq P_i(\langle T \rangle) \triangleq K' \) by 4.6.4, contrary to (a). Therefore:

\[ (b) \ e = 1. \]

If \( d > 1 \) set \( S_i = S_i \cap T \), and observe \( S_i \) consists of transvections. If \( d = 1 \) set \( S_i = T_i \). As \( e = 1 \), \( K' \) contains no transvections, so when \( d > 1 \), \( S_i \cap K' = \emptyset \). If \( d = 1 \), then \( S_i \cap K' = \emptyset \) by (a). Thus we can choose \( t \in S_i \) and \( j \in J \) so that the orbit \( O \) of \( U_j \) under \( (t) \) is of order \( s \geq 1 \). Let \( U = \langle O \rangle \). Then

\[ 1 = \dim(V/C_V(t)) \geq \dim(U/C_U(t)) \geq s - 1 \]

so \( s = 2 \) and \( t \) centralizes \( U_j \). Therefore \( t^2 \in K' \), so as \( S_i \cap K' = \emptyset \), we have \( |t| = 2 \). Hence as \( c \geq 5 \), we conclude from 4.6.3 that either \( p = 2 \) and \( d \geq 3 \), or \( c = 5 \) and \( |P_i| = 2 \).

Suppose the former case holds. Observe \( O \) is the set of members of \( E \) on which \( W = [V, t] \) projects nontrivially. As \( d \geq 3 \), the subgroup \( T \) of \( S_i \) of transvections with center \( W \) is of order at least 4. As \( T \) centralizes \( W \), it acts on the set \( O \) of nontrivial projections. As \( |O| = 2 \), the kernel \( T_0 \) of the action of \( T \) on \( O \) is of index 2 in \( T \), so \( 1 \not= T_0 \) centralizes \( U \), and acts on its complement \( U' = (E - O) \). As \( W = [V, T_0] \preceq U \), \( T_0 \) centralizes \( U' \), so \( V = U + U' \preceq C_V(T_0) \), a contradiction. This establishes (1) and (2).

Write \( H^E \) for the image of \( H \) in \( \text{Sym}(E) \). We've shown that for \( i \in I \), \( P_i = \{ t_i \} \) with \( t_i^E \) a transposition. Let \( Q = \{ P_i : i \in I \} \). Then \( Q \cong E_{2^k} \), \( t_i \) is a reflection, and \( V_i = \langle [V, t_i] \rangle \). As \( H \) is transitive on \( E \), \( t_i^E \) is a transposition, and \( Q \cong E_{2^k} \), it follows that there exists a positive integer \( r \) such that \( m = 2r \) is even, \( Q^E \cong E_{2^r} \), and the orbits of \( Q \) on \( E \) form a regular \((2, r)\)-partition of \( E \). Let \( t_i^E \) be the transposition \( (U_{i1}, U_{i2}) \). Then \( Q \) acts on \( U(i) = U_{i1} + U_{i2} \), so as \( V = [V, Q] \), also \( U(i) = \{ U(i), Q \} \). Hence there exists \( j(i) \in I \) with \( U(i) = V_i + V_{j(i)} \). We conclude that \( k = 2r \) is even, and there is a partition \( \Sigma \) of \( I \) consisting of \( r \) blocks of size 2, such that for \( \sigma = [i, l] \in \Sigma \), we have \( l = j(i) \), \( t_i^E = t_l^E \), and \( Q^E \cong E_{2^r} \) is the direct product of the groups \( P_\sigma^E = \langle \sigma^E \rangle \cong Z_2 \). For \( \sigma \in \Sigma \), let \( V_\sigma = \langle V_i : i \in \sigma \rangle \). Then \( B = \{ V_\sigma : \sigma \in \Sigma \} \subseteq D(H) \), and by construction \( B \subseteq D \cap E \). Thus (3) holds. \( \square \)

In the next lemma we use the following notation: Given groups \( X \) and \( Y \), write \( X \ast Y \) for the central product of \( X \) and \( Y \) with identified centers, and \( X/Y \) for an extension of \( Y \) by \( X \).

\[ (4.8). \] Assume Hypothesis 4.5 with \( c = 5 \). Assume \( H \preceq G \preceq \text{GL}(V) \) with \( G \) primitive on \( V \). Then

\( (1) \) One of the following holds:

(a) \( \text{SL}(V) \preceq G \) and \( |\text{GL}(V) : G| = |K_1 : P_1| \leq 2 \).

(b) \( \Omega(V, q) \preceq G \) for some quadratic form \( q \) on \( V \).

(c) \( G \) has a normal subgroup \( S \cong S_n \) acting naturally on \( V \), with \( n = k + 2 \) if \( k \equiv -2 \) mod 5, and \( n = k + 1 \) otherwise.

(d) \( k = 2 \) and \( O^2(G) \cong \text{SL}_2(3) \).

(e) \( k = 4 \), \( H \) is solvable, and \( O^2(G) \) is \( A_6/(Z_A \ast Q_3^2) \), \( A_5/Q_3D_8 \), or \( \text{SL}_2(3) \ast \text{SL}_2(3) \).

(f) \( k = 6, 7, \) or \( 8 \), and \( O^2(G) \cong O^2(W(E_k)) \) is the Weyl group of a root system of type \( E_k \).

(2) If \( N_{\text{Alt}(V)}(D) \preceq H \), then (a) holds, or (d) holds with \( G = N_{\text{GL}(V)}(O^2(G)) \), or (e) holds with \( G \cong S_6/(Z_A \ast Q_3^2) \). Further in cases (d) and (e), \( N_{\text{Alt}(V)}(D) = H \).

**Proof.** By hypothesis, \( c = p = 5 \). Then \( T_1 \) contains a reflection \( r \), so \( G \) is described in Appendix to [W], which lists all primitive subgroups \( G \) of \( \text{GL}(V) \) which contain a reflection.

Next \( \det(r) = -1 \), so \( r \not\in \text{SL}(V) \). Thus if \( \text{SL}(V) \preceq G \) then as \( \text{GL}(V)/\text{SL}(V) \cong Z_4 \), \( |\text{GL}(V) : G| \leq 2 \), with equality iff \( G \) contains no element of determinant of order 4 iff \( |K_1 : P_1| = 2 \). Thus we may assume \( \text{SL}(V) \not\preceq G \).
As \( r \) is an \( F_5 \)-reflection rather than an \( F_{25} \)-reflection, \( G \) is not an extension of a unitary group. All other cases appearing in the list in \([W]\) appear in the list in (1), so (1) is established.

Assume \( J = N_{\text{Alt}(V)}(D) \leq H \). In cases (b), (c) and (f), \( G \) induces a group of similarities of a quadratic form \( q \) on \( V \). Then \( [r, V] \) is a nonsingular point in the orthogonal space \((V, q)\). Further as \( f \leq H \), there is \( g \in P \) of order 4 faithful on \([V, r]\) with \( \dim([V, g]) = 2 \), so as \( g \) is a similarity, \( k = 2 \). But then (b) or (c) holds and \( G \) has a normal subgroup of order 3. This is impossible as \( Q_8 \rtimes Z_3 \cong J \leq H \leq G \).

Thus we may assume (d) or (e) holds. In case (d), \( J \) contains \( Z(\text{GL}(V)) \) and the reflection \( r \), so \( N_{\text{GL}(V)}(O^2(G)) = O^2(G)J \leq \text{Alt}(V) \), and hence (2) holds in this case.

Thus we may assume (e) holds. Then \( k = 4 \) and \( |J| = 2^{10} \cdot 3 \). As \( |J| \) divides \(|G|\), we conclude \( G \cong S_6/(Z_4 \rtimes Q_8^2) \) and \( G = J \circ O^2(G) \leq \text{Alt}(V) \), so again (2) holds. \( \Box \)

(4.9). Assume Hypothesis 4.5 with \( c > 5 \), and assume \( H \leq G \leq \text{GL}(V) \) with \( G \) primitive on \( V \). Then one of the following holds:

1. \( |\text{GL}(V) : G| \leq |K_1 : P_1| \leq 2 \), so in particular \( \text{SL}(V) \leq G \).
2. \( c = 9 \), \( p = 3 \), \(|K_1 : P_1| = 2 \), \( G = \text{Sp}(V) \) or \( G\text{Sp}(V) \), and the members of \( D \) are nondegenerate lines of the symplectic space \( V \).
3. \( c = p = 7 \) or \( c = p = 11 \), \( k = 2 = \dim(V) \), \(|K_1 : P_1| = |N_{\text{GL}(V)}(D) : H| = 2 \), and \( G = N_{\text{GL}(V)}(L) \) where
   - (i) \( p = 7 \), \( L \cong \text{SL}_2(3) \) and \( G/Z(\text{GL}(V)) \cong S_4 \); or
   - (ii) \( p = 11 \), \( L \cong \text{SL}_2(5) \), and \( G = Z(\text{GL}(V)) \circ L \).

Proof. Assume \( G \) is a counter example. By 4.6.3, \( e = |K_1 : P_1| \leq 2 \), so \( T_1 \cap P = \langle g_1 \rangle \), where \( g_1 \) is a pseudo-reflection with eigenvalue \( \lambda \) of order \( e = (p - 1)/\epsilon \). In particular \( \det(g_1) = \lambda \).

Suppose first that \( \text{SL}(V) \leq G \). Then as \( \det(g_1) = \lambda \), of order \( e = (p - 1)/\epsilon \), \( |\text{GL}(V) : G| \leq \epsilon \leq 2 \). Thus (1) holds if \( \text{SL}(V) \leq G \), so as \( G \) is a counter example:

(a) \( \text{SL}(V) \not\leq G \).

Suppose next that the set \( \mathcal{X} \) of transvections in \( G \) is nonempty. Let \( X = \langle \mathcal{X} \rangle \), \( X_i = \langle x_i \rangle \), and \( W_i = [V, X_i] \). By 6.5 in [A5], \( X \) is the direct product of the groups \( X_i \), \( V \) is the direct sum of the subspaces \( W_i \), and \( X_i \) is irreducible on \( W_i \). Hence as \( G \) is primitive on \( V \), \( n = 1 \) and \( X \) is irreducible on \( V \). Then by work of McLaughlin in [M1] and [M2] (cf. 6.6 in [A5]), either \( X = \text{SL}(V) \) or \( \text{Sp}(V) \), or \( p = 2 \) and \( X = O^+(V) \) or \( S_n \), with \( V \) the natural module. By (a), \( X \) is not \( \text{SL}(V) \). In the remaining cases \( X = \text{Sp}(V) \). If \( p \leq 3 \) then as \( c > 5 \), either \( c = 9 \) or there is a 3-dimensional subspace \( U \) of \( V_3 \) and \( P_1 \) contains a subgroup \( Q \) acting faithfully on \( U \) as \( \text{SL}(U) \) and with \( V = U \oplus C_V(Q) \). In the former case if \(|K_1 : P_1| = 2 \) then (2) holds, so we may assume \( K_1 = P_1 \). In this case, and when \( p > 3 \), let \( Q = \langle g_1 \rangle \) and \( U = [V, g_1] \). Then in each case, \( Q \) is irreducible on \( U \) of odd dimension, so \( U \) is a totally singular subspace of the symplectic space \( V \). Therefore the representation of \( Q \) on \( V/U \) is dual to its representation on \( U \), impossible as \( U = [V, Q] \). Thus the lemma holds when \( \mathcal{X} \neq \emptyset \), so as \( G \) is a counter example:

(b) \( \mathcal{X} \) is empty.

It follows from (b) and 4.6.3 that:

(c) \( c = p \) and \( k = \dim(V) \).

Let \( B \) be the set of conjugates of \( \langle g_1 \rangle \). Thus for \( B \in B \), \( V_B = [V, B] \) is of dimension 1. Let \( A \in B - \{B\} \) and let \( U = V_{A,B} = V_A + V_B \) and \( Y = \langle A, B \rangle \). We claim:

(d) \( \dim(U) = 2 \) and \( V = U \oplus C_V(Y) \).

If \( V_A = V_B \) then \( Y \) contains a transvection with center \( V_A \), contrary to (b). Thus \( V_A \neq V_B \), so from (c), \( U \) is of dimension 2, and as \( \dim(V/C_V(C)) = 1 \) for \( C \in B \), it follows that \( \dim(V/U_1) \leq 2 \),
where $U_1 = C_V(Y)$. Thus if $U_0 = U \cap U_1 = 0$ then (d) holds, so we may assume $U_0$ is a point of $V$. Then $V = BR$, where $R$ is the centralizer in $Y$ of the flag $0 = U_0 < U < V$, and hence $R$ is a $p$-group. As $B$ is cyclic of order dividing $p - 1$, $B$ acts nontrivially on some subgroup $R_0$ of $R$ of order $p$, and replacing $A$ by a member of $B^{R_0}$, we may assume $R = R_0$ is of order $p$. As $\dim(U) = 2$, $R$ does not act on $V_B$ so $R$ is nontrivial on $U$. By (b), $U = [V, R]$, so $R$ has one nontrivial Jordan block on $V$, and that block is of size $3$. Thus $V = W + C_V(Y)$ with $\dim(W) = 3$, and hence $Y$ is faithful on $W$. We may regard $R$ as the subgroup of an orthogonal group $\Sigma = O(W) \cong O_3(p)$ on $W$. Then in $\Sigma$ there is $\sigma$ of order $e$ acting faithfully on $R$ with eigenvalue $\lambda$ on $U_0$ and $\lambda^{-1}$ on $W/U$. Now $B = (b)$, and $\Aut_{GL(W)}(R) = \Aut(R)$ is cyclic, so we may choose $\sigma$ with $f = b\sigma^{-1} \in C_{GL(W)}(R)$. Then replacing $\sigma$ by a conjugate under $\Op(U(C_{GL(W)}(R)))$, we may assume $f$ is a $p'$-element in $C_{GL(W)}(R)$. Thus $f = \mu_1$ is a scalar map with eigenvalue $\mu$, so as $b$ centralizes $U_0$ and $W/U$, we conclude $\mu = \lambda = \lambda^{-1}$, a contradiction as $|\lambda| = e \geq (p - 1)/2$ and $p = c > 5$. This completes the proof of (d).

By (d), we may regard $Y$ as a subgroup of $GL(U) \cong GL_2(p)$. Recall $A$ and $B$ are cyclic of order $e = (p - 1)/e$ with $e \leq 2$, $p > 5$ by (c) and our hypothesis that $c > 5$, and $Y$ contains no transvections by (b). Further for $B = (b)$, $|\det(b)| = e$. It follows from Dickson’s Theorem (cf. A.1.3 in [ASm]) that:

(e) One of the following holds:

(i) $Y$ is abelian.

(ii) $p = 7$, $|B| = 3$, and $V \cong SL_2(3)$ or $Z_3 \times SL_2(3)$.

(iii) $p = 11$, $|B| = 5$, and $V \cong Z_3 \times SL_2(5)$.

Let $\theta(B) = \{A \in B: [A, B] \neq 1\}$. If $\theta(B) = \emptyset$, then $(\mathcal{B}^C)$ is an abelian normal subgroup of $G$ not contained in $Z$, contradicting $G$ primitive on $V$. Thus it follows from (e) that:

(f) $p = 7$ or $11$ and $\theta(B) \neq \emptyset$.

Suppose next that $k = 2$, so that $V = U$. Then from (e) and Dickson’s Theorem, $N_{GL(V)}(Y)$ contains all overgroups of $Y$ in $GL(V)$ which do not contain $SL(V)$, so it follows that $G \leq N_{GL(V)}(Y)$. Then as $H \leq G$, it follows that (3) holds in this case. Thus we have shown:

(g) $\dim(V) > 2$.

Suppose that $V(B) = V_{A,B}$ is independent of $A \in \theta(B)$, and let $B^\perp = \{A \in B: V(A) = V(B)\}$. Then for $C \in B - A^\perp$, $C$ centralizes $V(A)$ and $A^\perp$, so $\{V(A): A \in B\}$ is a system of imprimitivity for $G$ on $V$, a contradiction. Thus we may pick $A, C \in \theta(B)$ such that $V_C \not\subset V_{A,B}$. Hence $W = V_{A,B} + V_C$ is of dimension $3$. Set $M = \langle A, B, C \rangle$ and let $y$ be the involution in $Y$. We claim:

(h) $M$ acts faithfully on $W$, so we can view $M$ as a subgroup of $GL(W) \cong GL_3(p)$.

As in the proof of (d), $\dim(V)/C_Y(M) \leq 3$, so either $V = W \oplus C_Y(M)$ or $W_0 = C_W(M) \neq 0$. In the former case, (h) holds, so assume the latter. Now $R = C_M(W_0) \cap C_M(W/W_0) \cap C_M(V/W)$ is a $p$-group, so it follows from (e) that $A$, $B$, and $C$, and hence also $M$, centralize $R$. Thus as $[W, y] = U$ is a complement in $W$ to $W_0$, $y \in Z(M)$, and hence $M$ also acts on $U$. But then also $y$ is the involution in $Y'$ = $\langle B, C \rangle$, as that involution centralizes $W_0$ and inverts $W/W_0$. Then by symmetry, $V_{B,C} = [V, Y'] = [V, y] = U$, contradicting $\dim(W) = 3$. This completes the proof of (h).

As $Y$ and $Y'$ are irreducible on $V_{A,B}$ and $V_{A,C}$, respectively, either $M$ is irreducible on $W$, or $C_W(Y) = C_W(Y')$. The latter contradicts (b) and (h), so $M$ is irreducible on $W$. Next $U = [W, y]$ and $C_W(y) = C_W(y)$, so $C_M(y)$ acts on $U$ and $C_W(y)$, and then from the proof of (g), $Y \subset C_M(y)$. However $GL_3(p)$ contains no irreducible subgroup $M$ with an involution $y$ such that $Y \subset C_M(y)$. This completes the proof of the lemma. □

(4.10). Assume Hypothesis 4.5, and assume $H \leq G \leq GL(V)$ with $\mathcal{D} \not\subset \mathcal{D}(G)$, and $E = \{U_1, \ldots, U_m\} \in \mathcal{D}(G)$ with $E < \mathcal{D}$ and $E$ is maximal subject to this constraint. Let $m = |E|$, $s = k/m$, $U^i = \{U_j: j \neq i\}$, $K' = K(E)$, $f_i = C_K(U^i)$, $Z$ be the subgroup of $G$ inducing scalars on each $U_i$, and $R = G \cap K'$. Then either
(1) $c = p$ and $s \leq 4$, so that $H \cap K(E)$ is solvable, or
(2) $F^+(R) = L_1 \cdots L_m Z$, where $L_i \leq J_i$ is quasisimple and irreducible on $U_i$. Thus $R/L_1 \cdots L_m$ is solvable.

**Proof.** As $E \in \mathcal{D}(G)$ and $H \leq G$, $E \in \mathcal{D}(H)$. Let $G_1 = N_G(U_1)$, $H_1 = H \cap G_1$, and $D_1 = \{V_i : V_i \leq U_1\}$. As $E \leq D$, $D_1 \leq \mathcal{D}(H_1, U_1)$ and $|D_1| = s > 1$. We apply 4.7, 4.8, and 4.9 to $\rho = (U_1, \text{Aut}_H(U_1), D_1, \text{Aut}_P(U_1), \text{Aut}_G(U_1))$ in the role of $(V, H, D, P, G)$.

First, suppose that $A_1 \in \mathcal{D}(\text{Aut}_G(U_1))$. Then $A_1 \in \mathcal{D}(\text{Aut}_H(U_1))$, so we conclude from our application of 4.7 to $\rho$, that either $A_1 \leq D_1$, or $\rho$ satisfies the conclusions of 4.7. Let $A = A^\rho$, where $A \in A_1$. Then $A \in \mathcal{D}(G)$ with $E < A$, and if $A_1 \leq D_1$, then $A \leq D$, contrary to $D \notin \mathcal{D}(G)$ and the maximality of $E$. Therefore $A_1 \not\leq D_1$, so $\rho$ satisfies the conclusions of 4.7. Thus by 4.7, $|A| = s$ and $B_1 = D_1 \land A_1$ is of order $r = s/2$. As $B_1 < D$ and $B_1 \in \mathcal{D}(\text{Aut}_G(U_1))$, we have shown that $r = 1$, so that $s = 2$ and hence (1) holds.

Therefore we may assume that $G_1$ is primitive on $U_1$. Therefore by 4.8 and 4.9, either $c = p$ and $s \leq 4$, so that $\text{Aut}_H(U_1)$ is solvable, or $F^+(\text{Aut}_G(U_1)) = X_1 Y_1$, where $X_1$ is quasisimple and irreducible on $U_1$, and $Y_1$ is the group of scalars in $\text{Aut}_G(U_1)$. We may assume the latter case holds. Then $X_1 = [X_1, P_1]$, so the preimage $L_1$ in $(P_i^G_1)$ of $X_1$ is isomorphic to $X_1$ and normal in $G_1$. Then setting $L_i = L_i^{g_i}$ for $g_i \in H$ with $U_i^{g_i} = U_i$, (2) holds. \[ \square \]

(4.11).

(1) If $p = 2$ and $\dim(V) > 2$ then $GL(V) \leq \text{Alt}(V)$.
(2) If $p$ is odd then for all $\mathcal{D} \in \mathcal{D}(H)$, $K(\mathcal{D}) \not\leq \text{Alt}(V)$.

**Proof.** If $p = 2$ and $\dim(V) > 2$ then $GL(V) = O^2(GL(V))$, so (1) holds. Thus we may assume $p$ is odd and $\mathcal{D} \in \mathcal{D}(H)$. Let $U \in \mathcal{D}$ and $g \in N_{GL(V)}(U)$ centralize each member of $\mathcal{D} - \{U\}$, with $(g)$ regular on $U^\#$. Let $q = |U|$ and $k = |\mathcal{D}|$. Then $g$ has $(q^k - q^{k-1})/(q - 1) = q^k - 1$ cycles of length $q - 1$, so $g$ is an odd permutation on $V$. This establishes (2). \[ \square \]

(4.12). Let $G = GL(V)$ or $GL(V) \cap \text{Alt}(V)$. Assume $\mathcal{D} \in \mathcal{D}(H)$ such that for $U \in \mathcal{D}$, $c = |U| > 4$ and $H = N_G(\mathcal{D})$. Then

(1) $H$ acts as $\text{Sym}(\mathcal{D})$ on $\mathcal{D}$.
(2) Either
   (a) $\mathcal{D}(H) = \{\mathcal{D}\}$, or
   (b) $c = p = 5$, $\dim(V) = 2$, $H \cong Z_4 \times Q_8$, $\mathcal{D}(H)$ is of order $3$, and $N_G(H)$ is transitive on $\mathcal{D}(H)$. Further $G = GL(V) \cap \text{Alt}(V) = Z(GL(V)) \text{SL}(V) \cong Z_4 \times \text{SL}_2(5)$, so $O_G(H) = \{H, N_G(H), G\}$.
(3) Either
   (i) $H$ is maximal in $G$, or
   (ii) $G = \text{Alt}(V)$, $O_G(H) = \{H, M, G\}$, $M = N_G(O^2(M))$, and one of the following holds:
      (a) $c = 5, k = 2$, and $O^2(M) \cong \text{SL}_2(3)$.
      (b) $c = 7, k = 2$, and $O^2(M) \cong Z_3 \times \text{SL}_2(3)$.
      (c) $c = 11, k = 2$, and $O^2(M) \cong Z_5 \times \text{SL}_2(5)$.
      (d) $c = 9, k = 2$, and $M \cong \text{Sp}_{4}(3)$.
      (e) $c = 5, k = 4$, and $M \cong S_6/(Z_4 \times Q_8)$.

**Proof.** We first prove (1). If $G = GL(V)$ then as $H = N_G(\mathcal{D})$, (1) is clear, so by 4.11.1 we may assume that $p$ is odd and $G = GL(V) \cap \text{Alt}(V)$. But then by 4.11.2, $K(\mathcal{D}) \not\leq \text{Alt}(V)$, so $N_{GL(V)}(\mathcal{D}) = N_G(\mathcal{D}) K(\mathcal{D})$ and so again (1) holds.

Next suppose (a) fails and let $E \in \mathcal{D}(H) - \{\mathcal{D}\}$. Observe that Hypothesis 4.5 is satisfied with $P = H \cap K(E)$. Therefore by 4.7, either $E < D$ or the various conclusions of 4.7 hold. But by (1), $H$ is primitive on $D$, so $D(H)$ contains no $E$ with $E < D$. Therefore we conclude from 4.7 that $c = p = 5$, $|P_i| = 2$ for each $i \in I$, and $|E| = |\mathcal{D}| = k = \dim(V)$ is even. Moreover $B = D \land E$ is of order $r = k/2$, \[ \square \]
so as \( H \) is primitive on \( D \), we must have \( B = \{ V \} \), and hence \( k = 2 \). As \( 5 \equiv 1 \mod 4 \), each reflection in \( GL(V) \) is in \( Alt(V) \), so \( H \) is the subgroup \( Z_4 \ast Q_8 \) of index 2 in \( N_{GL(V)}(D) \cong Z_4 \ast Z_2 \) generated by its reflections. Thus \( H \) has three \( E_4 \)-subgroups \( E_i, 1 \leq i \leq 3 \), each is normal in \( H \), and the members of \( D(H) \) are the sets \( \mathcal{W}(E_i), 1 \leq i \leq 3 \), where \( \mathcal{W}(E_i) \) is the set of weight spaces of \( E_i \). As \( N_C(H) \) is transitive on the three \( 4 \)-subgroups, it is also transitive on \( D(H) \). Moreover \( Z(GL(V))SL(V) \) is the subgroup generated by the involutions in \( GL(V) \), and hence this subgroup is contained in \( G \). Then \( N_C(H) \cong Z_4 \ast SL_2(3) \) is the unique maximal overgroup of \( H \) in \( G \), completing the proof of (2).

Suppose \( H < M < G \). By (2), \( M \) is primitive on \( V \). If \( c = 5 \) then conclusion ii.a or ii.e holds by 4.8.2. Thus we may assume \( c > 5 \). Therefore by 4.9, \( M \) satisfies conclusion (2) or (3) of 4.9. If conclusion (2) holds, then \( c = 9 \) and \( M \) is \( Sp(V) \) or \( GSp(V) \), and the members of \( D \) are nondegenerate. But as \( H = N_C(D) \), for each pair of distinct \( U, W \in D \), there exists an involution \( t_{U,W} \in H \) inducing a reflection on \( U \) and \( W \), and centralizing all other members of \( D \). It follows that \( k = 2 \) and \( G = GSp(V) \cong GSp_4(3) \), so ii.d holds in this case.

Thus we may assume conclusion (3) of 4.9 holds. But then ii.b or ii.c holds, completing the proof of the lemma. □

5. The poset \( \mathcal{F} = \mathcal{F}(\Omega) \)

In this section we assume that \( \Omega \) is a finite set and let \( S = \text{Sym}(\Omega) \) be the symmetric group on \( \Omega \). Recall the definition of a regular \((m, k)\)-product structure on \( \Omega \) from Section 2. Let \( \mathcal{F} = \mathcal{F}(\Omega) \) be the set of such structures.

We begin by defining a partial order on \( \mathcal{F} \).

Let \( \mathcal{F} = \{ \Omega_i: i \in I \}, \tilde{\mathcal{F}} = \{ \tilde{\Omega}_j: j \in \tilde{I} \} \) be regular \((m, k)\), \((\tilde{m}, \tilde{k})\) product structures on \( \Omega \), respectively. Set \( I = \{1, \ldots, k\} \) and \( \tilde{I} = \{1, \ldots, \tilde{k}\} \), and define \( \mathcal{F} \leq \tilde{\mathcal{F}} \) if there exists a positive integer \( s \) with \( \tilde{k} = ks \) and a regular \((s, k)\)-partition \( \Sigma = \{\sigma_i: i \in I\} \) of \( I \) such that for each \( i \in I \) and each \( j \in \sigma_i, \tilde{\Omega}_j \leq \Omega_i \), as defined in Section 3; that is the partition \( \Omega_i \) is a refinement of the partition \( \tilde{\Omega}_j \).

\((5.1)\).

(1) If \( \mathcal{F} \leq \tilde{\mathcal{F}} \leq \tilde{\mathcal{F}} \) is a tower of product structures of type \((k, m)\), \((\tilde{k}, \tilde{m})\), with \( \tilde{k} = ks \) and \( \tilde{k} = \tilde{k}s \), then \( \tilde{k} = kt \) where \( t = ss \), and \( \mathcal{F} \leq \tilde{\mathcal{F}} \).

(2) The relation \( \leq \) is a partial order on \( \mathcal{F} \).

Proof. It suffices to prove (1). Trivially \( \tilde{k} = kt \). By definition of the relation \( \leq \), there exists \((s, k)\) and \((\tilde{s}, \tilde{k})\) partitions \( \Sigma = \{\sigma_i: i \in I\} \) and \( \Delta = \{\delta_j: j \in \tilde{I}\} \) of \( I \) and \( \tilde{I} \), such that \( \tilde{\Omega}_u \leq \Omega_i \leq \tilde{\Omega}_i \) for \( j \in \sigma_i \) and \( u \in \delta_j \). Then as \( \leq \) is a partial order on partitions of \( \Omega, \tilde{\Omega}_u \leq \Omega_i \). For \( i \in I \), define

\[
\lambda_i = \bigcup_{j \in \sigma_i} \delta_j \subset \tilde{I}.
\]

Then \( \Lambda = \{\lambda_i: i \in I\} \) is a \((t, k)\)-partition of \( \tilde{I} \), and we showed that for \( u \in \lambda_i \), \( \tilde{\Omega}_u \leq \Omega_i \). Thus \( \mathcal{F} \leq \tilde{\mathcal{F}} \). □

\((5.2)\). Assume \( \mathcal{F} = \{\Omega_i: i \in I\} \) and \( \tilde{\mathcal{F}} = \{\tilde{\Omega}_j: j \in \tilde{I}\} \) are regular \((m, k)\), \((\tilde{m}, \tilde{k})\) product structures on \( \Omega \), respectively, \( I = \{1, \ldots, k\}, \tilde{I} = \{1, \ldots, \tilde{k}\} \), and \( \Sigma = \{\sigma_i: i \in I\} \) is a regular \((s, k)\)-partition of \( I \), such that for each \( i \in I \) and each \( j \in \sigma_i, \tilde{\Omega}_j \leq \Omega_i \). Then

(1) \( m = \tilde{m}s \).

(2) For each \( i \in I \),

\[
\Omega_i = \bigvee_{j \in \sigma_i} \tilde{\Omega}_j.
\]
(3) For each $i \in I$ and $A \in \Omega_i$, 

$$A = \bigcap_{j \in \sigma_i, A \subseteq \Omega_j} B.$$ 

(4) If $\hat{F}$ is a regular $(\hat{m}, \hat{k})$-product structure on $\Omega$ with $\mathcal{F} \subseteq \hat{F}$, then there exists a unique $(s, k)$-partition $\Sigma(\mathcal{F}, \hat{F}) = \{\hat{\sigma}_i; i \in I\}$ on $\hat{I} = \{1, \ldots, \hat{k}\}$ such that $\hat{\sigma}_i \subseteq \Omega_i$ for all $i \in \hat{\sigma}_i$.

**Proof.** As $m^k = |\Omega| = \hat{m}^k$ and $k = ks$, (1) holds. Let $i \in I$ and 

$$\Delta = \bigvee_{j \in \sigma_i} \hat{\Omega}_j,$$ 

so that $\Delta \in \mathcal{P}(\Omega)$, and as $\Omega_j \subseteq \Omega_i$ for all $j \in \sigma_i$, we have $\Delta \subseteq \Omega_i$. As $\hat{F}$ is a regular $(\hat{m}, \hat{k})$-product structure on $\Omega$, by definition of $\Delta$ the blocks of $\Delta$ are of size $\hat{m}^{\hat{k} - s}$. Further as $k = ks$, (1) implies that $m^{k - 1} = \hat{m}^{s(k - 1)} = m^{k - s}$. Thus as the blocks of $\Omega_i$ are also of size $m^{k - 1}$, and as $\Delta \subseteq \Omega_i$, (2) and (3) follow.

Finally assume the hypothesis of (4). As $\hat{F}$ is a regular product structure, $\hat{F}$ is injective, so the map 

$$\sigma \mapsto \bigvee_{i \in \sigma} \Omega_i$$

is an injection of the power set of $\hat{I}$ into $\mathcal{P}(\Omega)$, so (2) implies (4). □

We next recall the notion of composition of regular product structures, appearing in 1.11 of [A4].

**Definition 5.3.** Let $m, k, \hat{m}, \hat{k}$ be integers with $m, \hat{m} \geq 5$ and $k, \hat{k} > 1$. Let $I = \{1, \ldots, k\}$, $\hat{I} = \{1, \ldots, \hat{k}\}$, and let $\Gamma$ be an $m$-set and $\hat{\Gamma}$ an $\hat{m}$-set. Let $\mathcal{F} = \{\Omega_i; i \in I\}$ be a regular $(m, k)$-product structure on $\Omega$ and $\hat{\mathcal{F}} = \{\hat{\Omega}_j; j \in \hat{I}\}$ be a regular $(\hat{m}, \hat{k})$-product structure on $\hat{\Gamma}$. Recall from Section 2 in [A4] that we can choose bijections $f : \Omega \rightarrow \Gamma^I$ and $\hat{f} : \hat{\Gamma} \rightarrow \hat{\Gamma}^\hat{I}$ so that $\mathcal{F} = \mathcal{F}(f)$ and $\hat{\mathcal{F}} = \hat{\mathcal{F}}(\hat{f})$. That is $f(\omega) = (f_1(\omega), \ldots, f_k(\omega))$ and $\Omega_i = \{f_i^{-1}(\gamma); \gamma \in \Gamma\}$, and similarly for $\hat{f}$. Define $\tilde{m} = m$, $\tilde{\Gamma} = \hat{\Gamma}$, and $\tilde{I} = I \times \hat{I}$. Thus $\hat{k} = |\hat{I}| = kk$. Define 

$$\tilde{f} : \Omega \rightarrow \tilde{\Gamma}$$

by $\tilde{f}(\omega) = (\hat{f}(f_1(\omega)), \ldots, \hat{f}(f_k(\omega)))$, for $\omega \in \Omega$. That is $\tilde{f} = (\tilde{f}_{i,j}; (i, j) \in \tilde{I})$ has coordinate functions $\tilde{f}_{i,j} = \hat{f}_j \circ f_i$ for $(i, j) \in \tilde{I}$.

Visibly $\tilde{f}$ is an informal regular $(\tilde{m}, \tilde{k})$-product structure on $\Omega$, as defined in Section 2, giving rise to the formal product structure $\tilde{\mathcal{F}} = \mathcal{F}(\tilde{f}) = \{\Omega_{i,j}; (i, j) \in \tilde{I}\}$, where $\Omega_{i,j} = \{f_{i,j}^{-1}(\alpha); \alpha \in \Gamma\}$. We call $\tilde{\mathcal{F}}$ a composition of $\mathcal{F}$ and $\hat{\mathcal{F}}$, and sometimes write $\tilde{\mathcal{F}} \circ \mathcal{F}$ for such a composition.

Alternatively, as in Section 2, pick indexings $\Omega_i = \{\Omega_{i,\gamma}; \gamma \in \Gamma\}$ and $\Gamma_j = \{\Gamma_{j,\alpha}; \alpha \in \hat{\Gamma}\}$, and for $(i, j) \in \tilde{I}$ define

$$\Omega_{i,j} = \{\Omega_{i,j,\alpha}; \alpha \in \Gamma\}, \quad \text{where} \quad \Omega_{i,j,\alpha} = \bigcup_{\gamma \in \Gamma_{j,\alpha}} \Omega_{i,\gamma}.$$

Then $\tilde{\mathcal{F}} = \{\Omega_{i,j}; (i, j) \in \tilde{I}\}$ is a regular $(\tilde{m}, \tilde{k})$-product structure on $\Omega$ and a composition of $\tilde{\mathcal{F}}$ with $\mathcal{F}$.
(5.4). Let \( \hat{\mathcal{F}} \) be a composition \( \hat{\mathcal{F}} \circ \mathcal{F} \) of regular product structures \( \hat{\mathcal{F}} \) and \( \mathcal{F} \), and adopt the notation of 5.3. Then

1. \( \mathcal{F} \subseteq \hat{\mathcal{F}} \).
2. \( \Sigma(\mathcal{F}, \hat{\mathcal{F}}) = \{ \sigma_i : i \in I \} \) is the regular \((s, k)\)-partition of \( \hat{I} \) determined by 5.2.4, where \( \sigma_i = \{(i, j) : j \in I\} \) and \( s = k \).
3. We may view \( \Omega \) as the set of tuples \( \omega = (\omega_{i,j} : (i, j) \in \hat{I}) \) with \( \omega_{i,j} \in \hat{\mathcal{F}} \). \( \Gamma \) as the set of tuples \( \gamma = (\gamma_1, \ldots, \gamma_{\hat{k}}) \) with \( \gamma_i \in \hat{\mathcal{F}} \). \( \mathcal{F} = (\Omega_{i,j} \in \Omega : \alpha \in \Gamma) \), \( \Omega_{i,j} = \{ \Omega_{i,j,\alpha} : \alpha \in \Gamma_i \} \), and
   a) for each \( \alpha \in \Gamma_i \), \( \Omega_{i,j,\alpha} = \{ \omega \in \Omega : \omega_{i,j} = \alpha \} \), and
   b) for each \( \gamma \in \Gamma_i \), \( \Omega_{i,j,\gamma} = \{ \omega \in \Omega : \omega_{i,j} = \gamma \} \) for all \( j \in \hat{I} \).

Proof. For \( i \in I \) let \( \sigma_i = \{(i, j) : j \in I\} \subseteq \hat{I} \). Then \( \Sigma = \{ \sigma_i : i \in I \} \) is a regular \((\hat{k}, k)\)-partition of \( \hat{I} \). By definition in 5.3, for \( (i, j) \in I \) and \( \alpha \in \Gamma_i \), \( \Omega_{i,j,\alpha} \) is the union of the blocks \( \Omega_{i,j,\gamma} \), \( \gamma \in \Gamma_{i,\alpha} \), of \( \Omega_i \), so \( \Omega_{i,j} \subseteq \Omega_i \). Thus \( \mathcal{F} \subseteq \hat{\mathcal{F}} \), establishing (1). By 5.2.4, \( \Sigma = \Sigma(\mathcal{F}, \hat{\mathcal{F}}) \), so (2) holds.

From 5.3 we may identify \( \Omega \) with \( \hat{\mathcal{F}} \), \( \Gamma \) with \( \hat{\mathcal{F}} \), \( \Omega_{i,j} \subseteq \Omega_i \). Similarly from 5.3 we may identify \( \Gamma \) with \( \Gamma_i \) with \( \hat{\mathcal{F}} \), \( \gamma \in \Gamma_{i,\alpha} \), of \( \Omega_i \), so \( \Omega_{i,j} \subseteq \Omega_i \). Thus \( \mathcal{F} \subseteq \hat{\mathcal{F}} \), establishing (1). By 5.2.4, \( \Sigma = \Sigma(\mathcal{F}, \hat{\mathcal{F}}) \), so (2) holds.

(5.5). Let \( \mathcal{F}, \hat{\mathcal{F}} \) be regular \((m, k)\), \((\tilde{m}, \tilde{k})\) product structures on \( \Omega \), with \( \mathcal{F} \subseteq \hat{\mathcal{F}} \). Then \( \hat{\mathcal{F}} \) is a composition \( \hat{\mathcal{F}} \circ \mathcal{F} \) for some regular product structure \( \mathcal{F} \) on an \( m \)-set \( \Gamma \).

Proof. As \( \mathcal{F} \subseteq \hat{\mathcal{F}} \) there is an \((s, k)\)-partition \( \Sigma = \{ \sigma_i : i \in I \} \) of \( \hat{I} = \{1, \ldots, \hat{k}\} \), such that for each \( i \in I \) and \( j \in \sigma_i \), \( \Omega_{i,j} \subseteq \Omega_i \). Pick an \( s \)-set \( \hat{I} \). From 5.2.1, \( m = \tilde{m}^s \).

Let \( i \in I \) and \( j \in \sigma_i \). As \( \Omega_{i,j} \) has \( \tilde{m} \) blocks, and the refinement \( \Omega_i \) has \( m \) blocks, it follows that

\[ \ast \quad \text{each } A \in \Omega_{i,j} \text{ is the union of a set } P_{i,j}(A) \text{ of } m/\tilde{m} = \tilde{m}^{s-1} \text{ blocks of } \Omega_i, \text{ so } P_{i,j} = \{ P_{i,j}(A) : A \in \Omega_{i,j} \} \text{ is an } (\tilde{m}^{s-1}, 1)\text{-partition of } \Omega_i. \]

Let \( \tilde{m} = \tilde{m}, \tilde{k} = s, \) and for \( i \in I \) let \( \tilde{\mathcal{F}}_i = \{ P_{i,j} : j \in \sigma_i \} \). By 5.2.3, the set \( \tilde{\mathcal{F}}_i \) of partitions of \( \Omega_i \) is injective, so \( \tilde{\mathcal{F}}_i \) is a regular \((\tilde{m}, \tilde{k})\)-product structure on \( \Omega_i \).

For \( i \in I \), pick a bijection \( \varphi_i : \hat{I} \rightarrow \sigma_i \); then the map \( (i, l) \mapsto \varphi_i(l) \) is a bijection of \( I \times \hat{I} \) with \( \hat{I} \). Let \( (i, l) \in I \times \hat{I}, \ j = \varphi_i(l), \) and define \( \Omega_{i,j} = \{ B : B \in P_{i,j} \} \), where \( B = \bigcup_{C \in B} C \). Claim \( \Omega_{i,j} = \Omega_{i,j} \). Namely the members \( B \) of \( P_{i,j} \) are of the form \( P_{i,j}(A) \) for the various \( A \in \Omega_{i,j} \), and by definition of \( P_{i,j}(A) \) in \( \ast \), \( B = A \), so the claim holds.

By the claim, \( \tilde{\mathcal{F}} = (\Omega_{i,j} : j \in \sigma_i) \) is the partition \( \{ \Omega_{i,j} : (i, l) \in I \times \hat{I} \} \) of \( \Omega \) defined in 1.12 of [A4]. Hence the lemma follows from 1.12.1 in [A4].

(5.6). Assume the set up of 1.6 in [A4]. Assume \( s \) and \( \tilde{m} \) are positive integers with \( \tilde{m} \geq 5 \). Let \( \hat{I} = \{1, \ldots, \hat{s}\} \), and assume for each \( i \in I \) that \( E_i = \{ E_{i,j} : j \in \hat{I} \} \) is a set of subgroups of a subgroup \( E_i \) of \( D_i \) such that \( D_i = D_{i,\omega} E_{i}\).

(5.7). Assume the set up of 1.6 in [A4]. Assume \( s \) and \( \tilde{m} \) are positive integers with \( \tilde{m} \geq 5 \). Let \( \hat{I} = \{1, \ldots, \hat{s}\} \), and assume for each \( i \in I \) that \( E_i = \{ E_{i,j} : j \in \hat{I} \} \) is a set of subgroups of a subgroup \( E_i \) of \( D_i \) such that \( D_i = D_{i,\omega} E_{i}\).

E is a direct product of the subgroups \( E_{i,j} \), \( j \in \hat{I} \), and for each \( j \in \hat{I} \), \( |E_{i,j} : E_{i,j,\omega}| = \tilde{m} \). Let \( \hat{I} = l \times \hat{I} \) and \( \mathcal{E} = (E_{i,j} : (i, j) \in \hat{I}) \). Let \( \mathcal{E} = (E_{i,j} : (i, j) \in \hat{I}) \). Then

1. \( E \) is a direct product of the subgroups \( E_{i,j} \), \( (i, j) \in \hat{I} \), and for each \( \omega \in \Omega_i \),

\[ E = \prod_{(i,j) \in \hat{I}} E_{i,j} \quad \text{and} \quad E_{\omega} = \prod_{(i,j) \in \hat{I}} E_{i,j,\omega}. \]
(2) $\mathcal{F}(E)$ is a regular $(\bar{m}, ks)$-product structure on $\Omega$.

(3) $\mathcal{F}(D) \leq \mathcal{F}(E)$.

**Proof.** In Example 1.6 in [A4], $I = \{1, \ldots, k\}$, $D$ is a transitive subgroup of $S$, and $\mathcal{D} = \{D_i : i \in I\}$ is a collection of subgroups of $D$ such that $|D_i : D_{i,\omega}| = m$ for each $i \in I$.

\[
D = \prod_{i \in I} D_i \quad \text{and} \quad D_{\omega} = \prod_{i \in I} D_{i,\omega}.
\]

Then by 1.6 in [A4], $\mathcal{F}(D) = \{\Omega_i : i \in I\}$ is a regular $(m, k)$-product structure on $\Omega$, where for $i \in I$, $\Omega_i$ is the set of orbits of $D_i = \prod_{j \neq i} D_j$.

As $D_i = D_{i,\omega}E_i$ for each $i \in I$ and $D$ is transitive on $\Omega$, it follows from (\*) that (1) holds. Then by (1) we may apply 1.6 in [A4] to $E$ to conclude that (2) holds. Indeed $\mathcal{F}(E) = \{\Omega_i : i \in \bar{I}\}$, where for $i = (i, j) \in \bar{I}$, $\Omega_i$ is the set of orbits of $E_i = \prod_{j \neq i} E_j$. Then $D_i \subseteq E_i$, so for $\omega \in \Omega$, $\omega D_i \subseteq \omega E_i$. Thus $\Omega_i \subseteq \Omega_i$, so (3) holds. $\square$

See Definition 2.3 in [A4] for the definition of the various types of primitive groups, including affine, semisimple, doubled, complemented, and diagonal primitive groups. The product structures $\mathcal{F}(H)$ and $\mathcal{F}^2(H)$ are defined in Notation 2.6 and Lemma 5.11.6 of [A4]. For example if $H$ is semisimple with components $\mathcal{L}$, then we saw in Section 3 that the partitions of $\mathcal{F}(H)$ are indexed by $\mathcal{L}$, and for $L \in \mathcal{L}$, the partition determined by $L$ consists of the orbits of $\langle L \rangle$ on $\Omega$. Further when $H$ is neither affine nor semisimple and product decomposable, $\mathcal{F}(H)$ is the greatest member of $\mathcal{F}(H)$, while when $H$ is product decomposable, $\mathcal{F}^2(H)$ is the greatest member. The product structures $\mathcal{F}(H, K)$ (appearing below in Lemma 5.9) are also defined in Notation 2.6 of [A4].

**Definition 5.7.** A primitive group $H$ on $\Omega$ is pseudo-semisimple if one of the following holds:

(i) $H$ is semisimple and product decomposable. In this case set $\mathcal{F}^+(H) = \mathcal{F}^2(H)$.

(ii) $H$ is affine, and there is a unique maximal member $D^*$ of $\mathcal{D}(H)$. In this case set $\mathcal{F}^+(H) = \mathcal{F}(D^*)$.

(iii) $H$ is semisimple and product indecomposable, but not almost simple; doubled with more than two components; diagonal but not strongly diagonal; or complemented. In each of these subcases, set $\mathcal{F}^+(H) = \mathcal{F}(H)$.

**5.8.** If $H$ is pseudo-semisimple and $M \in \mathcal{O}_G(H)^\prime$ then either

(1) $M$ is pseudo-semisimple, or

(2) $H$ and $M$ are affine, $F^*(H) = F^*(M)$, and $M$ is primitive on $F^*(M)$.

**Proof.** If $M$ is almost simple then so is $H$ by Proposition 1 in [A4]. Hence $H$ is product decomposable from 5.7. Then $M$ is also product decomposable by 8.31 in [A4], so (1) holds. Thus we may assume $M$ is not almost simple.

If $M$ is diagonal, doubled, or complemented, then (1) holds by Proposition 6, 8, or 10 of [A4], respectively. Therefore we may assume $M$ is affine or semisimple.

If $H$ is affine, then by Proposition 4 in [A4] and as $M$ is not almost simple, either $M$ is semisimple and (1) holds, or $M$ is affine and $F^*(M) = F^*(H)$. Thus if $M$ is primitive on $F^*(M)$ then (2) holds, so we may assume $\mathcal{D}(M) \neq \emptyset$. As $H \leq M$, $\mathcal{D}(M) \subseteq \mathcal{D}(H)$. Thus if $D, E \in \mathcal{D}(M)$ then as $H$ is pseudo-semisimple, $D \leq D^* \geq E$ in $\mathcal{D}(H)$. Then by 4.3, $D \lor E$ exists in $\mathcal{D}(M)$. It follows that $\mathcal{D}(M)$ has a unique maximal member, so (1) holds.

Thus we may assume $M$ is semisimple but not almost simple. Thus $M$ is pseudo-semisimple, so that (1) holds. $\square$
(5.9). Let $H$ be a primitive group on $\Omega$. Then

1. If $H$ is pseudo-semisimple then $\mathcal{F}^+(H)$ is the unique maximal member of $\mathcal{F}(H)$.
2. If $H$ is affine and imprimitive on $F^+(H)$ then the structures $\mathcal{F}(D)$, $D$ maximal in $D(H)$, are the maximal members of $\mathcal{F}(H)$.
3. If $H$ is affine and primitive on $F^+(H)$, then $\mathcal{F}(H) = \emptyset$.
4. If $H$ is neither affine nor pseudo-semisimple then $\mathcal{F}(H) = \emptyset$.
5. Assume $H$ is semisimple, product indecomposable, and not almost simple. Let $L$ be a component of $H$. Then the map $K \mapsto \mathcal{F}(H, K)$ is an isomorphism of the dual of $\mathcal{O}_H(N_H(L))'$ with the poset $\mathcal{F}(H)$.

Proof. Assume $H$ is pseudo-semisimple but not affine or semisimple and product decomposable. We claim that the map $K \mapsto \mathcal{F}(H, K)$ is a bijection of $\mathcal{O}_H(U)'$ with $\mathcal{F}(H)$, and that $\mathcal{F}(H, U) = \mathcal{F}^+(H)$, where $U = N_H(L)$ for $L$ the product of the components in some $\sigma \in \Sigma(H)$ if $H$ is diagonal, and $L$ is a component of $H$ otherwise. (See 2.6 in [A4] for the definition of the notation $\mathcal{F}(H, K)$.) The claim follows from Proposition N in [A4] for $H$ diagonal, semisimple, doubled, complemented, and with $N$ equal to 7, 5, 9, 11, respectively.

Set $E = 1^H$ and for $K \in \mathcal{O}_H(U)'$, set $L_K = (L^K)$ and $D_K = L_K^H$. Then $\mathcal{F}(H, K) = \mathcal{F}(D_K)$, $\mathcal{F}^+(H) = \mathcal{F}(E)$, and $D_K$, $E$ satisfy the hypothesis of 5.6. Therefore (1) follows from lemma 5.6 in this case, as does (5).

Next suppose $H$ is affine. Then 4.1 in [A4] says that (3) holds, and the map $D \mapsto \mathcal{F}(D)$ is a bijection of $\mathcal{D}(H)$ with $\mathcal{F}(H)$.

Then 5.6 completes the proof of (1) and (2) in this case.

Suppose $H$ is semisimple and product decomposable, and pick a component $L$ of $H$. Then by 5.11.6 in [A4], the map $K \mapsto \mathcal{F}^2(H, K)$ is a bijection of $\mathcal{O}_H(K_1)'$ with $\mathcal{F}(H)$, where $K_1$ is the subgroup of index 2 in $N_H(L)$ defined in 5.1.4 of [A4], and $\mathcal{F}^2(H) = \mathcal{F}^2(H, K_1)$, so by 5.7, $\mathcal{F}^+(H) = \mathcal{F}^2(H, K_1)$. Then from 5.11 in [A4], $\mathcal{F}(H, K) \leq \mathcal{F}^+(H)$, completing the proof of (1).

Finally assume the hypothesis of (4). Then $H$ is almost simple and product indecomposable, or $H$ is doubled with two components, or $H$ is strongly diagonal. Hence by Proposition 2, 9, or 7, respectively, $H$ is contained in no semisimple group which is not almost simple. That is (4) holds.

(5.10). Let $L$ be the stabilizer in $S$ of a regular $(m, k)$-product structure on $S$, $A$ the alternating group on $\Omega$, and $K$ the kernel of the action of $M$ on $\mathcal{F}$. Then

1. $K \leq A$ iff $m$ is even.
2. $M \leq A$ iff $m$ is even and either $k > 2$, or $k = 2$ and $m \equiv 0 \mod 4$.
3. If $k = 2$ and $m \equiv 2 \mod 4$, then $M \cap A = K$, so $M \cap A$ is not primitive on $\Omega$. Otherwise $M \cap A$ induces $\text{Sym}(\mathcal{F})$ on $\mathcal{F}$.

Proof. Let $I = \{1, \ldots, k\}$. From Section 2, we may regard $\Omega$ as the set of tuples $\mathbf{a} = (a_1, \ldots, a_k)$ with $a_k$ in an $m$-set $\Gamma$, and $\mathcal{F} = \langle \Omega_i : i \in I \rangle$ with $\Omega_i = \{\Omega_{i,a} : a \in \Gamma\}$, where $\Omega_{i,a} = \{\mathbf{a} : a_i = a\}$. Further from 1.8 in [A4], $\Gamma = K_1 \times \cdots \times K_k$, where $K_i$ fixes each member of $\Omega_j$ for $j \neq i$ and acts faithfully as $\text{Sym}(\Omega_i)$ on $\Omega_i$. Let $t$ be a transposition in $K_1$. We can view $t$ as a transposition $(a, b)$ on $\Gamma$ with $t : \mathbf{a} \mapsto (a_1, a_2, \ldots, a_k)$. Thus $t$ has $m^{k-1}$ cycles $((a, a_2, \ldots, a_k), (b, a_2, \ldots, a_k))$ of length 2 on $\Omega$, so $t$ is odd iff $m$ is odd. Then as each $K_i$ is generated by its transpositions, (1) holds.

Recall from 1.8 in [A4] that $M$ induces $\text{Sym}(\mathcal{F})$ on $\mathcal{F}$. Suppose that $m$ is odd. Then (2) follows from (1) in this case. Further as $|S : A| = 2$ and $K \not\leq A$ by (1), $M = (M \cap A)K$, so as $M$ induces $\text{Sym}(\mathcal{F})$ on $\mathcal{F}$, so does $M \cap A$, proving (3) in this case. Therefore during the rest of the proof we may assume that $m$ is even. In particular $K \leq A$ by (1).

Define $t \in M$ by $t : \mathbf{a} \mapsto (a_2, a_1, a_3, \ldots)$. Then $t$ induces the transposition $(\Omega_1, \Omega_2)$ on $\mathcal{F}$, and fixes the $m^{k-1}$ points $(a, a, a_3, \ldots)$, so $t$ has $(m^k - m^{k-1})/2 = m^k/(m - 1)/2$ cycles of length 2. Hence as $m$ is even, $t$ is odd iff $k = 2$ and $m \equiv 2 \mod 4$. Further as $K \leq A$, $|S : A| = 2$, and $M/K$ is generated by transpositions on $\mathcal{F}$, we conclude that $M \not\leq A$ iff these conditions hold, and in that event $M \cap A = K$, so $(M \cap A)_\omega$ is not maximal in $M \cap A$. Thus (2) and (3) hold.
(5.11) Let $G$ be $S$ or $F^*(S)$. Given a regular $(m,k)$-product structure $F = (\Omega_i; i \in I)$ on $\Omega$ with $I = \{1, \ldots, k\}$, define $M(F) = N_G(F)$, $D(F) = F^*(M(F))$, and for $i \in I$ (cf. 1.8 in [A4]), write $D_i(F)$ for the component of $M(F)$ fixing $\Omega_j$ pointwise for each $j \in I - \{i\}$. Thus from 1.8 in [A4], $D_i(F)$ acts faithfully as the alternating group of degree $m$ on $\Omega_i$.

(5.12). Let $F, \widetilde{F}$ be regular $(m,k)$, $(\bar{m}, \bar{k})$ product structures on $\Omega$, respectively, such that $F < \widetilde{F}$, and let $\Sigma = (\Sigma(F), \widetilde{\Sigma})$ be the corresponding $(s,k)$-partition on $\bar{I}$ determined by 5.2.4. Then

1. $D(\widetilde{F}) \leq D(F)$.
2. Represent $\bar{M} = M(\widetilde{F})$ on $\bar{I}$ so that the map $I \mapsto \bar{D}_i = D_i(\widetilde{F})$ is an equivalence of the representations of $\bar{M}$ on $\bar{I}$ and on the set of components of $\bar{M}$ via conjugation. Then $H = M(F) \cap \bar{M}$ is the stablizer in $\bar{M}$ of $\Sigma$.
3. $H$ is maximal in $\bar{M}$.
4. For each $i \in I$, there is a regular $(\bar{m}, \bar{k})$-product structure $\tilde{F}_i$ on $\Omega_i$ such that $H$ is the subgroup of $M(F)$ permuting $\tilde{F}_i = \{\tilde{F}_i; i \in I\}$, and the subgroup of $\bar{M}$ permuting $\tilde{F}$.
5. Let $K$ be the kernel of the action of $M = M(F)$ on $\tilde{F}$. Then exactly one of the following holds:

(a) $H$ is maximal in $M$.
(b) $m$ is even and $s = 2$ if $\bar{m} \equiv 2 \pmod 4$. In this case $H \cap K$ is maximal in $O^2(K)$, $M = HK$, and $O_{M}(H) = \{H\}$ is isomorphic to $M_2$, $T_2$ or $T_1$ for $k$ odd, $k > 2$ even, $k = 2$, respectively.

Proof. As $F \leq \widetilde{F}$, 5.5 says that $\bar{F}$ is a composition $\bar{F} \circ F$. Hence, adopting the notation in 5.3, it follows from 5.4 that $k = ks$, and setting $I = \{1, \ldots, s\}$, we may regard $\bar{I}$ as $I \times \bar{I}$, $\Omega$ as the set of tuples $\omega = (\omega_{ij}); \bar{I}$, with $\omega_{ij} \in \bar{I}$, $\bar{\gamma} = (\bar{\gamma}_j; j \in \bar{I})$ with $\bar{\gamma}_j \in \bar{I}$, $\bar{F} = (\Omega_i, \bar{j})$, $\Gamma$ as the set of tuples $\gamma = (\gamma_j; j \in \bar{I})$, and $\bar{F} = (\Omega_i; i \in I)$, where $\Omega_{ij} = \{\Omega_{ij, \alpha}; \alpha \in \Gamma\}$, $\Omega_i = \{\Omega_{i, \gamma}; \gamma \in \Gamma\}$, and (a) and (b) of 5.3.3 hold.

Let $\gamma = (\gamma_1, \ldots, \gamma_s) \in \bar{I}$, $\bar{I} = \{(u, v) \in \bar{I}\}$, and set $\bar{D}_i = \bar{D}_i(\bar{F})$. By definition of $\bar{D}_i$, $\bar{D}_i$ is trivial on $\Omega$ for $\bar{I} \neq I$, while $\bar{D}_i$ acts faithfully as the alternating group on $\bar{I}$, with the map $\alpha \mapsto \Omega_{i, \alpha}$ an equivalence of the representation of $\bar{D}_i$ on $\Gamma$ and $\Omega_i$. As $d \in D_i$ fixes $\Omega_{i, \alpha}$, and hence fixes $\Omega_i$ pointwise. Similarly for each $\alpha \in \Gamma$, $\bar{D}_i$ is not fixes $\Omega_{i, \gamma}$, and, by definition of the action of $\bar{D}_i$ on $\Gamma$, $\Omega_{i, \alpha, \gamma} = \Omega_{i, \alpha, \gamma, \delta}$, so we conclude from (a) and (b) of 5.3.3 that $d = \Omega_{i, \alpha, \delta}$, for $j \neq \alpha$, and $\delta_j = \gamma_{\delta_j}$ for $j \neq \alpha$, and $\delta_j = \gamma_{\delta_j}$. In particular $\bar{D}_i$ permutes $\Omega_{i, \alpha}$ and fixes $\Omega_i$ pointwise for $i \neq I$. It follows from 1.8 in [A4] that $\bar{D}_i \leq D_i(F)$, so that (1) holds.

Indeed represent $\bar{M}$ on $\bar{I}$ as in (2) and let $\bar{K}$ be the kernel of that representation. Assume for the moment that $G = S$. Then from 1.8 in [A4], $\bar{K}$ is the direct product of groups $\bar{K}_i, i \in I$, where $\bar{K}_i$ is trivial on $\Omega_i$ for $i \in I - \{I\}$ and acts faithfully on $\Omega_i$ as the symmetric group. The argument in the previous paragraph shows that $\bar{K} \leq M = N_S(\bar{F})$. Further by 1.8 in [A4], there is a complement $\bar{T}$ to $\bar{K}$ in $\bar{M}$ acting faithfully as the symmetric group of $\bar{I}$, and so that for $i \in I, \omega \in \Omega$, and $y \in \bar{I}$,

\[
(\omega y)_i = \omega_{i, \gamma}^{-1}.
\]

Let $\bar{Y}$ be the stablizer of $\Sigma$ in $\bar{T}$. We can represent $\bar{Y}$ on $\bar{I}$ so that the bijection $i \mapsto \sigma_i$ is an equivalence of the representations of $\bar{Y}$ on $\bar{I}$ and $\sigma_i \in \Sigma$. By 5.4.2, $\sigma_i = \{(i, j); j \in \bar{I}\}$, for $y \in Y$, (i, j) $y = (iy, j)$ for some $j' \in \bar{I}$. Therefore by (1), \((\omega y)_{i, j} = (\omega y)_{i, j'} = \omega_{i, j}^{-1}\). Then from 5.3.3, for $\gamma \in \bar{\gamma}_i, \Omega_{i, \gamma} = \Omega_{i, \gamma'}$ for some $\gamma' \in \Gamma$, so $\bar{Y}$ permutes $\bar{\Sigma}$. That is $\bar{Y} \leq M$, so the stablizer $\bar{H}$ of $\Sigma$ in $\bar{M}$ is contained in $H$. Similarly if $\bar{t} \in \bar{T} - \bar{Y}$ then there is $i \in I$ such that $\sigma_i t \notin \Sigma$, and setting $\gamma = 1, 1, \in \Gamma, \Omega_{i, \gamma} t \neq \Omega_{i, \gamma'}$ for any $i' \in I$ and $\gamma' \in \Gamma$. Thus $H = \bar{H}$, establishing (2) in the case $G = S$.

On the other hand suppose $G = F^*(S)$ and let $\bar{H}$ be the stablizer in $N_S(\bar{F})$ of $\Sigma$. Then

\[H = \bar{M} \cap M(F) = N_G(\bar{F}) \cap N_G(F) = N_S(\bar{F}) \cap N_S(F) \cap G = \bar{H} \cap G,\]
As above, \( \tilde{k} = ks \) with \( k > 1 < s, \tilde{k} > 2 \). Therefore by 5.10.3, \( \tilde{M} = \tilde{M}^I = \tilde{M}/\tilde{K} \) is Sym(\( \tilde{I} \)). By (2), \( \tilde{K} \leq H \), so \( H \) is maximal in \( M \) iff \( \tilde{H} = H/\tilde{K} \) is maximal in \( M \). Further by (2), \( \tilde{H} \) is the stabilizer in \( M \) of \( \tilde{\Sigma} \). Thus as \( M = \text{Sym}(\tilde{I}) \), \( \tilde{H} \) contains a transposition, so \( \tilde{H} \not\leq \text{Alt}(\tilde{I}) \), and hence \( \tilde{H} \) is maximal in \( \tilde{M} \) from the Main Theorem of [LPS2]. This completes the proof of (3).

For \( i \in I \), set \( \tilde{F}_i = \{ p_i : i \in \sigma_i \} \), where \( p_i = \{ p_i(A) : A \in \Omega_i \} \) and \( p_i(A) = \{ b \in \Omega_i : b \subseteq A \} \). From the proof of 5.5, \( \tilde{F}_i \) is a regular \(( m, s )\)-product structure on \( \Omega_i \). For \( h \in H \), \( p_i(A)h = \{ bh \in \Omega_i : bh \subseteq Ah \} = p_i(Ah) \), so \( p_i h = p_i h \tilde{F}_i \). Therefore \( H \) is contained in the stabilizer \( H' \) of \( C = \{ \tilde{F} : i \in I \} \) in \( M \). By (3), \( H' = H \), while by 1.12.2 in [A4], \( N_{M}(\tilde{F})(C) \leq M(\tilde{F}) \cap M = H \). Thus (4) holds.

Let \( M = M(\tilde{F}) \) and \( K \) the kernel of the action of \( M \) on \( \tilde{\Sigma} \). Suppose for the moment that \( G = S \). As above, \( K = K_1 \times \cdots \times K_k \) with \( K_i \) acting faithfully as \( \text{Sym}(\Omega_i) \) on \( \Omega_i \), and \( M = K\tilde{T} \), where \( T \) acts faithfully on \( \tilde{F} \) as \( \text{Sym}(\tilde{F}) \). By (4), \( H = N_{M}(C) \), so we may choose \( T \) so that \( H = (H_1 \times \cdots \times H_k)T \), where \( H_i \) is the stabilizer in \( K_i \) of \( \tilde{F}_i \). If \( H \not\leq A_i = O^{2}(K_i) \), then from the Main Theorem of [LPS2], \( H \) is maximal in \( K_i \), so that (a) holds. Further by 5.10.2, \( H_i \not\leq A_i \) iff \( m \) is even, and \( m \equiv 0 \) mod 4 if \( s = 2 \). In this case \( H_i \) is maximal in \( A_i \) by the Main Theorem of [LPS2], and \( H \cap K \) is selfnormalizing in \( K \), so each proper overgroup of \( H \) in \( M \) contains \( O^{2}(K) \). Hence \( O_{M}(H) - (H) \) is isomorphic to the lattices of \( T \)-invariant submodules of \( K/O^{2}(K) \), so \( K/O^{2}(K) \) is the \( F \)-permutation module of degree \( k \) for \( T \equiv S_{b} \), (b) holds in this case.

Next assume \( G = A \) is the alternating group on \( \Omega \). If \( m \) is even then \( K \leq A = G \) by 5.10.1. Then the argument above applies and shows that (a) or (b) holds. Thus we may assume \( m \) is odd. Write \( J = J_1 \times \cdots \times J_k \) for the kernel of the action of \( N_{G}(\tilde{F}) \) on \( \tilde{\Sigma} \), and let \( \pi : J \rightarrow J_1 \) be the projection map. From 5.10.1, \( |J : K_1| = 2 \), \( J_1 = K \pi \), and \( H \pi \) is the stabilizer in \( J_1 \) of \( \tilde{F}_1 \), and hence maximal in \( J_1 \). It follows that \( H \) is maximal in \( M \), so that (a) holds, and the proof of (5) is complete. \( \square \)

(5.13). Let \( \tilde{F}, \tilde{F} \) be regular \(( m, k ), ( \tilde{m}, \tilde{k} ) \)-product structures on \( \Omega \), respectively. Let \( M = M(\tilde{F}), \tilde{M} = M(\tilde{F}) \), and \( Y = M \cap M \). Then the following are equivalent:

1. \( \tilde{F} \leq \tilde{F} \).
2. \( D(\tilde{F}) \leq D(\tilde{F}) \) and \( Y \) is transitive on the components of \( \tilde{M} \).
3. \( Y \) contains a primitive pseudo-semisimple subgroup \( H \) of \( G \) with \( F^{+}(H) = \tilde{F} \).

Proof. First (1) implies (2) by parts (1) and (2) of 5.12, and 5.10.3.

Assume (2) holds. Then \( \tilde{D} = D(\tilde{F}) = F^{+}(Y) \) as \( \tilde{D} = F^{+}(M) \) by 1.8 in [A4]. Further by 1.8 in [A4], the components of \( M \) are the groups \( D_j = D_j(\tilde{F}), j \in I \), and \( D_j \) acts as the alternating group on \( \Omega_j \), so \( D_{j, \omega} \) is maximal in \( D_j \). Then as \( Y \) is transitive on the components of \( M, Y \) is primitive, semisimple, and product indecomposable, but not almost simple as \( \tilde{k} > 1 \). Thus \( Y \) is pseudo-semisimple. By construction, \( F^{+}(H) = F(\tilde{M}) = \tilde{F} \). Hence (2) implies (3).

Finally assume (3). As \( H \leq M, \tilde{F} = F(M) \in F(H) \), so by 5.9.1, \( \tilde{F} = F^{+}(H) \geq \tilde{F} \). That is (3) implies (1). \( \square \)

(5.14). Let \( \tilde{F}, \tilde{F} \) be regular \(( m, k ), ( \tilde{m}, \tilde{k} ) \)-product structures on \( \Omega \), respectively. Then \( \tilde{F} \leq \tilde{F} \) iff \( D(\tilde{F}) \leq D(\tilde{F}) \).

Proof. By 5.12.1, it suffices to assume \( E = D(\tilde{F}) \leq D(\tilde{F}) = X \), and to prove \( \tilde{F} \subseteq \tilde{F} \). Let \( M = M(\tilde{F}) \), adopt the notation in the first two paragraphs of Notation 5.2 in [A4] with \( r = k \), let \( \{ \tilde{E}_j : j \in I \} \) be the set of components of \( E \), and for \( r \leq I \) define \( Q_{r} \), as in 5.2 of [A4]. The proofs of parts (1) and (2) of 5.3 in [A4] go through under our hypothesis. Then the proof of 5.7.1 in [A4] also goes through.

Suppose each \( E_i \) is contained in a component of \( M \). Let \( \mu = \{ j \in I : E \subseteq X_1 \} \) of order \( s \) and \( \sigma : E \rightarrow E_{\mu} \) the projection map. Then \( E_{\mu} \leq \langle X_1 : i \neq 1 \rangle = \ker(\pi) \), so \( E_{\mu} \leq Q_{1} \). Hence \( Q_{1} \sigma = E_{\mu} \sigma = E_{\mu} \), so \( E_{\mu} = E_{\pi} \) is transitive on \( \Gamma_1 = \omega X_1 \) by 5.7.1 in [A4]. Thus \( m = |\Gamma_1| = |E_{\mu} : E_{\mu, \omega}| = \tilde{m}^2 \), so \( s \) is independent of the choice \( X_1 \) of component of \( M \). Hence \( \tilde{F} \subseteq \tilde{F} \) by 5.6. Thus it suffices to show \( L = E_1 \) is contained in some component of \( M \).
Choose notation so that \( L \pi_1 \neq 1 \). We may assume \( L \not\leq X_1 \), so \( L \pi_1 \not\leq X_1 \). For \( \gamma \in \{1, 1' \} \), \( 1 \neq L \omega \pi_\gamma \leq X_{1', \omega} \), so \( L \pi_\gamma \) is not semiregular on \( \Gamma_\gamma \). Let \( \sigma : E \to L \) be the projection map of 5.7 in [A4]. As \( L \pi_\gamma \) is not semiregular on \( \Gamma_\gamma \), \( P_\gamma = Q_\gamma \sigma_\gamma \neq L \) by 5.71 in [A4]. But by 5.31 in [A4], \( E = Q_1 Q_\gamma \), so \( L = P_1 P_\gamma \). However \( L_\omega \not\leq P_1 \cap P_\gamma \), and \( L \) acts as the alternating group \( A_\bar{m} \) on \( \omega L \) of order \( \bar{m} \), so \( L_\omega \) is maximal in \( L \). Thus \( L = P_1 \) for some \( \gamma \in \{1, 1' \} \), contrary to an earlier remark. □

(5.15). Assume \( H \leq G \) is pseudo-semisimple, set \( \tilde{F} = F^+ (H) \), and assume \( H = M \cap \tilde{M} \), where \( M = M (F) \) for some \( F \in \mathcal{F} (H) \). Then

1. \( H \) is maximal in \( \tilde{M} \).
2. \( H \) is product indecomposable semisimple with \( F^* (H) = F^* (\tilde{M}) \).
3. \( \mathcal{F} (H) = \{ \tilde{F}, \tilde{F} \} \).
4. \( \mathcal{M} (H) = \{ M, M \} \).

Proof. By 5.9.1, \( F \subseteq \tilde{F} \), so (1) follows from 5.12.3, while part (2) follows from 5.12.2.

Let \( F_i \in \mathcal{F} (H) \) \( - \{ \tilde{F} \} \) and \( M_1 = M (F_i) \). Then \( H \leq \tilde{M} \cap M_1 \), so \( H = \tilde{M} \cap M_1 \) by (1). Let \( \Sigma = \Sigma (F, \tilde{F}) \). Then, in the notation of 1.6 in [A4], and appealing to 1.7 in [A4], \( \mathcal{F} = \mathcal{F} (E) \), where in the notation of the proof of 5.12, \( E = \{ D_\sigma : \sigma \in \Sigma \} \) and \( D_\sigma = \prod_i \sigma_i D_i \). Similarly \( F_i = \mathcal{F} (E_i) \) where \( E_i = \{ D_\sigma : \sigma \in \Sigma_i = \Sigma (F_i, \tilde{F}) \} \). But by 5.12.2, \( H \) is the stabilizer in \( \Sigma \) and \( \Sigma_i \), so \( \Sigma_i = \Sigma \), and hence \( E_i = E_1 \) and \( \mathcal{F} = \mathcal{F} (E_i) = \mathcal{F} (E_i) = F_i \). Therefore (3) holds. Finally (4) follows from (2), (3), and Proposition 5 in [A4]. □

(5.16). Let \( F_i, i = 1, 2 \), be regular \((m_i, k_i)\) product structures on \( \Omega \), such that \( F_1 \) and \( F_2 \) are not comparable in \( \mathcal{F} \). Set \( M_1 = M (F_i) \) and assume \( M_1 \cap M_2 = M_1 \) contains a primitive pseudo-semisimple subgroup \( H \). Then

1. \( \mathcal{F} = F_1 \lor F_2 \) exists in \( \mathcal{F} \). Let \( \mathcal{F} \) be an \((m, k)\)-structure, \( l = \{1, \ldots, k\} \), \( M = M (\mathcal{F}) \), \( D = D (\mathcal{F}) \), and \( H_i = M_i \cap M \).
2. \( H_i \) is the stabilizer of a regular \((s_i, k_i)\)-partition \( \Sigma_i \) on \( l \), where \( s_i = k/k_i \). Further \( \rho = (\Sigma_1, \Sigma_2) \) is an injective regular rank 2 chamber system on \( l \).
3. Let \( K \) be the kernel of the action of \( M \) on \( l \). Then \( M/K \) acts faithfully as the alternating or symmetric group on \( l \), \( H_i \) is the stabilizer in \( M \) of \( \Sigma_i \), and \( M_1 \cap H_2 = H_1 \cap H_2 \) is the stabilizer of the chamber system \( \rho \).
4. \( D \subseteq D (F_1) \lor D (F_2) \).
5. \( M_1 \) is primitive semisimple with \( F^* (M_1, 2) = D \), so \( M_1 \) is pseudo-semisimple and \( \mathcal{F} (M_1, 2) = \mathcal{F} (F_1, \lor F_2) \). For \( F^+ (H) \).

Proof. Pick \( H \) so that \( F^+ (H) \) is minimal subject to \( H \leq M_1 \) and \( H \) primitive pseudo-semisimple, and with \( H \) maximal subject to this constraint. Set \( \mathcal{F} = F^+ (H) \), \( M = M (\mathcal{F}) \), \( D = D (\mathcal{F}) \), and \( H_i = M \cap M_i \). We eventually show that \( \mathcal{F} = F_1 \lor F_2 \), so that this notation is consistent with that in the statement of the lemma.

As \( H \leq M_1 \), \( \mathcal{F} (M_1, 2) \subseteq \mathcal{F} (H) \), and \( M_1 \) is pseudo-semisimple by 5.8. Thus \( F^+ (M_1, 2) \subseteq F^+ (H) \) by 5.9.1, so \( H = M_1 \) by minimality of \( F^+ (H) \) and maximality of \( H \).

By 5.9.1, \( F_i \subseteq F \) for \( i = 1, 2 \), so by 5.14, \( D \subseteq D (F_1) \lor D (F_2) = E \). Thus \( D \leq M_1 \), \( H \), so \( D = F^* (H) \). Thus as \( M \) is semisimple, so is \( H \), and the pair \( H, M \) satisfies Hypothesis 5.1 of [A4]. By 5.3.4 of [A4], \( H \) is transitive on the set of components of \( M \), and on \( l \). Hence (5) holds, modulo the verification that \( \mathcal{F} = F_1 \lor F_2 \).

By 5.12, \( k = k_1 s_1 \) for some \( 1 < s_1 \in \mathbb{Z} \), and \( H_i \) is the stabilizer in \( M \) of a regular \((s_i, k_i)\)-partition \( \Sigma_i \) on \( l \). As \( H \leq H_i \) and \( H \) is transitive on \( l \), we conclude that \( \Sigma_1 \lor \Sigma_2 = \Sigma \) is a regular partition of \( l \). If \( \Sigma = \infty \) then (2) follows from the definition of injective regular chambers systems in Section 3, again modulo the verification that \( \mathcal{F} = F_1 \lor F_2 \).

So assume \( \Sigma \neq \infty \), and let \( \sigma \in \Sigma \). By 5.9.5, \( \mathcal{F} (m, N_H (\sigma)) \subseteq \mathcal{F} (H) \) and as \( \Sigma \geq \Sigma_1, \mathcal{F}_i \subseteq \mathcal{F} \not\subseteq \mathcal{F} \). Thus \( D = D (F^*) \subseteq M_1 \cap M_2 = H \), and \( D \leq D' \) by 5.12.1. As \( \mathcal{F} \subseteq \mathcal{F} (H) \), \( D' \) is \( H \)-invariant, so \( D' = F^* (H) = D \), and hence \( D = D' \). But now \( M = N_G (D) = N_G (D') = N_G (F^*) \), contradicting \( F \not\subseteq \mathcal{F} \).
We've seen that $D \leq E$. But from the embedding of $H_i$ in $M_i$ in 5.12.4, $N_D(F_i)(D)$ is in the kernel of the action of $H_i$ on $\Sigma_i$. Thus $E = D(F_1) \cap D(F_2)$ is in the intersection of the kernels of $H$ on $\Sigma_1$ and $\Sigma_2$, which is $K$ as $\Sigma = \infty$. Therefore (4) holds, again modulo verifying that $F = F_1 \cup F_2$, which we now establish.

Suppose $U \in F$ with $F_i \leq U$ for $i = 1, 2$. To show $F = F_1 \cup F_2$, we must show $F \leq U$. Let $U = D(U)$. As $F_i \leq U$, $U \leq D(F_i)$ by 5.14. Then as $E \leq K$, $U = U^\infty \leq E^\infty = D$, so $F \leq U$ by 5.14. This completes the proof of (1), (2), (4), and (5), so it remains only to verify (3).

Finally the remainder of (3) follows from these remarks and (2), so the proof of the lemma is complete. □

(5.17). Assume $H \leq G$ is pseudo-semisimple. Then one of the following holds:

\begin{enumerate}
  \item $\mathcal{M}(H) = \{M(F): F \in F(H)\}$.
  \item $H$ is affine and $\mathcal{M}(H) = \{N_G(F^\ast(H)), M(F(D)): D \in D(H)\}$.
  \item $H$ is octal semisimple and $\mathcal{M}(H) = S_0(H) \cup S_5(H)$, where $S_0(H) = \{M(F): F \in F(H)\}$, $S_5(H) = \{N_G(R): R \in A(H)\}$, $A(H)$ is the set of $H$-invariant affine structures on $\Omega$, and $N_5(H)$ is transitive on $A(H)$.
\end{enumerate}

Proof. If $H$ is octal semisimple, then (3) holds by Proposition 5 in [A4]. Similarly if $H$ is affine then (2) holds by 4.1 in [A4]. Thus we may assume $H$ is neither octal semisimple nor affine.

Let $M \in \mathcal{M}(H)$ and set $Q = \{M(F): F \in F(H)\}$. Thus $Q \subseteq \mathcal{M}(H)$, so we may assume $M \notin Q$, and it remains to derive a contradiction.

Next as $H$ is pseudo-semisimple, $H$ is not almost simple unless $H$ is product decomposable, and in the latter case $M \in Q$ by 8.3.2 in [A4]. Thus $H$ is neither almost simple nor affine, so $M$ is not almost simple by Proposition 1 in [A4]. As $H$ is pseudo-semisimple but not affine, $M$ is pseudo-semisimple by 5.8. Thus $M$ is not strongly diagonal, and $M$ is not affine as pseudo-semisimple affine groups $M$ act imprimitively on $F^\ast(M)$. Therefore $M \in Q$ by 2.5 in [A4], completing the proof. □

6. The proof of Theorem C

In this section we prove Theorem C, so we assume the hypothesis and notation of that theorem. In addition set $S = \text{Sym}(\Omega)$ and $A = \text{Alt}(\Omega) = F^\ast(S)$.

By Theorem 13 in [A4], we may assume that either:

\begin{enumerate}
  \item $\mathcal{M} = \{N_G(F): F \in F(H)\}$, or
  \item $n = p^e$ is a prime power, $H$ is affine, and $\mathcal{M} = \{N_G(D), N_G(F(D)): D \in D(H)\}$.
\end{enumerate}

Assume first that (I) holds. We claim $H$ is pseudo-semisimple. Suppose otherwise. As $\Lambda$ is an $I$-lattice, $|\mathcal{M}| > 1$. Thus by 5.9.4, $H$ is affine. But then (cf. 2.7 in [A4]) either $N_G(D) \in \mathcal{M}$ or $G = S$ and $N_G(D) \leq A$, so that $A \in \mathcal{M}$. As (I) holds, both cases are impossible, establishing the claim.

Let $F = F^\ast(H)$ and $M = M(F)$. As $\Lambda$ is an $I$-lattice, $H = M \cap M$ for some $M \in \mathcal{M}$, and by (I), $M = M(F)$ for some $F \in F(H) - \{\varnothing\}$. Therefore by 5.15, $F(H) = \{F, F\}$ and $\mathcal{M} = \{M, M\}$. By 5.15.1, $H$ is maximal in $M$, so by 1.1, $\Lambda \cong T_0 \square \Delta$, where $\Delta = \mathcal{O}(M,H)$. By 5.12.5, $\Delta - (H) \cong T_{-1}$, $M_2$, $T_2$, or $T_1$, so $\Lambda \cong M_2, M_14, T_{14}$, or $T_{1,3}$. But now conclusion (2) of Theorem C holds. Therefore:

(6.1). We may assume $n = p^e$ is a prime power, $H$ is affine, and $\mathcal{M} = \{N_G(D), N_G(F(D)): D \in D(H)\}$.

As $\Lambda$ is an $I$-lattice, there exists $M \in \mathcal{M} - \{N_G(D)\}$ with $H = N_G(D) \cap M = N_M(D)$. By 6.1, $M = N_G(F)$, where $F = F(D)$ for some $D \in F(H)$.

Suppose $D(H) \neq \{D\}$. Then by 4.12.2, $n = 25$, $H_\omega \cong \mathbb{Z}_4 \ast Q_8$, $D(H)$ is of order 3, $N_G(H)$ is transitive on $D(H)$, $G = A$, $N_G(D) \omega \cong \mathbb{Z}_4 \ast SL_2(5)$, and $\mathcal{O}(N_G(D))(H) = \{H, N_G(H), N_G(D)\}$. 

As $G = A$, $M$ is of index 2 in $N_S(F) ≃ S_5 \wr \mathbb{Z}_2$ by 5.10. Then as $H = N_M(D)$, $H$ is maximal in $M$. Therefore as $O_{N_C(D)}(H) ≃ T_1$, it follows that $A ≃ T_2 * T_1 * T_1 ≃ T_2,1,1,1$, so that conclusion (3ii) of Theorem C holds. Therefore:

(6.2). We may assume $D(H) = \{D\}$, so that $M = \{N_C(D), M\}$.

(6.3).

(1) $\Lambda \cong O_{N_C(D)}(H) \oplus O_M(H)$.
(2) If $H$ is maximal in both $N_C(D)$ and $M$, then $\Lambda \cong M_2$.
(3) $F$ is a regular $(m, k)$-product structure on $\Omega$, where $|D| = k$ and $m = |D_1| = p^f \geq 5$ for $D_1 \in D$.

Proof. Part (1) follows from 6.2 and 1.1. Then (1) implies (2). Part (3) follows from the fact that $F = F(D)$, and the definition of $F(D)$ in 1.6 in [A4]. □

(6.4). Either:

(1) $p$ is odd and $H$ is maximal in $M$, and $\Lambda \cong M_2$ if $H$ is maximal in $N_C(D)$, or
(2) $p = 2$, $G = A$, $H$ is maximal in $N_C(D)$, and $\Lambda \cong M_{1,4}, T_{1,4}$ or $T_{1,3}$.

Proof. From Section 2, $N_S(F)$ is the wreath product of $S_m$ with $S_k$. Let $K$ be the kernel of the action of $N_S(F)$ on $F$. From Section 2, $K = K_1 \times \cdots \times K_k$ with $K_i \cong S_m$.

Suppose first that $p = 2$. By 6.3.3, $m = 2^f \geq 5$, so $m \geq 8$. Hence by 4.11.1, $N_C(D) \leq A$. Then $N_C(H) \leq N_C(D) \leq A$, so by 3.7 in [A4], $G = A$. By 4.12.3, the stabilizer $H$ in $N_C(D)$ of $D$ is maximal in $N_C(D)$.

Next by 5.10.2, $N_S(F) \leq A$, so $M = N_S(F)$. Now $H = N_M(D)$ with $D = \{D_1, \ldots, D_k\}$, and from 1.6 in [A4] we may choose notation so that $D_1 = K_i \cap D$. Again from 4.11.1, $N_K(D_i) \leq O^2(K_i)$, so $H \cap K = N_K(D) \leq O^2(K)$. Further from Proposition 4 in [A4], $N_{O^2(K)}(D_1)$ is maximal in $O^2(K)$, so $H \cap O^2(K) = N_{O^2(K)}(D_1)$ is maximal in $O^2(K)$. Thus there is a bijection between $O_M(H) - \{H\}$ and the lattice $\Delta$ of $H$-invariant subgroups of $K/2^2(K)$. As $M = HK$ and $K/2^2(K)$ is the $F_2$-permutation module of dimension $k$ for $M/K \cong S_k$, it follows that $\Delta$ is $M_2$, $T_2, T_1$ for $k$ odd, $k > 2$ even, $k = 2$, respectively. Hence (2) holds by 6.3.1.

So assume that $p$ is odd. We must show that (1) holds, so by 6.3.2, it remains to show that $H$ is maximal in $M$. Let $KM = K \cap M$. By 5.10.1, $|K : KM| = |S : G| = e \in \{1, 2\}$. In any event $Aut_{KS}(K) \cong S_m$, from the Main Theorem of [LPS2], $N_{Aut_{KS}(K)}(D_i)$ is maximal in $Aut_{KS}(K)$. In particular $N_{K_i}(D_i) \not\cong O^2(K_i)$, so as $K_i$ is transitive on its regular elementary abelian subgroups, $O^2(K_i)$ is also transitive on such subgroups. Thus $D^M = D^O^2(K)$, so by a Frattini argument, $M = HO^2(K)$. Then as $N_{Aut_{KS}(K)}(D_i)$ is maximal in $Aut_{KS}(K)$, $H$ is maximal in $M$, completing the proof. □

Observe that in case (2) of 6.4, conclusion (3iv) of Theorem C holds, while in case (1) of 6.4, conclusion (3i) of Theorem C holds when $H$ is maximal in $N_C(D)$. Therefore:

(6.5). We may assume that $p$ is odd and $H$ is not maximal in $N_C(D)$.

(6.6). $G = A$, $O_{N_C(D)}(H) \cong T_1$, and $n = 5^2$, $7^2$, $11^2$, $3^4$, or $5^4$.

Proof. This follows from 6.5 and 4.12.3. □
7. Almost simple groups

In this section we write $\mathcal{I}$ for the set of tuples $\xi = (H_1, \ldots, H_m; \Omega)$ such that $m$ is a positive integer, $\Omega$ is a finite set, $H_1 < H_2 < \cdots < H_m \leq \text{Sym}(\Omega)$, and $H_m$ does not contain the alternating group on $\Omega$. The length of the tuple $\xi$ is $m$ and we sometimes write $(H_1, \ldots, H_m; n)$ for $\xi$ if $\Omega$ is an $n$-set.

An isomorphism $\varphi : \xi \to \xi'$ of tuples is a bijection $\varphi : \Omega \to \Omega'$ such that $\xi$ and $\xi'$ have the same length and $H_i\varphi^* = H'_i$ for all $1 \leq i \leq m$, where $\varphi^* : \text{Sym}(\Omega) \to \text{Sym}(\Omega')$ is the isomorphism $\varphi^* : \sigma \mapsto \varphi^{-1}\sigma\varphi$. Write $[\xi]$ for the collection of tuples isomorphic to $\xi$, and write $[\mathcal{I}]$ for the set of isomorphism types of tuples. The following elementary observation shows that the quasiequivalence classes of faithful permutation representations of finite groups of degree $n = |\Omega|$, other than $A_n$ and $S_n$:

(7.1). Tuples $\xi = (H_1, \ldots, H_m; \Omega)$ and $\xi' = (H'_1, \ldots, H'_m; \Omega)$ in $\mathcal{I}$ are isomorphic iff there exists $g \in \text{Sym}(\Omega)$ with $H_i^g = H'_i$ for each $1 \leq i \leq m$.

Recall from Section 4 of [FGT] that if $C$ is a category, and for $i = 1, 2$, $\pi_i : G_i \to \text{Aut}(X_i)$ are representations of groups $G_i$ on objects $X_i$ in $C$, then a quasiequivalence of the representations is a pair $\alpha, \beta$, where $\beta : G_1 \to G_2$ and $\alpha : X_1 \to X_2$ are isomorphisms such that for each $g \in G$, $g\pi_1\alpha = \alpha g\pi_2$.

Write $\mathcal{P}_\mathcal{I}$ for the set of $\xi \in \mathcal{I}$ of length 1, and $[\mathcal{P}_\mathcal{I}]$ for the set of isomorphism types of members of $\mathcal{P}$. The following elementary observation shows that $[\mathcal{P}]$ is essentially the set of quasiequivalence classes of faithful permutation representations of finite groups of degree $n = |\Omega|$, other than $A_n$ and $S_n$:

(7.2). Tuples $\xi = (H, \Omega)$ and $\xi' = (H', \Omega')$ in $\mathcal{P}$ are isomorphic iff the representations of $H$ on $\Omega$ and $H'$ on $\Omega'$ are quasiequivalent.

Proof. This is a consequence of 7.1 and Exercise 1.7 in [FGT]. □

Define $\mathfrak{A}$ to be the collection of $(H, \Omega) \in \mathcal{P}$ such that $H$ is almost simple, primitive, and product indecomposable on $\Omega$, but not octal. Let $[\mathfrak{A}]$ be the set of $[\xi] \in \mathcal{P}$ such that $\xi \in \mathfrak{A}$.

(7.3). Let $\xi = (H, \Omega) \in \mathfrak{A}$, $L = F^*(H)$, and $S = \text{Sym}(\Omega)$. Then

1. The stabilizer $M(\xi)$ in $\text{Aut}(L)$ of the equivalence class of the representation of $L$ on $\Omega$ is isomorphic to $N_S(L)$. Equivalently, $M(\xi) = N_{\text{Aut}(L)}(L_{\omega})$, for $\omega \in \Omega$.
2. $N_{M(\xi)}(H) \cong N_S(H).
3. For each $H' \in \mathcal{O}_S(H)$, $(H', \Omega) \in \mathfrak{A}.$

Proof. As $(G, \Omega) \in \mathfrak{A}$, $G$ is primitive on $\Omega$, so $L$ is transitive on $\Omega$. Thus the quasiequivalence class, equivalence class of the representation $\pi$ of $L$ on $\Omega$ is determined by $L^L_{\omega}$, $L^L_{\omega}$ respectively, (Cf. 5.9 in [FGT]). Hence (1) follows from 1.12 in [SG]. Then (1) implies (2). Finally (3) is a consequence of 8.5 in [A4]. □

Notation 7.4. For $\tau = (H_1, \ldots, H_m; n) \in \mathcal{I}$, define $\phi(\tau) = (F^*(H_1), \ldots, F^*(H_m); n)$, where $F^*(H_i)$ is interpreted as the isomorphism class of the permutations group $(F^*(H_i), \Omega)$.

Define $\mathcal{S}$ to be the set of triples $([\xi_1], [\xi_2], n) \in \mathcal{I} \times \mathcal{I} \times \mathbb{Z}$ such that $(H_1, \Omega) = \xi_1$, $(H_1, H_2; \Omega) \in \mathcal{I}$, $H_1$ is maximal in $H_2$, and $n = |\Omega|$. Set $\phi([\xi_1], [\xi_2], n) = \phi(H_1, H_2, n)$, and $\Phi = \{(L_1, L_2, n) \in \phi(\mathcal{S}) : L_1 \neq L_2\}$. Tables III–VI in [LPS2] list the set $\Phi$. We refer to this collection of tables as the Tables.

Set $\mathcal{L} = \phi(\mathfrak{A})$. Define the relation $\preceq$ on $\mathcal{L}$ by $(L, n) \preceq (L', n')$ if $n = n'$ and $(L, L', n) \in \Phi$. Write $\preceq$ for the transitive extension of $\preceq$ on $\mathcal{L}$. Thus $\preceq$ is a partial ordering of $\mathcal{L}$. Write $\mathcal{L}^*$ for the maximal members of the poset $\mathcal{L}$. Write $\mathcal{L}_\lambda$ for the set of $\lambda \in \mathcal{L}$ such that there exists a unique $\mu \in \mathcal{L}$ with $\lambda \preceq \mu$, and further $\mu \in \mathcal{L}^*$.
For \( \lambda \in \mathcal{L} \), set

\[
\mathcal{O}(\lambda) = \{ \lambda' \in \mathcal{L} : \lambda \leq \lambda' \}, \quad \mathcal{M}(\lambda) = \mathcal{O}(\lambda) \cap \mathcal{L}^* \quad \text{and} \quad \Phi(\lambda) = \{ \lambda' \in \mathcal{L} : \lambda \leq \lambda' \}.
\]

(7.5) Let \( \lambda = (\text{Sp}_{2a}(q), n) \in \mathcal{L} \), where \( a, k \) are positive integers with \( a > 1 \), \( (a, k) \neq (2, 1) \), \( q = 2^k \), \( \epsilon = \pm 1 \), and \( n = q^a(q^\theta + \epsilon)/2 \). Then

(1) If \( q = 2 \) then \( \lambda \in \mathcal{L}^* \).
(2) If \( q > 2 \) then \( \Phi(\lambda) = \{ (\text{Sp}_{2ab}(2^{k/b}), n) : b \text{ is a prime divisor of } k \} \).
(3) If \( q > 2 \) then \( \mathcal{O}(\lambda) = \{ (\text{Sp}_{2ac}(2^{k/c}), n) : c \text{ is a divisor of } k \} \).
(4) \( \mathcal{M}(\lambda) = \{ (\text{Sp}_{2ak}(2), n) \} \).

Proof. We inspect the Tables for a triple \((\text{Sp}_{2a}(q), L, n)\). The only triples with first entry \(\text{Sp}_{2a}(2^k)\) appear in Table VI, and as \( n \) is even, the triple is not in row one. Further if \( (a, k) = (11, 1) \) or \( (3, 1) \) then \( n \) is not \( |\Omega^+_{2a}(2) : CO_1| \) or 960, respectively. Thus the triple appears in row two of Table VI. This implies (1) and (2), as the second entry in the row is \(\text{Sp}_{2ab}(2^{k/b})\) for some prime \( b \). Then (3) and (4) follow from (1) and (2) by induction on \( k \). \( \square \)

(7.6) Let \( \lambda = (P\Omega^+_{8}(q), q^3(q^2 - 1)/(2, q - 1)) \in \mathcal{L} \). Then either \( \lambda \in \mathcal{L}^* \), or \( q = 2 \) and \( \Phi(\lambda) = \{ (\text{Sp}_8(2), 120) \} \).

Proof. Let \( n = q^3(q^2 - 1)/(2, q - 1) \). We inspect the Tables for a triple \((P\Omega^+_{8}(q), L, n)\). The only triples with first entry \(P\Omega^+_{8}(q)\) appear in Table VI. As \( 4 \neq ab \) with \( a > 1, b \) an odd prime, the triple is not in row three. This forces \( q = 2 \), so that \( n = 120 \), and hence \((L, n) \neq (P\Omega^+_{8}(3), 1120)\). On the other hand the lemma holds in the only remaining case, where the triple appears in row five. \( \square \)

(7.7) Let \( \lambda = (A_m, n) \in \mathcal{L} - \mathcal{L}^* \). Then

(1) If \( m = 7 \) then \( \Phi(\lambda) = \{ (A_8, 15), (A_8, 120), (A_9, 120) \}, or \{ (A_8, 35) \} \).
(2) If \( m = 8 \) then \( \Phi(\lambda) = \{ (\text{Sp}_6(2), 28) \} \text{ or } \{ (\text{Sp}_6(2), 120) \} \).
(3) If \( m = 9 \) then \( \Phi(\lambda) = \{ (\Omega^+_{8}(2), 120) \} \text{ or } \{ (A_{10}, 126) \} \).

Proof. The only triples in the Tables with first entry \(A_m\) are \((A_m, L, n)\) where \((L, n)\) is listed in the lemma in the part corresponding to \( m \). \( \square \)

(7.8) Let \( \lambda = (U, n) \in \mathcal{L} - \mathcal{L}^* \) with \( U = P\Omega^+_{2m}(q) \) and \( m \geq 3 \). Then either

(1) \( q \) is even and \( \lambda \in \mathcal{L}^* \) or
(2) \( \lambda = (P\Omega^+_{2m}(3), 1080) \) and \( \mathcal{O}(\lambda) = \{ (U, n), (L, n), (K, n) \} \) where \( L = \Omega_7(3) \) and \( K = P\Omega^+_{2m}(3) \).

Proof. As \( \lambda \notin \mathcal{L}^* \), there exists \( \tau = (U, L, n) \in \Phi \). Let \( \mu = (L, n) \). From the Tables there are four possible cases:

(i) \( \mu = (\text{Sp}_{2m}(q), n) = |\text{Sp}_{2m}(q)/|\text{Sp}_{2a}(q)|b| \), with \( q \) even, \( ab = a > 1, b \) prime, and \( (b, \epsilon) \neq (2, +) \).
(ii) \( \epsilon = -1, q \) and \( m \) are even, \( \mu = (\text{Sp}_{2m}(q), n) = |\text{Sp}_{2m}(q)|/2|\text{Sp}_{m}(q)|^2 \).
(iii) \( q = 2, \epsilon = -1, \mu = (\text{Sp}_{2m}(2), 2^{m-1}(2^m - \epsilon)) \).
(iv) \( U = P\Omega^+_{2m}(3) \) and \( \mu = (P\Omega^+_{2m}(3), 1120) \).
(v) \( U = P\Omega^+_{2m}(3) \) and \( \mu = (\Omega_7(3), 1080) \).

In case (v), from the discussion in the proof of 7.10, \( \Phi(\mu) = \{ (P\Omega^+_{2m}(3), n) \} \), so (2) holds by 7.6. Thus we may assume \( q \) is even. Observe also that the value of \( n \) is different in cases (i)-(iv), so to show \( \lambda \notin \mathcal{L}^* \), it remains to show \( \mu \notin \mathcal{L}^* \) in each case.
In case (iii), \( \mu \in \mathfrak{L}^* \) by 7.5.1. In case (iv), \( \mu \in \mathfrak{L}^* \) as 3 is odd. In cases (i) and (ii), \( L \cong \text{Sp}_{2m}(q) \), inspecting the Tables for entries \( \tau = (L, K, n') \) with \( L \cong \text{Sp}_{2m}(q) \) and \( m \geq 3 \), we determine that \( n' \neq n \) for each such entry, completing the proof. \( \Box \)

\[(7.9) \text{. Let } \lambda = (U, n) \in \mathfrak{L} - \mathfrak{L}^* \text{ with } U \cong P\Omega_{2r-1}(q), r \geq 4. \text{ Then one of the following holds:}
\]

1. \( \lambda \in \mathfrak{L}_+ \). Moreover if \( r = 4 \) and \( n = q^3(q^4 - 1)/(2, q - 1) \), then \( \Phi(\lambda) = \{(P\Omega_8^+(q), n)\} \).

2. \( q \) is even and \( \lambda = (\text{Sp}_{2a}(q^b), \text{Sp}_{2ab}(q, q^{ab} + q^a + q^b + 2)) \) for some \( a > 1 \) and prime \( b \).

3. \( \lambda = (\text{Sp}_6(2), 120) \) and \( \Phi(\lambda) = \{(\Omega_8^+(2), 120)\} \).

**Proof.** As \( \lambda \notin \mathfrak{L}^* \) there is \( \tau = (U, L, n) \in \Phi \). Set \( \mu = (L, n) \). From the Tables there are seven possible cases:

1. \( L = P\Omega^+_{2r}(q) \) and \( n = \prod_{i=1}^{r-1}(q^i + 1) \).
2. \( r = 4, L \cong P\Omega^+_{8}(q) \), and \( n = q^2(q^4 - 1)/(2, q - 1) \).
3. \( r = 4, q = 3, L \cong P\Omega^+_{8}(3) \), and \( n = 28431 \).
4. \( q \) even and \( \tau = (\text{Sp}_{2a}(q^b), \text{Sp}_{2ab}(q), q^{ab} + 2) \) for some \( a > 1 \) and prime \( b \).
5. \( q = 2, \Omega^+_{24} \), \( n \) where \( n = |\Omega^+_{24}: \text{Co}_1| \).
6. \( q = 2, \tau = (\text{Sp}_6(2), \Omega^+_{8}(2), 960) \).

Suppose first that \( q \) is odd, so that one of cases (i)–(iii) hold. Observe for fixed \( q \), that \( n \) is different in each of the three cases. Further by 7.8, \( \mu \in \mathfrak{L}^* \), so that (1) holds in this case. Therefore we may assume \( q \) is even, so case (iii) does not hold. Recall \( \Omega_{2r-1}(q) = \text{Sp}_{2r-2}(q) \).

In case (v), (2) holds, and in case (ii), (3) holds when \( q = 2 \), so we may assume these cases do not hold. Again we check that in each of the remaining five cases, \( \lambda \) is different.

In case (i), \( n \) is odd, while we check that whenever \( (\Omega^+_{2r}(q), K, n') \in \Phi, n' \) is even. Thus \( \Phi(\mu) = \emptyset \) in case (i), so (1) holds.

In case (ii), \( \Phi(\mu) = \emptyset \) as \( q \neq 2 \) in this case. Thus (1) holds in this case. In case (iv), \( m \geq 3 \) so \( \mu \in \mathfrak{L}^* \) by the discussion in case (vi) in 7.15, so (1) holds. Finally we check in cases (vi) and (vii) that \( \Phi(\mu) = \emptyset \), so (1) holds. The proof is complete. \( \Box \)

\[(7.10) \text{. Let } \lambda = (A_7, 120) \in \mathfrak{L}. \text{ Then}
\]

1. \( \mathcal{O}(\lambda) = \{(L, 120); L \in \mathcal{L}\}, \) where \( \mathcal{L} = \{A_7, A_8, A_9, \text{Sp}_6(2), \Omega^+_{8}(2), \text{Sp}_8(2)\} \).
2. \( \mathcal{M}(\lambda) = \{\text{Sp}_8(2), 120)\} \).
3. \( \mathcal{M}(L, 120) = \{\text{Sp}_8(2), 120)\} \) for each \( L \in \mathcal{L} \).

**Proof.** Part (1) follows from 7.5–7.7 and 7.9, which also show that \( \text{Sp}_8(2) \) is the unique \( L \in \mathcal{L} \) such that \( \Phi(L, 120) = \emptyset \), so (2) also follows. Then (1) and (2) imply (3). \( \Box \)

\[(7.11) \text{. Let } \lambda = (A_m, n) \in \mathfrak{L}. \text{ Then one of the following holds:}
\]

1. \( \lambda \in \mathfrak{L}^* \).
2. \( \lambda \in \mathfrak{L}_+ \).
3. \( (m, n) = (6, 15), (10, 2520), \) or \( (22, |A_{24}: M_{24}|) \), and \( \mathcal{O}(\lambda) = \{(A_m, n) : |A_{m+1}, n) < (A_{m+2}, n)\} \).
4. \( n = 120 \) and \( (A_7, n) \leq (A_m, n) \) with \( 7 \leq m \leq 9 \).

**Proof.** We may assume neither (1) nor (2) hold, so:

(a) There exists \( \tau = (A_m, L, n) \in \Phi \), and
(b) either \( \mu = (L, n) \notin \mathfrak{L}^* \) or \( |\Phi(\lambda)| > 1 \).
By inspection of the Tables, either $\tau$ appears in Table III, so that $L = A_r$ for some $r$, or $\tau = (A_{12}, \Omega_{10}^+(2), 495)$, or $\tau = (A_8, \text{Sp}_6(2), 28)$, or $n = 120$ and $8 \leq m \leq 10$. Indeed $\Phi(A_{10}, 120) = \langle \text{Sp}_8(2), 120 \rangle$, so in the fourth case, (4) holds by 7.10. In the second case $n = 495$ is odd, so there is no $(\Omega_{10}^-(2), K, n)$ in row 3, 4, or 5 of Table VI, and hence $\mu = (\Omega_{10}^-(2), n) \in \mathfrak{L}^*$. Further there is no $(A_{12}, A_r, 495)$ in Table III, contrary to (b). Similarly in the third case $\mu \in \mathfrak{L}^*$ by 7.5.1, again contrary to (b). Thus we may assume:

(c) $\tau$ appears in Table III, so $L = A_r$ for some $r$.

Observe next that:

(d) If $m = r - 1$ and $r = 2^{d-1} \pm 2^{d-1}$ with $d \geq 3$, then $m \equiv -1 \text{ mod } 4$ and $m \geq 28$.

(e) The lemma holds if $m$ is even, so we may assume $m$ is odd.

For assume $m$ is even. By (c), (d), and inspection of Table III, $m = 6, 10, 12, 22$. For future reference, note that there are two more cases where $(A_m, L, n) \in \Phi$ with $m$ even; namely $m = 8$ and 12 appear in Table VI, and these cases were treated during the proof of (c). In particular for $\mu = (A_{12}, n') \in \mathfrak{L}$, we showed that either $\mu \in \mathfrak{L}^*$ or $n' = 495$ and $\mu \in \mathfrak{L}_e$. Similarly $(A_{24}, n^*) \in \mathfrak{L}^*$ for all $n^*$.

If $m = 6$ then (3) holds by Table III and 7.8. If $m = 10$, then by Table III and the discussion of $A_{12}$ above, (3) holds.

Thus we may assume $m = 22$. Here from Table III, $n = |A_{24} : M_{24}|$. Let $\eta = (A_{23}, n)$. As $23 < 28$ and $n \neq 21!$, it follows from (d) and inspection of the Tables that $\Phi(\eta) = \{ (A_{24}, n) \}$. Thus (3) holds in this case, completing the proof of (e).

(f) $m > 7$ and $L = A_{m+1}$, so $\Phi(\lambda) = \{(L, n)\}$.

By (e), $m$ is odd, so by inspection of the Tables, either (f) holds or $m = 7$. But in the latter case (4) holds by 7.7.1 and 7.10.

By (e) and (f), $L = A_{m+1}$ and $m + 1$ is even. But we showed during the proof of (c) and (e) that if $\mu = (A_{m+1}, n)$ with $m + 1$ even, then either $\mu \in \mathfrak{L}^*$ or $m + 1 \in \{ 6, 8, 10, 12, 22 \}$, so $m \in \{ 5, 7, 9, 11, 21 \}$. By (f), $m = 9, 11, 21$, or 22, so by inspection of the Tables and (d), either $m = 11$ and $n = 2520$ or 9!, or $n = \binom{23}{2}$, where $2l = m + 1$. But now by inspection of the Tables, $\mu \in \mathfrak{L}^*$, contrary to (f) and (b). Thus the proof is complete. □

(7.12). Let $\lambda = (U, n) \in \mathfrak{L} - \mathfrak{L}^*$ with $U = G_2(q)'$. Then one of the following holds:

1. $\lambda \in \mathfrak{L}_e$, and $\Phi(\lambda) = \{(L, n)\}$, where $(L, n)$ is one of the following:
   - (a) $(\Omega_7(q), n)$ with $q$ odd or $q = 2, n = q^2(q^3 + 1)/2$, and $\epsilon = \pm 1$.
   - (b) $q$ is odd and $(L, n)$ is $(\Omega_7(q), n)$ with $n = (q^6 - 1)/(q - 1)$.
   - (c) $q > 2$ is even and $(L, n)$ is $(\text{Sp}_6(q), q^4(q^6 - 1)/(q^2 - 1))$.
   - (d) $q = 3$ and $(L, n) = (\Omega_7(3), 3159)$.
2. $q$ is even, $n = (q^6 - 1)/(q - 1)$, and $\mathcal{O}(\lambda) = \{(U, n) < (L, n) < (K, n)\}$, where $(U, L, K)$ is $(G_2(q), \text{Sp}_6(q), L_5(q))$.
   - (3) $q = 2^k > 2$ is even, $n = q^3(q^3 + 1)/2$, and $\mathcal{M}(\lambda) = \{(\text{Sp}_{6k}(2), n)\}$.

Proof. As $\lambda \notin \mathfrak{L}^*$, there exists $\tau = (U, L, n) \in \Phi$. Let $\mu = (L, n)$. From the Tables there are four possible cases:

   (i) $\mu = (\Omega_7(q), q^3(q^3 + 1)/2), \epsilon = \pm 1$.
   (ii) $\mu = (\Omega_7(q), (q^6 - 1)/(q - 1))$.
   (iii) $\mu = (\text{Sp}_6(q), q^4(q^6 - 1)/(q^2 - 1))$ with $q > 2$ even.
   (iv) $q = 3$ and $\mu = (\Omega_7(3), 3159)$.
If $\mu \notin \mathcal{L}^*$, then from the discussion in the proof of 7.9, one of the cases listed there occurs: $n = n'$, where $n'$ is $(q^3 + 1)(q^3 + 1)(q + 1)$ or $q^2(q^4 - 1)/(2, q - 1)$; $q = 3$ and $n = n'$, where $n' = 28431$; $q = 2^k$ is even (so that $L = S_\Phi(q)$) and either $\mu \leq \eta = (L_8(q), (q^8 - 1)/(q - 1))$ or (using 7.6.3) $\mathcal{M}(\mu) = \{v\}$, where $v = (S_{\Phi_6}(2), q^4(q^3 + \epsilon)/2)$; or $q = 2$ and $\mu \leq \eta = (\Omega_8^+(2), 960)$. However $n \neq n'$, so $q$ is even and one of the latter three cases holds. Similarly if $q = 2$ then $n \neq 960$. This leaves case (ii), where $\lambda \Subset \mu \leq \eta$, and case (i). Further in case (ii), from the Tables, $\eta \in \mathcal{L}^*$, so (2) holds in this case. Finally we saw that (1a) or (3) holds in case (i).

We have shown that either (2) or (3) holds, or $\mu \in \mathcal{L}^*$ (so that $\lambda \in \mathcal{L}^*$) and one of the subcases of (1) holds. □

(7.13). Let $\lambda = (U, n) \in \mathcal{L} \setminus \mathcal{L}^*$ with $U \cong L^\ell_m(q), m \neq 4$, and $U \neq L_2(5)$. Then one of the following holds:

(1) $\lambda \in \mathcal{L}_e$. 
(2) $\lambda = (L_2(7), 28)$ and $\Phi(\lambda) = ((A_8, 28), (U_3(3), 28))$; $\mathcal{M}(\lambda) = \{S_{\Phi_6}(2), 28\}$, and $O(\lambda) = \{\lambda\} \cup \Phi(\lambda) \cup \mathcal{M}(\lambda)$. 
(3) $O(\lambda) = \{(L_2(11), 55) \leq (M_{11}, 55) \leq (A_{11}, 55)\}$. 
(4) $O(\lambda) = \{(L_2(23), 276) \leq (L_24, 276) \leq (A_{24}, 276)\}$. 
(5) $O(\lambda) = \{(L_m(2, n) \leq (\Omega_2^+(2), 2n) \leq (S_{2m}(2), n))$, where $n = 2^{m-1}(2^{m-1} - 1) \geq 5$.

Proof. As $\lambda \notin \mathcal{L}^*$ there is $\tau = (U, L, n) \in \Phi$. Set $\mu = (L, n)$. From the Tables there are eight possible cases:

(i) $m = 2$ and $\mu = (A_{q+1}, q(q+1)/2)$. 
(ii) $m = 2$, $q = 11$, and $\mu = (M_{11}, 11)$, $(M_{11}, 55)$, or $(M_{12}, 12)$. 
(iii) $m = 2$, $q = 23$, and $\mu = (M_{24}, 24)$ or $(M_{24}, 276)$. 
(iv) $U \cong U_4(4)$ and $\mu = (G_2(4), 416)$. 
(v) $q = 2$ or $(3, \epsilon = \pm 1$, and $\mu = (P\Omega_2^+(2), qm+1(qm+1)/(2, q - 1))$. 
(vi) $m = 2, q = 17$, and $\mu = (S_{\Phi}(2), 136)$. 
(vii) $m = 3, q = 4, \epsilon = \pm 1$, and $\mu = (U_3(3), 280)$. 
(viii) $m = 2, q = 7, \epsilon = \pm 1$, and $\mu = (U_3(3), 28)$. 

Suppose first that $\lambda = (L_2(7), 28)$. Then as $L_2(7) \cong L_2(2)$, we are in case (i), (v), or (viii). Now $\mu \in \mathcal{L}_e$, with $\Phi(\mu) = (S_{\Phi_6}(2), 28))$; this is a consequence of 7.7.2 in cases (i) and (v), and of 7.12 in case (vii). Therefore (2) holds in this case.

In cases (vii), (vi), and (iv), (1) holds by 7.8, 7.5.1, and 7.12, respectively.

Suppose (i) holds. We handled the case $q = 7$, so we may take $q \geq 9$. Hence $n \geq 45$. We claim $\mu \in \mathcal{L}^*$, so that (1) holds. If $q$ is odd, then from the proof of (c) and (e) in 7.11, $q + 1 \in \{6, 8, 10, 12, 22\}$, so $q = 9$ or 11. Then from the proof of 7.11, $n = 120$ or 2520 if $q + 1 = 10$, while $n = 495$ if $q + 1 = 12$, a contradiction. Therefore $q$ is even. We may assume $(K, n) \in \Phi(\mu)$. By 7.11f, $K \equiv A_{q+2}$. Then $q + 2 \equiv 2$ mod 4, so case (d) of 7.11 does not hold. Also $n \neq (q + 1)!$ and $q + 1 \neq 2^d - 1$. Now by inspection of Table III, we have a contradiction. This establishes the claim, and shows (1) holds in case (i) when $q > 7$.

In case (ii), we conclude from inspection of the Tables that (1) or (3) holds. Similarly in case (iii), (1) or (4) holds by inspection of the Tables.

It remains to treat case (v), where we may assume $U$ is not $L_3(2) \cong L_2(7)$, since we treated this case earlier by 7.8, one of the following holds:

(i) $\mu \in \mathcal{L}^*$. 
(ii) $q$ is even and $\mu \in \mathcal{L}_e$. 
(iii) $q = 3, m = 3, \text{ and } n = 1080$.

As $n = q^{m-1}(q^m - 1)/(2, q - 1)$, case (iii) is out. Similarly in case (ii), comparing $n$ to the cases arising in the proof of 7.8, we conclude $q = 2$ and $\Phi(\mu) = \{\eta\}$, where $\eta = (S_{2m}(2), n)$. Thus (5) holds in this case. Finally in case (i), (1) holds. This completes the proof of the lemma. □
(7.14). Let $\lambda = (U, n) \in \mathcal{L} - \mathcal{L}^*$ with $U \cong \text{PSp}_4(q)$. Then either

(1) $n = (q^d - 1)/(q - 1)$, $\lambda \in \mathcal{L}_s$, and $\Phi(\lambda) = \{(L_4(q), n)\}$, or
(2) $q > 2$ is even and $n = q^2(q^2 + \epsilon)/2$, $\epsilon = \pm 1$.

**Proof.** As $\lambda \notin \mathcal{L}^*$ there is $\tau = (U, L, n) \in \Phi$. Set $\mu = (L, n)$. From the Tables there are two possible cases:

(i) $L = P\Omega_6^+(q)$ and $n = (q^d - 1)/(q - 1)$.
(ii) $q = rb$ with $r$ even and $b$ prime, $L \cong \text{Sp}_{ab}(r)$, and $n = q^2(q^2 + \epsilon)/2$.

In case (ii), (2) holds, so we may assume case (i) holds. In this case as $q > 2$, $\Phi(\mu) = \emptyset$ by inspection of the Tables, so (1) holds. $\Box$

(7.15). Let $\lambda = (U, n) \in \mathcal{L} - \mathcal{L}^*$ with $U \cong \text{PSp}_{2m}(q)$, $q$ odd, $m > 2$. Then $\lambda \in \mathcal{L}_s$ and $\Phi(\lambda) = \{(L_{2m}(q), (q^{2m} - 1)/(q - 1))\}$.

**Proof.** As $\lambda \notin \mathcal{L}^*$ there is $\tau = (U, L, n) \in \Phi$. Set $\mu = (L, n)$. From the Tables, $L = L_{2m}(q)$ and $n = (q^{2m} - 1)/(q - 1)$. From 7.13, $\mu \in \mathcal{L}^*$, so the lemma holds. $\Box$

(7.16). Let $\lambda = (U, n) \in \mathcal{L}$ with $U$ sporadic. Then one of the following holds:

(1) $\lambda \in \mathcal{L}^*$.
(2) $\lambda \in \mathcal{L}_s$.
(3) $n = 66, 495, 2016$, or $2^{11}(2^{12} - 1)$, and $\mathcal{O}(\lambda) = \{(U, n) < (L, n) < (K, n)\}$, where $(U, L, K)$ is $(M_{11}, M_{12}, A_{12}), (M_{12}, A_{12}, \Omega^-_{10}(2)), (J_2, G_2(4), \text{Sp}_6(4))$, or $(\text{Co}_1, \Omega^+_2(2), \text{Sp}_{24}(2))$, respectively.

**Proof.** We may assume neither (1) nor (2) holds, so:

(a) There exists $\tau = (U, L, n) \in \Phi$, and
(b) either there exists $(L, K, n) \in \Phi$ or $|\Phi(\lambda)| > 1$.

Set $\mu = (L, n)$. We first show:

(c) $L$ is not an alternating group.

Assume $L = A_m$, so that $\tau$ appears in Table III. Then from the Tables, $U$ is $k$-transitive on a set $\Delta$ of degree $m$ with $k \geq 2$, and $\tau = (U, L, \Omega)$ with $\Omega$ the set of $l$-subsets of $\Delta$ for some $2 \leq l \leq k$.

Suppose $\mu \notin \mathcal{L}^*$. If $m$ is even, then from the proof of parts (c) and (e) of 7.11, either $m = 12$ and $n = 495$, or $m = 22$ and $n = |A_{24} : M_{24}|$. But then as $n = \binom{m}{l}$, it follows that $m = 12, l = 4$, and (3) holds as $\mu \in \mathcal{L}_s$ from 7.12 and its proof. Therefore we may assume $m$ is odd, so $m = 11$ or 23. From the proof of 7.11f, and the following discussion, $m = 11$ and $n = 2520, 9!$, or $\binom{12}{6}/2$. But then $n \neq 55$ or 165, a contradiction.

Therefore $\mu \in \mathcal{L}^*$, so by (b), there is $\eta = (X, n) \in \Phi(\lambda) - \{\mu\}$. We have shown that $\tau$ is the unique member of $\Phi$ of the form $(U, L', n')$ with $L'$ alternating, so $X$ is not an alternating group. But now by inspection of the Tables, there is no $(U, X, n) \in \Phi$ with $n = \binom{m}{l}$. This completes the proof of (c).

(d) $L$ is not a sporadic group.

Assume $L$ is sporadic, so that $\tau$ appears in Table IV. Suppose first that $\mu \notin \mathcal{L}^*$, and let $(L, K, n) \in \Phi$. If $K = A_m$ is alternating, then comparing $n$ to the values in Table III, we conclude that $(U, L, K, n) = (M_{11}, M_{12}, A_{12}, 66)$. Further from the proof of (c), $(L, 66) \in \mathcal{L}_s$, so (3) holds in this case.
Suppose $K$ is sporadic, so the $(L, K, n)$ appears in Table IV. Then as $(U, L, n)$ also appears in Table IV, we obtain a contradiction, since the only time an $n$ appears twice, one of the first entries is not sporadic.

If $K$ is an exceptional group of Lie type, then from Table V, $(L, n) = (J_2, 2016)$, which is inconsistent with Table IV. Therefore $K$ is classical, so from Table VI, $(L, K, n)$ is $(C_{01}, \Omega^+(2), 2^{11}(2^{12} - 1))$, $(J_3, U_9(2), 43605)$, or $(M_{22}, U_6(2), 672)$. As $(U, L, n)$ appears in Table IV, this is a contradiction.

Therefore $\mu \in \mathfrak{L}^*$, so by (b), there is $\eta = (X, n) \in \Phi(\lambda) - \{\mu\}$. By (c), $X$ is not alternating. Inspecting for two instances of the same $n$ in Table IV, we conclude $X$ is not sporadic. From Table IV, $L$ is not $J_2$, so from Table V, $X$ is not exceptional. Then from Table VI, $L$ is $C_{01}$ or $J_3$, or $(L, X, n)$ is $(M_{22}, U_6(2), 672)$. In each case, inspection of Table IV supplies a contradiction. This completes the proof of (d).

(e) We may assume $L$ is classical.

Assume otherwise. Then by (c) and (d), $L$ is exceptional, so $\tau$ appears in Table V. Therefore $\tau = (J_2, G_2(4), 2016)$. But then (3) holds by 7.12, since $2016 = 4^3(4^3 - 1)/2$. So (e) is established.

By (e), $L$ is classical, so from Table VI:

(f) $\tau$ is $(C_{01}, \Omega^+(2), 2^{11}(2^{12} - 1))$, $(J_3, U_9(2), 43605)$, or $(M_{22}, U_6(2), 672)$.

Suppose $n = 2^{11}(2^{12} - 1)$. By 7.9, $\mu = (\Omega^+(2), n) \in \mathfrak{L}$, and by inspection of the Tables, $\Phi(\mu) = \{\text{Sp}_1(2), n\}$, so (3) holds in this case.

If $n = 43605$ or 672, then by inspection of the Tables, $\mu \in \mathfrak{L}^*$, so (2) holds in these cases. Hence the proof of the lemma is complete. □

(7.17). Let $\lambda = (U, n) \in \mathfrak{L} - \mathfrak{L}^*$ with $U$ an exceptional group of Lie type other than $G_2(q)$. Then $q$ is even and either

1. $\lambda = (C^2D_4(q), q^6(q^6 + q^4 + 1)) \in \mathfrak{L}$, or
2. $\lambda = (Sz(q), n) \in \mathfrak{L}$ with $n = q^2(q^2 + 1)/2$ and $q = 2^k$ for some odd $k \geq 3$. Further $\Phi(\lambda) = \{\mu, \eta\}$ where $\mu = (A_{q^2+1}, n)$ and $\eta = (Sp_4(q), n)$, $M(\lambda) = \{\mu, \nu\}$, where $\nu = (Sp_{4k}(2), n)$, and $O(\eta) = \{(Sp_{4k}(2k/c), n) \in \mathfrak{L}; c$ is a divisor of $k\}$.

Proof. Let $\tau = (U, L, n) \in \Phi$. Set $\mu = (L, n)$. From the Tables there are three possible cases:

(i) $\tau = (Sz(q), A_{q^2+1}, q^2(q^2 + 1)/2)$.
(ii) $q$ is even and $\tau = (C^2D_4(q), F_4(q), q^6(q^8 + q^4 + 1))$.
(iii) $\tau = (Sz(q), Sp_4(q), q^2(q^2 + 1)/2)$.

In case (ii), by inspection of the Tables, $\mu \in \mathfrak{L}^*$, so (1) holds. Thus we may assume $\lambda = (Sz(q), n)$ with $n = q^2(q^2 + 1)/2$. Suppose $L = A_{q^2+1}$. Observe $q^2 + 1 = (q + (2q)^{1/2} + 1)(q - (2q)^{1/2} + 1)$ is odd but not prime or $2^d - 1$, $q^2 + 1 \geq 65$, and $n \neq (q^2 + 1)/2$. Thus by inspection of the Tables, $\mu \in \mathfrak{L}^*$ in this case.

Next suppose $L = Sp_4(q)$. Now $q = 2^k$ with $k \geq 3$ odd, so by parts (3) and (4) of 7.5, $M(\eta) = \{(Sp_{4k}(2), n) \in \mathfrak{L}; c$ is a divisor of $k\}$. Thus (2) holds in this case. □

Theorem 7.18. Let $\lambda = (U, n) \in \mathfrak{L}$. Then either

1. $|M(\lambda)| = 1$, or
2. $\lambda = (Sz(q), n)$ with $n = q^2(q^2 + 1)/2$ and $q = 2^k$ for some odd $k \geq 3$. Further $\Phi(\lambda) = \{\mu, \eta\}$ where $\mu = (A_{q^2+1}, n)$ and $\eta = (Sp_4(q), n)$, and $M(\lambda) = \{\mu, \nu\}$, where $\nu = (Sp_{4k}(2), n)$.
Proof. Suppose \( \lambda \) is a counterexample. Then \( \lambda \not\in \mathfrak{L}^* \cup \mathfrak{L}_+ \). If \( U \cong A_\lambda \) is an alternating group, then by 7.11, \( n = 120 \) and \( (A_7, n) \leq (A_m, n) \) with \( 7 \leq m \leq 9 \). But then \( \mathcal{M}(\lambda) = \{ \text{Sp}_8(2) \} \) by 7.10, a contradiction.

By 7.16, \( U \) is not sporadic. By 7.12 and 7.17, \( U \) is not exceptional. Therefore \( U \) is a classical group.

By 7.8, \( U \) is not \( P\Omega_{2m}^\e(q) \), with \( m \geq 3 \). Hence \( U \) is not \( L_3^\e(q) \). Then by 7.13, \( U \) is not \( L_m^\e(q) \). Suppose \( U \cong P\Omega_{2r-1}^\e(q) \) with \( r \geq 4 \). Then by 7.9, case (2) or (3) of that lemma holds, contrary to 7.5.4. By 7.14 and 7.5.4, \( U \) is not \( \text{PSp}_q(2) \), while by 7.15, \( U \) is not \( PSp_{2m}(q) \) with odd and \( m > 2 \). This completes the proof of the theorem. \( \square \)

8. The proof of Theorem A

In this section we continue the notation of the previous section, and we assume:

Hypothesis 8.1. Assume \( \xi = (H, \Omega) \in \mathfrak{A}, \) and set \( n = |\Omega|, \ S = \text{Sym}(\Omega), \ A = \text{Alt}(\Omega), \) and \( \phi(\xi) = \lambda = (U, \Omega). \)

(8.2). Assume

(a) \( \mathcal{M}(\lambda) = \{ v \}, \) with \( v = (V, \Omega) \) and \( U \leq V. \) Set \( K = N_S(V). \)

(b) \( U^S \cap V = U^K. \)

(c) \( N_S(U) \cong N_K(U). \)

Then \( \mathcal{M}_S(H) = \{ K \}. \)

Proof. By (c), \( N_S(U) = N_K(U) \leq K. \) By (b) and a Frattini argument (cf. 5.21 in [FGT]), \( O_S(U) \cap V^S = V^{N_S(U)} \), so as \( N_S(U) \leq K, \) we have \( O_S(U) \cap V^S = \{ V \}. \)

Let \( B \in O_S(H). \) By 7.3.3, \( \beta = (B, \Omega) \in \mathfrak{A}. \) Set \( \phi(\beta) = \phi = (D, \Omega). \) Then \( \lambda \leq \delta \leq \xi, \) so by (a), \( \delta \leq v. \)

Thus using 7.1, there exists \( g \in S \) with \( D \leq V^g. \) Then for \( b \in B, \) \( U \leq D = D^b \leq V^{gb}, \) so as \( O_S(U) \cap V^S = \{ V \}, \) it follows that \( V^g = V = V^b. \) Therefore \( B \leq N_S(V) = K, \) completing the proof. \( \square \)

(8.3). Assume \( \mathcal{M}_S(H) = \{ K \} \) is of order 1, \( H \leq G \in \{ A, S \}, \) and \( O_G(H) \) is an I-lattice. Then \( G = S, \ H \leq A, \ K = N_S(H) \) so that \( F^*(H) = F^*(K) \), and \( O_G(H) = \{ H, A, N_S(H), S \}. \)

Proof. Unless \( G = S \) and \( H \leq A, \ O_G(H) = \{ G \} \), so that \( K \cap G \) is the unique maximal of \( O_G(H) \) on an I-lattice. Therefore \( G = S, \) and \( H \leq A. \)

By 3.7 in [A4], \( N_S(H) \) is a maximal subgroup of \( S \) and \( H = N_A(H). \) Then as \( \{ K \} = \mathcal{M}_S(H), \ N_S(H) \leq K. \) Thus \( K = N_S(H) \not\leq A \) by maximality of \( N_S(H). \) Also \( H = N_A(H) \cap A, \) so \( H \) is maximal in \( A \) as \( \{ K \} = \mathcal{M}_S(H). \) Now 3.8 in [A4] completes the proof. \( \square \)

(8.4). If \( \lambda \in \mathfrak{L}^* \) then \( \mathcal{M}_S(H) = \{ N_S(U) \}. \)

Proof. Observe hypotheses (a)-(c) of 8.2 are satisfied with \( v = \lambda \) and \( V = U, \) so the lemma follows from 8.2. \( \square \)

(8.5). Assume \( \mathcal{M}_S(H) = \{ M \} \) and \( \xi = (H', \Omega). \) Then \( \mathcal{M}_S(H') = \{ M \}. \)

Proof. This follows as \( \mathcal{M}_S(H') \subseteq \mathcal{M}_S(H). \) \( \square \)

(8.6). Assume \( \lambda = (\text{Sp}_{2ak}(q), n) \in \mathfrak{L}, \) where \( a, k \) are positive integers with \( a > 1, \ (a, k) \neq (2, 1), \ q = 2^k, \ e = \pm 1, \) and \( n = q^a(q^a + e)/2. \) Then \( \mathcal{M}_S(H) = \{ K \}, \) where \( \text{Sp}_{2ak}(2) \cong K \leq S. \)

Proof. By 7.5.4, \( \mathcal{M}(\lambda) = \{ v \}, \) where \( v = (K, \Omega) \) with \( U \leq K \cong \text{Sp}_{2ak}(2). \) Let \( W \) be a 2ak-dimensional symplectic space over \( F_2 \) with \( K = \text{Sp}(W) \) the isometry group of \( W. \) From the treatment in [LPS2]
of the triple \((U, K, \Omega) \in \Phi\) (when \(k\) is prime), \(N_K(U)\) is the stabilizer of an \(F_q\)-structure \(W_{F_q}\) on \(W\). From part (1) of Theorem A in [A1], \(N_K(U)\) is the extension of \(U\) by the group of field automorphisms of \(U\) of order \(k\). Hence \(N_K(U) \cong \text{Aut}(U)\), unless \(a = 2\), where \(N_K(U)\) is the stabilizer \(M\) in \(\text{Aut}(U)\) of the class of maximal parabolics stabilizing a point of \(W_{F_q}\).

Claim \(N_K(U) = N_5(U)\). By 7.3.1, we must show \(N_K(U) = M(\xi)\). If \(N_K(U) = \text{Aut}(U)\) this is trivial, so assume \(a = 2\). From [LPS2], the stabilizer \(U_\omega\) of \(\omega \in \Omega\) is isomorphic to \(O_4^+(q)\). By 7.3.1, \(M(\xi) = N_{\text{Aut}(U)}(U_\omega)\), so \(M(\xi)\) is \(U\) extended by field automorphisms, and hence indeed \(M(\xi) = N_K(U)\), establishing the claim.

By Theorem B.3.3 in [A1], \(K\) is transitive on \(F_q\)-structures on \(W\) stabilized by a copy of \(U\). Further if \(U \cong U' \leq K\) then as \(2ak\) is the minimal dimension of a faithful \(F_2U\)-module, and each such module is quasiequivalent to the representation of \(U\) on \(W\), \(U'\) stabilizes an \(F_q\)-structure on \(U\). Thus \(U^S \cap K = U^K\). Therefore the lemma follows from 8.2. □

\((8.7)\). Assume \(U \cong P \Omega_q^E(2m)(q)\) with \(m \geq 3\). Then \(|M_5(H)| = 1\).

Proof. By 8.4, we may assume \(\lambda \notin \Sigma^*\). Therefore by 7.8, \(M(\lambda) = \{v\}\), for some \(v = (V, \Omega)\) with \(U \subseteq V\). Further the possibilities for \(\lambda\) and \(v\) are listed in cases (i)-(v) of the proof of 7.8.

Suppose case (v) holds. Then \(U = \rho \Omega_q(3), n = 1080 = 2^5 \cdot 3^3 \cdot 5\), and \(V \cong \rho \Omega_q(3)\). Let \(W\) be an 8-dimensional orthogonal space over \(F_2\) and \(\hat{V} = \rho(W) \cong \Omega_q^+(3), V = \hat{V} / Z(\hat{V})\). From 7.9, \(\lambda \leq \beta \leq \nu\) with \(\beta = (B, \Omega), B = \Omega_7(3)\). Let \(W_0\) be a 7-dimensional orthogonal space with \(B = \rho(W_0) \cong \Omega_7^+(3), V = \text{Spin}_8^+(3)\) the universal covering group of \(V\), and \(Z = Z(\hat{V})\). From [LPS2], \(U\) is the stabilizer of \(B\) in a nonsingular point of \(W_0\), and there is a trityl automorphism \(\rho\) of \(\hat{V}\), such that \(BZ\) is the stabilizer of \(V\) of a point of the orthogonal space \(W_1 = W_0^\rho\), and \(\Omega\) is an orbit under \(V\) of nonsingular points of \(W\). Now \(U\) stabilizes a nondegenerate 2-dimensional subspace \(W_2\) of \(W_1\) of sign \(-1\), and from [LPS2], \(H\) contains \(\tau \subseteq B\) inducing a graph automorphism on \(U\), which must then interchange the two singular points of \(W_2\). As \(\rho^{-1}\) maps singular points of \(W_1\) to maximal totally singular subspaces of \(W\) (cf. 15.1 in [A1]), \(U\) stabilizes a decomposition \(W = W_2 \oplus W_5\) as the sum of maximal totally singular subspaces, and \(\tau\) interchanges \(W_4\) and \(W_5\).

By 7.3.1, \(N_5(V) = K\) is the stabilizer in \(\text{Aut}(V)\) of \(V_\omega^\nu\), so \(K = P\Omega(V)\) of index 6 in \(\text{Aut}(V)\) (cf. 2.1 and 2.2 in [A6]). As \(\text{Aut}(V, U) = \text{Aut}(U)\) (cf. 2.1 and 2.2 in [A6]), it follows from 7.1 that \(N_5(U) \cong N_K(U)\).

Let \(J \in U^S \cap V\). We recall that:

\((*)\) The only nontrivial irreducibles for \(J\) of dimension at most 8 are of degree 4 and 6.

Suppose \(J\) fixes a point \(W_0\) of \(W\). Then \(J\) is faithfully embedded in \(N_5(W_0) \cong O_3(N_5(W_0))\) acting faithfully as \(\Omega_q^+(3)\) on the 6-dimensional orthogonal space \(W_6 = W_0^\perp / W_0\), or as \(O_7^+(3)\) on the 7-dimensional orthogonal space \(W_7^\perp\), for \(W_0\) singular, nonsingular, respectively. Hence by \((*)\), \(J\) is irreducible on \(W_6\) if \(W_0\) is singular, while the composition factors for \(J\) on \(W_7^\perp\) are of dimension 1 and 6 if \(W_0\) is nonsingular. Then as \(H^1(W_6, J) = 0\), in either case \(J\) centralizes a nondegenerate line of \(W\), so \(J\) fixes a point of \(\Omega\). This is impossible as \(J \in U^S\), so \(J\) is transitive on \(\Omega\).

Therefore \(C_W (J) = 0\), so it follows from \((*)\) that \(J\) acts irreducibly on a maximal totally singular subspace \(W_4\) of \(W\). Now \(E = O_3(N_5(W_4)) \cong W_6\) as a \(J\)-module, so again as \(H^1(W_6, J) = 0\), \(J\) is conjugate to a Levi factor of \(N_5(W_4)\), and hence acts on a complement \(W_5\) to \(W_4\) in \(W\). Further by Witt's lemma, \(K\) is transitive on decompositions of the form \(W = W_4 \oplus W_5\), so \(U^S \cap V = U^K\). Therefore the lemma holds in this case by 8.2.

We thus may assume one of the remaining cases holds, so \(q\) is even. Suppose next that case (iv) holds, so that \(U = \Omega_q(2) \cong \Omega_q(2) \cong \Omega_q(2) \cong N_K(U)\) and the representation of \(U\) on \(W\) is determined up to quasiequivalence, so \(U^S \cap V = U^K\). So once again the lemma holds by 8.2.
In the remaining three cases, \( V \cong \text{Sp}_{2m}(q) \). Let \((W, f)\) be the \(2m\)-dimensional symplectic space over \( F_q \) such that \( V = O(W, f) \). In each case from [LPS2], \( U = O(W, Q)^{\infty} \) for some quadratic form \( Q \) on \( W \) with associated bilinear form \( f \), and as \( V \) is transitive on quadratic forms on \( W \) associated to \( f \) of sign \( \epsilon \), \( U^T \cap V = U^V \).

Suppose case (iii) holds, so that \( q = 2, n = 2^{m-1}(2^m - \epsilon) \), and from [LPS2], \( V_{\omega} = O(W, Q)^{\infty} \cong O_{2m}(2) \) for a quadratic form \( Q' \) of sign \( -\epsilon \) on \( W \) with associated symplectic form \( f \). Hence \( K = \text{Aut}(V) \) is \( V \) extended by its group of field automorphisms, so unless \((n, \epsilon) = (4, +)\), \( \text{Aut}_K(U) = \text{Aut}(U) \). In the exceptional case, \( U_{\omega} \cong \text{Sp}_6(2) \), so from 7.4.1, \( N_5(U) \cong N_{\text{Aut}(U)}(U_0^{\Upsilon}) \cong O_6^+(2) \). Thus in any event, \( N_5(U) \cong \text{Aut}_K(U) \), so the lemma follows from 8.2 in case (iii).

Suppose case (ii) holds. Then \( \epsilon = -1, m \) is even, and from [LPS2], \( \Omega \) is the set of decompositions \( W = W_+ \oplus W_\perp \) as the orthogonal direct sum of two \( m/2 \)-dimensional nondegenerate subspaces. In particular again \( K = \text{Aut}(V) \), so \( \text{Aut}_K(V) = \text{Aut}(U) \) as \( \epsilon = -1 \), and the lemma follows from 8.2 in case (ii).

Finally assume case (i) holds. Then \( m = ab \) for some prime \( b \) and \( a \geq 1 \), \((b, \epsilon) \neq (2, +)\), and setting \( F = F_{n_b} \), from [LPS2], we may take \( \Omega \) to be the set of \( F \)-structures \( W_F \) on \( F \). Hence \( K = \text{Aut}(V) \), so as \((m, \epsilon) \neq (4, +)\), as usual the lemma follows from 8.2. This completes the proof. \( \square \)

(8.8) Assume \( n = 120 \) and \( U \in \{A_7, A_8, A_9, \text{Sp}_6(2), \Omega_8^+(2), \text{Sp}_8(2)\} \). Then \( \mathcal{M}_5(H) = \{K\} \), where \( K \cong \text{Sp}_8(2) \).

Proof. By 7.11, \( \mathcal{M}(\lambda) = \{v\} \), where \( v = (V, \Omega) \) with \( U \subseteq V \), and \( V \cong \text{Sp}_8(2) \). Further there is \( \lambda' = (H', \Omega) \) with \( S_7 \cong H' \) and \( H' \leq H \). Thus by 8.5, we may assume \( \lambda = \lambda' \), and it remains to verify conditions (b) and (c) of 8.2.

Next there exists \( \tau = (H_1, \ldots, H_5; \Omega) \in \tau \) with \( H_1 \equiv H_2 \equiv S_8 \equiv O^+_5(2), H_3 \equiv \text{Sp}_6(2), H_4 \equiv \Omega_8^+(2) \), and \( V = H_5 \equiv \text{Sp}_8(2) \). Let \((W, Q)\) be an 8-dimensional orthogonal space over \( F_2 \) with \( H_4 = O(W, Q)^{\infty} \). From [LPS2], we may take \( \Omega \) to be the set of nonsingular points of \((W, Q)\), \( \rho \) a triality automorphism of \( H_4 \), and \( W_0 = W^\rho \) with \( H_3 \) the stabilizer in \( H_4 \) of a nonsingular point of \( W_4 \). Further \( H_2^\infty \) is the centralizer of a 2-dimensional nondegenerate subspace \( W_2 \) of \( W_4 \), and \( \tau \in H_1 - H_2^\infty \) interchanges the two singular points in \( W_2 \). Then as in the proof of 8.7, the image of these points under \( \rho^{-1} \) defines a decomposition \( W = W_4 \oplus W_5 \) with \( W_4 \) and \( W_5 \) \( H_2^\infty \)-invariant maximal totally singular subspaces and \( W_4^T = W_5 \). Moreover \( U \) is irreducible on \( W_4 \) with \( W_4 \cong W_4^* \) as a \( U \)-module.

Let \( B \) be a 6-dimensional symplectic space for \( H_3 \) and \( A_7 \cong J \leq H_3 \). The stabilizer of no proper subspace of \( B \) contains a copy of \( J \), so \( J \) is irreducible on \( B \). Then as \( \dim(B) = 6 \), \( B \) is the core of the 7-dimensional permutation module for \( J \). In particular \( J \) preserves a unique symplectic form on \( B \), so \( J \) is determined up to conjugacy in \( H_3 \).

Next let \( J \) be the symplectic form associated to \( Q \); we may take \( H_5 = O(W, f) \). By [LPS2], we may view \( \Omega \) as \( V/V_{\omega} \), where \( V_{\omega} \cong O_5^- \). Let \( A_7 \cong J \leq H_5 \). If \( J \) fixes a point \( W_0 \) of \( W \), then \( J \) is faithfully embedded in \( N_{H_5}(W_0) \), so \( J \) acts irreducibly on \( W_1 = W_0^J/W_0 \), so by the previous paragraph, \( J \) is irreducible on \( W_1 \). Then as \( H_1^J(W_1, J) = 0 \), \( J \) centralizes a nondegenerate line \( W_2 \) of \( W \) and preserves a quadratic form on \( W_2^J \), so \( J \) is contained in \( H_2 \)-conjugate of \( V_\omega \). But if \( J \in U^S \) then \( J \) is transitive on \( \Omega \), a contradiction. Therefore if \( J \in U^S \) then \( C_W(J) = 0 \), so as the only irreducibles for \( J \) of dimension at most 8 are of degree 4 and 6, it follows that \( J \) acts irreducibly on a maximal totally singular subspace \( W_4 \) of \( W \). Now \( E = O_3(N_{H_5}(W_4)) \) has two chief factors \( E_m, m = 4, 6 \), for \( N_{H_5}(W_4)/E \cong L_4(2) \) with \( \dim(E_m) = m \), and these chief factors remain irreducible under \( J \). Then as \( H_1^J(E_m, J) = 0 \) for \( m = 4, 6 \), \( J \in U^E \). Finally as \( H_5 \) is transitive on its maximal totally singular subspaces, \( U^S \cap V = U^V \). Further \( H = \text{Aut}(U) = \text{Aut}_V(U) \), so 8.2 completes the proof. \( \square \)

(8.9) Assume \( U \cong P\Omega_2^{2r-1} \) with \( r \geq 4 \). Then \( |\mathcal{M}_5(H)| = 1 \).

Proof. By 8.4, we may assume \( \lambda \notin \Omega^* \). Therefore by 7.18, \( \mathcal{M}(\lambda) = \{v\} \), for some \( v = (V, \Omega) \) with \( U \subseteq V \). Further the possibilities for \( \lambda \) are listed in cases (i)-(vii) of the proof of 7.9. In case (v), the
lemma is a consequence of 8.6, while if \( \lambda = (\text{Sp}_6(2), 120) \), the lemma follows from 8.8. Thus we may assume neither of these cases holds, so by 7.9, \( \lambda \in \mathcal{L}_s \), so that \( \Phi(\lambda) = \{v\} \).

Assume first that one of cases (i)–(iii) holds. Then \( V = P \Omega_{2r}^+(q) \). Let \((W, Q)\) be a \(2r\)-dimensional orthogonal space with \( V = \tilde{V} / Z(\tilde{V}) \) and \( \tilde{V} = O(W, Q)\). From [LPS2], we may choose \( W \) so that \( U = \tilde{V}^\infty_{W_1} \) for some nonsingular point \( W_1 \) of \( W \).

Suppose case (i) holds. Then from [LPS2], \( n = \prod_{i=1}^{28431} (q^i + 1) \) and we may take \( \Omega \) to be an orbit of \( \Omega \) on maximal totally singular subspaces of \( W \). Thus by 7.3.1, \( K = N_S(V) = N_{\text{Aut}(V)}(V^\infty) \). Hence if \( r \neq 4 \) then \( K \) is \( K_0 \) extended by the field automorphisms of \( V \), where \( K_0 \) is the normalizer in the group of similarities of \((W, Q)\) of \( \Omega \).

In particular if \( r \neq 4 \) then \( K_0 \) is transitive on nonsingular points of \( W \).

Next either the representation of \( U \) on \( W_{2r}^+, W_{2r}^+/W_1 \) for \( q \) odd, even, respectively, is the unique nontrivial representation of \( U \) of degree at most \( 2r \), or \( r = 4 \) and that representation and the spin representation of degree 8 are the only such representations. It follows that if \( j \in U \cap K_0 \) then either \( j \in U \cap K_0 \) or \( r = 4 \). When \( r = 4 \) it is probably best to shift our point of view by adopting a triality automorphism of \( V \), and regard \( \Omega \) as the set of singular points of \( W \). Now \( K \) is \( K_0 \) extended by field automorphisms, where \( K_0 \) is the full group of similarities of \( W \), and the subgroups of \( V \) transitive on \( \Omega \) are those acting in the spin representation, so again \( K_0 \) is transitive on such subgroups. Therefore for all \( r, U^S \cap V = U \cap \Omega \). Further \( \text{Aut}_K(U) = \text{Aut}(U) \), as \( \text{Aut}(U) = PO_{2r-1}(q) \) extended by field automorphisms. Thus the lemma holds in this case by 8.2.

Assume (ii) holds. Then \( r = 4 \) and \( n = q^4(q^4 - 1)/(2, q - 1) \). From [LPS2], we may take \( \Omega \) to be a class of nonsingular points in \( W^\rho \) under \( V \), for a triality automorphism \( \rho \) of \( V = \text{Spin}^9(2) \). Then \( N_S(V) = K \) is the stabilizer in the similarity group of \( W^\rho \) of \( \Omega \), so \( K = PO(W^\rho) \) extended by field automorphisms. Arguing as above, \( U^K = U^S \cap V \) is the set of \( \Omega_7(q) \)-subgroups acting in the spin representation on \( W^\rho \), and \( \text{Aut}(U) = \text{Aut}_K(U) \). So the lemma follows from 8.2 in case (ii).

Suppose case (iii) holds. Then \( r = 4 \), \( q = 3 \), and \( n = 28431 \). By [LPS2], we may take \( \Omega = V / J \), where \( \Omega^+_{28431}(3) \trianglelefteq J \trianglelefteq V \). By 7.4.1, \( K = N_{\text{Aut}(V)}(J^V) \), so \( K = V N_{\text{Aut}(V)}(J) \) and \( K / V \equiv S_3 \). In particular \( K \) has two orbits on \( U^\text{Aut}(V) \), with representatives \( U \), \( \tilde{U} \) the stabilizers of representatives \( W_1 \), \( W_1 \) of the two \( V \)-classes of nonsingular points of \( W^\rho \). Further the stabilizer in \( N_K(J) \) of \( U(t) \), where \( t \) acts as a reflection on \( W \). From Lemma C in 5.115 of [LPS1], \( N_J(W_1) = U \cap J \cong A_2 / E_{64} \). However \( N_J([W, t]) = C_J(t) \cong \text{Sp}_6(2) \), \( [W, t] \not\in W_1^J = W_1^V \), and hence \( [W, t] \not\in U \). Thus \( U^S \cap V = U^K \). Further \( t \) does not centralize \( U \), so we may pick \( t \) to induce a reflection on \( U \), and hence \( \text{Aut}_K(U) = PO_7(3) = \text{Aut}(U) \). Thus the lemma holds in case (iii) by 8.2.

Therefore we may assume one of cases (iv), (vi), or (vii) holds. In particular \( q \) is even, so \( U \cong \text{Sp}_{2m}(q) \), where \( m = r - 1 \). Suppose first that case (iv) holds. Then \( V \cong L_{2m}(q) \). Let \( W \) be the natural module for \( \tilde{V} = SL_{2m}(q) \) with \( V = \tilde{V} / Z(\tilde{V}) \). From [LPS2], \( \Omega \) is the set of points of \( W \). Then by 7.3.1, \( K = N_S(V) = N_{\text{Aut}(V)}(W) = P^G(L(W)) \). Further the representation of \( U \) on \( W \) is determined up to quasiequivalence, so \( U^S \cap V = U^K \). Finally \( \text{Aut}(U) = \text{Aut}_K(U) \) as \( r \geq 4 \), so the lemma holds in case (iv) by 8.2.

Therefore we may assume (vi) or (vii) holds, so \( q = 2 \) and \( V = \Omega^+_{2m}(2) \). As \( q = 2 \), \( U = \text{Aut}(U) \). Let \((W, Q)\) be an orthogonal space with \( V = O(W, Q) \). In case (vi), we find in [LPS2] that \( U \) is the stabilizer of a nonsingular point of \( W \), so \( U^S \cap V = U^V \) by our usual argument, and hence the lemma follows from 8.2 in this case.

Therefore we may assume case (vii) holds, so from [LPS2], \( \Omega = V / J \), where \( A_9 \cong J \trianglelefteq V \). Conjugating in \( \text{Aut}(V) \), we may assume \( V \) is the natural module for \( J \), so \( N_{\text{Aut}(V)}(J) \cong J(t) \cong S_9 \), where \( t \) is a transposition inducing a transvection on \( W \). Therefore by 7.3.1, \( K = N_S(V) = V(t) = O(W, Q) \).

Next by 5.15.1a in [LPS1], \( J_U = J \cap U \cong \text{Aut}(L_2(8)) \). As \( W \) is the natural module for \( J \), \( W \) is the core of the 9-dimensional permutation module for \( J_U \), so \( C_W(J_U) = 0 \). Therefore for \( U^S \in U^S \cap V \), \( U^S \) is not the stabilizer of a point of \( W \), so from 5.15.1a in [LPS1], \( W \) is the spin module for \( U^S \). Then as \( K = O(W, Q) \), \( U^S \cap V = U^K \), so 8.2 completes the proof.

\((8.10)\). Assume \( U \cong A_m \) is an alternating group. Then \( |M_5(H)| = 1 \).
**Proof.** By 8.4, we may assume $\lambda \not\in \Omega^*$. By 7.18, $\mathcal{M}(\lambda) = \{v\}$, for some $v = (V, \Omega)$ with $U \leq V$. From 7.11, one of cases (2)–(4) of 7.11 holds. In case (4) of 7.11, the lemma follows from 8.8, so we may assume this case does not hold.

Suppose case (3) holds, so that $V = A_{m+2}$, and $U$ is the stabilizer in $V$ of two points in an $(m+2)$-set permuted by $V$. Thus $U^5 \cap V = U^V$ and $\text{Aut}_V(U) \cong S_m$, so either $\text{Aut}_V(U) = \text{Aut}(U)$, or $m = 6$, and applying 8.2, we may assume the latter. Therefore $n = 15$, so $\Omega$ is the set of points in the natural module $W$ for $V = GL(W)$. Then by 7.3.1, $N_5(U) = N_{\text{Aut}(U)}(U^m_0) \cong S_6 \cong \text{Aut}_V(U)$, so again 8.2 completes the proof.

Therefore we may assume case (2) of 7.11 holds, so $\{v\} = \Phi(\lambda)$. Hence the various possibilities for $\tau = (U, V, \Omega)$ are considered in the proof of 7.11. First suppose $\tau$ does not appear in Table III of [LPS2]. Then by the proof of (c) in the proof of 7.11, $\tau$ is $(A_{12}, \Omega_{10}(2), 495)$ or $(A_8, Sp_6(2), 28)$. As $A_8 \cong \Omega^+_6(2)$, the latter case was handled in 8.7, so we may assume the former holds. Let $(W, Q)$ be an orthogonal space for $V$. From [LPS2], we may take $\Omega$ to be the set of singular points of $W$, and $W$ the natural module for $U$. Thus by 7.4.1, $K = N_5(V) = O(W, Q)$, so $\text{Aut}_K(U) \cong S_{12} \cong \text{Aut}(U)$. As the representation of $U$ on $W$ is determined up to quasiequivalence, $U^5 \cap V = U^K$, so 8.2 completes the proof in this case.

Thus we may assume $\tau$ appears in Table III. If $m$ is even, then from the proof of (e) in the proof of 7.11, and as $\tau$ appears in Table III, $\lambda$ is in one of the cases treated in case (3) of 7.11 above.

Thus we may assume $m$ is odd. Further by 8.5 and our treatment of case (3), we may assume $\lambda \not\in \Phi(\lambda^\prime)$ for some $\lambda^\prime$ treated in case (3). From 8.8, we may assume $n \neq 120$. Thus by inspection of Table III, $V = A_{m+1}$. In particular $U^5 \cap V = U^V$, so setting $K = N_5(V)$, by 8.2 it remains to show that $\text{Aut}_K(U) = \text{Aut}_5(U)$. Moreover if $K \cong S_{m+1}$, then $\text{Aut}_5(U) \cong S_6 \cong \text{Aut}(U)$, so we may assume either $K = V$ or $m = 5$. Let $I$ be an $m+1$-set permuted by $V$.

Suppose $\Omega$ is the set of regular $((m+1)/2, 2)$-partitions of $I$. If $m \neq 5$ then $K = N_5(V) \cong S_{m+1}$, contrary to the previous paragraph, so $m = 5$. Then $V$ acts on $\Omega$ of order 10 as $L_2(9)$, so $K \cong \text{Aut}(U)$ and $\text{Aut}_K(U) \cong S_5 \cong \text{Aut}(U)$, and once again 8.2 completes the proof. So we may assume this case does not hold.

By the previous paragraph and inspection of Table III, $m > 5$. If $m = 7$ or 9, then by 7.7 and our exclusions, $\Omega$ is the set of $((m+1)/2, 2)$-partitions of $I$, contrary to the previous paragraph. Therefore $m \geq 11$.

Suppose next that $m = 175$ or 275 and $\Omega = V/J$, where $J \cong HS$ or $Co_3$, respectively. Then $U \cap J = J_x$ for $x \in I$, and $J_x$ is isomorphic to $Z_2/U_3(5)$ or $\text{Aut}(Mv)$, respectively. In particular $|\text{Out}(F^*(U \cap J))| = 2$, so $O^2(N_{\text{Aut}(U)}(U \cap J)) \leq U$, and hence $U = N_5(U) \cong \text{Aut}(U)$, so the lemma holds in these two cases by 8.2.

Similarly suppose $n = 2d-1(d^2 + \epsilon)$ for some $d \geq 3$ and $\epsilon = \pm 1$. Then from [LPS2], there exists $Sp_{2d}(2) \cong \Omega \leq V$ with $\Omega = V/J$ and for $i \in I$, $J_i \cong O^2_6(2)$. Thus choosing $U = V_i$, $J_i \cap U$. Now $|\text{Out}(F^*(J_i))| = 2$, so arguing as above, $U = N_5(U)$ and hence $\text{Aut}_5(U) \cong U = \text{Aut}(U)$, so 8.2 completes the proof.

This leaves the case where $m$ is prime and $\Omega = V/J$, where $J \cong L_2(2)$. Now $N_{\text{Sym}(I)}(J) \cong PGL_2(m)$ contains an odd permutation, so $K = \text{Sym}(I)$, contrary to an earlier observation.

This finally completes the proof of the lemma. $\Box$

**Remark.** Assume $U \cong G_2(q)^\prime$. Then either

1. $|\mathcal{M}_5(H)| = 1$, or
2. $H \cong G_2(3)$, $n = 3159$, $|\mathcal{M}_5(H)| = 3$, $\text{Aut}(H) \cong N_5(U) \in \mathcal{M}_5(U)$, and $N_5(U)$ is transitive on $\mathcal{M}_5(H) - \{N_5(U)\}$, $V \in \mathcal{M}_5(H) - \{N_5(H)\}$ is isomorphic to $\Omega_7(3)$, and $H$ is maximal in $V$.

**Proof.** By 8.4, we may assume $\lambda \not\in \Omega^*$. By 7.12, $\mathcal{M}(\lambda) = \{v\}$, for some $v = (V, \Omega)$ with $U \leq V$. The possibilities for $n$ and the members $\mu = (L, n)$ of $\Phi(\lambda)$ are listed in (1)–(iv) of the proof of 7.12. In particular in each case, $L \cong \Omega_7(q)$. Let $(W, Q)$ be a 7- orthogonal or 6-dimensional symplectic space for $L$ for $q$ odd or even respectively. The representation of $U$ on $W$ is determined up to quasiequivalence and $Q$ is determined up to a scalar, so $U^5 \cap L = U^M$, where $M$ is the similarity group of $(W, Q)$.
Suppose for the moment that (iv) does not hold. We will show that $N_5(L) = \Gamma = \Gamma(W, Q) \geq M$, and either $\text{Aut}(U) = \text{Aut}_\tau(U)$ or $q$ is power of 3. If $q$ is a power of 3 and $\tau \in \text{Aut}(U) - \text{Aut}_\tau(U)$, we will show that $\tau$ does not act on $U^\omega$, $\omega \in \Omega$, so by 7.3.1, $N_5(U) \leq \Gamma = N_5(L)$, and hence $N_5(U) \cong N_5(U)$. Therefore (modulo verifying $N_5(L) = \Gamma$ and $\tau$ does not act on $U^\omega$) (1) holds by 8.2 in cases (i)–(iii) when $\lambda \in \mathcal{E}_\omega$.

Suppose $\mu$ appears in case (i) of the proof of 7.12. Then from 7.12, $n = q^3(q^3 + \epsilon)/2$, and either $q = 2^k > 2$ and $V \cong \text{Sp}_{q+2}(2)$, or $\mu = \nu$. From [LPS2], we may take $\Omega$ to be a $V$-class of nonsingular points of $W$, so indeed $N_5(L) = \Gamma$ by 7.3.1. If $q$ is a power of 3, then $\tau$ interchanges the two classes of root groups of $U$, so $\tau$ does not act on $U^\omega \cong \mathbb{Z}_2/\mathbb{Z}_3^\epsilon(q)$. Thus by the previous paragraph we may assume suppose $\mu \neq \nu$, and hence $q = 2^k > 2$. In this case write $\hat{W}$ for $W$ viewed as an $\mathbb{F}_2$-space, let $T : \mathbb{F}_q \to \mathbb{F}_2$ be the trace map, and $f = QC$ the composition, so from [A1], $(\hat{W}, f)$ is a 6k-dimensional symplectic space and $\Gamma$ is the stabilizer in $V = O(\hat{W}, f)$ of the $\mathbb{F}_q$-structure $W$ on $\hat{W}$. Then $N_5(U) \cong N_5(U) = N_5(U)$. Further from [A1], $V$ is transitive on the $\mathbb{F}_q$-structures on $\hat{W}$, so as $U^S \cap L = U^\Gamma$, also $U^S \cap V = U^V$. So (1) holds in case (i) by 8.2.

Next suppose case (ii) holds, then by 7.12, $\Phi(\lambda) = \mu$, and $n = (q^6 - 1)/(q - 1)$. From [LPS2], we may take $\Omega$ to be the singular points of $W$, so again $N_5(L) = \Gamma$ by 7.3.1. When $q$ is a power of 3, $\tau$ does not act on this class of parabolics, so by an earlier remark we may assume $\mu \neq \nu$. Hence by 7.12, $q$ is even and $V \cong L_6(q)$. As $\Omega$ is the set of points of $W$, $K = N_5(V) = P^F(W)$ by 7.3.1, so $\Gamma \subseteq K$. Then as $K$ is transitive on symplectic forms on $W$, as $U^S \cap L = U^\Gamma$, and as $N_5(U) = N_5(U)$, (1) holds in case (ii) by 8.2.

In the remaining cases, $U \in \mathcal{E}_\omega$. Suppose (iii) holds. Then $q$ is even and by [LPS2], we may take $\Omega$ to be the set of nondegenerate lines of $W$, so that $\Gamma = N_5(L)$ by 7.3.1. Thus the lemma holds in case (iii) by earlier remarks.

This leaves case (iv), where $q = 3$ and $n = 3159$. Now [LPS2] says that $\Omega = V / J$, where $J \cong \text{Sp}_6(2)$. We have $\Gamma = P\Omega(3)$ and $N_5(V) = V$, so $V = N_5(V)$ by 7.3.1. From the discussion in 5.1.14 in [LPS1], $U^\omega \cong L_3(2)/E_8$. Further from Corollary 11 in [G2], $U^\omega$ is the unique class of such subgroups, so $\tau$ acts on this. Thus by 7.3.1, $\text{Aut}(U) \cong N_5(U)$, so in particular $N_5(U) \not\subseteq V$. Therefore to show that (2) holds in this case, it remains to show that $U^S \cap V = U^V$. As $|\Gamma : V| = 2$ and $U = N_5(U)$, $V$ has two orbits on $U^S$, while the representation of a $G_2(3)$-subgroup of $GL(W)$ is determined up to quasiequivalence, so each such subgroup is in $U^\Gamma$. Thus it suffices to show:

(*) $U^S \not\subseteq U^S$ for $g \in \Gamma - V$.

Let $X$ be the set of $L(3)(2)/E_8$-subgroups of $V$. Each $X \in \mathcal{X}$ determines $Y = Y(X) = \{Y_1, \ldots, Y_7\} \in \mathcal{Y}$, the set of orthogonal direct sum decompositions of $W$ via isometric points, where $Y(X)$ is the set of weight spaces of $O_2(X)$. Observe $V$ is transitive on $\mathcal{Y}$ and the stabilizer $N_5(Y(X))$ in $\Sigma = O(W, Q)$ of $Y \in \mathcal{Y}$ is $EL$, where $E = E_X \cong E_{27}$ is the kernel of the action of $\theta = N_5(Y)$ on $Y$ and $L$ acts faithfully as $\text{Sym}(Y)$ on $Y$. Of course $N_5(Y(X)) \cong L_7(2)/E_6$ and $E = (E \cap V) \times Z$ where $Z = Z(\Sigma)$. Observe also that $L_3(2)$-subgroups of $S_7$ are selfnormalizing, so $N_9(XE) = XE$.

Next one class of maximal parabolics of $J$ is conjugate to $R = X(E \cap V)$. By (*) we may assume $g \in \Sigma - V Z$ with $U^S \subseteq U^S$. Then $X = U^S \cap J \in \mathcal{X}$, so $X = XE_X \subseteq R^L$. Then conjugating in $J$, we may assume $R = R \leq \theta$. As $U^S$ is transitive on $U^S \cap \mathcal{X}$, we may assume $X^S = X$, so $(XE)^S = XE_X \subseteq XE$, and hence $E = E^S$ so $g \in N_5(E) = \theta$. This is impossible as $N_9(XE) \subseteq V Z$ and $g \in \Sigma - V Z$. Hence the proof is at last complete. \(\square\)

**8.12.** Assume $U$ is $k$-transitive on a set $\Delta$ for some $k \geq 2$ and $|\Delta| > 6$, and $\Omega$ is the set of $k$-subsets of $\Delta$. Let $V = \text{Alt}(\Delta)$, $K = \text{Sym}(\Delta)$, and assume:

(a) For $\omega \in \Omega$ and $\delta \in N_5(U)U^\omega \cap \text{Aut}(U)$, $N_5(U)U^\omega \cap \text{Aut}(U)$.

(b) If $\rho : U \to K$ is a faithful transitive representation of $U$ on $\Delta$, then $\rho$ is quasiequivalent to the inclusion map $U \subset K$. 

Then

(1) \( N_5(U) \cong N_K(U) \), and

(2) \( U^S \cap V = U^K \).

Proof. By (a) and two applications of 7.3.1, \( N_5(U) \cong N_{\text{Aut}(U)}(U^U) = N_{\text{Aut}(U)}(U^U) = N_K(U) \), so (1) holds.

If \( s \in S \) with \( U^s \leq V \) then the conjugation map \( c_s : U \to K \) is a faithful transitive representation of \( U \) on \( \Delta \), so by (b), \( c_s \) is quasiequivalent to the inclusion \( U \subseteq K \), and hence \( U^s = Uc_s \in U^K \) by 7.1 and 7.2. \( \square \)

(8.13). Assume \( U \cong L^q_m(q) \) with \( m \neq 4 \) and \( U \neq L_2(5) \). Then one of the following holds:

(1) \( |\mathcal{M}_5(H)| = 1 \).

(2) \( H \cong L_2(q) \), with \( q \in \{11, 23\} \), \( n = q + 1 \), \( |\mathcal{M}_5(H)| = 3 \), \( PGL_2(q) = N_5(U) \in \mathcal{M}_5(U) \) and \( N_5(U) \) is transitive on \( \mathcal{M}_5(H) - \{N_5(U)\} \), and if \( V \in \mathcal{M}_5(H) - \{N_5(U)\} \) is isomorphic to the Mathieu group \( M_n \), and \( H \) is maximal in \( V \).

(3) \( H \cong L_2(17), n = 136 \), \( |\mathcal{M}_5(H)| = 3 \), \( PGL_2(17) \cong N_5(U) \in \mathcal{M}_5(U) \) and \( N_5(U) \) is transitive on \( \mathcal{M}_5(H) - \{N_5(U)\} \), \( V \in \mathcal{M}_5(H) - \{N_5(U)\} \) is isomorphic to \( Sp_8(2) \), and \( H \) is maximal in \( V \).

(4) \( \lambda = (L_3(4), 280) \), \( |\mathcal{M}_5(U)| = 4 \), \( Aut(U) \cong N_5(U) \in \mathcal{M}_5(U) \) and \( N_5(U) \) is transitive on \( \mathcal{M}_5(U) - \{N_5(U)\} \), and if \( K \in \mathcal{M}_5(H) - \{N_5(U)\} \) is isomorphic to \( \text{Sp}_3(3) \), \( |K| = 3 \), \( PGL_2(3) = N_5(U) \in \mathcal{M}_5(U) \) and \( N_5(U) \) is transitive on \( \mathcal{M}_5(H) - \{N_5(U)\} \), \( K \in \mathcal{M}_5(H) - \{N_5(U)\} \) is isomorphic to \( S_{11} \), and \( \mathcal{O}_K(H) = \{ H < L < V < K \} \) with \( L \cong M_{11} \) and \( V \cong A_{11} \).

Proof. By 8.4, we may assume \( \lambda \notin \mathbf{L}^s \). By 7.13, \( \mathcal{M}(\lambda) = \{ v \} \), for some \( v = (V, \Omega) \) with \( U \subseteq V \). The possibilities for \( n \) and the members \( \mu = (n, n) \) of \( \Phi(\lambda) \) are listed in (i)--(viii) of the proof of 7.13.

In case (i) when \( \lambda \in \mathcal{L}^s \), and in cases (3) and (4) of 7.13, \( \Omega \) is the set of 2-subsets of a set \( \Delta \) of order \( m \), with \( V = \text{Alt}(\Delta) \) and \( K = N_5(V) = \text{Sym}(\Delta) \) by 7.4.1. In (i) and (4), \( m = q + 1 \) and \( PGL_2(q) = N_K(U) \). Then as \( PGL_2(q) = Aut(U) \), condition (a) of 8.12 holds. As \( U^u_{\text{Aut}(U)} \) is the unique \( Aut(U) \)-class of subgroups of index \( n \), condition (b) of 8.12 holds. Now 8.2 and 8.12 say that (1) holds in these cases.

Suppose case (3) of 7.13 holds. From the discussion below of case (ii) when \( n = 11 \), \( \mathcal{O}_K(H) = \{ H < L < V < K \} \) with \( L \cong M_{11} \) and \( V \cong A_{11} \). Also \( U^w \cong D_{12} \) is determined up to conjugacy in \( Aut(U) \), so \( PGL_2(11) \cong Aut(U) = N_{\text{Aut}(U)}(U^U) \cong N_5(U) \) by 7.4.1.

We saw \( U = N_K(U) \) and \( K \) is transitive on its \( L_2(11) \)-subgroups, so \( H = K \cap N_5(U) \) and \( N_5(U) \) is transitive on \( K^5 \cap \mathcal{O}_S(U) \). Therefore (5) holds in this case.

Suppose case (2) or (5) of 7.13 holds. In (2), \( \lambda = (L_2(7), 28) = (L_2(2), n) \), where \( n = 2^2(2^3 - 1) \) and \( V \cong Sp_6(2) \), while in (5), again \( \lambda = (L_2(2), n) \) where \( n = 2^{m-1}(2^m - 1) \), and \( V \cong Sp_{2m}(2) \). From [LPS2], \( H = Aut(U) \). Let \( W \) be the symplectic space with \( V = Sp(W) \). Then \( H \) is the stabilizer in \( V \) of a decomposition \( W = W_1 \oplus W_2 \) of maximal totally singular subspaces \( W \), and \( V \) is transitive on such decompositions by Witt’s Lemma, so \( U^S \cap V = U^V \). Then as \( Aut(U) = H \subseteq V \), (1) holds in this case by 8.2.

We have treated cases (2)–(5) of 7.13, so we may assume case (1) of 7.13 holds, where \( \lambda \in \mathcal{L}^s \), so that \( \{ v \} = \Phi(\lambda) \). We have also treated case (i) from the proof of 7.13. In case (ii), \( \lambda = (L_2(11), n) \), and as \( \lambda \in \mathcal{L}^s \), \( n = 11 \) or 12 and \( V = M_n \) is a Mathieu group. Similarly in case (iii) we may take \( \lambda = (L_2(23), 24) \) and \( V = M_{24} \).

Suppose (ii) or (iii) holds with \( n = q + 1 \). Now \( V = N_{\text{Aut}(V)}(V^V) \), so \( K = V \). Further \( V \) is transitive on its \( L_2(q) \)-subgroups, so \( U^S \cap V = U^V \). Finally \( U \) is maximal in \( V \), while \( PGL_2(q) = Aut(U) = N_5(U) \) by 7.3.1. Thus (2) holds in this case.

Suppose (ii) holds with \( n = 11 \). Then \( U^w \cong A_5 \) is selfnormalizing in \( PGL_2(11) \), so \( N_5(U) = U \) by 7.3.1. Therefore as \( V \) is transitive on its \( L_2(11) \)-subgroups, (1) holds by 8.2.

Assume case (iv) holds, so that \( \lambda = (U_3(4), 416) \) and \( V \cong G_2(4) \). From [LPS2], \( \Omega = V / J \), where \( J \cong J_2 \). From Corollary 11 in [G2], \( V \) is transitive on its \( J_2 \)-subgroups, so \( Aut(V) = V N_{\text{Aut}(V)}(J) \) and
hence $K = \text{Aut}(V)$ by 7.3.1. Similarly from Corollary 11 in [G2], $V$ is transitive on its $U_3(4)$-subgroups and $N_K(U) \cong \text{Aut}(U)$. Therefore (1) holds by 8.2.

Assume case (v) holds. As $\lambda \in \mathcal{L}^*$, $\lambda = (L_{2m}(3), n)$, $V = P\Omega^+_{2m}(3)$, and $n = 3^{m-1}(3^m - 1)/2$. By [LPS2], $\Omega$ is an orbit of $V$ on the singular points of the orthogonal space $W$ such that $V = 0(W)$ and $V = \hat{V}/Z(\hat{V})$. Then as $m \neq 4$, $K = P\Omega^+_{2m}(3)$. Again from [LPS2], $H$ contains an element inducing a graph automorphism on $U$, so $H \cap V$ contains the stabilizer in $V$ of a decomposition $W = W_1 \oplus W_2$ via maximal totally singular subspaces. As $K \cong \Omega^+_{2m}(3)$, $K$ is transitive on such decompositions, so $U^S \cap V = U^K$. Also $\text{Aut}(U) = \text{Aut}_K(U)$, so (1) holds by 8.2.

Assume case (vi) holds. Then $\lambda = (L_2(17), 136)$ and $V \cong \text{Sp}_8(2)$. As $\text{PGL}_2(17) \not\leq L_6(2)$, $U = \text{Aut}_V(U)$. The representation of $U$ on the symplectic space $W$ for $V$ is determined up to quasiequivalence, and $\text{End}_{E_7}(U) = F_2$, so $U$ preserves a unique symplectic form on $W$, and hence $U^S \cap V = U^V$. Finally by an order argument, $U_\omega \cong D_{18}$, so by 7.3.1, $N_S(U) = N_{\text{Aut}(U)}(U_\omega^U) = \text{Aut}(U) \cong \text{PGL}_2(17)$. It follows that (3) holds.

Assume case (vii) holds. Then $\lambda = (L_3(4), 280)$ and $V \cong U_4(3)$. Let $W$ be a unitary space with $SU(W) = \hat{V} \cong SU_4(3)$ and $V = \hat{V}/Z(\hat{V})$. From [LPS2], we may take $\Omega$ to be the set of singular points of $W$, so $K = \text{Aut}(V)$. By 5.2.7 in [LPS1], $U_\omega \cong \text{Sym}_6(E_6)$. Thus by 7.3.1, $N_S(U) \cong \text{Aut}(U)$. Finally $|\text{Aut}(U): \text{Aut}_K(U)| = 3$, so we conclude that (4) holds in this case.

Finally case (viii) in the proof of 7.13 leads to conclusion (2) of 7.13, a case we have already treated, so the proof of the lemma is complete. \(\square\)

(8.14). Assume $U \cong \text{PSp}_4(q)$, $q > 2$, or $\text{PSp}_{2m}(q)$ with $m > 2$ and $q$ odd. Then $|\mathcal{M}_S(H)| = 1$.

**Proof.** By 8.4, we may assume $\lambda \notin \mathcal{L}^*$. By 7.18, $\mathcal{M}(\lambda) = \{v\}$, for some $v = (V, \Omega)$ with $U \leq V$. If case (2) of 7.14 holds, then the lemma follows from 8.6, so if $U$ is $\text{PSp}_4(q)$ then we may assume case (1) of 7.15 holds, where we set $m = 2$. Thus by 7.14 and 7.15, $n = (q^{m-1} - 1)/(q - 1)$ and $V \cong L_{2m}(q)$. From [LPS2], we may take $\Omega$ to be the set of points in the $E_7$-module $W$ for $\hat{V} = \text{SL}(W)$ with $\hat{V}/Z(\hat{V}) = V$. From 7.3.1, $K = N_S(V) = P\Gamma L_{2m}(q)$ and $N_S(U)$ is the subgroup $\Gamma$ of $K$ which is $\text{PGSp}_{2m}(q)$ extended by field automorphisms. Thus $N_S(U) \cong N_K(U)$, and as the representation of $U = \text{Sp}(W)$ on $W$ is determined up to quasiequivalence, $U^S \cap V = U^K$. Hence the lemma follows from 8.2. \(\square\)

(8.15). Assume $U$ is sporadic. Then either

1. $|\mathcal{M}_S(H)| = 1$, or
2. $H = U \cong H_5$, $n = (\binom{176}{2}) = 15400$, $|\mathcal{M}_S(H)| = 3$, $\text{Aut}(U) \cong N_S(U) \in \mathcal{M}_S(U)$ and $N_S(U)$ is transitive on $\mathcal{M}_S(H) - \{N_S(U)\}$, $K \in \mathcal{M}_S(H) - \{N_S(U)\}$ is isomorphic to $S_{176}$, and $U$ is maximal in $V$.

**Proof.** By 8.4, we may assume $\lambda \notin \mathcal{L}^*$. By 7.16, $\mathcal{M}(\lambda) = \{v\}$ for some $v = (V, \Omega)$ with $U \leq V$. Set $K = N_S(V)$.

Suppose that case (3) of 7.16 holds; then there are four possibilities for $n$. If $n = 66$ then $U \cong M_{11}$, $V \cong A_{11}$, and the hypotheses of 8.12 are satisfied with $k = 2$ and $\Delta$ an 11-set. Namely $U = \text{Aut}(U)$, so condition (a) of 8.12 is satisfied, and $M_{11}$ has a unique class of subgroups of index 11. Thus (1) holds in this case by 8.12 and 8.2.

If $n = 495$ then $U \cong M_{12}$ and $V \cong \Omega^+_{10}(2)$. Let $W$ be the orthogonal space with $V = O(W)^\infty$. By [LPS2], we may take $\Omega$ to be the set of singular points of $W$, so by 7.3.1, $K = O(V)$. As $U$ has a unique equivalence class of faithful 10-dimensional modules, and preserves a quadratic form $Q$ on one such module $W$, with $\text{End}_{E_7}(W) \cong F_2$, $Q$ is the unique $U$-invariant form on $W$, so $\text{Aut}(U) \leq O(W)$, and $U^S \cap V = U^K$. Thus (1) follows from 8.2.

Suppose $n = 2016$, so that $U \cong J_2$ and $V \cong \text{Sp}_6(4)$. Now $\lambda \leq \mu \leq v$ with $\mu = (L, n)$ and $L \cong G_2(4)$. Also $\text{Aut}(L)$ is $L$ extended by a field automorphism, so $\text{Aut}(L) = \text{Aut}_K(L)$. We saw during the proof of 8.13 that $\text{Aut}(U) = \text{Aut}_{\text{Aut}(L)}(U)$, so $\text{Aut}(U) \cong \text{Aut}_K(U)$, and as the representation of $U$ on $W$ is determined up to quasiequivalence, $U^S \cap V = U^K$, so (1) holds in this case by 8.2.
This leaves the case $n = 2^{11}(2^{12} - 1)$, where $U \cong \text{Co}_1$ and $V \cong \text{Sp}_{24}(2)$. Then $U = \text{Aut}(U)$ and the Leech lattice is the unique faithful 24-dimensional $F_2U$-module, so as usual (1) holds in this case by 8.2.

Thus we may assume case (2) of 7.16 holds, so $U \in \mathcal{L}_e$ and hence $\{v\} = \Phi(\lambda)$ and $\tau = (U, V, n) \in \Phi$.

Assume $U$ is $k$-transitive on a set $\Delta$ for some $k \geq 2$ and $|\Delta| > 6$, $\Omega$ is the set of $k$-subsets of $\Delta$, and $V = \text{Alt}(\Delta)$. Then the list of possibilities for $\tau$ appears in the bottom half of Table III of [LPS2]. To show that (1) holds, it suffices to verify conditions (a) and (b) of 8.12. Inspecting the list in Table III, we find that condition (b) holds in each case. If $U = \text{Aut}(U)$ then condition (a) is trivial, so we may assume $U = M_{12}, M_{22},$ or HS. Suppose $U \cong M_{12}$. Then $U = N_{\text{Aut}(U)}(U_5)$ for $\delta \in \Delta$. By 8.5 and our treatment of case (3) of 7.16, we may take $k = 3$ and $n = 220$. Then for $\omega \in \Omega$, $U_\omega \cong \text{GL}_2(3)/E_9$ is the normalizer of a 9-group. Then $U = N_{\text{Aut}(U)}(U_\omega)$, so (a), and therefore (1) holds.

If $U = M_{22}$ then $U = N_V(U)$, so (a), and hence also (1) holds. This leaves the case where $U \cong HS, k = 2$, and $|\Delta| = 176$. Here $U_8 \cong \text{Z}_2/U_3(5)$ and $U = N_{\text{Aut}(U)}(U_8)$, so $U = N_K(U)$. Further $U_\omega \cong \text{Z}_2 \times \text{Aut}(A_6)$ is the centralizer in $U$ of a non-2-central involution, so from 7.3.1, $N_3(U) = \text{Aut}(U)$. Thus (2) holds in this case.

There is a unique class of $M_{23}$-subgroups of $M_{24}$ and $\text{Out}(M_{23}) = 1$, so (1) holds by 8.2 if $(U, V) = (M_{23}, M_{24})$.

Suppose $\tau = (M_{22}, HS, 176)$. We just saw that $V = K$. Further HS has a unique class of $M_{22}$-subgroups, so $U^5 \cap V = U^V$. From 6.7 in [LPS1], $U_\omega \cong A_7$, and then from [GLS3], $U = N_{\text{Aut}(U)}(U_\omega)$, so (1) holds in this case by 8.2.

Suppose $\tau = (J_2, G_2(4), 2016)$. From [LPS2], $\Omega = V/\Gamma$ where $\Gamma \cong \text{Aut}(U_4(4))$. From Corollary 11 in [G2], $V$ is transitive on its $J_2$-subgroups and subgroups isomorphic to $J$, so (1) holds in this case by 8.2.

Suppose $\tau = (J_3, U_9(2), 43605)$. Let $W$ be the natural module for $U = SU_9(2)$ with $V = \hat{V}/Z(\hat{V})$. From [LPS2], we may take $\Omega$ to be the set of singular points of $W$, so from 7.4.1, $\text{Aut}(V) = K$. Let $U$ be the covering group of $U$. Up to conjugation under $\text{Gal}(F_4/F_2)$, there is a unique faithful 9-dimensional $F_4\hat{U}$-module, so $N_K(U) = \text{Aut}(U)$ and $U^5 \cap V = U^V$. Therefore (1) follows from 8.2.

Suppose $\tau = (M_{22}, U_6(2), 672)$. Let $W$ be the natural module for $V = SU_6(2)$ with $V = \hat{V}/Z(\hat{V})$. From [LPS2], we may take $\Omega$ to be the set of nonsingular points of $W$. Then from 7.3.1, $\text{Aut}(V) = K$. Let $U$ be the covering group of $U$. Up to conjugation under $\text{Gal}(F_4/F_2)$, there is a unique faithful 6-dimensional $F_4\hat{U}$-module, so $N_K(U) = \text{Aut}(U)$ and $U^5 \cap V = U^V$. Therefore (1) follows from 8.2. This completely the proof of the lemma. □

(8.16). Assume $U$ is an exceptional group of Lie type other than $G_2(q)$. Then either

1. $|\mathcal{M}_S(H)| = 1$, or
2. $U \cong \text{Sz}(q), q = 2^k, n = q^2(q^2 + 1)/2, \mathcal{M}_S(U) = \{K_1, K_2\}$ where $K_i = N_5(V_i) \cong \text{Aut}(V_i), V_1 \cong A_{q^2+1}, V_2 \cong \text{Sp}_{4(2)}$, and $N_5(U) \cong \text{Aut}(U)$ is maximal in $V_1$.

Proof. By 8.4, we may assume $\lambda \notin \Sigma^e$.

Suppose that case (1) of 7.17 holds. Then $\mathcal{M}(\lambda) = \Phi(\lambda) = \{v\}$ for some $v = (V, \Omega)$ with $U \leq V, U \cong \hat{3}D_4(q), q = 2^s$ even, and $V \cong F_4(q)$. From [LPS2], $\Omega = V/\Gamma$, where $\Gamma \cong \text{Sp}_{8}(q)$. The discussion in subsection B of the introduction to [LPS1], $K = N_{\text{Aut}(V)}(V_\omega)$ is the subgroup of $\text{Aut}(F_4)$ trivial on the Dynkin diagram of $V$, and $V$ is transitive on the $3D_4(q)$-subgroups $U'$ with $U'J = V$. Thus $U^5 \cap V = U^V$ and $K$ is the extension of $V$ by the group $Z_e$ of field automorphisms. However from the Main Theorem of [LS], $|N_V(U) : U| = 3$, so $|\text{Out}_K(U)| = 3e$. Finally from 2.5.12 in [GLS3], $|\text{Out}(U)| = 3e$, so $\text{Aut}_K(U) = \text{Aut}(U)$. Therefore (1) holds in this case by 8.2.

Thus we may assume case (2) of 7.17 holds. Therefore $U \cong \text{Sz}(q)$ with $q = 2^k$ and $\mathcal{M}(\lambda) = \{\mu, v\}$ where $v = (V, n)$ with $V \cong \text{Sp}_{4(2)}, \mu = (V_\mu, n)$ with $V_\mu \cong A_{q^2+1}, \mu \in \Phi(\lambda)$, and $n = q^2(q^2 + 1)/2$. Now $U$ is 2-transitive on a set $\Delta$ of order $q^2 + 1, V_\mu = \text{Alt}(\Delta)$, and we may view $\Omega$ as the set of 2-subsets of $\Delta$. Condition (a) of 8.12 is satisfied with $\text{Aut}(U) = N_{\text{Aut}(U)}(U_\delta)$ for $\delta \in \Delta$, and condition (b) is satisfied as $U_\delta$ is a Borel subgroup of $U$. Therefore, as in the proof of 8.12, $K_\mu = N_5(V_\mu)$ is the unique member of $\mathcal{M}_S(U)$ with $F^*(K_\mu) \cong A_{q^2+1}$. 


Next let $W$ be a symplectic space over $F_2$ with $V_\pi = \text{Sp}(W)$. Then $U$ is contained in the stabilizer $X$ of an $F_2$-structure $W_{\mathcal{F}_2}$ on $W$, $X$ is $\text{Sp}_4(q)$ extended by the group $\mathbb{Z}_k$ of field automorphisms, and $V_\pi$ is transitive on its $F_q$-structures. Further $X$ is transitive on is $Sz(q)$-subgroups, so we conclude $U^S \cap V_\pi = U^{V_\pi}$ and $\text{Aut}(K, U) = \text{Aut}(U)$. Thus, as in the proof of 8.12, $K_\pi$ is the unique member of $\mathcal{M}_5(U)$ with $\mathcal{F}(K_\pi) \cong \text{Sp}_{4k}(2)$. Therefore $\mathcal{M}_5(U) = \{K_\mu, K_\nu\}$. That is (2) holds in this case, completing the proof of the lemma. \qed

We are now in a position to prove Theorem A. Assume the hypothesis of Theorem A. By 8.5 in [A4], all members of $O_5(H)$ are almost simple, product indecomposable, and not octal. Observe Hypothesis 8.1 is satisfied, so we can appeal to the lemmas in this section. Assume conclusion (2a) of Theorem A holds. Thus we may assume that $U$ is of Lie type.

Suppose $U$ is exceptional. If $U$ is $G_2(q)'$ then (2b) holds by 8.11, while if $U$ is not $G_2(q)'$ then (4) holds by 8.16. Thus we may assume $U$ is a classical group.

By 8.7, $U$ is not $PO_2m(q)$ with $m \geq 3$, so that $U$ is not $L_4(q)$. Hence if $U \cong L_m^n(q)$ then (2), (3), or (5) holds by 8.13.

By 8.9, $U$ is not $PO_{2r-1}(q)$ with $r \geq 4$. Then by 8.14, $U$ is not a symplectic group of degree at least 4. We have considered all the classical groups, so the proof of Theorem A is complete.

9. The proof of Theorem B

In this section we prove Theorem B, so we assume the hypothesis of Theorem B. Observe Hypothesis 8.1 is satisfied with $\xi = (H, \Omega)$. Thus we continue the notation of the previous section. In particular $U = F^*(H)$. Pick $\omega \in \Omega$ and write $\mathcal{M}$ for the maximal overgroups of $H$ in $G$.

We begin by considering the various cases appearing in Theorem A.

(9.1) Assume case (2a) of Theorem A holds. Then

1. $N_5(U) \leq A \geq K$.
2. $G = A$.
4. $A \cong T_{1,2,2}$.

Proof. From Theorem A, in case (2a), $U = H \cong HS$, $V \cong A_m$ with $m = 176$, $n = \binom{m}{2}$, $N_5(U) \cong \text{Aut}(U)$, and $\mathcal{M}_5(H) = K_{N_5(U)} \cup \{N_5(U)\}$ is of order 3. As $U$ is HS, $|\text{Aut}(U) : U| = 2$. As $n = \binom{m}{2}$, $V_\omega$ is the global stabilizer in $V$ of a 2-subset of the $m$-set $\Delta$ permuted by $V$, so by 7.3.1, $K = N_5(V) = \text{Sym}(\Delta) = \text{Aut}(V)$, and hence $|K : V| = 2$. Hence (3) is established.

Next, if $N_5(U) \leq A$ then $G = A$ by 3.7 in [A4]. Then in addition $K \leq G$, $M = K_{N_5(U)} \cup \{N_5(U)\}$ is of order 3, with $U$ maximal in $N_5(U)$ and $O_K(U) = \{U, V, K\}$. Therefore (4) and the lemma hold in this case, so it remains to verify (1).

Let $t \in K$ be a transposition on $\Delta$ and $s \in N_5(U) - U$ an involution. To show $K \leq A$, $N_5(U) \leq A$, amounts to showing $t$ and $s$ are even permutations on $\Omega$.

First, $m = 176$ and $t$ has $m - 2$ cycles of length 2 on $\Omega$, so $t$ is indeed in $A$.

Next from the proof of 8.15, $U_\omega$ is the centralizer of a non-2-central involution of $U$, so we may view $\Omega$ as the set of non-2-central involutions in $U$. From [GLS3], we may choose $s$ so that $C_U(s) \cong S_8$, with $C_2(s)$ the involutions in $C_U(s) - E(C_U(s))$. Therefore $|\text{Fix}_\Omega(s)| = 448 \equiv n \mod 4$, and hence $s \in A$, completing the proof. \qed

(9.2) Assume case (2b), (2c), or (2d) of Theorem A holds. Then

1. $N_5(U) \not\leq A$.
3. If $G = S$ then $A \cong M_{1,3}$.
4. If $G = A$ then $A \cong M_2$. 

Proof. From Theorem A, \( U = H, N_S(U) \cong \text{Aut}(U) \), and \( \mathcal{M}_S(H) = K^{N_S(U)} \cup \{N_S(U)\} \) is of order 3. Further \( U \) is \( G_2(3) \) or \( L_2(q) \), \( q \in \{11, 23, 27\} \), so \( \text{Aut}(U):U = 2 \). From 8.11 and 8.13, \( K = V \), so (2) holds.

Suppose (1) holds. If \( G = S \) then as \( H = U \leq A \geq K \) and \( \mathcal{M}_S(U) = K^{N_G(S)} \cup \{N_G(S)\} \) we conclude that \( \mathcal{O}_C(H) = K^{N_G(S)} \cup \{H, A, N_S(U), G\} \), so \( A \cong M_{1,3} \) and (3) holds. On the other hand if \( G = A \) then \( \mathcal{O}_C(S) = K^{N_G(S)} \cup \{H, G\} \), so (4) holds. Therefore it remains to establish (1).

In case (c), 8.12 says that \( n = q + 1 \) and \( N_S(U) \cong \text{PGL}_2(q) \), so the stabilizer in \( N_S(U) \) of 2 points is generated by a \( (q - 1) \)-cycle, which is odd. Thus (1) holds in this case.

Let \( X = N_S(U) \) and \( s \in X - U \) an involution. We must show \( s \) is an odd permutation on \( \Omega \). Suppose case (b) holds. From the proof of 8.11, \( U_\omega \cong L_3(2)/E_8 \), so \( X_\omega = N_X(U_\omega) = U_\omega(s) \cong L_3(2)/E_{16} \) and \( C_{U_\omega}(s) \cong F_{21}/E_8 \). In particular \( U_\omega \) is the set of involutions in \( X_\omega - U \), so \( C_\omega(s) \cong 2G_2(3) \cong \text{Aut}(L_2(8)) \) is transitive on \( \text{Fix}_\Omega(s) \), with the 9 Sylow 2-subgroups of \( C_\omega(s) \) in 1–1 correspondence with \( \text{Fix}_\Omega(s) \). Then \( n = 3159 \) is not congruent to 9 modulo 4, so \( s \) is indeed odd.

Finally suppose (d) holds. Then \( X \cong \text{PGL}_2(17) \) and from the proof of 8.13, \( X_\omega \cong D_{36} \). Thus we may view \( \Omega \) as \( s^I \) of order 136, so as \( |C_{\omega}(s)| = 10 \), \( s \) is odd. \( \square \)

(9.3). Assume case (3) of Theorem A holds. Then

1. \( N_S(U) \not\leq A \).
2. If \( H/U \) is not a 2-group then \( G = S, H = N_A(U) \), and \( A \cong M_2 \).
3. If \( H/U \) is a 2-group then \( |H:U| = |N_G(U):U|_2 \), and \( A \cong T_{1,2} \).

Proof. From Theorem A, \( n = 280, U \cong L_3(4), N_S(U) \cong \text{Aut}(U), V \cong U_4(3), K = N_S(V) \cong \text{Aut}(V) \), and \( \mathcal{M}_S(H) = K^{N_S(U)} \cup \{N_S(U)\} \) is of order 4.

Let \( X = N_S(U) \). There are three orbits of \( X \) on involutions in \( X - U \) with representatives \( f, \tau, \sigma \). From the proof of 8.13, we may take \( X_\omega = f \times (\sigma) = C_X(\sigma) \) to be the normalizer of a Sylow 3-subgroup of \( U \), with \( J \cong \text{GL}_2(3)/E_9 \). Thus we may view \( \Omega \) has \( \sigma^U \), so as \( C_X(f) \cong \text{Z}_2 \times \text{PGL}_2(7) \) and \( C_X(\tau) \cong \text{Z}_2 \times S_5 \), we calculate that \( |C_{\omega}(f)| = |\text{PGL}_2(7):D_{12}| = 28 \) and \( |C_{\omega}(\tau)| = |S_5:D_{12}| = 10 \). Then as \( n = 0 \equiv 28 \mod 4 \) and \( 10 \equiv 2 \mod 4 \), we conclude that \( f \in A \) but \( \sigma \notin A \). In particular (1) holds.

As \( |K^X| = 3 \), we can choose notation so that \( N_K(U) = Y = U(f, \tau) \). Suppose \( H/U \) is not a 2-group. Then \( \mathcal{M}_S(H) = \{N_S(U)\} \). But \( A \) is an I-lattice, so \( |\mathcal{M}| \geq 1 \) and hence \( A \in \mathcal{M} \), so \( G = S \) and \( \mathcal{M} = \{A, N_S(U)\} \). Then by 3.7.2 in [A4], \( H = N_A(U) \), so \( H \) is maximal in \( A \), and then \( A \cong M_2 \) by 3.8 in [A4], so that (2) holds.

Thus we may assume during the remainder of the proof that \( H/U \) is a 2-group, and hence that \( H \leq Y \). As \( A \) is an I-lattice, \( N_C(U) \cap M = H \) for some \( M \in \mathcal{M} \). Further either \( M = K^X \cap G \) for some \( x \in X \), or \( G = S, M = A \), and \( H = N_A(U) \). The latter case contradicts \( H/U \) a 2-group, so the former holds. Then \( H = X \cap K^X \cap G = Y^H \cap G \), so as \( H \leq Y \cap G \) it follows that \( Y^H \cap G = H = Y \cap G \). Therefore \( \mathcal{O}_G(H) = \{H, HV, K \cap G, N_G(U), G\} \cong T_{1,2} \), so (3) holds. \( \square \)

(9.4). Assume case (4) of Theorem A holds. Then

1. \( N_S(U) \leq A \).
2. \( G = A \).
3. \( H = N_S(U) \).
4. The minimal members of \( \mathcal{O}_G(H) \) are \( V_1 \cong A_{q^2 + 1} \) and \( LH \), where \( L \cong \text{Sp}_4(q) \).
5. \( \mathcal{O}_{V_2}(LH) \cong \Gamma(k) \).
6. \( A \cong T_{-1} \circ \Gamma(k) \).

Proof. From Theorem A, \( U \cong \text{Sz}(q) \) with \( q = 2^k \geq 8, n = q^2(q^2 + 1)/2 \), \( \mathcal{M}_S(U) = \{K_1, K_2\} \) where \( K_1 = N_S(V_1) \cong \text{Aut}(V_1), V_1 \cong A_{q^2 + 1}, V_2 \cong \text{Sp}_{4k}(2) \), and \( \text{Aut}(U) \cong N_S(U) \) is maximal in \( V_1 \).

Let \( X = N_S(U) \). Then \( X \) is extended by the group of field automorphisms of order \( k \), and \( k \) is odd, so \( X = O_2(X) \leq A \), establishing (1). Therefore \( H \leq X \leq A \), so (2) follows from 3.7 in [A4].
By the previous paragraph, $V_1$ is the alternating group on a set $\Delta$ of order $q^2 + 1$, $K_1 = \text{Sym}(\Delta)$, and $\Omega$ is the set of 2-subsets of $\Delta$. Let $t$ be a transposition in $K_1$. Then $t$ has $q^2 - 1$ cycles of length 2 on $\Omega$, so as $q$ is even, $K_1 \cap G = K_1 \cap A = V_1$. Of course $K_2 = V_2 \leq A = G$.

As $\Lambda$ is an 1-lattice, $H = K_1 \cap K_2 = X$, establishing (3). Then by 1.1, $\Lambda \cong \Omega_{K_1}(H) \cap \Omega_{K_2}(H) \cong T_0 \cap \Omega_{K_2}(H)$ as $H = X$ is maximal in $V_1$.

Proceeding as in the last paragraph of the proof of 8.16, let $W$ be a symplectic space over $F_2$ for $V_2 = \text{Sp}_2$. From that proof, $U$ is contained in the stabilizer $\gamma$ of $\text{aut}(H)$ on $W$, and $\gamma$ is $L \cong \text{Sp}_4(q)$ extended by the group $\mathbb{Z}_q$ of field automorphisms of $L$. Thus $\gamma = L$. Further from that proof, $L$ is transitive on its $S_2(q)$-subgroups, so as $H = \text{aut}(U)$, the proof of 8.2 shows that $\gamma$ is the unique overgroup $\gamma'$ of $H$ in $V_2$ with $F^*(\gamma') \cong L$. By 7.17, $\phi(\lambda) = \langle \mu, \eta \rangle$, where $\mu = (V_1, n)$ and $\eta = (L, n)$. Thus (4) follows from the uniqueness of $Y$. Further by (4) and as $\Lambda \cong T_0 \cap \Omega_{K_2}(H)$, also $\Lambda \cong T_1 \cap \Omega_{K_2}(Y)$. Thus to establish (5) and complete the proof of the lemma, it remains to show $\Omega_{K_2}(Y) \cong \Gamma(k) = \Gamma$.

From Section 1, $\Gamma'$ is the set of positive divisors of $k$, partially ordered by $d \leq e$ if $d$ divides $e$. Let $F = F_2$. Then the map $d \mapsto F_{2d}$ is an isomorphism of $\Gamma'$ with the set $\Delta$ of subfields of $F$. Restricting $W_F$ to the subfield $F_{2d}$, we obtain an $F_{2d}$-structure $W(d)$ on $W$. Let $Y(d)$ be the stabilizer of this structure. Then $Y(d) = L(d)H$ where $L(d) \cong \text{Sp}_{4k/d}(2^d)$, and $Y(d)$ is $L(d)$ extended by the group of field automorphisms of order $d$. The map $d \mapsto Y(d)$ is an isomorphism of the dual of $\Gamma'$ with a sublattice $\Theta$ of $\Omega_{V_2}(Y)$, so as $\Gamma'$ is selfdual, $\Theta \cong \Gamma'$. To complete the proof it remains to show that $\Theta = \Omega_{V_2}(Y)$. But if $Y' \in \Omega_{V_2}(Y)$ then $\alpha = \phi(Y', n) \in \Omega(n)$, so by 7.18, $\alpha = (L', n)$ with $L' \cong \text{Sp}_{4k/d}(2^d)$ for some $d \in \Gamma'$, and $[\alpha] = ([L(d), n])$. Then as $V_2$ is transitive on such subgroups and $L(d) \cap L^{V_2} = L^{(d)}$, the argument in 8.2 shows $L' = L(d)$, completing the proof. □

(9.5). Assume case (5) of Theorem A holds. Then

1. $N_{S}(U) \not\cong A$.
2. $K \not\cong A$.
3. For $t \in N_{S}(H) - H$, $K \cap K^t = H$.
4. If $G = A$ then $\Lambda = \{H, L, V, L^t, V^t, A\} \cong \Omega_{2.2}$.
5. If $G = S$ then $\Lambda \cong T_1 \ast H_7$.

Proof. From Theorem A, $H \cong L_{2}(11)$, $n = 55$, $\mathcal{M}(H) = \{N_{S}(H), K, K^t\}$ is of order 3, where $N_{S}(H) \cong \text{PGl}_{2}(11)$, $K \cong S_{11}$, and $\Omega_{K}(H) = \{H < L < V < K\}$, with $L \cong M_{11}$ and $V \cong A_{11}$.

Let $X = N_{S}(U)$. A subgroup $P$ of $H$ of order 55 is regular on $\Omega$, and there is an involution $t \in N_{K}(P) - H$. Now $|\text{Fix}_{G}(t)| = |C_{P}(t)| = 5$, so $t$ has $(55 - 5)/2 = 25$ cycles of length 2, establishing (1).

Similarly if $s$ is a transposition in $K$, then $s$ has $11 - 2 = 9$ cycles of length 2 on $\Omega$, so (2) holds.

As $H = X \cap K = N_{K}(H)$ and $K \cap K^t \in \Omega_{K}(H)$ is invariant under $X = H(t)$, (3) follows.

Suppose $G = A$. Then by (1) and (2), $\Lambda = \{H, L, V, L^t, V^t, A\}$, and then by (3), $\Lambda$ is the hexagon $T_{2.2}$.

Finally suppose $G = S$. Then by (1) and (2), and as $\mathcal{M}(S) = \{X, K, K^t\}$, we have $\mathcal{M} = \{X, K, K^t, A\}$. Further $H = X \cap A$ is maximal in $X$, $\Omega_{K}(H) = \{H < L < V < K\}$, $V = A \cap K$, and $K \cap K^t = H$ by (3). Thus (5) follows. □

We can now complete the proof of Theorem B. By Theorem A, one of the conclusions of Theorem A holds. If conclusion (1) of Theorem A holds, then $\Lambda \cong M_{2}$ by 8.3. We have considered each of the remaining cases arising in Theorem A in lemmas 9.1 through 9.5. Recalling from Section 1 that $T_{2.2} \cong T_{1.2} \cup \Omega^{(3)}$, the lattices appearing in Theorem B are precisely those listed in those lemmas. Hence the proof is complete.

References


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