For a primitive generalized tournament matrix, we present upper and lower bounds on an entry in its Perron vector in terms of the corresponding row sum of the matrix. The bounds are then used to help prove that if \( n \) is even and sufficiently large, any tournament matrix of order \( n \) which maximizes the Perron value must be almost regular. Throughout, we use both analytic and combinatorial techniques.

1. INTRODUCTION AND PRELIMINARIES

A directed graph on \( n \) vertices is called a tournament if it is loop free and has the property that for any pair of distinct vertices \( i \) and \( j \) with \( 1 \leq i, j \leq n \) either \( i \) dominates \( j \) or \( j \) dominates \( i \), but not both. A tournament matrix is just the adjacency matrix of a tournament; equivalently, \( T \) is a tournament matrix if and only if \( T \) is \((0, 1)\) and satisfies \( T + T^T = J - I \), where \( J \) is the all ones matrix and \( I \) is the identity matrix.

Since a tournament matrix is an example of a nonnegative matrix, an application of the Perron-Frobenius theorem (see [4]) shows that its spectral...
radius is an eigenvalue, called the **perron value**. One of the standard facts concerning tournament matrices is that the perron value of any tournament matrix of order $n$ is bounded above by $(n - 1)/2$, and that if equality holds then necessarily $n$ must be odd (see [1]). We remark that whenever $n$ is odd, there exists a tournament matrix of order $n$ with perron value $(n - 1)/2$. This raises the following natural question: if $n$ is even, which tournament matrix (or matrices) of order $n$ yield the maximum perron value?

In 1983, Brualdi and Li [2] conjectured that the tournament matrix of even order with largest perron value can be described as

$$
\begin{bmatrix}
T & T^T \\
T^T + I & T
\end{bmatrix},
$$

where $T$ is the matrix of order $n/2$ with zeros on and below the main diagonal and ones above the main diagonal. Friedland [3] has since named that matrix the **Brualdi-Li** matrix. The following result, which appears in [5], discusses the asymptotic behavior of the perron value of the Brualdi-Li matrix. [We remark that throughout this paper, the notation $O(n^{-k})$ is used to denote a sequence $x_n$ such that $n^k x_n$ is a bounded sequence.]

**Proposition 1.** Let $r_n$ be the perron value of the Brualdi-Li matrix of order $n$. Then

$$
 r_n = \frac{n - 1}{2} - \frac{e^2 - 1}{2(e^2 + 1)n} + O(n^{-3}),
$$

where $e$ is the constant $2.718 \ldots$.

Observe that half of the row sums of the Brualdi-Li matrix are $(n - 2)/2$, while the other half are $n/2$. Any tournament matrix of even order with those row sums is called **almost regular**. In this paper we prove the following theorem (which is weaker than the Brualdi-Li conjecture):

**For all sufficiently large even $n$, a tournament matrix of order $n$ with maximum perron value is almost regular.**

In particular, this improves upon a result of Friedland [3] which states that for even $n$, a tournament matrix of order $n$ with maximum perron value has at least one row whose sum is either $n/2$ or $(n - 2)/2$. 


In some of what follows, we will work with the notion of a generalized tournament matrix, that is, a square nonnegative matrix $M$ such that $M + M^T = J - I$. For a generalized tournament matrix $M$, its score vector is the vector of row sums, $MI$, where $I$ is the all ones vector. Observe that if $M$ is $(0,1)$, so that it is a tournament matrix, then the $i$th entry in its score vector, the score of vertex $i$, is just the number of vertices dominated by vertex $i$ in the corresponding tournament. Suppose that we have a primitive generalized tournament matrix $M$ of order $n$ with score vector $s$, perron value $p$, and right perron vector $v$. Using the equation $M - M^T = J - I$, it is not difficult to show that $s^T I = n(n - 1)/2$, that $(2p + 1)v^Tv = (v^T I)^2$, and that $v^T s = (n - 1 - p)v^T I$; we will make use of those facts in the sequel.

2. MAIN RESULTS

Our first result shows how an entry in the perron vector of a tournament matrix can be bounded in terms of the perron value and the score of the corresponding vertex.

**Theorem 1.** Suppose that $T$ is a primitive generalized tournament matrix of order $n$. Let $p$ be the perron value of $T$, and let $v$ be the corresponding perron vector, normalized so that $1^Tv = 1$. If the $i$th row sum of $T$ is $s_i$, then

$$\left[ (n - 1)p^2 + (2p + 1)s_i + s_i(n - s_i - 1) \right] v_i^2$$

$$- 2(p + 1)s_i v_i + \frac{2p - (n - 2)}{2p + 1} s_i + \frac{s_i^2}{2p + 1} \leq 0. \quad (2)$$

**Proof.** By simultaneously permuting the rows and columns of $T$ if necessary, we can assume that $i = 1$. We partition off the first row and column of $T$ to give

$$T = \begin{bmatrix} 0 & r^T \\ I - r & A \end{bmatrix},$$
and we similarly partition $v$ as

$$
\begin{bmatrix}
  v_1 \\
  w
\end{bmatrix}
$$

From the eigenvalue-eigenvector relation, we have $r^Tw = \rho v_1$ and $v_1(1 - r) + Aw = \rho w$, which yield $(1/\rho)[A + (1/\rho)(I - r)r^T]w = w$. Letting $M = (1/\rho)[A + (1/\rho)(1 - r)r^T]$, we see that $M + M^T + (1/\rho)I = (1/\rho)J + (1/\rho^2)(1 - r)r^T + r(1^T - r^T)$. Pre- and postmultiplying this last equation by $w^T$ and $w$ respectively, and using the fact that $Mw = \rho w$, we find that $(2\rho + 1)w^Tw = (w^TI)^2 + (2/\rho)w^T(1 - r)r^Tw$.

Next we consider the vector

$$
x = w - \frac{w^TI}{n - 1} - \frac{n - 1}{(n - 1)r^Tr - s_1^2} \left( w^Tr - \frac{s_1}{n - 1}w^TI \right) \left( r - \frac{s_1}{n - 1}I \right),
$$

and note that $x$ is the projection of $w$ onto the orthogonal complement of the subspace spanned by $I$ and $r$. Since $s_1 = r^TI \geq r^Tr$, we find that

$$
\begin{align*}
  w^Tw - \frac{(w^TI)^2}{n - 1} - \frac{n - 1}{s_1(n - 1 - s_1)} \left( w^Tr - \frac{s_1}{n - 1}w^TI \right)^2 \\
  \geq w^Tw - \frac{(w^TI)^2}{n - 1} - \frac{n - 1}{(n - 1)r^Tr - s_1^2} \left( w^Tr - \frac{s_1}{n - 1}w^TI \right)^2 = x^Tx \geq 0.
\end{align*}
$$

Consequently,

$$
\begin{align*}
  w^Tw - \frac{(w^TI)^2}{n - 1} - \frac{n - 1}{s_1(n - 1 - s_1)} \left( w^Tr - \frac{s_1}{n - 1}w^TI \right)^2 \geq 0.
\end{align*}
$$
Using the facts that \((2p + 1)w^Tw = (w^T1)^2 + (2/p)w^T(1 - r)r^Tw, w^Tr = \rho v_1, \) and \(w^T1 = 1 - v_1,\) it now follows that

\[
(1 - v_1)^2 \left( \frac{1}{2p + 1} - \frac{1}{n - 1} \right) + \frac{2}{\rho(2p + 1)}(1 - v_1 - \rho v_1)\rho v_1 \\
- \frac{(n - 1)}{s_1(n - 1 - s_1)} \left( \rho v_1 - \frac{s_1}{n - 1}(1 - v_1) \right)^2 \geq 0.
\]

This last inequality can now be rearranged to yield (2). \(\blacksquare\)

**Remark.** An analysis of the proof of Theorem 1 reveals that equality holds in (2) if and only if \(r\) is a \((0, 1)\) vector (since we must have \(r^Tr = s_1\)) and \(w\) is a linear combination of \(1\) and \(r\) (since \(x\) must be the zero vector). The example below illustrates that situation.

**Example 1.** Consider the tournament matrix given by

\[
T = \begin{bmatrix}
0 & 0^T & 1^T \\
1 & R_1 & 0 \\
0 & J & R_2
\end{bmatrix},
\]

where \(R_1\) and \(R_2\) are regular tournament matrices of orders \(n - 1 - k\) and \(k\) respectively (note that necessarily \(n\) must be odd). It can be verified that the perron value \(\rho\) of \(T\) satisfies \(\rho^3 - [(n - 3)/2] \rho^3 + [(k - 1)(n - k - 2)/4] \rho - k(n - k - 1) = 0,\) and that the corresponding perron vector is given by

\[
v = \alpha \begin{bmatrix}
1 \\
[\rho - (n - k - 2)/2]^{-1}1 \\
(\rho/k)1
\end{bmatrix},
\]

where \(\alpha = [\rho - (n - k - 2)/2]/[\rho^2 - \rho(n - k - 2)/2 + (n - k)/2].\) In
particular, since the vector consisting of the last $n - 1$ rows of $v$ can be written as a linear combination of

$$ I \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}, $$

we see that equality holds in (2).

We continue with some useful consequences of Theorem 1.

**Corollary 1.1.** Let $T$ be as in Theorem 1. For $1 \leq i \leq n$, we have

$$ s_i(\rho + 1) + \frac{\sqrt{s_i(n - 1 - s_i)}}{2\rho + 1} [s_i(n - 1 - s_i) - 2\rho^3 + (n - 2)\rho^2] \geq v_i $$

$$ \frac{(n - 1)\rho^3 + (2\rho + 1)s_i + s_i(n - 1 - s_i)}{(n - 1)\rho^2 + (2\rho + 1)s_i + s_i(n - 1 - s_i)} \geq s_i(\rho + 1) - \frac{\sqrt{s_i(n - 1 - s_i)}}{2\rho + 1} [s_i(n - 1 - s_i) - 2\rho^3 + (n - 2)\rho^2] $$

In particular, for $1 \leq i \leq n$, one has $s_i(n - 1 - s_i) \geq 2\rho^3 - (n - 2)\rho^2$.

**Proof.** The upper and lower bounds on $v_i$ come from an application of the quadratic formula to the inequality (2), while the lower bound on $s_i(n - 1 - s_i)$ follows from observing that the discriminant of that quadratic must be nonnegative.

**Remark.** It follows from a theorem of Ostrowski (see [6]) that for a generalized tournament matrix $T$, we have (using the notation of Corollary 1.1) $\max_{1 \leq i \leq n} s_i(n - 1 - s_i) \geq \rho^2$. The inequality obtained in Corollary 1.1 bears some resemblance to the Ostrowski inequality, since it implies that $\min_{1 \leq i \leq n} s_i(n - 1 - s_i) \geq \rho^2[2\rho - (n - 2)]$. 
Corollary 1.2. Let $T$ be a primitive generalized tournament matrix of order $n$ with score vector $s$ and perron value $\rho$. Then

\[ s^T s = \frac{n(n-1)^2}{4} - \frac{n}{4} \]

\[ \leq \left( \frac{n-1}{2} - \rho \right) \left( 6\rho^2 - (n-8)\rho + \frac{n^2 - 5n + 8}{2} \right) - \frac{n}{4} \]

\[ + \frac{2}{n(n-1)} \left( \frac{n-1}{2} - \rho \right) (-2\rho + n - 2) \]

\[ \times \left( -2\rho^2 + 2(n-3)\rho + \frac{(n-1)(n+2)}{2} \right). \]

In particular, if

\[ \rho \geq \frac{n-1}{2} - \frac{e^2-1}{2(e^2+1)n} + O(n^{-3}), \]

then for all $i$ we have

\[ \left| s_i - \frac{n-1}{2} \right| = O(n^{1/2}). \]

Moreover, if $T$ is a tournament matrix of even order $n$ and $\rho \geq (n-1)/2 - \frac{e^2-1}{2(e^2+1)n} + O(n^{-3})$, then there are at most

\[ \frac{e^2 - 2}{4(e^2 + 1)} n + O(1) \]

rows $i$ such that

\[ \left| s_i - \frac{n-1}{2} \right| \neq \frac{1}{2}. \]
Proof. Let \( v \) be the right Perron vector of \( T \), normalized so that \( v^T 1 = 1 \). We sum (2) over all \( i \) to find that

\[
(n - 1)\rho^2 v^T v + (2\rho + 1) \sum_{i=1}^{n} s_i v_i^2 + \sum_{i=1}^{n} s_i(n - s_i - 1)v_i^2
\]

\[
- 2(\rho + 1)s^T v + \frac{2\rho - (n - 2)n(n - 1)}{2\rho + 1} + \frac{s^T s}{2\rho + 1} \leq 0.
\]

We note that by Corollary 1.1, \( s_i(n - 1 - s_i) \geq 2\rho^3 - (n - 2)\rho^2 \); also, it follows from the Kantorovich inequality (see [6], for example) that \( \sum_{i=1}^{n} s_i v_i^2 \geq (\sum_{i=1}^{n} s_i v_i)^2 / \sum_{i=1}^{n} s_i \). We thus find that

\[
(n - 1)\rho^2 v^T v + \frac{(2\rho + 1)(s^T v)^2}{n(n - 1)/2} + \left[2\rho^3 + (n - 2)\rho^2\right] v^T v
\]

\[
- 2(\rho + 1)(s^T v) + \frac{2\rho - (n - 2)n(n - 1)}{2\rho + 1} + \frac{s^T s}{2\rho + 1} \leq 0.
\]

Using the facts that \( v^T v = 1/(2\rho + 1) \) and that \( s^T v = n - 1 - \rho \), we arrive at

\[
\frac{(n - 1)\rho^2}{2\rho + 1} + \frac{2(2\rho + 1)(n - 1 - \rho)^2}{n(n - 1)} + \frac{2\rho^3 - (n - 2)\rho^2}{2\rho + 1}
\]

\[
- 2(\rho + 1)(n - 1 - \rho) + \frac{2\rho - (n - 2)n(n - 1)}{2\rho + 1} + \frac{s^T s}{2\rho + 1} \leq 0.
\]

A few manipulations now yield the desired upper bound on \( s^T s - n(n - 1)^2/4 - n/4 \).

Note that

\[
s^T s - \frac{n(n - 1)^2}{4} = \left( s^T - \frac{n - 1}{2} I^T \right) \left( s - \frac{n - 1}{2} \right),
\]
so that for any $i$ we have $|s_i - (n - 1)/2|^2 \leq s^T s - n(n - 1)^2/4$. From our arguments above, we find that if $\rho \geq r_n$, where $r_n$ is given by (1), then we must have

$$s^T s - \frac{n(n - 1)^2}{4} \leq \frac{3(e^2 - 1)}{4(e^2 + 1)} n + O(1),$$

which implies that $|s_i - (n - 1)/2| = O(n^{1/2})$.

Now we suppose that $T$ is a tournament matrix (so that each $s_i$ is an integer), that $n$ is even, and that $\rho \geq r_n$. Under these hypotheses, we have

$$s^T s - \frac{n(n - 1)^2}{4} = \left( s^T - \frac{n - 1}{2} I^T \right) \left( s - \frac{n - 1}{2} I \right) \geq \frac{n}{4}$$

and

$$\frac{e^2 - 1}{2(e^2 + 1)n} + O(n^{-3}) \geq \frac{n - 1}{2} - \rho \geq 0.$$

Consequently, we find that

$$0 \leq s^T s - \frac{n(n - 1)^2}{4} - \frac{n}{4}$$

$$\leq \left( \frac{n - 1}{2} - \rho \right) \left( 6\rho^2 - (n - 8)\rho + \frac{n^2 - 5n + 8}{2} \right) - \frac{n}{4} + O(n^{-1})$$

$$\leq \left( \frac{n - 1}{2} - \rho \right) \left[ \frac{3}{4} n^2 + O(n) \right] - \frac{n}{4} + O(n^{-1})$$

$$\leq \frac{e^2 + 2}{2(e^2 + 1)} n + O(1).$$

Suppose that there are $k$ rows of $T$ such that for the corresponding scores, $|s_i - (n - 1)/2| \neq 1/2$; note that for such a score, we necessarily have $|s_i - (n - 1)/2| \geq 3/2$. Hence $s^T s - n(n - 1)^2/4 - n/4 \geq k(9/4) + (n - k)(1/4) - n/4 = 2k$. Thus, we find that

$$k \leq \frac{e^2 - 2}{4(e^2 + 1)} n + O(1).$$
The following result, while somewhat computational, is a key component in the proof of Theorem 2. We remark that our interest in the quantity \( v_iw_j - v_jw_i \) arises from its connection to perturbation results for the perron value.

**Lemma 1.** Suppose that \( T \) is a primitive generalized tournament matrix of order \( n \) whose perron value \( \rho \) is at least \( (n - 1)/2 - (e^2 - 1)/2(e^2 + 1)n + O(n^{-3}) \). Let the right and left perron vectors be \( v \) and \( w^T \) respectively, normalized so that \( v^Tv = w^Tw = 1 \). If \( T \) has rows sums \( s_i - t \) and \( s_j + t \) for some \( 0 \leq t \leq 1 \), where \( s_i \geq s_j + 2 \), then

\[
v_iw_j - v_jw_i \geq \frac{8n^2}{(n - 1)^5} \left[ 1 - \left( \frac{e^2 - 1}{4(e^2 + 1)} \right)^{1/2} - t \right] + O(n^{-7/2}).
\]

**Proof.** We apply Corollary 1.1 and the fact that \( (n - 1)^2/4 \geq (s_i - t)(n - 1 - s_i + t) \) to find that

\[
v_i \geq \frac{(s_i - t)(\rho + 1) - \sqrt{\frac{(n - 1)^2}{4(2\rho + 1)} \left( \frac{n - 1}{2} - \rho \right) \left( 2\rho^2 + \rho + \frac{n - 1}{2} \right)}}{(n - 1)\rho^2 + (2\rho + 1)(s_i - t) + (s_i - t)(n - 1 - s_i + t)}.
\]

Since Corollary 1.1 applies to \( w^T \) as well, we also have

\[
w_j \geq \frac{(n - 1 - s_j - t)(\rho + 1) - \sqrt{\frac{(n - 1)^2}{4(2\rho + 1)} \left( \frac{n - 1}{2} - \rho \right) \left( 2\rho^2 + \rho + \frac{n - 1}{2} \right)}}{(n - 1)\rho^2 + (2\rho + 1)(n - 1 - s_j - t) + (s_j + t)(n - 1 - s_j - t)},
\]

\[
v_j \leq \frac{(s_j + t)(\rho + 1) + \sqrt{\frac{(n - 1)^2}{4(2\rho + 1)} \left( \frac{n - 1}{2} - \rho \right) \left( 2\rho^2 + \rho + \frac{n - 1}{2} \right)}}{(n - 1)\rho^2 + (2\rho + 1)(s_j + t) + (s_j + t)(n - 1 - s_j - t)},
\]

\[
w_i \leq \frac{(n - 1 - s_i + t)(\rho + 1) + \sqrt{\frac{(n - 1)^2}{4(2\rho + 1)} \left( \frac{n - 1}{2} - \rho \right) \left( 2\rho^2 + \rho + \frac{n - 1}{2} \right)}}{(n - 1)\rho^2 + (2\rho + 1)(n - 1 - s_i + t) + (s_i - t)(n - 1 - s_i + t)}.
\]
Consequently, we find that

\[ v_iw_j \geq \left( \rho + 1 \right)^2 (s_i - t)(n - 1 - s_j - t) - (\rho + 1)(n - 1 + s_i - s_j - 2t) \]

\[ \times \sqrt{\frac{(n - 1)^2}{4(2\rho + 1)} \left( \frac{n - 1}{2} - \rho \right) \left( 2\rho^2 + \rho + \frac{n - 1}{2} \right) + O(n^2)} \]

\[ \times \left\{ \left[ (n - 1)^2 \rho^2 + (s_i - t)(n - 1 - s_i + t) \right] \right. \]

\[ \times \left[ (n - 1)^2 \rho^2 + (s_j + t)(n - 1 - s_j - t) \right] \]

\[ + (n - 1)^2 (2\rho + 1)(n - 1 + s_i - s_j - 2t) + O(n^4) \right\}^{-1} \]

\[ = \left[ \left( \rho + 1 \right)^2 (s_i - t)(n - 1 - s_j - t) - (\rho + 1)(n - 1 + s_i - s_j - 2t) \right] \]

\[ \times \sqrt{\frac{(n - 1)^2}{4(2\rho + 1)} \left( \frac{n - 1}{2} - \rho \right) \left( 2\rho^2 + \rho + \frac{n - 1}{2} \right) \}

\[ \times \left\{ \left[ (n - 1)^2 \rho^2 + (s_i + t)(n - 1 - s_i - t) \right] \right. \]

\[ \times \left[ (n - 1)^2 \rho^2 + (s_j + t)(n - 1 - s_j - t) \right] \]

\[ + (n - 1)^2 (2\rho + 1)(n - 1 + s_i - s_j - 2t) \right\}^{-1} + O(n^{-4}). \]
A similar computation shows that

$$v_j w_i \leq \left[ (\rho + 1)^2 (s_j + t)(n - 1 - s_i + t) + (\rho + 1)(n - 1 - s_i + s_j + 2t) \right.$$

$$\times \sqrt{\frac{(n - 1)^2}{4(2\rho + 1)}} \left( \frac{n - 1}{2} - \rho \right) \left( 2\rho^2 + \rho + \frac{n - 1}{2} \right) \left[ (n - 1)\rho^2 + (s_i - t)(n - 1 - s_i + t) \right]$$

$$\times \left[ (n - 1)\rho^2 + (s_j + t)(n - 1 - s_j - t) \right]$$

$$\left. + (n - 1)\rho^2 (2\rho + 1)(n - 1 - s_i + s_j + 2t) \right]^{-1} + O(n^{-4}).$$

Thus we find that

$$v_i w_j - v_j w_i \geq \frac{1}{d_1 d_2} \left( \left\{ (n - 1)\rho^2 + (s_i - t)(n - 1 - s_i + t) \right\} \right.$$
where

\[
a_1 = (\rho + 1)^2(s_i - t)(n - 1 - s_j - t) - (\rho + 1)(n - 1 + s_i - s_j - 2t)
\]

\[
\times \sqrt{\frac{(n - 1)^2}{4(2\rho + 1)} \left( \frac{n - 1}{2} - \rho \right) \left( 2\rho^2 + \rho + \frac{n - 1}{2} \right)},
\]

\[
a_2 = (\rho + 1)^2(s_j + t)(n - 1 - s_i + t) + (\rho + 1)(n - 1 - s_i - s_j + 2t)
\]

\[
\times \sqrt{\frac{(n - 1)^2}{4(2\rho + 1)} \left( \frac{n - 1}{2} - \rho \right) \left( 2\rho^2 + \rho + \frac{n - 1}{2} \right)},
\]

\[
d_1 = [(n - 1)\rho^2 + (s_i - t)(n - 1 - s_i + t)]
\]

\[
\times [(n - 1)\rho^2 + (s_j + t)(n - 1 - s_j - t)]
\]

\[
+ (n - 1)\rho^2(2\rho + 1)(n - 1 - s_i + s_j + 2t),
\]

\[
d_2 = [(n - 1)\rho^2 + (s_i - t)(n - 1 - s_i + t)]
\]

\[
\times [(n - 1)\rho^2 + (s_j + t)(n - 1 - s_j - t)]
\]

\[
+ (n - 1)\rho^2(2\rho + 1)(n - 1 + s_i - s_j - 2t).
\]

We note that \(a_1\) and \(a_2\) are \(O(n^4)\), while \(d_1d_2\) is \(O(n^{12})\); from Corollary 1.2 we have \(s_j - s_i = O(n^{1/2})\). It now follows that

\[
v_iw_j - v_jw_i \geq \frac{a_1 - a_2}{d_1d_2} \left\{ [(n - 1)\rho^2 + (s_i - t)(n - 1 - s_i + t)]
\]

\[
\times [(n - 1)\rho^2 + (s_j + t)(n - 1 - s_j - t)]
\]

\[
+ (2\rho + 1)(n - 1)^2\rho^2 \right\} + O(n^{-7/2}).
\]
Now

\[ a_1 - a_2 = (\rho + 1)^2 (n - 1) (s_i - s_j - 2t) \]

\[ \geq 2(n - 1)(\rho + 1)^2 \]

\[ \times \left[ 1 - \frac{1}{\rho + 1} \sqrt{\frac{(n - 1)^2}{4(2\rho + 1)} \left( \frac{n - 1}{2} - \rho \right) \left( 2\rho^2 + \rho + \frac{n - 1}{2} \right)} - t \right] \]

the inequality following from the fact that \( s_i \geq s_j + 2 \). As a result, we have

\[ v_i w_j - v_j w_i \]

\[ \geq \frac{1}{d_1 d_2} \left\{ \left[ (n - 1) \rho^2 + (s_i - t)(n - 1 - s_i + t) \right] \right. \]

\[ \times \left[ (n - 1) \rho^2 + (s_j + t)(n - 1 - s_j - t) \right] \]

\[ + (2\rho + 1)(n - 1)^2 \rho^2 \}

\[ \times \left[ 2(n - 1)(\rho + 1)^2 \right] \]

\[ \times \left[ 1 - \frac{1}{\rho + 1} \sqrt{\frac{(n - 1)^2}{4(2\rho + 1)} \left( \frac{n - 1}{2} - \rho \right) \left( 2\rho^2 + \rho + \frac{n - 1}{2} \right)} - t \right] \]

\[ + O(n^{-7/2}) \]
Since \((n - 1)/2 \geq \rho \geq r_n\), where \(r_n\) is given by (1), a straightforward computation shows that

\[
\frac{1}{\rho + 1} \sqrt{\frac{(n - 1)^2}{4(2\rho + 1)} \left( \frac{n - 1}{2} - \rho \right) \left( 2\rho^2 + \rho + \frac{n - 1}{2} \right)} \leq \sqrt{\frac{n}{2} \left( \frac{n - 1}{2} - \rho \right)} \leq \left( \frac{e^2 - 1}{4(e^2 + 1)} \right)^{1/2} + O(n^{-2}).
\]

Estimating \([2(\rho + 1)^2]/[(n - 1)\rho^4]\), we find that it is bounded below by \(8n^2/(n - 1)^5\). We thus arrive at the desired inequality:

\[
v_iw_j - v_jw_i \geq \frac{8n^2}{(n - 1)^5} \left( 1 - \left( \frac{e^2 - 1}{4(e^2 + 1)} \right)^{1/2} \right) - t \right) + O(n^{-7/2}).
\]

Having established Lemma 1, we are ready to prove one of the main results of this paper.
THEOREM 2. Suppose that \( T \) is a tournament matrix of order \( n \), and that the perron value \( \rho \) of \( T \) is at least
\[
\frac{n - 1}{2} - \frac{e^2 - 1}{2(e^2 + 1)n} + O(n^{-3}).
\]

Let the score vector of \( T \) be \( s \), and suppose that \( s_i > s_j + 2 \) for some indices \( i \) and \( j \) such that \( t_{ij} = 1 \). Construct the tournament matrix \( M \) from \( T \) by setting \( m_{ji} = 1 \), \( m_{ij} = 0 \), and letting \( M \) agree with \( T \) in all other positions. If \( n \) is sufficiently large, then the perron value of \( M \) is bounded below by
\[
\rho + \frac{8n^3}{(n - 1)^5} \left[ \frac{1}{2} - \left( \frac{e^2 - 1}{4(e^2 + 1)} \right)^{1/2} \right] + O(n^{-5/2}).
\]

Proof. First, we note that \( T \) is necessarily primitive whenever \( n \) is sufficiently large, since the spectral radius of a tournament matrix which is not primitive is bounded above by \( (n - 2)/2 \). For each \( 0 \leq t \leq 1 \), let \( T(t) = tM + (1 - t)T = T + t(e_{i}^t e_{j}^T - e_{i} e_{j}^T) \), and note that \( T(0) = T \), while \( T(1) = M \). For \( t \) as above, let \( \rho(t) \) be the perron value of \( T(t) \), and denote the right and left perron vectors by \( v(t) \) and \( w(t) \) respectively, where the vectors are normalized so that \( v(t)^Tv(t) = w(t)^Tw(t) = 1 \).

We note that \( \rho(t) \) is continuous for \( 0 \leq t \leq 1 \). A standard perturbation result (see [7], for example) asserts that if \( \lambda \) is a simple eigenvalue of a matrix \( A \) with right eigenvector \( x \) and left eigenvector \( y^* \), then \( d\lambda/da_{ij} = y_i^* x_j/y^* x \). Since \( T(t) \) is certainly primitive for \( 0 \leq t < 1 \) [so that \( \rho(t) \) is simple], we find that \( \rho \) is differentiable for such \( t \), and it follows that
\[
\frac{d\rho(t)}{dt} = \frac{v_i(t)w_j(t) - v_j(t)w_i(t)}{[v(t)]^T w(t)}.
\]

Fix \( t \) with \( 0 \leq t \leq 1 \), and suppose that
\[
\rho(t) \geq \frac{n - 1}{2} - \frac{e^2 - 1}{2(e^2 + 1)n} + O(n^{-3}).
\]
Since
\[
[v(t)]^T w(t) = \frac{1}{n} + \left( [v(t)]^T - \frac{1}{n} I^T \right) \left( w(t) - \frac{1}{n} l \right),
\]
we find from the Cauchy-Schwarz inequality that
\[
\left| [v(t)]^T w(t) - \frac{1}{n} \right| \leq \left( [v(t)]^T v(t) - \frac{1}{n} \right)^{1/2} \left( w(t) - \frac{1}{n} \right)^{1/2}
\]
\[
= \frac{n - 1 - 2 \rho(t)}{n[2 \rho(t) + 1]} = O(n^{-3}).
\]
Since \( v(t) - (1/n) l = O(n^{-3/2}) \), it now follows that
\[
\frac{v_i(t)w_j(t) - v_j(t)w_i(t)}{[v(t)]^T w(t)} = n \left( [v_i(t)w_j(t) - v_j(t)w_i(t)] + O(n^{-3}) \right).
\]

Next, suppose that for some \( 0 < t \leq 1 \) we have \( \rho(t) \leq \rho(0) \), and let \( \tau = \inf\{t \mid 0 < t \leq 1 \text{ and } \rho(t) \leq \rho(0)\} \); note that necessarily \( \tau > 0 \) because when \( t = 0 \), we find from Lemma 1 that
\[
\frac{d \rho(t)}{dt} \left| [v(t)]^T w(t) \right|_{t=0} = v_i(0)w_j(0) - v_j(0)w_i(0)
\]
\[
\geq \frac{8n^2}{(n - 1)^5} \left[ 1 - \left( \frac{e^2 - 1}{4(e^2 + 1)} \right)^{1/2} \right] + O(n^{-7/2}),
\]
and this last member is positive when \( n \) is sufficiently large. Then for all \( 0 \leq t \leq \tau \) we have \( \rho(t) \geq \rho(0) \geq r_n \), where \( r_n \) is given by (1). Thus,
\[
0 = \rho(\tau) - \rho(0) = \int_0^\tau \frac{v_i(t)w_j(t) - v_j(t)w_i(t)}{[v(t)]^T w(t)} dt
\]
\[
= n \int_0^\tau [v_i(t)w_j(t) - v_j(t)w_i(t)] dt + O(n^{-3}).
\]
From Lemma 1, we have

\[ v_i(t)w_j(t) - v_j(t)w_i(t) \geq \frac{8n^2}{(n-1)^5} \left[ 1 - \left( \frac{e^2 - 1}{4(e^2 + 1)} \right)^{1/2} - t \right] \]

\[ + O(n^{-7/2}) \]

whenever \( 0 \leq t \leq \tau \), so that

\[ 0 = \rho(\tau) - \rho(0) \]

\[ \geq n \int_0^\tau \frac{8n^2}{(n-1)^5} \left[ 1 - \left( \frac{e^2 - 1}{4(e^2 + 1)} \right)^{1/2} - t \right] dt + O(n^{-5/2}) \]

\[ = \frac{8n^3}{(n-1)^5} \tau \left[ 1 - \left( \frac{e^2 + 1}{4(e^2 + 1)} \right)^{1/2} - \frac{\tau}{2} \right] + O(n^{-5/2}) , \]

a contradiction, since this last quantity is positive for sufficiently large \( n \).

As a result, we must have \( \rho(t) > \rho(0) \) for all \( 0 \leq t \leq 1 \). Using Lemma 1 again to estimate \( v_i(t)w_j(t) - v_j(t)w_i(t) \) from below, it now follows that

\[ \rho(1) - \rho(0) = \int_0^1 \frac{d\rho(t)}{dt} dt \geq \frac{8n^3}{(n-1)^5} \left[ \frac{1}{2} - \left( \frac{e^2 - 1}{4(e^2 + 1)} \right)^{1/2} \right] \]

\[ + O(n^{-5/2}) . \]

Next we use Theorem 2 and Corollary 1.2 to obtain the following result.

**Theorem 3.** Suppose that \( n \) is even and that \( T \) is a tournament matrix of order \( n \) with maximum Perron value. If \( n \) is sufficiently large, \( T \) must be almost regular.

**Proof.** By Proposition 1, since \( T \) has maximum Perron value, that Perron value must be at least \( r_n \), where \( r_n \) is given by (1). Suppose that \( T \) is not almost regular. If \( T \) has two scores \( s_i \) and \( s_j \) with \( s_i \geq s_j + 2 \), then by Theorem 2, \( t_{ji} \) must be 1; otherwise, by Theorem 2, for \( n \) sufficiently large
we could find another tournament matrix whose perron value exceeded that of $T$. So if $T$ has a score $s_k$ with $s_k > n/2 + 2$, we see that in the tournament corresponding to $T$, vertex $k$ can only dominate a vertex whose score exceeds $n/2 + 1$, and that there must be at least $n/2 + 2$ such vertices. But then $T$ has at least $n/2 + 2$ rows $i$ such that $|s_i - (n - 1)/2| 
eq 1/2$, which is a contradiction to Corollary 1.2 when $n$ is sufficiently large. A similar argument applies if $T$ has a score $s_k$ with $s_k \leq (n - 2)/2 - 2$, so we find that $T$ can only have scores $n/2 + 1$, $n/2$, $(n - 2)/2$, and $(n - 2)/2 - 1$.

Suppose that $T$ has a score of $n/2 + 1$. It follows that at least $n/2 + 1$ scores of $T$ are either $n/2 + 1$ or $n/2$. But in that case, there can be no score equal to $(n - 2)/2 - 1$; otherwise the corresponding vertex would have to dominate all vertices with score $n/2 + 1$ or $n/2$, of which there are at least $n/2 + 1$, a contradiction. Hence $T$ can only have scores $n/2 + 1$, $n/2$, and $(n - 2)/2$. If there are $m$ scores equal to $n/2 + 1$, it follows that there must be $n/2 - 2m$ scores equal to $n/2$, and $n/2 + m$ scores equal to $(n - 2)/2$. Since a vertex with score $n/2 + 1$ must be dominated by every vertex with score $(n - 2)/2$, we find that a vertex with score $n/2 + 1$ can dominate at most $n/2 - m$ vertices, another contradiction.

Hence $T$ has no score of $n/2 + 1$, and a similar argument shows that $T$ can have no score of $(n - 2)/2 - 1$. Consequently, $T$ must be almost regular, as desired.

REFERENCES


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