Derived Equivalences for the 3-Dimensional Special Unitary Groups in Non-defining Characteristic

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1. INTRODUCTION

Let $G$ be a finite group. Let $k$ be an algebraically closed field of prime characteristic $p$. In [1], Broué has made a conjecture concerning the derived categories of blocks of finite groups.

Conjecture (Broué [1, 6.2 Question; 2, 4.9 Conjecture]). Let $P$ be a Sylow $p$-subgroup of $G$. If $P$ is abelian, then is it true that the principal $p$-block of $G$ and the principal $p$-block of $N_G(P)$ are derived equivalent?

The conjecture has been checked for the cases of blocks of $p$-solvable groups, certain blocks of finite reductive groups [13], blocks with cyclic defect groups [15, 18, 19], principal 3-blocks with elementary abelian defect group of order 9 (for example, [7, 9]), defect 2 blocks of symmetric groups [3], the principal $p$-block of SL$(2,p^n)$ [4, 12, 18], and the principal 2-block of the smallest Janko group [6].

The purpose of this paper is to show the following.
Theorem 1.1. Let \( G = SU(3, q^2) \) be the 3-dimensional special unitary group over the field with \( q^2 \) elements where \( q \) is a power of some prime. Let \( k \) be an algebraically closed field of characteristic \( r \). We assume that \( r \geq 5 \) and \( r \) divides \( q + 1 \). Let \( P \) be a Sylow \( r \)-subgroup of \( G \). Then the principal blocks of \( kG \) and \( kN_G(P) \) are derived equivalent.

For blocks of finite groups, Rickard introduced a stronger notion of a derived equivalence in [17]. This equivalence is called a splendid equivalence. The equivalences in the cases stated before Theorem 1.1 are splendid, and actually so is our case (see Remark 5.5).

In Section 2 we explain results of Okuyama [9] and Gollan and Okuyama [6], which are important for a proof of our main theorem. In Section 3 we list known and needed results on \( r \)-local subgroups of \( SU(3, q^2) \) and structures of their blocks. In Section 4 we discuss the image of simple modules under a stable equivalence between the principal blocks of \( G \) and \( N_G(P) \). In Section 5 using the results of Section 4 and applying the method in Section 2, we prove our main theorem.

Let \( G \) be a finite group. Let \( k \) be an algebraically closed field of characteristic \( p \) and \( A \) a finite dimensional \( k \)-algebra. Throughout this paper, all modules are finitely generated right modules unless otherwise stated. We denote by \( K^b(\text{proj} A) \) the homotopy category of bounded complexes of projective \( A \)-modules. We denote by \( B_k(G) \) the principal block of \( kG \). Let \( V \) be a \( kG \)-module. We denote by \( P(V) \) the projective cover of \( V \) and denote by \( V^* \) the dual of \( V \). We consider \( V^* \) as a left \( kG \)-module. For a \( p \)-subgroup \( R \) of \( G \), let \( Br^R_r(V) = V^R / \sum_{R' < R} Tr^R_{R'}(V^R) \), where \( V^R \) is the set of \( R \)-fixed points in \( V \) and \( Tr^R_{R'} : V^R \to V^R \) is the trace map. Let \( \Delta(G) = \{(g, g) \mid g \in G\} \leq G \times G \), the diagonal subgroup of \( G \times G \).

For basic results on derived equivalences, splendid equivalences, and stable equivalences of Morita type, we refer to [1, 2, 8, 9, 14, 16–19].

2. Preliminaries

Let \( k \) be an algebraically closed field of characteristic \( p \). Let \( A \) be a finite dimensional symmetric \( k \)-algebra. We denote by \( \{S(i) \mid i \in I\} \) the set of the simple \( A \)-modules and denote by \( P(i) \) the projective cover of \( S(i) \). Take a subset \( I_0 \) of \( I \). For \( i \in I \setminus I_0 \), let \( V(i) \) be the largest factor module of \( P(i) \) such that each composition factor of \( V(i) \) lies in \( \{S(j) \mid j \in I \setminus I_0\} \), and let \( \cdots \to R(i) \to P(i) \to V(i) \to 0 \) be a minimal projective resolution of \( V(i) \).
For \( i \in I \), let

\[
P(i)^{\prime} := \begin{cases} \cdots \rightarrow 0 \rightarrow P(i) \rightarrow 0 \rightarrow 0 \rightarrow \cdots & \text{if } i \in I_0 \\ \cdots \rightarrow 0 \rightarrow R(i) \rightarrow P(i) \rightarrow 0 \rightarrow 0 \cdots & \text{if } i \notin I_0 \end{cases}
\]

and set \( P(I_0)^{\prime} = \bigoplus_{i \in I} P(i)^{\prime} \). Then \( P(I_0)^{\prime} \) is a tilting complex for \( A \) by a result of Rickard (see [9, Proposition 1.1]).

Set \( A_i = \operatorname{End}_{k^i} (P(I_i)) \). Let \( P^i(i) \) be the projective indecomposable \( A_i \)-module corresponding to the direct summand \( P(i)^{\prime} \) of \( P(I_0)^{\prime} \), and \( S^i(i) = P^i(i)/\operatorname{Rad} P^i(i) \). Since \( A \) and \( A_i \) are derived equivalent (see [14, Theorem 6.4]), there exists an \((A, A_i)\)-bimodule \( L \) which induces a stable equivalence of Morita type between \( A \) and \( A_i \) (see [16, Corollary 5.5]). In particular, we may assume that the bimodule \( L \) has the following property.

**Lemma 2.1** (Okuyama [9, Lemma 2.1]). (i) For \( i \in I \setminus I_0 \), \( S(i) \otimes_k L \equiv S^i(i) \) in the stable category.

(ii) For \( i \in I_0 \) let \( U(i) \) be the factor module of \( P(i) \) satisfying the following conditions.

(a) each composition factor of \( \operatorname{Soc}(U(i)) \) lies in \( \{ S(j) \mid j \in I_0 \} \)

(b) each composition factor of \( \Omega(U(i))/S(i) \) lies in \( \{ S(j) \mid j \in I \setminus I_0 \} \).

Then \( U(i) \otimes_k L \equiv S^i(i) \) in the stable category.

Let \( A' \) be a symmetric \( k \)-algebra which is stably equivalent of Morita type to \( A \). Let \( N \) be an \((A', A)\)-bimodule inducing this equivalence. We assume that \( A' \) and \( A \) have no semisimple part and that the set of the simple \( A \)-modules is \( \{ T(i) \mid i \in I \} \). In [9], Okuyama gave the following method to prove that \( A' \) and \( A \) are derived equivalent. We choose a sequence of subsets \( I_0, I_1, \ldots, I_m-1 \) of \( I \). Then we can define \( A_i = \operatorname{End}_{k^i} (P(I_i)^{-}) \) for \( l = 2, \ldots, m \) inductively. Let \( L_{l-1} \) be an \((A_{l-1}, A_i)\)-bimodule which induces a stable equivalence of Morita type between \( A_{l-1} \) and \( A_i \). Then since each \( A_{l-1} \) is symmetric (see [16, Corollary 5.3]), we calculate \( T(i) \otimes_k N \otimes_k L \cdots \otimes_k L_{m-1} \) using Lemma 2.1. If \( T(i) \otimes_k N \otimes_k L \cdots \otimes_k L_{m-1} \) is simple for all \( i \in I \), then \( A' \) and \( A_m \) are Morita equivalent by a theorem of Linckelmann [8, Theorem 2.1], so that \( A' \) and \( A \) are derived equivalent.

**Remark 2.2.** If we can show existence of a derived equivalence between \( A' \) and \( A \) using the above method, then we can also construct a Rickard complex for \( A' \) and \( A \) whose degree zero term is isomorphic to a direct sum of \( N \) and a projective \((A', A)\)-module and other terms are projective \((A', A)\)-modules (see [10, Sect. 3]).
Next let $G$ be a finite group with abelian Sylow $p$-subgroup $P$ and $H = N_G(P)$. In order to apply the above method to the case $A = B_0(G)$ and $A = B_0(H)$, we recall a result on stable equivalences of Morita type between these blocks discussed in [6, Sect. 1].

Let $Q$ be a fixed subgroup of $P$ satisfying $C_G(Q) = Q \times G_1$, for some subgroup $G_1$ of $G$. Set $H_1 = H \cap G_1$. For a $k[G_1 \times H_1]$-module $Z$, we denote by $\tilde{Z}$ the $k[\Delta(Q)(G_1 \times H_1)]$-module which is equal to $Z$ as a $k[G_1 \times H_1]$-module and the action of $\Delta(Q)$ is trivial on $\tilde{Z}$. For complexes of $k[G_1 \times H_1]$-modules we use the same notation. Let $X^0$ be the Green correspondent of $B_0(G)$ with respect to $(G \times G, \Delta(P), G \times H)$.

PROPOSITION 2.3 (Gollan and Okuyama [6, Proposition 1.6, Lemma 1.7, and Corollary 1.8]). Assume the following.

(i) For $R < P$ and $R \ntriangleleft H$, the bimodule $Br_{\Delta(R)}(X^0)$ induces a Morita equivalence between $B_0(C_G(R))$ and $B_0(C_H(R))$.

(ii) For $1 < Q' \leq Q$, $C_G(Q') = N_G(Q')$.

(iii) There exists a splendid tilting complex

$$Z^- : 0 \to Z^0 \to 0$$

of $(B_0(G_1), B_0(H_1))$-modules satisfying

(a) The $k[G_1 \times H_1]$-module $Z^0$ is the Green correspondent of $B_0(G_1)$ with respect to $(G_1 \times G_1, \Delta(Q_1), G_1 \times H_1)$, where $Q_1$ is a Sylow $p$-subgroup of $H_1$ and $G_1$.

(b) The $k[G_1 \times H_1]$-module $Z^- 1$ is projective.

Then $B_0(G)$ and $B_0(H)$ are stably equivalent of Morita type.

The $k[G \times H]$-module $N$ which induces a stable equivalence of Morita type between $B_0(G)$ and $B_0(H)$ is constructed as follows. We set $Y^- = \tilde{Z} \cdot \tau_{1 C_G(Q) \times C_H(Q)}$. Then the complex $Y^-$ is a splendid tilting complex between $B_0(C_G(Q))$ and $B_0(C_H(Q))$ (see [6]). Let $Y^0$ be the degree $-1$ term of $Y^-$, and let $X^0$ be the Green correspondent of $Y^0$ with respect to $(G \times H, \Delta(Q), C_G(Q) \times C_H(Q))$. Then we can obtain the complex $X^- : 0 \to X^1 \to X^0 \to 0$ satisfying $Br_{\Delta(Q)}(X^-) = Y^-$, and we define the desired bimodule $N$ by the exact sequence

$$0 \to X^1 \xrightarrow{(\gamma, \tau)} X^0 \oplus \text{proj} \to N \to 0. \quad (1)$$
3. \( r \)-LOCAL STRUCTURES OF \( SU(3, q^2) \)

In the rest of this paper, let \( G = SU(3, q^2) = \{ A \in SL(3, q^2) \mid A \cdot \overline{A} = I \} \), the 3-dimensional special unitary group over the finite field with \( q^2 \) elements. Let \( r \geq 5 \) be a prime which divides \( q + 1 \). In particular let \( r^a \) be the \( r \)-part of \( q + 1 \) and \( s = (q + 1)/r^a \). Let \( k \) be an algebraically closed field of characteristic \( r \).

Let

\[
h(x_1, x_2, x_3) = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix},
\]

where \( x_i^{q+1} = 1 \) for \( i = 1, 2, 3 \) and \( x_1x_2x_3 = 1 \).

Then \( P = \{ h(x_1, x_2, x_3) \mid x_i^{r^a} = 1, i = 1, 2, 3 \} \) is a Sylow \( r \)-subgroup of \( G \) and \( P \cong \mathbb{Z}_{r^a} \times \mathbb{Z}_{r^a} \). Let \( C = \{ h(x_1, x_2, x_3) \mid x_i^{q+1} = 1, i = 1, 2, 3 \} \), and

\[
u = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},
\]

Then,

\[
C = C_G(P) \cong \mathbb{Z}_{q+1} \times \mathbb{Z}_{q+1} \quad \text{and} \quad N_G(P) = \langle C, \nu \rangle \cong (\mathbb{Z}_{q+1} \times \mathbb{Z}_{q+1}) \rtimes S_3.
\]

We set \( H = N_G(P) \).

**Lemma 3.1.** (i) Let \( Q = \{ h(x, x, x^{-2}) \mid x^{r^a} = 1 \} \). Then \( Q \cong \mathbb{Z}_{r^a} \) and

\[
C_G(Q) = \left\{ A = \begin{pmatrix} A_1 & 0 \\ 0 & \alpha \end{pmatrix} \right\} A_1 \in U(2, q^2), (\det A_1) \alpha^2 = 1 \}
\]

Moreover let

\[
G_1 = \left\{ A = \begin{pmatrix} A_1 & 0 \\ 0 & \alpha \end{pmatrix} \in C_G(Q) \mid \alpha^{r^a} = 1 \}
\]

Then \( C_G(Q) = Q \times G_1 \).

(ii) For \( 1 \neq R \preceq P \) and \( R \not\preceq H \) \( Q \), we have \( C_G(R) = C_H(R) = C \).

(iii) For \( 1 \neq Q' \leq Q \), we have \( N_G(Q') = C_G(Q') = C_G(Q) = N_G(Q) \).

We set \( H_1 = G_1 \cap H \). Since a Sylow \( r \)-subgroup \( Q_1 \) of \( G_1 \) is cyclic of order \( r^a \), \( B_0(G_1) \) and \( B_0(H_1) \) are splendidly equivalent by [19].
Lemma 3.2. Let $k_{G_i}$ and $U$ be the simple $B_0(G_i)$-modules and let $k_{H_i}$ and $Y$ be the simple $B_0(H_i)$-modules, where $\dim U = q - 1$. There exists a splendid tilting complex of $k(G_1 \times H_1)$-modules of the form

$$0 \rightarrow P(U)^{\ast} \otimes P(Y) \rightarrow Z^0 \rightarrow 0,$$

where $Z^0$ is the Green correspondent of $B_0(G_i)$ with respect to $(G_1 \times G_1, \Delta(Q), G_1 \times H_1)$.

Next we describe the basic algebra structure of $B_0(H)$. We can choose the generators $\alpha$ and $\beta$ of $kP$ such that

$$\alpha^u = \beta, \quad \beta^u = \alpha, \quad \alpha^v = \omega \alpha, \quad \beta^v = \omega^3 \beta,$$

where $\omega \neq 1$ and $\omega^3 = 1$. Set $e_0 = (1 + u)/(1 + v + v^2)/6$, $e_1 = (1 - u)(1 + v + v^2)/6$, $e_2 = (1 + \omega^2 v + \omega v^2)/3$, and $e = e_0 + e_1 + e_2$. Set $x_i = e_i a e_2$, $y_i = e_2 a e_i$ for $i = 0, 1$ and $z = e_2 a e_2$.

Lemma 3.3 [11, Lemma 3.5]. The elements $e_0, e_1, e_2, x_0, x_1, y_0, y_1, a$ and $z$ generate the basic algebra $eB_0(H)e$ of $B_0(H)$ and satisfy relations $x_0y_1 = 0$, $x_1y_0 = 0$, $y_0x_0 + y_1x_1 = z^2$, and

- if $r^a \equiv 1 \pmod{3}$ then
  $$zz^{(r^a - 1)/3}_\beta = 0$$
  $$z^{(r^a - 1)/3}_\beta y_i = 0, \quad i = 0, 1$$
  $$x_i z^{(r^a - 1)/3}_\alpha = 0, \quad i = 0, 1$$

- if $r^a \equiv 2 \pmod{3}$ then
  $$z^{(r^a - 2)/3}_\beta (y_0x_0 - y_1x_1) = 0$$
  $$z^{(r^a - 2)/3}_\beta y_i = 0, \quad i = 0, 1$$
  $$x_i z^{(r^a - 2)/3}_\alpha z = 0, \quad i = 0, 1,$$

where $z_\alpha = z(y_0x_0 - y_1x_1)$ and $z_\beta = (y_0x_0 - y_1x_1)z$.

We denote by $k_{H_1}, 1_{H_1},$ and $2_{H_1}$ the simple $B_0(G)$-modules corresponding to the idempotents $e_0, e_1,$ and $e_2$, respectively. Let $S_H(Q)$ be the Scott module in $H$ corresponding to $Q$. 
Lemma 3.4. (i) [11, Lemma 3.2]. The Scott module $S_H(Q)$ has the following Loewy and socle series

$$(k_H \ 2_H)$$

$$2_H \ 1_H$$

$$k_H \ 2_H$$

$$\vdots \ \vdots$$

$$2_H \ 1_H$$

$$k_H \ 2_H$$

where the Loewy length is $r^a$.

(ii) Let $(\gamma_0, \tau_0): S_H(Q) \to P(k_H) \oplus P(2_H)$ be an injective hull mapping of $S_H(Q)$. Then $\ker \gamma_0 = (1^n)$ and $\ker \tau_0 = k_H$.

(iii) The Cartan matrix of $B_0(H)$ is

$$\begin{pmatrix}
(r^a - 1)(r^a - 2)/6 + r^a & (r^a - 1)(r^a - 2)/6 \\
(r^a - 1)(r^a - 2)/6 & (r^a - 1)(r^a - 2)/6 + r^a \\
(r^a - 1)(r^a - 2)/3 + r^a - 1 & (r^a - 1)(r^a - 2)/3 + r^a - 1
\end{pmatrix}$$

Proof. (ii) Since there is no uniserial module whose Loewy layers are

$$(2_H \ 1_H \ 2_H)$$

there are not any submodules of Loewy length 3 with a simple socle in $S_H(Q)$. Since $\ker \gamma_0 \cap \ker \tau_0 = 0$, these modules have a simple socle. This means that the Loewy length of both modules are 2 at most. So (ii) follows.

4. STABLE EQUIVALENCES

In this section we discuss the image of simple $B_0(G)$-modules under the stable equivalence between $B_0(G)$ and $B_0(H)$. 
We denote by $k_G$, $S$, and $T$ the simple $B_0(G)$-modules, where $\dim_k S = q^2 - q$ and $\dim_k T = (q - 1)(q^2 - q + 1)$ (see [5, 11]). The Green correspondent $f(S)$ of $S$ is determined in [11].

**Lemma 4.1.** (i) [11, Proposition 3.6]. There is an exact sequence

$$0 \rightarrow S_H(Q)/k_H \rightarrow \Omega^2(k_H) \rightarrow f(S) \rightarrow 0,$$

where $f(S)$ is the Green correspondent of $S$.

(ii) The composition factors of $f(S)$ are $[(r^a - 1)(r^a - 5)/6] \times k_H + [(r^a - 1)(r^a + 1)/6] \times 1_H + [(r^a - 1)(r^a - 2)/3] \times 2_H$.

(iii) $f(S)/\text{Rad}(f(S)) \cong 1_H$.

(iv) The natural inclusion $\Omega f(S) \rightarrow P(1_H)$ induces an isomorphism $\text{Hom}_{k_H} (1_H \otimes S_H(Q), \Omega f(S)) \cong \text{Hom}_{k_H} (1_H \otimes S_H(Q), P(1_H))$.

**Proof.** In this proof, we use the same notation for $B_0(H)$-modules and $eB_0(H)e$-modules. So $P(k_H) = e_0B_0(H)e$, $P(1_H) = e_1B_0(H)e$, and $P(2_H) = e_2B_0(H)e$.

(ii) Since the top of $\Omega(k_H)$ is isomorphic to $2_H$, we get this directly by Lemma 3.4 (iii).

(iii) Using $\tau_0$ in Lemma 3.4 (ii), we can see $S_H(Q)/k_H$ as a submodule of $P(2_H)$. We set

$$\delta = \begin{cases} 
\alpha^a(r^a-1)/3, & r^a \equiv 1 \pmod{3} \\
\alpha^a(r^a-2)/3, & r^a \equiv 2 \pmod{3}.
\end{cases}$$

From the proof of [11, Proposition 3.6], we know $\Omega^2(k_H) = \langle y_1, \delta \rangle$. Since $\Omega^{-1}(1_H) = \langle y_1 \rangle$, the Loewy length of $\langle y_1 \rangle$ is $2(r^a - 1)$. From Lemma 3.4 (iii), $\Omega^2(k_H)/\langle y_1 \rangle \cong 2_H$. So $\text{Rad}(\Omega^2(k_H)) = \text{Rad}(\langle y_1 \rangle)$. In particular, $\text{Rad}^{r-2}(\Omega^2(k_H)) = \text{Rad}^{r-2}(\langle y_1 \rangle) = \text{Soc}^{r}(\langle y_1 \rangle)$. Since $\delta \in \Omega^2(k_H) \cap \text{Soc}^{r}(\Omega^2(k_H))$, $\Omega^2(k_H)/\text{Rad}^{r-2}(\Omega^2(k_H)) \cong \langle y_1 \rangle/\text{Rad}^{r-2}(\langle y_1 \rangle) \oplus 2_H$.

Denote $P(1_H)/\text{Rad}^{r-2}(P(1_H))$ by $V$. Since the Loewy length of $S_H(Q)$ is $r^2$, we have $(\ker \pi) \cap \langle y_1 \rangle \subseteq \text{Rad}^{r-2}(\langle y_1 \rangle)$. Therefore $f(S)/\text{Rad}^{r-2}(f(S)) \cong \pi(\Omega^2(k_H))/\pi(\text{Rad}^{r-2}(\Omega^2(k_H))) \cong V \oplus X$ where

$$X = \begin{cases} 2_H & \text{if ker } \pi \subseteq \text{Rad}^{r-2}(\Omega^2(k_H)) \\
0 & \text{else.}
\end{cases}$$
Since the Loewy length of $S_H(Q)$ is $r^a$,

\[
\dim \text{Hom}_{kH}(V, S_H(Q) \oplus (1_H \otimes S_H(Q))) \\
= \dim \text{Hom}_{kH}(P(1_H), S_H(Q) \oplus (1_H \otimes S_H(Q))) - 2.
\]

The number of composition factors which are isomorphic to $1_H$ of $S_H(Q) \oplus (1_H \otimes S_H(Q))$ are $r^a$. Since $S_H(Q) \oplus (1_H \otimes S_H(Q))$ is a direct summand of $k_Q^{1_H}$, we get

\[
\dim \text{Hom}_{kH}(V, k_Q^{1_H}) \\
\geq \dim \text{Hom}_{kH}(P(1_H), S_H(Q) \oplus (1_H \otimes S_H(Q))) - 2 = r^a - 2.
\]

From the proof of [11, Proposition 3.6], $f(S)_{\mathcal{Q}}$ is isomorphic to a direct sum of $r^a - 2$ copies of $\Omega(k_Q)$. Therefore we have

\[
r^a - 2 = \dim \text{Hom}_{kQ}(f(S)_{\mathcal{Q}}, kQ) \\
= \dim \text{Hom}_{kH}(f(S), k_Q^{1_H}) \\
\geq \dim \text{Hom}_{kH}(f(S) / \text{Rad}^{r^a - 2}(f(S)), k_Q^{1_H}) \\
= \dim \text{Hom}_{kH}(V, k_Q^{1_H}) + \dim \text{Hom}_{kH}(X, k_Q^{1_H}) \\
\geq r^a - 2 + \dim \text{Hom}_{kH}(X, k_Q^{1_H}).
\]

So we get $X = 0$ and (iii) follows.

Let $L$ be a subgroup of $H$ such that $Q \leq L$ and $L \equiv (\mathbb{Z}_{q+1} \times \mathbb{Z}_n) \rtimes \mathbb{Z}_2$. Then $\mathbb{Z}_q^{1_L} = 1_H \otimes S_H(Q)$ for 1-dimensional simple $kL$-module $1_L$. Since $f(S)_{\mathcal{Q}} = (r^a - 2)\Omega(k_Q)$, the $kL$-module $f(S)_{\mathcal{Q}}$ has no non-zero projective summand. In particular the sequence

\[
0 \rightarrow \Omega f(S)_{\mathcal{Q}} \rightarrow P(1_H)_{\mathcal{Q}} \rightarrow f(S)_{\mathcal{Q}} \rightarrow 0
\]

is a minimal injective resolution of $\Omega f(S)_{\mathcal{Q}}$. Hence

\[
\text{Hom}_{kH}(1_H \otimes S_H(Q), \Omega f(S)) \equiv \text{Hom}_{kL}(1_L, \Omega f(S)_{\mathcal{Q}}) \\
\equiv \text{Hom}_{kL}(1_L, P(1_H)_{\mathcal{Q}}) \\
\equiv \text{Hom}_{kH}(1_H \otimes S_H(Q), P(1_H))
\]

and (iv) follows. 

By Lemmas 3.1, 3.2, and Proposition 2.3, $B_0(G)$ and $B_0(H)$ are stably equivalent of Morita type. Let $X^{-1}$ be the Green correspondent of
Lemma 4.3. $\ker \gamma = (\gamma_0^H)$, $\ker \tau = 1_H$, $\Omega f(S)/\text{Im} \gamma = k_H$.

Proof. Let $\gamma_0, \tau_0: 1_H \otimes S_H(Q) \to P(1_H) \oplus P(2_H)$ be an injective hull mapping of $1_H \otimes S_H(Q)$. We set $M_0 = \text{Im} \gamma_0$ and $M = \Omega f(S)$. It follows from Lemma 4.1 (iv) that $M_0 \subset M$. Moreover we have $M/M_0 \cong k_H$ by Lemmas 3.4 (iii) and 4.1 (ii). Let $\sigma \in \text{End}_{k_H}(1_H \otimes S_H(Q))$ such that $\ker \sigma = \text{Soc}^2(1_H \otimes S_H(Q))$. Since $\ker \gamma_0 \subset \ker \sigma$ by Lemma 3.4, the homomorphism $\sigma$ induces a homomorphism $\sigma_0 : M_0 \to M_0$. Thus we have a
commutative diagram

\[
\begin{array}{ccc}
1_H \otimes S_H(Q) & \xrightarrow{\gamma_0} & M_0 \\
\sigma \downarrow & & \sigma_0 \\
1_H \otimes S_H(Q) & \xrightarrow{\gamma_0} & M_0.
\end{array}
\]

We set \( \gamma_n = \gamma_0 \circ \sigma^n \in \text{Hom}_{kH}(1_H \otimes S_H(Q), M_0) \), for \( n = 1, 2, \ldots, (r^s - 1)/2 \). Then \( \sigma_0 \circ \gamma_n = \gamma_n \circ \sigma \). We can consider \( \gamma_n \in \text{Hom}_{kH}(1_H \otimes S_H(Q), M) \). Note that \( \{ \gamma_n | n = 0, 1, 2, \ldots, (r^s - 1)/2 \} \) is a \( k \)-basis of \( \text{Hom}_{kH}(1_H \otimes S_H(Q), M) \). We also consider \( \sigma_0 \) as a homomorphism from \( M_0 \) to \( P(1_H) \). Hence \( \text{Im} \sigma_0 \) is extendible to a \( kH \)-homomorphism from \( M \) to \( P(1_H) \). We denote it by \( \rho \). We claim that \( \text{Im} \rho \subset M \). We put \( M' = \text{Im} \rho + M_0 \). Suppose \( \text{Im} \rho \subset M \). Suppose \( M' \cap M = M_0 \) and \( (M' + M)/M_0 \equiv kH \). However, \( \text{Soc}(P(1_H))/M_0 \) is a direct summand of \( \text{Soc}(P(1_H)/M) \oplus kH = 1_H \oplus kH \), which is a contradiction. Therefore \( \text{Im} \rho \subset M_0 \) or \( M' = M \), that is, \( \text{Im} \rho \subset M \). Hence we have the commutative diagram

\[
\begin{array}{ccc}
1_H \otimes S_H(Q) & \xrightarrow{\gamma_n} & M \\
\sigma \downarrow & & \rho \\
1_H \otimes S_H(Q) & \xrightarrow{\gamma_n} & M.
\end{array}
\]

In the sequence (3), suppose \( \text{Im} \gamma \neq \text{Im} \gamma_0 \). Then \( \tau \) must be a monomorphism since \( \ker \gamma \supset \text{Soc}(1 \otimes S_H(Q)) \). Therefore there exists \( \rho_1 \in \text{End}_{kH}(R) \) such that \( \rho_1 = \tau \circ \sigma \). Note that \( \gamma \) is a linear combination of \( \{ \gamma_n | n \geq 1 \} \). Hence we have the following commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & 1_H \otimes S_H(Q) \xrightarrow{(\gamma, \tau)} M \oplus R \rightarrow \Omega h(S) \rightarrow 0 \\
\sigma \downarrow & & \tilde{\rho} \downarrow \gamma \\
0 & \rightarrow & 1_H \otimes S_H(Q) \xrightarrow{(\gamma, \tau)} M \oplus R \rightarrow \Omega h(S) \rightarrow 0
\end{array}
\]

where \( \tilde{\rho} = (\rho_1 \circ \rho \circ \rho) \). Since \( \text{Hom}_{B(H)}(\Omega h(S), \Omega h(S)) \equiv k \) and \( \eta \) is not an isomorphism, \( \eta \) is a projective homomorphism. Therefore there exists a \( kH \)-homomorphism \( \lambda = (\lambda_1, \lambda_2) : M \oplus R \rightarrow 1_H \otimes S_H(Q) \) such that \( \lambda \circ (\gamma, \tau) = \sigma \). Since \( \text{Im} \tau \subset \text{Soc}^{r_s}(R) \) and \( \ker \lambda_2 \supset \text{Rad}^{r_s}(R) = \text{Soc}^{r_s-1}(R) \), we have \( \text{Im} \lambda_2 \circ \tau \subset \text{Soc}(1_H \otimes S_H(Q)) = \text{Rad}^{r_s-1}(1_H \otimes S_H(Q)) \). On the other hand, we have \( \text{Im} \sigma \subset \text{Im}(\lambda_1 \circ \gamma) \) since \( \ker \sigma \subset \ker \gamma \). Hence \( \text{Im}(\lambda_1 \circ \gamma + \lambda_2 \circ \tau) \subset \text{Im}(\lambda_1 \circ \gamma) + \text{Im}(\lambda_2 \circ \tau) \subset \text{Im} \sigma \); this is a contradiction. Therefore we have \( \text{Im} \gamma = \text{Im} \gamma_0 \). Hence by Lemmas 3.4 and 4.1, \( \ker \gamma = (\frac{1}{n^2}) \) and \( M/\text{Im} \gamma \equiv kH \), and so \( R = P(2_H) \). It follows from the structure of \( 1_H \otimes S_H(Q) \) that \( \ker \tau = 1_H \).
Lemma 4.4. The module \( \Omega h(S) \) has a submodule \( W \) such that \( \Omega h(S)/W \cong k_H \) and \( W \cong \Omega^{-1}(k_H) \).

Proof. We set \( Y = 1 \otimes S_H(Q) \) and \( Y_1 = \ker \tau \oplus \ker \gamma \subset Y \). We also set \( (\gamma, \tau)(Y_1) = M_1 \oplus M_2 \), where \( M_1 \subset \Omega f(S) \), \( M_2 \subset R \). The \( kH \)-homomorphism \( (\gamma, \tau) \) induces a \( kH \)-homomorphism \( (\gamma, \tau) : Y/Y_1 \to \Omega f(S)/M_1 \oplus R/M_2 \). We note that \( \gamma \) and \( \tau \) are both monomorphisms. Therefore there exists an isomorphism \( \alpha : \Im \gamma \to \Im \tau \) such that \( \alpha \gamma = \tau \). Then we have the following commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \to & Y/Y_1 & \xrightarrow{(\gamma, \tau)} & \Im \gamma/M_1 \oplus R/M_2 & \xrightarrow{(-\alpha, \text{id})} & \Omega^{-1} \left( \begin{array}{c} k_H \\ 2_H \end{array} \right) & \to & 0 \\
0 & \to & Y/Y_1 & \xrightarrow{(\gamma, \tau)} & \Omega f(S)/M_1 \oplus R/M_2 & \to & \Omega h(S) & \to & 0.
\end{array}
\]

So we have \( \Omega^{-1}(k_H) \subset \Omega h(S) \) and

\[
\Omega h(S)/\Omega^{-1} \left( \begin{array}{c} k_H \\ 2_H \end{array} \right) \cong (\Omega f(S)/M_1 \oplus R/M_2)/(\Im \gamma/M_1 \oplus R/M_2) \cong k_H.
\]

Remark 4.5. (i) We easily see \( k_G \otimes_{B_0(G)} N \cong k_H \), since \( k_G \otimes_{kG} P(U)^* = 0 \).

(ii) We do not have any information about the Green correspondent of the simple \( B_0(G) \)-module \( T \). However, we know that

\[
\Omega^{-2}(k_G) \cong \begin{pmatrix}
S \\
T \\
S \\
k_G \\
S
\end{pmatrix}
\]

from [11, Lemma 3.4 (3)]. Therefore we consider the complex

\[
X_T : 0 \to \begin{pmatrix} S \\ k_G \\ S \end{pmatrix} \to \Omega^{-2}(k_G) \to S \to 0
\]

instead of the simple module \( T \) (note that \( X_T \equiv T \) in the derived category).
5. PROOF OF THEOREM 1.1

Let \( A_0 = eB_0(H)e \) be the basic algebra of \( B_0(H) \). Set

\[ P^0(i) = e_i A_0 \quad \text{and} \quad S^0(i) = P^0(i)/\text{Rad } P^0(i) \quad \text{for } i \in I = \{0, 1, 2\}. \]

We choose subsets \( I_0 = \{1, 2\}, I_1 = \{1\}, \) and \( I_2 = \{1, 2\} \) of \( I \), and define \( A_I = \text{End}_{K^*(\text{proj } A_{I-1})}(P(I_{I-1})) \) inductively as in Section 2.

In this section we give a proof of Theorem 1.1. In fact, it suffices to prove the following lemma.

**Lemma 5.1.** \( B_0(G) \) and \( A_3 \) are Morita equivalent.

For \( l = 1, 2, 3 \), we use the following notation. We denote by \( L_{l-1} \) the \((A_{l-1}, A_l)\)-bimodule which induces a stable equivalence of Morita type between \( A_{l-1} \) and \( A_l \) as in Section 2. For \( i \in I \), we denote by \( P^{l-1}(i) \) the direct summand of the tilting complex \( P(I_{l-1}) \) for \( A_{l-1} \), and denote by \( P^l(i) \) the corresponding projective indecomposable \( A_l \)-module and \( S^l(i) = P^l(i)/\text{Rad } P^l(i) \). If the \( A_l \)-modules \( U_1 \) and \( U_2 \) are isomorphic in the stable category then we write \( U_1 \equiv U_2 \).

To show Lemma 5.1, we investigate the form of the complex \( P^{l-1}(i) \), the image of the simple \( B_0(G) \)-modules via the functor \( - \otimes_{B_0(G)} N \otimes_{A_3} L_0 \otimes \cdots \otimes_{A_{l-1}} L_{l-1} \) and some information on extensions of simple \( A_r \)-modules for each \( l = 1, 2, 3 \).

**Lemma 5.2.**

(i) The complex \( P^0(0) \) has the form

\[ P^0(0) : \cdots \to 0 \to P^0(2) \xrightarrow{x_0} P^0(0) \to 0 \to \cdots. \]

(ii) \( \Omega h(S) \otimes_{A_3} L_0 = (S^{l(0)}, S^{l(2)}). \)

(iii) It holds

\[
\dim \text{Ext}^1_{A_3}(S^1(i), S^1(1)) = \begin{cases} 
1, & \text{if } i = 0 \\
0, & \text{if } i = 1, 2.
\end{cases}
\]

**Proof.**

(i) This follows from Lemma 3.3.

(ii) From Lemma 3.3, we can easily check that \( \Omega^{-1}(S^{l(0)}) \) satisfies the conditions (a) and (b) of Lemma 2.1 (ii). Therefore we have \( \Omega^{-1}(S^{l(0)}) \otimes_{A_3} L_0 \equiv S^1(2) \). Therefore this follows from Lemma 4.4.
(iii) We have $\Omega^{-1}(S^0(1)) \otimes_{A_0} L_0 \equiv S'(1)$ by Lemmas 2.1 and 3.3. Hence
\[
\text{Ext}^1_{A_i}(S^1(0), S^1(1)) \equiv \text{Ext}^1_{A_0}(S^0(0), \Omega^{-1}(S^0(1))) \\
\equiv \text{Hom}_{A_0}(\Omega^2(S^0(0)), S^0(1)) \equiv k \\
\text{Ext}^1_{A_i}(S^1(1), S^1(1)) \equiv \text{Ext}^1_{A_0}(S^0(1), S^0(1)) = 0
\]

\[
\text{Ext}^1_{A_i}(S^1(2), S^1(1)) \equiv \text{Ext}^1_{A_0}\left(\begin{pmatrix} S^0(0) \\ S^0(2) \end{pmatrix}, S^0(1)\right) \\
\equiv \text{Hom}_{A_0}\left(\Omega\left(\begin{pmatrix} S^0(0) \\ S^0(2) \end{pmatrix}\right), S^0(1)\right) = 0.
\]

**Lemma 5.3.** (i) The complexes $P^1(0)'$ and $P^1(2)'$ have the forms
\[
P^1(0)' : \cdots \rightarrow 0 \rightarrow P^1(1) \rightarrow P^1(0) \rightarrow 0 \rightarrow \cdots \\
P^1(2)' : \cdots \rightarrow 0 \rightarrow c \times P^1(1) \rightarrow P^1(2) \rightarrow 0 \rightarrow \cdots,
\]
where $c$ is a non-negative integer.

(ii) $\Omega h(S) \otimes_{A_0} L_0 \otimes_{A_1} L_1 \equiv (S^2_{i(2)}).$

(iii) It holds
\[
\dim \text{Ext}^1_{A_i}(S^2(0), S^2(i)) = \begin{cases} 
0, & \text{if } i = 0 \\
1, & \text{if } i = 2.
\end{cases}
\]

**Proof.** (i) By Lemma 5.2 (iii) we have $\Omega^{-1}(S^1(1)) \subset P^1(0)$. Let $b_{ij} = \dim \text{Hom}_{A_0}(P^1(i), P^1(j))$. Then $b_{10} = (r^2 - 1)(r^2 - 2)/6 + r^2 - 1$ and $b_{11} = (r^2 - 1)(r^2 - 2)/6 + r^2$ by Lemmas 3.4 (iii) and 5.2 (i). Hence $P^1(0)/\Omega^{-1}(S^1(1))$ does not contain $S'(1)$ as a composition factor. Therefore the assertion for $P^1(0)'$ follows.

(ii) This follows from Lemmas 2.1 and 5.2 (ii).

(iii) By Lemma 3.3 we have
\[
\text{Ext}^1_{A_i}(S^2(0), S^2(0)) \equiv \text{Ext}^1_{A_0}(S^0(0), S^0(0)) = 0
\]
\[
\text{Ext}^1_{A_i}(S^2(0), S^2(2)) \equiv \text{Ext}^1_{A_0}\left(\begin{pmatrix} S^0(0) \\ \Omega^{-1}S^0(2) \end{pmatrix}\right) \\
\equiv \text{Hom}_{A_0}\left(\Omega^2(S^0(0)), \begin{pmatrix} S^0(0) \\ S^0(2) \end{pmatrix}\right) \equiv k.
\]
LEMMA 5.4. (i) The complex $P^2(0)$ has the form

$$P^2(0) : \cdots \to 0 \to P^2(2) \oplus d \times P^2(1) \to P^2(0) \to 0 \to \cdots,$$

where $d = \dim \text{Ext}^1_{A_3}(S^2(0), S^3(1))$.

(ii) $h(S) \otimes_{A_0} L_0 \otimes_{A_1} L_1 \otimes L_2 = S^3(2)$.

(iii) It holds

$$\dim \text{Ext}^1_{A_3}(S^2(i), S^3(0)) = \begin{cases} 0 & \text{if } i = 0 \\ d & \text{if } i = 1 \\ 1 & \text{if } i = 2. \end{cases}$$

(iv) The multiplicity of $S^3(1)$ as a composition factor of $\Omega^{-2}(S^3(0))$ is 1.

Proof. (i) This follows from Lemma 5.3 (iii).

(ii) We have

$$\text{Hom}_{A_3}\left(S^2(0), \Omega^{-1}\left(\begin{array}{c} S^2(0) \\ S^2(2) \end{array} \right) \right) \cong \text{Ext}^1_{A_3}\left(S^2(0), \begin{array}{c} S^2(0) \\ S^2(2) \end{array} \right) = 0$$

by Lemma 5.3 (iii). Therefore we have

$$h(S) \otimes_{A_0} L_0 \otimes_{A_1} L_1 \otimes L_2 = \Omega^{-1}\left(\begin{array}{c} h(S) \otimes_{A_0} L_0 \otimes_{A_1} L_1 \otimes L_2 \\ \end{array} \right)$$

by Lemmas 2.1 and 5.3 (ii).

(iii) This follows from the similar calculations to Lemma 5.2 (iii).

(iv) From (iii) we have an exact sequence

$$0 \to \Omega^{-1}(S^3(0)) \to P^2(2) \oplus d \times P^2(1) \to \Omega^{-2}(S^3(0)) \to 0.$$

Let $c_{ij} = \dim \text{Hom}_{A_3}(P^2(i), P^3(j))$ and $d_{ij} = \dim \text{Hom}_{A_3}(P^3(i), P^3(j))$. Then $c_{10} = b_{11} - b_{10} = 1$ and so $d_{10} = c_{12} + d \cdot c_{11} - 1$, $d_{11} = c_{11}$ and $d_{12} = c_{12}$. Therefore the multiplicity of $S^3(1)$ as a composition factor of $\Omega^{-2}(S^3(0))$ is $d_{12} + d \cdot d_{11} - d_{10} = 1$.

Proof of Lemma 5.1 and Theorem 1.1. By a theorem of Linckelmann [8, Theorem 2.1], the $(B_0(G), A_3)$-bimodule $N \otimes_{A_0} L_0 \otimes_{A_1} L_1 \otimes A_2 L_2$ has a unique indecomposable non-projective direct summand $\mathcal{M}$, which induces a stable equivalence of Morita type between $B_0(G)$ and $A_3$. We have
already seen $S \otimes_{B_0(G)} \mathcal{N} \cong S^3(2)$. Moreover we know $k_G \otimes_{B_0(G)} \mathcal{N} \cong S^3(0)$ from Remark 4.5 (i) and the choice of subsets $I_0$, $I_1$, and $I_2$ of $I$. Hence we also know that the top and socle of $T \otimes_{B_0(G)} \mathcal{N}$ are direct sums of some copies of $S^3(1)$. We consider the complex $X_T^*$ in Remark 4.5 (ii) and apply $- \otimes_{B_0(G)} \mathcal{N}$ to $X_T^*$. Then we have the complex

$$0 \to \begin{pmatrix} S^3(2) \\ S^3(0) \\ S^3(2) \end{pmatrix} \xrightarrow{(\sigma_1, \sigma_2)} \Omega^{-2}(S^3(0)) \oplus V \xrightarrow{(\lambda_1, \lambda_2)} S^3(2) \to 0,$$

where $V$ is a projective $A_3$-module. Note that $\sigma_1$ is monic and $\lambda_1$ is epic. Moreover we may consider $\sigma_2$ and $\lambda_2$ are both zero. Thus the complex

$$0 \to \begin{pmatrix} S^3(2) \\ S^3(0) \\ S^3(2) \end{pmatrix} \xrightarrow{\sigma_2} \Omega^{-2}(S^3(0)) \xrightarrow{\lambda_1} S^3(2) \to 0$$

is a direct summand of the complex $X_T^* \otimes_{B_0(G)} \mathcal{N}$. Therefore it follows from Lemma 5.4 (iv) that $T \otimes_{B_0(G)} \mathcal{N}$ has only one copy of $S^3(1)$ as a composition factor. So we have $T \otimes_{B_0(G)} \mathcal{N} \cong S^3(1)$. Hence the bimodule $\mathcal{N}$ sends each simple $B_0(G)$-module to a simple $A_3$-module, so that $\mathcal{N}$ induces a Morita equivalence between $B_0(G)$ and $A_3$ by [8, Theorem 2.1]. Therefore we can conclude $B_0(G)$ and $B_0(H)$ are derived equivalent.

From the above proof, the composition factors of $\Omega^{-2}(S^3(0))$ must be $S^3(0) + S^3(1) + 3 \times S^3(2)$. Hence we have $c = 2$ and $d = 0$ by calculating the Cartan matrices of $A_2$ and $A_3$.

**Remark 5.5.** From Remark 2.2 and the definition of the $(B_0(G), B_0(H))$-bimodule $N$ (see the sequence (1)), we can also conclude that $B_0(G)$ and $B_0(H)$ are splendidly equivalent.

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