Decay of Solutions of Wave Equations in a Bounded Region with Boundary Dissipation

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An energy decay rate is obtained for solutions of wave type equations in a bounded region in $\mathbb{R}^n$ whose boundary consists partly of a nontrapping reflecting surface and partly of an energy absorbing surface.

1. Introduction and Statement of Results

Let $\Omega$ be a bounded, open, connected set in $\mathbb{R}^n$ ($n \geq 2$) having a boundary $\Gamma$ which is of class $C^2$ and which consists of two parts, $\Gamma_0$ and $\Gamma_1$, with $\Gamma_1 \neq \emptyset$ and relatively open in $\Gamma$. $\Gamma_0$ is assumed to be either empty or to have a nonempty interior. We wish to consider the question of energy decay of solutions of wave type equations within $\Omega$ when $\Gamma_0$ is a reflecting surface and $\Gamma_1$ an energy absorbing surface. A specific example of the sort of problem to be considered is

\[
\frac{\partial^2 w}{\partial t^2} - \Delta_n w = 0 \quad \text{in } \Omega \times (0, \infty),
\]

\[
w(x, t) = 0 \quad \text{on } \Gamma_0 \times [0, \infty),
\]

\[
\frac{\partial w}{\partial v} + a(x) \frac{\partial w}{\partial t} = 0 \quad \text{on } \Gamma_1 \times [0, \infty),
\]

where $v$ is the unit normal of $\Gamma$ pointing towards the exterior of $\Omega$, and $a \in C^1(\overline{\Gamma_1})$ with $a(x) \geq a_0 > 0$ on $\Gamma_1$. We shall also treat variable coefficient situations analogous to (1.1)-(1.3).

A decay rate for (1.1)-(1.3) is a function $f(t)$ satisfying $f(t) \to 0$ as $t \to +\infty$ and

\[
E(w, t) \leq f(t) E(w, 0), \quad t \geq 0,
\]
where \( E(w, t) \) is the energy of the solution at time \( t \):

\[
E(w, t) = \frac{1}{2} \int_\Omega \left[ |w_t(x, t)|^2 + |\nabla w(x, t)|^2 \right] dx.
\]

The main purpose of this paper is to show that there is an exponential decay rate if \( \Gamma_0 \) and \( \Gamma_1 \) are subject to certain restrictions.

**Theorem 1.** Assume there is a vector field \( \ell(x) = (\ell_1(x), ..., \ell_n(x)) \) of class \( C^2(\overline{\Omega}) \) such that

(i) \( \ell \cdot v \leq 0 \) a.e. on \( \Gamma_0 \);

(ii) \( \ell \cdot v \geq \gamma > 0 \) a.e. on \( \Gamma_1 \);

(iii) \( \left( \frac{\partial \ell_1}{\partial x_1} + \frac{\partial \ell_1}{\partial x_1} \right) \) is uniformly positive definite on \( \overline{\Omega} \). Then there are positive constants \( C, \delta \), such that

\[
E(w, t) \leq Ce^{-\delta t}E(w, 0), \quad t \geq 0,
\]

for every solution of \((1.1)-(1.3)\) for which \( E(w, 0) < +\infty \).

The condition \( E(w, 0) < +\infty \) means that \( w(\cdot, 0) \in H^1(\Omega) \), \( w_t(\cdot, 0) \in L^2(\Omega) \), and \( w(x, 0) = 0 \) on \( \Gamma_0 \) if \( \Gamma_0 \neq \emptyset \). It should be noted that conditions (i) and (ii), together with the previous requirement that \( \partial \Omega \) be of class \( C^2 \), force

\[
\Gamma_0 \cap \Gamma_1 = \emptyset.
\]

Thus Theorem 1 cannot apply to simply connected regions \( \Omega \) unless \( \Gamma_0 = \emptyset \). If this is the case, the smoothness condition on \( \partial \Omega \) may be replaced by: \( \Omega \) is convex. For the remainder of this paper, (*) will be assumed to hold. (See the Remark below and also Section 4 for additional comments on this point).

The key to the proof of Theorem 1 is the following result which may be of independent interest. This is the analog for the problem \((1.1)-(1.3)\) of a result of W. Strauss [13] concerning solutions of the Dirichlet problem in a region exterior to a bounded obstacle.

**Theorem 2.** For every \( \varepsilon > 0 \) there is a number \( C_\varepsilon \) such that for every \( \beta > 0 \),

\[
\int_0^\infty \int_\Omega e^{-\beta t}(w - I(w_0))^2 \, dx \, dt \\
\leq C_\varepsilon E(w, 0) + \varepsilon \int_0^\infty \int_\Omega e^{-\beta t}w_0^2 \, dx \, dt,
\]

\((1.5)\)
for every solution of (1.1)-(1.3) for which $E(w, 0) < +\infty$, where

$$I(w_0) = 0, \quad \Gamma_0 \neq \emptyset,$$

$$= \frac{1}{\text{meas}(\Omega)} \int_{\Omega} w(x, 0) \, dx, \quad \Gamma_0 = \emptyset.$$

Another consequence of Theorem 2 is a simple and direct proof of energy decay in the absence of restrictions on $\partial \Omega$.

**Corollary 1.** If $w$ is a solution of (1.1)-(1.3) with $E(w, 0) < +\infty$, then

$$E(w, \infty) \equiv \lim_{t \to \infty} E(w, t) = 0.$$

This result was first obtained in [8] using a compactness argument and the Holmgren uniqueness theorem. Our proof is based on the following stronger consequence of Theorem 2.

**Corollary 2.** For every $\varepsilon > 0$ there is a constant $C_\varepsilon$ such that for every $\beta > 0$,

$$\int_0^\infty e^{-\beta t} E(w, t) \, dt \leq C\varepsilon \left[ E(w, 0) + E(w_\ell, 0) \right]$$

$$+ e \int_0^\infty \int_{\Omega} e^{-2\beta t} w_\ell^2 \, dx \, dt, \quad (1.6)$$

for every solution of (1.1)-(1.3) with $E(w_\ell, 0) < +\infty$.

The condition $E(w_\ell, 0) < +\infty$ means $w(\cdot, 0) \in H^2(\Omega)$, $w_\ell(\cdot, 0) \in H^1(\Omega)$, $w(x, 0) = w_\ell(x, 0) = 0$ on $\Gamma_0$, and $w_\ell(x, 0) + a(x) w_\ell(x, 0) = 0$ on $\Gamma_1$.

Conditions (i) and (ii) of Theorem 1 comprise a nontrapping hypothesis on the reflecting surface $\Gamma_0$ and form the vector field condition (V) of W. Strauss [13] (cf. [5]) adapted to a bounded region. Clearly some hypothesis of this sort is necessary if there is to be any hope of obtaining a decay rate for (1.1)-(1.3). Condition (ii) is an assumption about the rate of energy absorption in $\Gamma_1$ in the directions $l(x)$, $x \in \Gamma_1$. It has been conjectured [10, p. 696] that a decay rate obtains in the absence of any such restriction, but we are unable to verify this. In this regard, Quinn and Russell [8] have proved that if (i) holds with $l(x) = x - x_0$ for some $x_0$ in the exterior of $\Omega$, then $E(w, t) = O(t^{-1})$ as $t \to +\infty$, provided $E(w_\ell, 0) < +\infty$. G. Chen [2] subsequently proved (1.4) if in addition (ii) holds for this special form of $l(x)$. In the general case, Chen [3] obtained (1.4) assuming that $l \in C^3(\Omega)$ satisfies (i), (ii), and the following two additional conditions:
(iii)' \( (\ell_{ij} + \ell_{ji} - \delta_{ij}) \) is uniformly positive definite on \( \tilde{\Omega} \), and

(iv) there is an \( \eta > 0 \) sufficiently small (depending on \( \Omega \) and the greatest lower bound for the matrix in (iii)') such that

\[
|\ell_{ijij}| \leq \eta \quad \text{on} \quad \Omega,
\]

\[
|\ell_{ijij}| \leq \eta \quad \text{a.e. on} \quad \Gamma_1.
\]

In (iii)' and (iv), the additional subscripts denote differentiations of the vector field \( \ell \), e.g., \( \ell_{ij} = \partial \ell_i / \partial x_j \), and summation convention has been assumed.

Condition (iv) is clearly very restrictive and unnatural. It was the desire to remove this restraint which led to the present work.

An example of a region for which there is a vector field \( \ell \) satisfying (i)-(iii) but not one of the form \( \ell(x) = x - x_0 \) for any \( x_0 \) in the exterior of \( \Omega \) is suggested by an example in [7, p. 448]. Let \( B \) be the body that remains when a corkscrew is drilled part way through a ball with a standard corkscrew having a slightly rounded tip. Take for \( \Omega \) the region exterior to \( B \) and interior to a sphere concentric to the first ball and of larger radius.

Remark. The results of Quinn and Russell [8] and Chen [2, 3] cited above are claimed to hold in regions with less smoothness than those considered here. In particular, \( \Gamma \) is assumed only piecewise smooth and condition (*) above is not imposed, hence simply connected, star-shaped polyhedra are allowed. However, the proofs concerning decay of solutions of (1.1)-(1.3) given in each of those papers are open to question unless \( \Gamma \) is required to satisfy (*) and either the smoothness condition stated in the opening paragraph or a convexity condition. The difficulty is related to the lack of regularity of solutions of (1.1)-(1.3) in the absence of such conditions. In particular, solutions of (1.1)-(1.3) may not have square summable second derivatives in \( \Omega \) for fixed \( t > 0 \), no matter how smooth the solution is initially, in disagreement with what is claimed in [2, 3, 8]. We shall return to this matter in Section 4.

There is a duality between the existence of a decay rate for (1.1)-(1.3) and the following problem of exact controllability of solutions of (1.1): Given arbitrary functions \( w_0 \in H_1(\Omega) \), \( w_1 \in L^2(\Omega) \), with \( w_0 = 0 \) on \( \Gamma_0 \), and given a sufficiently large time \( T \) independent of \( (w_0, w_1) \), find a control function \( f \in L^2(\Omega_1 \times (0, T)) \) such that the solution of (1.1), (1.2) with data

\[
w(x, 0) = w_0(x), \quad \frac{\partial w}{\partial t}(x, 0) = w_1(x) \text{ in} \quad \Omega,
\]

\[
\frac{\partial w}{\partial v} + a \frac{\partial w}{\partial t} = f \quad \text{on} \quad \Gamma_1 \times (0, T),
\]
satisfies $E(w, T) = 0$. That is, given any finite energy initial state find a control $f$ which drives this state to the zero energy state in time $T$. It turns out that the existence of a solution to this control problem which depends continuously on the energy of the initial state is equivalent to the existence of a decay rate for (1.1)–(1.3). We shall not pursue the controllability problem here, but refer to [10] for details of this duality.

Both Theorem 1, and Theorem 2 and its corollaries, extend to the generalized wave process

$$\rho(x) \frac{\partial^2 w}{\partial t^2} - \nabla \cdot (A(x) \nabla w) + q(x)w = 0 \quad \text{in } \Omega \times (0, \infty), \quad (1.7)$$
$$w(x, t) = 0 \quad \text{on } \Gamma_0 \times [0, \infty), \quad (1.8)$$
$$v \cdot (A(x) \nabla w) + a(x) \frac{\partial w}{\partial t} = 0 \quad \text{on } \Gamma_1 \times [0, \infty), \quad (1.9)$$

if one assumes that the real-valued coefficients of (1.7) satisfy the following: $\rho, q$ are $C^1(\overline{\Omega})$ and $\rho(x) \geq \rho_0 > 0$, $q(x) \geq 0$ in $\Omega$; $A(x) = (A_{ij}(x))_1^n$ is symmetric, its entries have Lipschitz continuous second derivatives, and

$$\xi \cdot A(x)\xi \geq A_0 |\xi|^2, \quad \forall \xi \in \mathbb{R}^n,$$

for some constant $A_0 > 0$. In this case the energy functional is defined as

$$E(w, t) = \frac{1}{2} \int_{\Omega} \left[ \rho w_t^2 + \nabla w \cdot (A \nabla w) + q w^2 \right] dx, \quad (1.10)$$

and one has

$$\frac{d}{dt} E(w, t) = \int_{\Gamma} \frac{\partial w}{\partial t} v \cdot (A \nabla w) \, d\sigma \leq 0$$

for every solution of (1.7)–(1.9). With $E$ defined by (1.10), Theorem 2 and its corollaries carry over verbatim. The conclusion of Theorem 1 likewise remains valid if one postulates the existence of a vector field $\ell = (\ell_1, ..., \ell_n)$ of class $C^2(\overline{\Omega})$ which satisfies

(a) $\ell \cdot v \leq 0$ a.e. on $\Gamma_0$;
(b) $\ell \cdot v \geq \gamma > 0$ a.e. on $\Gamma_1$;
(c) $\ell \cdot \nabla \rho \geq \alpha \rho$ in $\Omega$ for some $\alpha > -1$;
(d) the matrix $[2A_{ik}(\partial \ell_j/\partial x_k) - \delta_{jk} - (\partial A_{ij}/\partial x_k) \ell_k] \geq \alpha A$ in $\Omega$.

The proofs given below carry over to this more general situation with only minor modifications which we will describe in the last section of this paper.
Theorem 1 is proved in the next section, and Theorem 2 and its corollaries in Section 3. For related results on decay of solutions of hyperbolic problems in bounded regions with boundary dissipation see [9, 12] in addition to the references cited above.

2. PROOF OF THEOREM 1

We first prove Theorem 1 under the assumption \( \Gamma_0 \neq \emptyset \). The modifications necessary to handle the case \( \Gamma_0 = \emptyset \) will be discussed at the end of the proof.

For \( k \) a positive integer let \( H^k(\Omega) \) be the Sobolev space of real valued functions with \( L^2(\Omega) \) derivatives to order \( \leq k \). We define \( H^k_{\Gamma_0}(\Omega) = L^2(\Omega) \),

\[
H^k_{\Gamma_0}(\Omega) = \{ u \in H^k(\Omega) \mid u = 0 \text{ on } \Gamma_0 \}, \quad k = 1, 2, \ldots ,
\]

\[ \mathcal{H}^1 = H^1_{\Gamma_0}(\Omega) \times L^2(\Omega), \]

and

\[ \mathcal{H}^k = \left\{ (u, v) \in H^k_{\Gamma_0}(\Omega) \times H^{k-1}_{\Gamma_0}(\Omega) \left| \frac{\partial u}{\partial v} + av = 0 \text{ on } \Gamma_1 \right. \right\}, \]

\[ k = 2, 3, \ldots . \]

\( \mathcal{H}^k \) is given the usual product norm. For \( u \in H^k_{\Gamma_0}(\Omega) \) with \( k \geq 1 \) Poincaré's inequality is valid,

\[
\int_{\Omega} u^2 \, dx \leq C(\Omega) \int_{\Omega} |\nabla u|^2 \, dx,
\]

so that the norm on \( \mathcal{H}^1 \) is equivalent to the energy norm

\[
\|(u, v)\|_E = \left[ \int_{\Omega} (|\nabla u|^2 + v^2) \, dx \right]^{1/2}.
\]

Suppose now that \( (w(\cdot, 0), w_r(\cdot, 0)) \in \mathcal{H}^1 \), that is \( F(w, 0) < +\infty \). It is proved in [8] that (1.1)–(1.3) has a unique solution with \( (w, w_r) \in C([0, \infty); \mathcal{H}^1) \). The map \( (w(\cdot, 0), w_r(\cdot, 0)) \rightarrow (w(\cdot, t), w_r(\cdot, t)) \) therefore defines a strongly continuous semigroup of linear contractions on \( \mathcal{H}^1 \) with respect to \( \| \cdot \|_E \). When \( \bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset \), its generator is the dissipative operator

\[
\mathcal{A} = \begin{pmatrix}
0 & I \\
A_n & 0
\end{pmatrix},
\]

with domain \( D(\mathcal{A}) = \mathcal{H}^2 \). (This is not true when \( \bar{\Gamma}_0 \cap \bar{\Gamma}_1 \neq \emptyset \), in which case
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$D(\mathcal{A}) \subseteq \mathbb{R}^2$ in general. See Sect. 4.) We shall assume that 
\((w(\cdot, 0), w_t(\cdot, 0)) \in \mathbb{R}^2\), which is sufficient to ensure that 
\((w, w_t) \in C([0, \infty); \mathbb{R}^2) \cap C'([0, \infty); \mathbb{R}^1)\).

The general case $E(w, 0) < +\infty$ is then handled by a simple limiting process.

If $w$ is a solution of (1.1) with initial data as described above, the following identity is readily verified:

\[
\frac{\partial}{\partial t} \left[ t(w_t^2 + |\nabla w|^2) + 2w_t(\ell \cdot \nabla w) + (\ell_{ij} - 1) w w_t \right] = \text{div} \left[ 2t w_t \nabla w \| 2(\ell \cdot \nabla w) \nabla w \right.
\]

\[
+ w_t^2 \ell + (\ell_{ij} - 1)w \nabla w - |\nabla w|^2 \ell \right)
\]

where $w_t = \partial w/\partial x_t$, $\ell_{ij} = \partial \ell_i/\partial x_j$, etc., and summation convention is assumed.

As in [3], we define

\[
Q(t) = \frac{1}{2} \int_{\Omega} (w_t^2 + |\nabla w|^2) \, dx + \int_{\Omega} [2w_t(\ell \cdot \nabla w) + (\ell_{ij} - 1) w w_t] \, dx
\]

and suppose $w$ is a solution of (1.1)–(1.3). Then from the above identity

\[
\dot{Q}(t) = -\frac{1}{2} \frac{\partial}{\partial t} \int_{\Omega} t(w_t^2 + |\nabla w|^2) \, dx
\]

\[
+ \int_{\Omega} \left[ 2(\delta_{ij} - \ell_{ij}) w_t w_j - \ell_{ij} w w_j \right] \, dx
\]

\[
+ \int_{\Gamma_1} \left[ 2(\ell \cdot \nabla w) \frac{\partial w}{\partial v} - |\nabla w|^2 \ell \cdot v \right] \, d\sigma + \int_{\Gamma_1} \left[ 2t w_t \frac{\partial w}{\partial v} + 2(\ell \cdot \nabla w) \frac{\partial w}{\partial v} \right.
\]

\[
+ w_t^2 \ell \cdot v + (\ell_{ij} - 1)w \frac{\partial w}{\partial v} - |\nabla w|^2 \ell \cdot v \right] \, d\sigma.
\]

By multiplying $\ell$ by a sufficiently large positive constant, we may assume that $\delta_{ij} - \frac{1}{2}(\ell_{ij} + \ell_{ji})$ is negative definite in $\Omega$. Since also $\nabla w = (\partial w/\partial v)v$ on $\Gamma_0$, it follows that

\[
\dot{Q}(t) \leq -\frac{1}{2} \int_{\Omega} (w_t^2 + |\nabla w|^2) \, dx - \int_{\Omega} \ell_{ij} w w_j \, dx + \int_{\Gamma_0} \left( \ell \cdot v \right) \left( \frac{\partial w}{\partial v} \right)^2 \, d\sigma
\]

\[
+ \int_{\Gamma_1} \left[ -\frac{t}{a} \left( \frac{\partial w}{\partial v} \right)^2 + 2(\ell \cdot \nabla w) \frac{\partial w}{\partial v} \right.
\]

\[
+ \frac{1}{a^2} \left( \frac{\partial w}{\partial v} \right)^2 \ell \cdot v + (\ell_{ii} - 1)w \frac{\partial w}{\partial v} - |\nabla w|^2 \ell \cdot v \right] \, d\sigma.
\]
Let $\varepsilon > 0$ be fixed. We have the following estimates:

\[
\int_{\Omega} \ell_{ij} w w_j \, dx \leq D_\varepsilon \int_{\Omega} w^2 \, dx + \varepsilon \int_{\Omega} |\nabla w|^2 \, dx,
\]

\[
\left| \int_{\Gamma_1} \left( \frac{\partial w}{\partial v} \right)^2 \ell \cdot v \, d\sigma \right| \leq E \int_{\Gamma_1} \left( \frac{\partial w}{\partial v} \right)^2 \, d\sigma,
\]

\[
2 \left| \int_{\Gamma_1} (\ell \cdot \nabla w) \frac{\partial w}{\partial v} \, d\sigma \right| \leq A_\varepsilon \int_{\Gamma_1} \left( \frac{\partial w}{\partial v} \right)^2 \, d\sigma + \varepsilon \int_{\Gamma_1} |\nabla w|^2 \, d\sigma,
\]

\[
\left| \int_{\Gamma_1} (\ell_{ii} - 1) w \frac{\partial w}{\partial v} \, d\sigma \right| \leq \varepsilon \int_{\Omega} |\nabla w|^2 \, dx + B_\varepsilon \int_{\Gamma_1} \left( \frac{\partial w}{\partial v} \right)^2 \, d\sigma + C_\varepsilon \int_{\Omega} w^2 \, dx,
\]

where $A_\varepsilon, D_\varepsilon, E$ are constants depending on $\Omega$ and, where indicated, on $\varepsilon$. Substitution into the last inequality for $Q$ gives

\[
Q(t) \leq -\frac{1}{2} \int_{\Omega} w^2 \, dx + \left( 2\varepsilon - \frac{1}{2} \right) \int_{\Omega} |\nabla w|^2 \, dx
\]

\[
+ \left( C_\varepsilon + D_\varepsilon \right) \int_{\Omega} w^2 \, dx + \int_{\Gamma_1} (\varepsilon - \ell \cdot v) |\nabla w|^2 \, d\sigma
\]

\[
+ \left( -\frac{t}{a_1} + A_\varepsilon + B_\varepsilon + \frac{E}{a_0^2} \right) \int_{\Gamma_1} \left( \frac{\partial w}{\partial v} \right)^2 \, d\sigma,
\]

where $a_1$ is an upper bound for $a(x)$ on $\Gamma_1$. If we choose $\varepsilon > 0$ sufficiently small and then choose $t$ sufficiently large we obtain

\[
Q(t) \leq -\frac{1}{4} \int_{\Omega} \left( w^2 + |\nabla w|^2 \right) \, dx + C \int_{\Omega} w^2 \, dx, \quad t \geq t_0,
\]

for some constants $C$ and $t_0 > 0$.

Let $\beta > 0$ be fixed. Multiply the last inequality by $e^{-2\beta t}$ and integrate from $t_0$ to $+\infty$. Since $|Q(t)| \leq (\text{const.})(t + 1) E(w, 0)$ (using Poincaré’s inequality) we obtain after an integration by parts,

\[
2\beta \int_{t_0}^{\infty} e^{-2\beta t} Q(t) \, dt + \frac{1}{4} \int_{t_0}^{\infty} \int_{\Omega} e^{-2\beta t} (w^2 + |\nabla w|^2) \, dx \, dt
\]

\[
\ll C_1 E(w, 0) + C_2 \int_{t_0}^{\infty} \int_{\Omega} e^{-2\beta t} w^2 \, dx \, dt,
\]
where $C_1$, $C_2$ are independent of $\beta$. Since $Q(t) \geq 0$ for $t$ sufficiently large, it follows that

$$
\int_0^\infty \int_\Omega e^{-2\beta t}(w_t^2 + |\nabla w|^2) \, dx \, dt \leq C_1 E(w, 0) + C_2 \int_0^\infty \int_\Omega e^{-2\beta t} w^2 \, dx \, dt,
$$

(2.1)

with constants $C_1$, $C_2$ independent of $\beta$. We now invoke Theorem 2 and conclude that

$$
\int_0^\infty \int_\Omega e^{-2\beta t}(w_t^2 + |\nabla w|^2) \, dx \, dt \leq CE(w, 0),
$$

(2.2)

with $C$ independent of $\beta$. This last estimate has been obtained for solutions for which $(w(., 0), w(., 0)) \in \mathcal{H}$, but clearly extends to data in $\mathcal{H}$ since the former space is dense in the latter with respect to the energy norm. Letting $\beta \downarrow 0$, we obtain

$$
\int_0^\infty E(w, t) \, dt \leq CE(w, 0),
$$

and from this follows that for any $\varepsilon > 0$ there is a time $T_\varepsilon \in (0, C/\varepsilon)$ ($T_\varepsilon$ may also depend on $w$) such that

$$
E(w, T_\varepsilon) \leq \varepsilon E(w, 0).
$$

(2.3)

Let $t > 0$ be fixed and set $\varepsilon = C/t$. Since $E(w, t)$ is nonincreasing it follows from (2.3) that

$$
E(w, t) \leq (C/t) E(w, 0), \quad t > 0.
$$

(2.4)

The estimate (1.4) now follows in a standard way from (2.4) and the semigroup property of the map $(w(., 0), w(., 0)) \rightarrow (w(., t), w(., t))$. This completes the proof if $\Gamma_0 \neq \emptyset$.

If $\Gamma_0 = \emptyset$ the above proof breaks down when we try to bound $Q(t_0)$ in terms of $E(w, 0)$, since that estimate made use of Poincaré's inequality. Let $w$ be a solution of (1.1), (1.3) with $E(w, 0) < +\infty$, and set $\tilde{w} = w - I(w_0)$. Then $\tilde{w}$ satisfies (1.1), (1.3), $E(\tilde{w}, t) = E(w, t)$, and

$$
\int_\Omega \tilde{w}(x, 0) \, dx = 0,
$$

hence Poincaré's inequality is valid for $\tilde{w}(x, 0)$. From this fact and

$$
\tilde{w}(x, t) = \tilde{w}(x, 0) + \int_0^t \tilde{w}_t(x, s) \, ds,
$$
it follows that
\[ \int_{\Omega} |\tilde{w}(x,t)|^2 \, dx \leq C(t) \, E(\tilde{w}, 0). \]

If we now define \( Q \) as above but with \( w \) replaced by \( \tilde{w} \), then
\[ Q(t_0) \leq C(t_0) \, E(\tilde{w}, 0) = C(t_0) \, E(w, 0), \]
and we again obtain the estimate (2.1) but with \( w \) replaced by \( \tilde{w} \) in the integral on the right. Theorem 2 now applies once again to give (2.2) for every solution with \( E(w, 0) < +\infty \). The remainder of the proof proceeds as above.

3. PROOF OF THEOREM 2

For the present we shall assume \( \Gamma_0 \neq \emptyset \) and then remove this restriction at the end of the proof.

Let \( T > 0 \) be fixed and \( \phi \in C^\infty(R) \) satisfy \( \phi(0) = \phi'(0) = 0, \phi(t) = 1 \) for \( t \geq T \). Let \( u = \phi w \), so that \( u \) satisfies
\[
\begin{align*}
\partial_t u - \Delta u &= 2\phi w, + \phi'' w \equiv g \\
u(x, 0) &= u_t(x, 0) = 0 \\
u &= 0 \\
\partial_t u + a \partial_t u &= a \phi w \equiv h \\
\partial_n u &= 0 \\
u &= 0 \\
\partial_n u &= 0 \\
\partial_n u &= 0
\end{align*}
\]
We see that \( g \in C([0, \infty); L^2(\Omega)), h \in C([0, \infty); H^{1/2}(\Gamma_1)) \) and, since \( g = h = 0 \) for \( t \geq T \),
\[
\begin{align*}
\int_0^T \int_\Omega g^2 \, dx \, dt &\leq C \int_0^T \int_\Omega (w_t^2 + w^2) \, dx \, dt \leq CE(w, 0), \\
\int_0^\infty \|h\|^2_{H^{1/2}(\Gamma_1)} \, dt &\leq C \int_0^T \|w\|^2_{H^{1/2}(\Omega)} \, dt \leq CE(w, 0),
\end{align*}
\]
by Poincaré's inequality. (Throughout this section, \( C \) will denote various numbers which do not depend on \( \epsilon \) or \( t \).)

We also note that the function
\[
\tilde{h} = h \quad \text{on} \quad \Gamma_1 \times [0, \infty),
\]
\[
= 0 \quad \text{on} \quad \Gamma_0 \times [0, \infty).
\]
belongs to $C([0, \infty); H^{1/2}(\Gamma))$, since $\tilde{h}$ is just $a(x)$ times the restriction to $\Gamma$ of $\phi_w$. We may therefore assume that $h$ is of class $C([0, \infty); H^{1/2}_0(\Gamma))$.

Let $\omega$ be a complex parameter with $\text{Im} \, \omega < 0$ and $U$ be the Fourier transform of $u$:

$$U(x, \omega) = \int_0^\infty e^{-i\omega t} u(x, t) \, dt.$$ 

The integral clearly converges, and $U$ satisfies

$$A_\pi U + \omega^2 U = G \quad \text{in } \Omega, \quad (3.3)$$

$$U = 0 \quad \text{on } \Gamma_0, \quad (3.4)$$

$$\frac{\partial U}{\partial n} + i\omega u = H \quad \text{on } \Gamma_1, \quad (3.5)$$

where $G \in L^2(\Omega)$, $H \in H^{1/2}_0(\Gamma)$ denote the Fourier transforms of $g$ and $h$, respectively. Equation (3.3)–(3.5) is to be understood in the variational sense: $U \in H^1_{r_0}(\Omega)$ and

$$b_\omega(U, V) = \int_{\Gamma_1} H\tilde{V} \, d\sigma - \int_{\Omega} G\tilde{V} \, dx$$

for every $V \in H^1_{r_0}(\Omega)$, where

$$b_\omega(U, V) = \int_{\Omega} \left( \nabla U \cdot \nabla \tilde{V} + \omega^2 U \tilde{V} \right) \, dx + i\omega \int_{\Gamma_1} aU\tilde{V} \, d\sigma.$$ 

(Here $H^1_{r_0}(\Omega)$ is the closed subspace of the complex space $H^1(\Omega)$ defined as in Section 2.)

We now consider the problem (3.3)–(3.5) for general complex values of $\omega$ and arbitrary $G \in L^2(\Omega)$, $H \in H^{1/2}_0(\Gamma)$. Theorem 2 will be deduced from the following result.

**Lemma 1.** For every $\omega_0 > 0$ there is a number $\delta(\omega_0) > 0$ such that if $|\text{Re} \, \omega| \leq \omega_0$ and $|\text{Im} \, \omega| \leq \delta(\omega_0)$, then (3.3)–(3.5) has a unique solution, and

$$\|U\|_{L^2(\Omega)} \leq C(\omega_0) [\|G\|_{L^2(\Omega)} + \|H\|_{H^{1/2}_0(\Gamma)}].$$ 

**Proof.** We consider $b_\omega(U, V)$ as a sesquilinear form in $L^2(\Omega)$ with dense domain $D(b_\omega) = H^1_{r_0}(\Omega)$. Then $b_\omega$ is a holomorphic family of type (a) [5, p. 395] in some neighborhood of the real axis in the complex $\omega$-plane. This means that (i) each $b_\omega$ is sectorial and closed with constant dense domain,
and (ii) \( b_\omega(V, V) \) is holomorphic in \( \omega \) for each \( V \in D(b_\omega) \). That (ii) holds is obvious. To check (i), write \( \omega = \alpha + i \beta \). Then

\[
\text{Re} \; b_\omega(V, V) = \int_\Omega |\nabla V|^2 \, dx - (\alpha^2 - \beta^2) \int_\Omega |V|^2 \, dx - \beta \int_{\Gamma_1} a |V|^2 \, ds, \tag{3.6}
\]

\[
\text{Im} \; b_\omega(V, V) = -2\alpha \beta \int_\Omega |V|^2 \, dx + \alpha \int_{\Gamma_1} a |V|^2 \, ds. \tag{3.7}
\]

Since

\[
\int_{\Gamma_1} a |V|^2 \, ds \leq C \| V \|_{H^1(\Omega)},
\]

if \( |\beta| \) is sufficiently small one has

\[
\text{Re} \; b_\omega(V, V) + \alpha^2 \| V \|_{L^2(\Omega)}^2 \geq C \| V \|_{H^1(\Omega)}^2,
\tag{3.8}
\]

for some positive \( C \), and

\[
|\text{Im} \; b_\omega(V, V)| \leq C(\alpha)[\text{Re} \; b_\omega(V, V) + \alpha^2 \| V \|_{L^2(\Omega)}^2]. \tag{3.9}
\]

From (3.8) and (3.9) it follows that \( b_\omega \) is sectorial (the values of \( b_\omega(V, V) \) lie in the sector \( |\arg(\zeta + \alpha^2)| \leq \theta \), where \( 0 < \theta < \pi/2 \) is given by tan \( \theta = C(\alpha) \), and (3.8) implies that \( b_\omega \) is closed, since \( b_\omega \) is continuous as a form on \( H^1(\Omega) \).

Associated with \( b_\omega \) is an \( m \)-sectorial operator \( B_\omega \) in \( L^2(\Omega) \) such that

\[
(B_\omega U, V)_{L^2(\Omega)} = b_\omega(U, V)
\]

for every \( U \in D(B_\omega) \) and \( V \in H^1_{\Gamma_0}(\Omega) \). In fact, \( B_\omega = -(\Delta + \omega^2) \) with

\[
D(B_\omega) = \left\{ U \in H^1_{\Gamma_0}(\Omega) \mid U \in H^2(\Omega) \text{ and } \frac{\partial U}{\partial v} + \im \omega U = 0 \text{ on } \Gamma_1 \right\}.
\]

Since \( b_\omega \) is holomorphic of type (a) in a neighborhood \( \mathcal{A} \) of the real axis, \( B_\omega \) is a holomorphic family of closed operators in \( \mathcal{A} \) [5, Theorem 4.2, p. 395]. Clearly each \( B_\omega \) has compact resolvent and consequently either zero is an eigenvalue of each \( B_\omega \) or \( B_\omega^{-1} \) exists as a bounded operator on \( L^2(\Omega) \) for all \( \omega \in \mathcal{A} \) with the possible exception of a finite number of values in each compact subset of \( \mathcal{A} \) [5, Theorem 1.10, p. 371]. We now show that zero cannot be an eigenvalue of \( B_\omega \) if \( \omega \) is real. From this it will follow that \( B_\omega^{-1} \) exists for all \( \omega \) in \( |\text{Re} \; \omega| \leq \omega_0, |\text{Im} \; \omega| \leq \delta(\omega_0) \) for some suitably small \( \delta(\omega_0) \).
Thus suppose \( \omega \) is real and that \( U \in H^1_{\Gamma_0}(\Omega) \) satisfies

\[
b_\omega(U, V) = 0, \quad \forall V \in H^1_{\Gamma_0}(\Omega).
\]

If \( \omega = 0 \) then (3.6) implies that \( U = 0 \), since \( \Gamma_0 \neq \emptyset \). If \( \omega \neq 0 \) then, from (3.7), \( \text{Im} \, b_\omega(U, U) = 0 \) implies that \( U = 0 \) on \( \Gamma_1 \), so that \( U \) satisfies \( U \in H^1_{\Gamma_0}(\Omega) \),

\[
\int_{\Omega} (\nabla U \cdot \nabla \overline{V} - \omega^2 UV) \, dx = 0, \quad \forall V \in H^1_{\Gamma_0}(\Omega). \tag{3.10}
\]

Let \( x_0 \in \Gamma_1 \) and \( S_\rho \) be a ball centered at \( x_0 \). Choose \( \rho \) so small that \( S_\rho \) contains no point of \( \Gamma_0 \). Extend \( U \) into \( \Omega_{\rho} = \Omega \cup S_\rho \) by setting \( U = 0 \) in \( \Omega_{\rho} - \Omega \). Then \( U \in H^1(\Omega_{\rho}) \) (since \( U = 0 \) on \( \Gamma_1 \)) and is a solution of \( \Delta_n U + \omega^2 U = 0 \) in \( \Omega_{\rho} \), since if \( V \in C^0(\Omega_{\rho}) \) its restriction to \( \Omega \) is in \( H^1_{\Gamma_0}(\Omega) \), and from (3.10)

\[
\int_{\Omega_\rho} (\Delta_n \overline{V} + \omega^2 \overline{V}) \, dx = 0.
\]

\( U \) is therefore analytic in \( \Omega_\rho \) and hence \( U = 0 \) there since \( U \) vanishes on the open subset \( \Omega_{\rho} - \Omega \).

We now know that \( B_\omega^{-1} \) exists as a bounded operator on \( L^2(\Omega) \) if \( |\text{Re} \, \omega| \leq \omega_0, \ |\text{Im} \, \omega| \leq \delta(\omega_0) \) for \( \delta > 0 \) sufficiently small. This is in fact the conclusion of the lemma if \( \Gamma \equiv 0 \). Suppose that \( H \in H^{1/2}_{\Gamma_0}(\Gamma) \) is not identically zero. By the Trace Theorem there is a function \( W \in H^2(\Omega) \) such that, on \( \Gamma \), \( W = 0 \), \( \partial W/\partial n = H \), and

\[
\| W \|_{H^3(\Omega)} \leq C \| H \|_{H^{1/2}(\Gamma)}.
\]

\( W \) therefore satisfies (3.4) and (3.5). Let \( Z \) be the unique solution in \( D(B_\omega) \) of (3.3)--(3.5) with \( H = 0 \) and \( G \) replaced by \( G - (\Delta_n W + \omega^2 W) \). It is then obvious that \( U = Z + W \) satisfies (3.3)--(3.5) and

\[
\| U \|_{L^2(\Omega)} \leq C \| G \|_{L^2(\Omega)} + \| W \|_{H^3(\Omega)} \]

\[
\leq C \| G \|_{L^2(\Omega)} + \| H \|_{H^{1/2}(\Gamma)}.
\]

**Remark.** We see that the solution \( U \in H^2_{\Gamma_0}(\Omega) \) and therefore satisfies (3.3)--(3.5) in the pointwise sense. Moreover, \( \| U \|_{L^2(\Omega)} \) may be replaced by \( \| U \|_{H^3(\Omega)} \) in the lemma.

**Lemma 2.** For every \( \varepsilon > 0 \) there is a number \( C_\varepsilon \) such that for every \( \beta > 0 \),

\[
\int_0^\infty \int_{\Omega} e^{-\gamma t} u^2 \, dx \, dt \leq C_\varepsilon E(w, 0) + \varepsilon \int_0^\infty \int_{\Omega} e^{-\gamma t} u^2 \, dx \, dt. \tag{3.11}
\]
Proof. It suffices to establish (3.11) for $0 < \beta < \beta_0$ for some $\beta_0 > 0$, since for $\beta \geq \beta_0$ the left-hand side does not exceed $CE(w, 0)/2\beta$.

Write $\omega = \alpha - i\beta$ for $\beta > 0$ and small. Then

$$e^{-\beta t}u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\alpha t} U(x, \omega) d\alpha,$$

$$e^{-\beta t}u_t(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\alpha t} (i\omega) U(x, \omega) d\alpha.$$

By Parseval's equality,

$$\int_{-\infty}^{\infty} \int_{\Omega} e^{-2\beta t} u^2 \, dx \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{\Omega} \omega U(x, \omega)^2 \, dx \, d\alpha,$$

$$\int_{-\infty}^{\infty} \int_{\Omega} e^{-2\beta t} u_t^2 \, dx \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{\Omega} \omega U(x, \omega)^2 \, dx \, d\alpha.$$

Let $A > 0$ be so large that $A^{-2} \leq \varepsilon$. For $|\alpha| \leq A$ and $|\beta| \leq \delta(A)$ we obtain from Lemma 1

$$\int_{-A}^{A} \int_{\Omega} |U|^2 \, dx \, d\alpha \leq C(A) \left[ \int_{-\infty}^{\infty} \int_{\Omega} |G|^2 \, dx \, d\alpha + \int_{-\infty}^{\infty} \|H\|_{H^{1/2}(\Gamma)}^2 \, d\alpha \right]. \tag{3.12}$$

We also have

$$\int_{|\alpha| > A} \int_{\Omega} |U|^2 \, dx \, d\alpha \leq \frac{A^2}{|\alpha|^2} \int_{\Omega} |U|^2 \, dx \, d\alpha$$

$$\leq \varepsilon \int_{-\infty}^{\infty} \int_{\Omega} |\omega U|^2 \, dx \, d\alpha$$

$$\leq 2\pi \varepsilon \int_{\Omega} \int_{-\infty}^{\infty} e^{-2\beta t} u_t^2 \, dx \, dt. \tag{3.13}$$

Adding (3.12) and (3.13) gives

$$2\pi \int_{0}^{\infty} \int_{\Omega} e^{-\beta t} u^2 \, dx \, dt$$

$$\leq C(A) \left[ \int_{-\infty}^{\infty} \int_{\Omega} |G|^2 \, dx \, d\alpha + \int_{-\infty}^{\infty} \|H\|_{H^{1/2}(\Gamma)}^2 \, d\alpha \right]$$

$$+ 2\pi \varepsilon \int_{0}^{\infty} \int_{\Omega} e^{-2\beta t} u_t^2 \, dx \, dt.$$
DECAY OF SOLUTIONS ...

as long as $0 < \beta < \delta(A)$. The Lemma now follows from (3.1) and (3.2).

The proof of Theorem 2 in the case $I_0 \neq \emptyset$ is now completed as follows. Recall that $w = u$ for $t \geq T$. Therefore,

$$
\int_0^\infty \int_\Omega e^{-\beta t} w^2 \, dx \, dt \leq \int_0^T \int_\Omega w^2 \, dx \, dt + \int_0^\infty \int_\Omega e^{-\beta t} u^2 \, dx \, dt
$$

$$
\leq C_\varepsilon E(w, 0) + \varepsilon \int_0^\infty \int_\Omega e^{-\beta t} u^2 \, dx \, dt.
$$

The estimate of Theorem 2 now follows immediately.

If $I_0 = \emptyset$, the above proof breaks down at places where Poincaré's inequality is used, namely, in (3.2) and in estimating $\|w\|_{L^\infty(\Omega)}$. However, these difficulties are easily avoided by working with the solution $\tilde{w} = w - I(u_0)$ rather than $w$, and by setting $u = \phi \tilde{w}$. The above proof then leads to the estimate of Theorem 2. In solving (3.3), (3.5) in this case, one must impose the requirement

$$
\int_\Omega U(x, 0) \, dx = 0
$$

to ensure that (3.3), (3.5) has no nontrivial solution when $\omega = 0$.

Proof of Corollary 1. It suffices to assume $E(w_t, 0) < +\infty$. Since

$$
\frac{d}{dt} E(w, t) = -\int_{\Gamma_1} a^{-1} \left( \frac{\partial w}{\partial n} \right)^2 \, d\sigma \leq 0,
$$

we have

$$
E(w, t) = E(w, \infty) + F(w, t) \quad (3.14)
$$
with $F(w, t) \geq 0$. Substituting (3.14) into (1.6) gives

$$\frac{E(w, \infty)}{2\beta} \leq C_\varepsilon [E(w, 0) + E(w, 0)]$$

$$+ \varepsilon \int_0^\infty e^{-2\beta t} \int_\Omega w_t^2 \, dx \, dt$$

$$\leq C_\varepsilon [E(w, 0) + E(w, 0)] + \frac{\varepsilon}{2\beta} E(w, 0).$$

Multiply through by $2\beta$ and then let $\beta \downarrow 0$. There results

$$E(w, \infty) \leq \varepsilon E(w, 0), \quad \forall \varepsilon > 0.$$

**Proof of Corollary 2.** Suppose $\Gamma_0 \neq \emptyset$. From Theorem 2 we have

$$\int_0^\infty \int_\Omega e^{-\beta t} w_t^2 \, dx \, dt \leq C_\varepsilon E(w, 0) + \varepsilon \int_0^\infty \int_\Omega e^{-\beta t} w_{tt}^2 \, dx \, dt. \quad (3.15)$$

Also

$$\int_0^\infty \int_\Omega e^{-2\beta t} |\nabla w|^2 \, dx \, dt$$

$$= \int_0^\infty e^{-2\beta t} \left[ \int_\Omega (-w \Delta w) \, dx + \int_{\Gamma_1} w \frac{\partial w}{\partial \nu} \, d\sigma \right] \, dt$$

$$= -\int_0^\infty e^{-2\beta t} \left[ \int_\Omega w w_t \, dx + \frac{1}{2} \int_{\Gamma_1} a \frac{\partial}{\partial t} (w^2) \, d\sigma \right] \, dt$$

$$= - \left[ e^{-2\beta t} \int_\Omega w w_t \, dx + \frac{1}{2} \int_{\Gamma_1} w^2 \, d\sigma \right]_{t=0}^{\infty}$$

$$+ \int_0^\infty \int_\Omega \frac{\partial}{\partial t} (e^{-2\beta t} w) w_t \, dx \, dt - \beta \int_0^\infty \int_{\Gamma_1} a e^{-2\beta t} w_t^2 \, d\sigma \, dt$$

$$\leq CE(w, 0) + \int_0^\infty \int_\Omega e^{-2\beta t} (w_t^2 + w^2) \, dx \, dt.$$

Equation (1.6) now follows from (1.5), (3.15).

The case $\Gamma_0 = \emptyset$ can be handled with minor modifications as above, since the estimate (3.15) continues to hold in this case provided $E(w, 0)$ is added to $E(w, 0)$ on the right. In the last estimate, $w$ must be replaced by $\bar{w} = w - I(w_0)$ in the integrals on the right.
4. Additional Comments

The requirements placed on $\partial \Omega$ in the opening section of this paper are very restrictive and exclude such simple regions as rectangles, unless $\Gamma_0 = \emptyset$. They are needed, however, to ensure that solutions of (1.1)–(1.3) can be described by a contraction semigroup on $\mathcal{H}^1$ whose generator in the dissipative operator

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ A_n & 0 \end{pmatrix}$$

with domain $D(\mathcal{A}) = \mathcal{H}^2$. This same property is also utilized extensively in [2, 3, 8]. When this obtains $\mathcal{A} - \lambda \mathcal{I}$ (the identity on $\mathcal{H}^1$) must map $\mathcal{H}^2$ one-to-one onto $\mathcal{H}^1$ whenever $\text{Re} \lambda > 0$, hence $(\mathcal{A} - \lambda \mathcal{I})^{-1}$ is a compact operator on $\mathcal{H}^1$. Since $\lambda = 0$ is not an eigenvalue of $\mathcal{A}$ when $\Gamma_0 \neq \emptyset$, it follows that $\mathcal{A}$ itself must map $\mathcal{H}^2$ one-to-one onto $\mathcal{H}^1$. It is not difficult to see that this is equivalent to the following regularity statement: If $G \in L^2(\Omega)$, $H \in H^{1/2}_0(\Gamma)$, and if $U \in H^1(\Omega)$ is a solution in the variational sense (cf. Section 3) to the problem

$$\begin{align*}
A_n U &= G & \text{in } \Omega, \\
U &= 0 & \text{on } \Gamma_0, \\
\frac{\partial U}{\partial v} &= H & \text{on } \Gamma_1,
\end{align*}$$

then $U \in H^1(\Omega)$.

When $\Gamma_0 \cap \Gamma_1 = \emptyset$, the square summability of the second derivatives of solutions of (4.1)–(4.3) in $\Omega$ follows from standard elliptic regularity theory. But in general $U$ will not have this property if $\Gamma_0 \cap \Gamma_1 \neq \emptyset$, even if $\Gamma_0$ and $\Gamma_1$ meet on a smooth part of $\partial \Omega$, as the following example shows.

**Example.** Let $\Omega$ be the semicircle

$$\Omega = \{(x, y) \mid x^2 + y^2 < 1, y > 0\},$$

Let

$$\Gamma_1 = \{(x, y) \mid y = 0, -1 < x < 0\},$$

and let $\Gamma_0$ be the remainder of $\partial \Omega$. Let $H = 0$ in (4.3) and

$$G = -A_2 \left( r^k \sin \frac{\theta}{2} \right), \quad k \geq 2,$$
in (4.1), where \((r, \theta)\) are polar coordinates in \(\Omega\). Since \(A_2(r^{1/2} \sin \theta/2) = 0\) in \(y > 0\), the solution of (4.1)–(4.3) with this choice of \(G, H\) is

\[
U = r^{1/2} \sin \frac{\theta}{2} - r^k \sin \frac{\theta}{2},
\]

and it is easy to see that \(U \in H^1_{T_0}(\Omega)\) but \(U_{xx} \notin L^2(\Omega)\), no matter how smooth \(G\) is, i.e., no matter how large \(k\) is. Note, however, that \(U_{xx} \in L^p(\Omega)\) for \(p < 4/3\). From the results of E. Shamir [11] we know that in general this is the most regularity that one can expect for solutions of (4.1)–(4.3) when \(\bar{T}_0 \cap \bar{T}_1 \neq \emptyset\). Similarly, the presence of nonconvex corners in \(\partial \Omega\) will also prevent solutions from having the required regularity (see, e.g., M. S. Hanna and K. T. Smith [4]).

With \(\Omega, \Gamma_0, \Gamma_1\) as in the last example, a solution of (1.1)–(1.3) (with \(a(x) \equiv 0\)) for which \((u(\cdot, 0), u_t(\cdot, 0)) \in \mathbb{H}^2\) but \((u(\cdot, t), u_t(\cdot, t)) \notin \mathbb{H}^2\) for almost all \(t > 0\) is provided by

\[
u(x, t) = r^{-1/2} \sin \pi r \sin \pi t \sin \theta/2.
\]

5. Generalization

In this section we detail the modifications of the proofs given above needed to treat the generalized wave process (1.7)–(1.9). Here \(\ell\) is a vector field satisfying the conditions (a)–(d) listed in the same paragraph as (1.7)–(1.9).

In the proof of Theorem 1, the functional denoted by \(Q(t)\) should be defined as (when \(\Gamma_0 \neq \emptyset\))

\[
Q(t) = (\alpha + 1) tE(w, t) + \int_{\Omega} \left[ 2\rho w_i(\ell \cdot \nabla w) + (\ell_{ii} - 1) \rho w_{ii} \right] dx,
\]

with \(E(w, t)\) defined by (1.10). The calculation of \(\dot{Q}(t)\) is now based on the identity

\[
\frac{\partial}{\partial t} \left[ 2tE(w, t) + 2\rho w_i(\ell \cdot \nabla w) + (\ell_{ii} - 1) \rho w_{ii} \right]
= \text{div} \left[ 2tw_iA_{ij}w_j + 2(\ell \cdot \nabla w) A_{ij}w_i \ight. \\
+ (\ell_{ii} - 1) wA_{jk}w_k + \rho w_i^2\ell - qw^2\ell \\
- (A_{ij}w_iw_j)\ell \right] - (\ell \cdot \nabla \rho) w_i^2 + (\ell \cdot \nabla q) w^2 \\
+ 2qw^2 - (\ell_{ii}A_{ij}w_k)w \\
- [2A_{ik}(\ell_{jk} - \delta_{jk}) - A_{ijk}\ell_k] w_iw_j,
\]
valid for sufficiently smooth solutions of (1.7). Using hypotheses (a)–(d) of Section 1 and noting that on $\Gamma_0$ one has $v \cdot (A\nabla w) = (A_{ij} v_i v_j) \delta w/\delta v$, one can show as before that

$$\dot{Q}(t) \leq -\delta E(w, t) + C \int_{\Omega} w^2 \, dx, \quad t \geq t_0,$$

for some positive constants $\delta$, $C$, and $t_0$. The remainder of the proof of Theorem 1 for the general case proceeds as before.

In the proof of Theorem 2, the result of taking the Fourier transform of $u = \phi w$ is the boundary value problem

\begin{align*}
\nabla \cdot (A(x) \nabla U) + \omega^2 \rho(x) U - q(x) U &= G \quad \text{in } \Omega, \quad (5.1) \\
U &= 0 \quad \text{on } \Gamma_0, \quad (5.2) \\
v \cdot (A \nabla U) + i\omega U &= H \quad \text{on } \Gamma_0 \quad (5.3)
\end{align*}

rather than (3.3)–(3.5). The corresponding bilinear form $b_\omega(U, V)$ on $L^2(\Omega)$ with domain $D(b_\omega) = H^1_0(\Omega)$ is now given by

$$b_\omega(U, V) = \int_{\Omega} [(A \nabla U) \cdot \nabla \bar{V} - \omega^2 \rho U \bar{V} + q U \bar{V}] \, dx + i\omega \int_{\Gamma_0} a U \bar{V} \, d\sigma.$$

Lemma 1, which is the key step in the proof of Theorem 2, is proved as before, with one change. In verifying that (5.1)–(5.3) with $G = H = 0$ has no real eigenvalues, we can no longer appeal to analyticity of solutions of (5.1) to conclude that $U \equiv 0$ from the fact that $U = 0$ in the open set $\Omega_{\rho} - \Omega$. Instead, we invoke the unique continuation theorem of N. Aronszajn [1], which is valid under the conditions on the coefficients in (5.1) stated in Section 1. The remainder of the proof of Theorem 2 is unchanged.

Corollaries 1 and 2 are proved as before except for obvious substitutions, e.g., $\nabla \cdot (A \nabla)$ for $\Delta$.

**References**


