On graphs with small number of Laplacian eigenvalues greater than two

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Abstract
All connected graphs with exactly one or two Laplacian eigenvalues greater than two are determined.
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1. Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V = V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = E(G)$. The adjacency matrix of $G$ is denoted by $A(G) = (a_{ij})$, where $a_{ij} = 1$ if $v_i$ and $v_j$ are adjacent and $a_{ij} = 0$ otherwise. The degree diagonal matrix of $G$ is denoted by $D(G) = \text{diag}(d_1(G), d_2(G), \ldots, d_n(G))$, where $d_i(G)$ is the degree of $v_i$. Without loss of generality, we can assume that $d_1(G) \geq d_2(G) \geq \cdots \geq d_n(G)$. Then $L(G) = D(G) - A(G)$ is the Laplacian matrix of $G$. It is known that $L(G)$ is a singular, positive semidefinite symmetric matrix. The eigenvalues of $L(G)$ are called the Laplacian eigenvalues of $G$, and are denoted by $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G) = 0$. It is proved in [1, Theorem 1] that if $\lambda$ is a Laplacian

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eigenvalue of $G$, then $0 \leq \lambda \leq n$ and the multiplicity of $0$ as a Laplacian eigenvalue of $G$ equals the number of components of $G$, and the multiplicity of $n$ equals one less than the number of components of the complement of $G$.

Let $I$ be an interval of the real line. The number of Laplacian eigenvalues of $G$, multiplicities included, that belong to $I$, is denoted by $m_G(I)$. Especially, if $I = [\lambda]$, then $m_G(\lambda)$ is just the multiplicity of $\lambda$ as a Laplacian eigenvalue of $G$. Grone et al. [2] and Merris [3] study the bounds of $m_G(I)$ for some certain $I$’s, especially for $I = (2, n]$. Ming and Wang [6] give a lower bound for $m_G(2, n]$ in terms of the matching number of $G$ when $G$ has no perfect matchings. Recently, Petrović et al. [9] consider a connected bipartite graph $G$ with exactly two Laplacian eigenvalues greater than two, that is, $m_G(2, n] = 2$, and determined all those graphs. A special motivation for their study is discovered connection between photoelectron spectra of saturated hydrocarbons (alkanes) and the Laplacian eigenvalues of the underlying molecular graphs [7, 8]. The results they obtained in the work can, in principle, be of interest in the photoelectron spectroscopy of organic compounds [9]. In this paper, we shall extend their results to connected graphs $G$ with $m_G(2, n] = 2$ and determine all those graphs. We also determine all graphs $G$ such that $m_G(2, n] = 1$.

In order to present our results, we need some notations. Let $G = (V, E)$ be a simple graph on $n$ vertices. A subset $M$ of $E$ is called a matching in $G$ if any two edges in $M$ are not incident. The maximum of cardinalities of all matchings of $G$ is commonly known as its matching number denoted by $\mu(G)$. A matching $M$ is called a perfect matching in $G$ if each vertex of $G$ must be incident to some edge in $M$. Denote by $G + e$ the graph obtained from $G$ by inserting a new edge $e$. The complement of $G$, is $G^c = (V, E^c)$, where $e \in E(G^c) = E^c$ if and only if $e \notin E$. If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are graphs on disjoint sets of $r$ and $s$ vertices, respectively, their union is the graph $G_1 + G_2 = (V_1 \cup V_2, E_1 \cup E_2)$, and their join is $G_1 \vee G_2 = (G_1^c + G_2^c)^c$, the graph on $r + s$ vertices obtained from $G_1 + G_2$ by inserting new edges from each vertex of $G_1$ to every vertex of $G_2$.

It follows from Courant–Weyl inequalities (see e.g. [10, Theorem 2.1]) that following is true.

**Theorem 1.1.** Let $G$ be a graph on $n$ vertices. Then the Laplacian eigenvalues of $G$ interlace those of $G + e$, that is,

$$\lambda_1(G + e) \geq \lambda_1(G) \geq \lambda_2(G + e) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G + e) = \lambda_n(G) = 0.$$ 

**Theorem 1.2** [2]. If $T$ is a tree on $n$ vertices with diameter $d$, then

$$m_T(0, 2) \geq \lfloor d/2 \rfloor \leq m_T(2, n),$$

where square brackets indicate the greatest integer function.

**Theorem 1.3** [6]. Let $G$ be a connected graph on $n$ vertices. If $n > 2\mu(G)$, then $m_G(2, n] \geq \mu(G)$. 

2. Main results

Denote by $H_1, H_2, H_3, H_4, H_5, H_6, H_7, H_8 = H_8(p, m, q) (p \geq 0, m \geq 1, q \geq 0), H_9 = H_9(p, m, q) (p \geq 0, m \geq 0, q \geq 0)$ the graphs displayed in Fig. 1. Denote by $S$ the set consisting of the following graphs:

$P_2 = H_9(p, 0, 0), P_3 = H_9(1, 1, 0) = H_9(0, 1, 1), C_4 = H_9(0, 1, 0)$, $S_1, p + 1 = H_8(p, 0, 0) (p \geq 2), S_1, q + 1 = H_8(0, 0, q) (q \geq 2)$.

Theorem 2.1. The graphs $H_1 - H_9$ shown in Fig. 1, except those in $\mathcal{S}$, hold the property that $m_{H_i}(2, n) = 2$ for each $i = 1, 2, \ldots, 9$.

Proof. By [9, Theorem 5] and direct calculation, the graphs $H_1 - H_6$, and $H_9 = H_9(p, 0, q)$, except those in $\mathcal{S}$, hold the property that $m_{H_i}(2, n) = 2$ for each $i = 1, 2, \ldots, 9$.

For the graph $H_9 = H_9(p, m, q) (m \geq 1)$, if $p = q \geq 1$, then $d_1 = d_1(H_9) = d_2(H_9) = m + p + 1 \geq 3$. By [11, Lemma 4] and its proof, $\lambda_1(H_9) \geq \lambda_2(H_9) \geq d_2(H_9) \geq 3$ and $\lambda_3(H_9) = \frac{1}{2} \left( d_1 + 2 - \sqrt{d_1^2 - 4p} \right)$. If $\lambda_3(H_9) > 2$, then $p + 1 \leq m + p = d_1 - 1 < p$, impossible.

For the other cases, $H_9$ contains a cycle $C_3$ as its subgraph. By Theorem 1.1, $\lambda_2(H_9) \geq \lambda_2(C_3) = 3$. We can construct a new graph $H = H_9(p + 1, m, p + 1)$ from $H_9$ by inserting some new vertices and edges. By Theorem 1.1 and the discussion above, $\lambda_3(H_9) \leq \lambda_3(H) \leq 2$. The result follows. \qed
Let $G$ be a connected graph with the property

$$\lambda_3(G) \leq 2.\quad (1)$$

Property (1) is hereditary because, as a direct consequence of Theorem 1.1, for any (not necessarily induced) subgraph $H \subset G$, $H$ also satisfies (1). The hereditarity of property (1) implies that there are minimal connected graphs that do not obey (1); such graphs are called forbidden subgraphs. It is easy to verify that the three graphs shown in Fig. 2 are forbidden subgraphs. They will play an important role in our discussion.

**Lemma 2.2.** Let $G$ be a connected graph on $n \leq 6$ vertices with the property $P$: $m_G(2,n) \leq 2$. Then $G$ is one of the following graphs: $H_1 - H_7$, and $H_8$, $H_9$ with at most 6 vertices, shown in Fig. 1, except those in $\mathcal{F}$.

**Proof.** By [9, Theorem 5], it is sufficient to consider the case for $G$ containing a cycle with odd length at following. If $n = 3$, $C_3 = H_9(0,1,0)$ is the unique graph with the property $P$. If $n = 4$, $G$ contains a cycle $C_3$. Since $G_1 = K_4$ is forbidden, the only two graphs with the property $P$ are $H_9(1,1,0), H_9(0,2,0)$. If $n = 5$, the only five graphs with the property $P$ are $H_9(2,1,0), H_9(1,1,1), H_9(1,2,0), H_9(0,3,0)$ and $H_4$ since $G_1, G_2$ are forbidden. For the last case of $n = 6$, since $G_1, G_2, G_3$ are forbidden, the only nine graphs with the property $P$ are $H_9(3,1,0), H_9(2,1,1), H_9(2,2,0), H_9(1,2,1), H_9(1,3,0), H_9(0,4,0)$ and $H_5, H_6, H_7$. The result follows. \[\Box\]

**Lemma 2.3.** Let $T$ be a tree on $n$ vertices. If $n \geq 7$ and $m_T(2,n) \leq 2$, then $T$ must be one of the following graphs: $S_{1,n-1}, H_8(p,1,n-p-3), (1 \leq p \leq n-4), H_9(p,0,n-p-2), (1 \leq p \leq n-3)$.

**Proof.** Let $d$ be the diameter of $T$. By Theorem 1.2, $d \leq 5$. If $d = 1$, then $T = P_2$, a contradiction. If $d = 2$, then $T = S_{1,n-1}$. If $d = 3$, there exists a path $P_3 = \{u_1, u_2, u_3, u_4\}$ as a subgraph of $T$, where $u_1, u_4$ are the vertices with degree 1 in the path. Any other vertex of $T$ except those of the path is adjacent to $u_2$ or $u_3$; otherwise $d \geq 4$, a contradiction. So $T = H_9(p,0,n-p-2)(1 \leq p \leq n-3)$ at this case. If $d = 4$, there exists a path $P_3 = \{u_1, u_2, u_3, u_4, u_5\}$ as a subgraph of $T$, where $u_1, u_5$ are the vertices with degree 1 in the path. If there exists some vertex $v$ of $T$, except
those of the path, adjacent to $u_3$, then $T$ has a connected subgraph $T_1$ with 7 vertices which contains $v$ and the vertices of the path, and has a matching with three edges. By Theorem 1.1 and Theorem 1.3, $m_T(2, n) \geq m_T(2, n) \geq 3$, a contradiction. So $T = H_8(p, 1, n - p - 3) (1 \leq p \leq n - 4)$ at this case. For the last case of $d = 5$, there exists a path $P_6$ in $T$, and a vertex of $T$, except those of the path, adjacent to one vertex of the path. By a similar discussion above, this is impossible. The result follows.

The following is our main result in this paper.

**Theorem 2.4.** Let $G$ be a connected graph on $n$ vertices. Then $m_G(2, n) = 2$ if and only if $G$ is one of graphs shown in Fig. 1, except those in $\mathcal{F}$.

**Proof.** The sufficiency holds by Theorem 2.1. We consider the necessity at following. If $n \leq 6$, the necessity holds by Lemma 2.2. Suppose $n \geq 7$ at following, and let $T$ be one of the spanning trees of $G$. By Theorem 1.1, $\lambda_3(T) \leq \lambda_3(G) \leq 2$. So $m_T(2, n) \leq 2$, and consequently $T$ is one of graphs in Lemma 2.3.

If $T = S_{1,n-1}$, then $G$ can be written as the form $G = \{u\} \cup \{G_1 + \cdots + G_k\}$ ($k \geq 1$), where $G_i$ is connected for each $i = 1, 2, \ldots, k$. If $G_1$, $G_2$, $\ldots$, $G_k$ are all isolated vertices, then $G = T$, and $m_G(2, n) = 1$, a contradiction. If there is exactly one graph (let it be $G_1$) among $G_1$, $G_2$, $\ldots$, $G_k$ with at least two vertices, then by [4, Theorem 2], $n, \lambda_1(G_1) + 1, \lambda_2(G_1) + 1$ are Laplacian eigenvalues of $G$. By [11, Lemma 2], $d_2(G_1) + 1 \geq \lambda_2(G_1) + 1 \geq 2$. So $d_2(G_1) = 1$, and $G_1$ is one of $P_2$, $P_3$ and $S_{1,m} (m \geq 3)$. Also $\lambda_1(G_1) \geq d_1(G_1) + 1 \geq 2$ [5]. So the other cases can not happen and $G = H_9(n - m - 2, m, 0) (m \geq 1)$.

If $T = H_8(p, 1, n - p - 3) (1 \leq p \leq n - 4) = F_1$ shown in Fig. 3, then the subgraph of $G$ induced by vertex set $V(K_{r}^e) \cup V(K_{n-p-3}^e)$ has no edges. Otherwise, $G$ has a connected subgraph with seven vertices which contains a matching with three edges, and consequently $m_G(2, n) \geq 3$. Since $G_2$ is forbidden, $w$ can not be adjacent to any vertex in $V(K_{p}^e) \cup V(K_{n-p-3}^e)$. So the possible edges in $G$, except those of $T$, are those between $u$ and some vertices in $V(K_{r}^e)$, $v$ and some vertices in $V(K_{n-p-3}^e)$. Therefore, $G$ is one graph of $H_8$ with $n \geq 7$ vertices.
If \( T = H_0(p, 0, n - p - 2) \) (1 \( \leq p \leq n - 3 \)) = \( F_2 \) shown in Fig. 3, we just need to discuss it at the following two cases:

(A) \( p = 1, n - p - 2 > 1 \), \hspace{1cm} \( \text{B) } p > 1, n - p - 2 > 1 \).

**Case (A).** We know that the induced subgraph of \( G \) by \( V(K_{n-p-2}^c) \) has no edges because \( G_2 \) is forbidden. Let \( V(K_p^c) = \{w\} \). If there exists some vertex in \( V(K_{n-p-2}^c) \) adjacent to \( u \), then \( w \) can not be adjacent to any vertex of \( V(K_{n-p-2}^c) \) because \( G_2 \) is forbidden. So \( G = H_0(0, m, n - m - 2) \) (\( m \geq 2 \)) or \( G = H_0(1, m, n - m - 3) \) (\( m \geq 1 \)). Otherwise, the possible edges in \( G \), except those of \( T \), are those between \( w \) and \( v \), \( w \) and some vertices in \( V(K_{n-p-2}^c) \). So \( G = H_0(0, m, n - m - 2) \) (\( m \geq 1 \)) or \( G = H_0(0, m, n - m - 2) \) (\( m \geq 1 \)).

**Case (B).** Both of the subgraphs of \( G \) induced by \( V(K_p^c) \) and \( V(K_{n-p-2}^c) \) have no edges because \( G_2 \) is forbidden. Also there are no edges between some vertices of \( V(K_p^c) \) and some of \( V(K_{n-p-2}^c) \); otherwise \( G \) would have a connected subgraph with seven vertices which contains a matching with three edges, impossible. So the possible edges of \( G \), except those of \( T \), are those between \( u \) and some vertices of \( V(K_{n-p-2}^c) \), \( v \) and some of \( V(K_p^c) \). Hence, \( G = H_0(t, 0, n - t - 2) \) (\( 2 \leq t \leq n - 4 \)) or \( G = H_0(t, m, n - t - m - 2)(m \geq 1) \).

The result follows. □

Finally, we shall discuss a connected graph \( G \) with exactly one eigenvalue greater than two, that is, \( m_G(2, n) = 1 \).

**Theorem 2.5.** Let \( G \) be a connected graph on \( n \geq 3 \) vertices. Then \( m_G(2, n) = 1 \) if and only if \( G \) is one of graphs \( P_3, P_4, C_4, S_{1,n-1}(n \geq 4) \), that is, \( G \) is one graph in \( \mathcal{F} \), except \( P_2 \).

**Proof.** The sufficiency holds obviously. Let \( T \) be a spanning tree of \( G \). Then \( m_T(2, n) = 1 \) because \( \lambda_1(T) \geq d_1(T) + 1 \geq 3 \) [5]. By Theorem 1.2, the diameter \( d \) of \( T \) satisfies \( d \leq 3 \). If \( d = 1 \), then \( T = P_2 \), impossible. If \( d = 3 \) and \( n \geq 5 \), there exists a connected subgraph with five vertices of \( T \) which contains a matching with two edges. It is impossible by Theorem 1.3. So \( G \) is a graph on four vertices, and \( G = P_4 \) or \( G = C_4 \) at this case. For the last case of \( d = 2 \), then \( T = S_{1,n-1} \), and consequently \( G = \{u\} \lor \{G_1 + \cdots + G_k\} \) (\( k \geq 1 \)), where \( G_i \) is connected for each \( i = 1, 2, \ldots, k \). If there exists some \( G_i \) among \( G_1, G_2, \ldots, G_k \) with at least 2 vertices, then by [4, Theorem 2], \( n \geq 3, \lambda_1(G_i) + 1 \) are eigenvalues of \( G \). So \( d_1(G_i) + 2 \leq \lambda_1(G_i) + 1 \leq 2 \) [5]. Consequently, \( d_1(G_i) = 0 \), a contradiction. Therefore, \( G = P_3 \) or \( G = S_{1,n-1}(n \geq 4) \) at this case. The result follows. □

**Corollary 2.6.** Let \( G \) be a connected graph on \( n \geq 2 \) vertices. Then \( m_G(2, n) = 1 \) or 2 if and only if \( G \) is one graph shown in Fig. 1, except \( P_2 \).
References