Riccati Type Transformations for Second-Order Linear Difference Equations, II

MAN KAM KWONG

Department of Mathematics, Northern Illinois University,
DeKalb, Illinois 60115

AND

JOHN W. HOOKER AND WILLIAM T. PATULA

Department of Mathematics, Southern Illinois University at Carbondale,
Carbondale, Illinois 62901

Submitted by G. Milton Wing


1. INTRODUCTION

Oscillation and non-oscillation criteria and comparison theorems for linear difference equations have been investigated in several recent papers [3–8]. Such results have also been obtained as specializations of results concerning generalized differential equations which contain both differential equations and difference equations [9–10]. In this paper we continue our study of the second-order linear difference equation

\[ c_n x_{n+1} + c_{n-1} x_{n-1} = b_n x_n, \quad n = 1, 2, \ldots \]  

with \( c_n > 0 \) for all \( n \geq 0 \). We employ various Riccati type transformations found in [6] to transform (1.1) into first order Riccati type equations.

Section 1 consists of a brief introduction and review of relevant material. Section 2 discusses a necessary condition for non-oscillation and then presents several sufficient conditions for oscillation. Section 3 deals with oscillation results related to the Leighton–Wintner criterion for ordinary differential equations. Section 4 presents some comparison theorems.
Equation (1.1) is equivalent to the self-adjoint equation

$$-\Delta(c_{n-1}\Delta x_{n-1}) + a_n x_n = 0, \quad n = 1, 2, \ldots, \tag{1.2}$$

where $a_n = b_n - c_n - c_{n-1}$ and the forward difference operator $\Delta$ is defined by $\Delta x_n = x_{n+1} - x_n$. A non-trivial solution of (1.1) or (1.2) is called oscillatory if for every $N > 0$ there exists an $n \geq N$ such that $x_n x_{n+1} \leq 0$. If one non-trivial solution of (1.1) or (1.2) is oscillatory then all non-trivial solutions are oscillatory, so the equation itself can be classified as being oscillatory or non-oscillatory. For these and additional properties of difference equations and other relevant information, we refer the reader to the books of Atkinson [1] and Fort [2] and to [6]. We note here also that the letters $m, n, M, N, P, i, j$ below always represent non-negative integers.

Elementary consideration of signs in (1.1) implies that if $b_n < 0$ for a sequence $n_k \to \infty$, then (1.1) is oscillatory. See [7, Lemma 3] or [6, Theorem 1]. Therefore we always assume the following condition in addition to our assumption that $c_n > 0$ for all $n \geq 0$.

**ASSUMPTION.** In (1.1) or (1.2), $b_n > 0$ for $n > 0$.

Suppose that $\{x_n\}, n \geq 0$, is a solution of (1.1) such that $x_n > 0$ for all $n \geq N$ for some $N$. The substitution $r_n = x_{n+1}/x_n$, $n \geq N$, leads to the first order non-linear difference equation

$$c_n r_n + c_{n-1}/r_{n-1} = b_n, \quad n > N. \tag{1.3}$$

Similarly, if we let $z_n = c_n x_{n+1}/x_n$, $n \geq N$, then $z_n$ satisfies

$$z_n + c_{n-1}^2/z_{n-1} = b_n, \quad n > N. \tag{1.4}$$

If we let $s_n = (b_{n+1} x_{n+1})/(c_n x_n), n \geq N$, then $s_n$ satisfies

$$q_n s_n + 1/s_{n-1} = 1, \quad n > N, \tag{1.5}$$

where $q_n = c_n^2/(b_n b_{n+1})$. In [6] the following theorem is proved.

**Theorem 1.1.** The following conditions are equivalent:

(i) Equation (1.1) is non-oscillatory.

(ii) Equation (1.3) has a positive solution $\{r_n\}, n \geq N$, for some $N > 0$.

(iii) Equation (1.4) has a positive solution $\{z_n\}, n \geq N$, for some $N > 0$.

(iv) Equation (1.5) has a positive solution $\{s_n\}, n \geq N$, for some $N > 0$. 

Equations (1.3), (1.4), and (1.5) we designate as difference equations of Riccati type, or, more simply, as Riccati difference equations related to (1.1). For further discussion of this terminology, see [6, p. 452–453].

2. OSCILLATION AND NON-OSCILLATION CRITERIA

Consider Eq. (1.1) written in the form (1.5), where \( q_n = \frac{c_n^2}{b_n b_{n+1}} \) for all \( n \geq 0 \). In [6, Theorems 5 and 6] the following two results were proved.

**THEOREM 2.1.** If \( q_n \geq \frac{1}{4-\varepsilon} \) for some \( \varepsilon > 0 \) and for all sufficiently large \( n \), then (1.1) is oscillatory.

**THEOREM 2.2.** If \( q_n \leq \frac{1}{4} \) for all sufficiently large \( n \), then (1.1) is non-oscillatory.

Theorems 2.1 and 2.2 together imply that the constant \( 1/4 \) is the best possible. An example [6, p. 455–6] also shows that the condition of Theorem 2.1 cannot be weakened to \( q_n \geq \frac{1}{4-\varepsilon_n} \) with \( \varepsilon_n \to 0 \) as \( n \to \infty \).

Even though Theorems 2.1 and 2.2 seem to be nearly complementary, it turns out that more can be said. Specifically, we have the following necessary condition for non-oscillation.

**THEOREM 2.3.** Suppose (1.1) is non-oscillatory. Then there exists \( N > 0 \) such that for any \( n \geq N \) and any \( m \geq 0 \),

\[
q_n q_{n+1} \cdots q_{n+m} < 4^{-m}. \tag{2.1}
\]

**Proof.** Let (1.1) be non-oscillatory and let \( \{x_n\} \) be a solution of (1.1) such that \( x_n \neq 0 \) for \( n \geq N \). Let \( z_n = c_n x_{n+1}/x_n \), \( n \geq N \). Then from (1.4) we may write

\[
b_n b_{n+1} = (z_n + c_{n-1}^2/z_{n-1})(z_{n+1} + c_n^2/z_n)
= c_n^2 (1 + 1/\alpha_{n-1})(1 + \alpha_n),
\]

where \( \alpha_n = z_n x_{n+1}/c_n^2 > 0 \).

Similarly, \( b_{n+1} b_{n+2} = c_{n+1}^2 (1 + \alpha_n)(1 + \alpha_{n+1}) \). It follows that

\[
b_n b_{n+1} b_{n+2} = c_n^2 c_{n+1}^2 (1 + 1/\alpha_{n-1})(1 + \alpha_n)(1 + 1/\alpha_n)(1 + \alpha_{n+1})
\geq c_n^2 c_{n+1}^2 (1 + 1/\alpha_{n-1}) 4(1 + \alpha_{n+1}).
\]
Proceeding inductively, we obtain

\[ b_n b_{n+1} \cdots b_{n+m} b_{n+m+1} \geq c_n^2 \cdots c_{n+m}^2 (1 + 1/x_{n-1}) 4^m (1 + x_{n+m}) \]

\[ > 4^m c_n^2 \cdots c_{n+m}^2. \]

Since this is equivalent to (2.1), the proof is complete.

Theorem 2.1 now becomes a corollary of Theorem 2.3. Specifically, if \( q_n > 1/(4-\epsilon) \) for all \( n \geq N \), then \( q_n q_{n+1} \cdots q_{n+m} \geq 1/(4-\epsilon)^{m+1} > 4^{-m} \), if \( n \geq N \) and \( m \) is large enough. Thus Eq. (1.1) is oscillatory.

Another corollary is the following result which is related to Theorem 8 of [6].

**Corollary 2.4.** If \( \lim \inf c_n 4^{-n} = 0 \) and \( \prod_{i=1}^{n} b_i / \prod_{i=1}^{n} c_i \) is bounded, say by \( M \), as \( n \to \infty \), then (1.1) is oscillatory.

**Proof.** If (1.1) is non-oscillatory, Theorem 2.3 implies that for some \( N > 0 \) and all \( m \geq 0 \),

\[ \frac{b_{N+m} \cdots b_{N+1} b_{N}}{c_{N+m} \cdots c_{N+1} c_{N}} \geq 4^m. \quad (2.2) \]

However, the left side of (2.2) is bounded above by \( M^2 c_{N+m+1}/b_N \). Thus we have

\[ M^2 c_{N+m+1}/b_N \geq 4^m, \quad m \geq 0, \quad (2.3) \]

which implies \( c_{N+m+1} 4^{-(N+m+1)} \geq b_N/(M^2 4^{N+1}) \) for all \( m \geq 0 \), a contradiction of our hypotheses.

The argument above also affords an alternate proof of Theorem 8 of [6], which is identical with Corollary 2.4 except that the condition \( \lim \inf \ c_n 4^{-n} = 0 \) is replaced by the condition \( \sum_{n=1}^{\infty} c_n^{-1} = \infty \). Specifically, (2.3) above implies \( 1/c_{N+m+1} \leq (M^2/b_N) 4^{-m} \), which contradicts \( \sum_{n=1}^{\infty} c_n^{-1} = \infty \).

Condition (2.1) is a necessary condition for non-oscillation. The following example shows that it is not a sufficient condition.

**Example 2.1.** Consider (1.1) with \( c_n = 1 \) for all \( n \geq 0 \), \( b_{2n} = 2^{-1} 4^{2-n} \), and \( b_{2n-1} = 4^{n-1} \) for all \( n \geq 1 \). It is easy to see that \( q_{2n-1} = 1/2 \) and \( q_{2n} = 1/8, n \geq 1 \). We also note that for any positive \( n \) and any \( m \geq 0 \), condition (2.1) is satisfied. However, we claim that with this definition of \( b_n \) and \( c_n \), Eq. (1.1) is oscillatory.

Suppose not. Then (1.5) has a positive solution \( \{ s_n \} \) defined for all \( n \) sufficiently large. By (1.5), \( 1/s_{n-1} \leq 1 \), hence

\[ s_{n-1} \geq 1. \quad (2.4) \]
Since \( q_{2n} = 1/8 \) and \( q_{2n-1} = 1/2 \), (1.5) implies that
\[
s_{2n} = 8(1 - 1/s_{2n-1}),
\]
and
\[
s_{2n-1} = 2(1 - 1/s_{2n-2}).
\]
Substitution of (2.6) in (2.5) yields
\[
s_{2n} = 4\left(\frac{(s_{2n-2} - 2)}{(s_{2n-2} - 1)}\right) = 4 - 4/(s_{2n-2} - 1).
\]
Since \( s_{2n-2} > 1 \) and \( s_{2n} > 1 \) by (2.4), then (2.7) implies that \( s_{2n-2} > 2 \) and \( s_{2n} < 4 \). Thus
\[
2 < s_i < 4, \quad \text{if } i \text{ is even and sufficiently large.} \quad (2.8)
\]
It follows from (2.7) and (2.8) that
\[
4 - 4/(s_i - 1) > 2
\]
for all even \( i \) sufficiently large. It is readily shown that (2.9) implies \( s_i > 3 \), hence, from (2.7),
\[
4 - 4/(s_i - 1) > 3 \quad (2.10)
\]
for all even \( i \) sufficiently large; (2.10) in turn implies that \( s_i > 5 \), which contradicts (2.8). Thus (1.1) in this example must be oscillatory.

We turn now to a corollary of Theorem 2.1 which is related to Theorem 3 of \cite{6}.

**Corollary 2.5.** If \( b_n \leq c_{n-1} \) and \( c_n/c_{n-1} \geq 1/(4 - \varepsilon) \) for some \( \varepsilon > 0 \) and for all sufficiently large \( n \), then (1.1) is oscillatory.

**Proof.** We have
\[
q_n = \frac{c_n^2}{b_nb_{n+1}} - \frac{c_n^2c_{n-1}}{b_nb_{n+1}c_{n-1}} - \frac{c_n}{b_{n+1}} - \frac{c_{n-1}}{b_n} - \frac{c_n}{c_{n-1}} = \frac{1}{4 - \varepsilon}.
\]
The result now follows from Theorem 2.1.

It is also of interest to compare this corollary with Theorems 7 and 8 and Corollaries 3 and 4 of \cite{7}, since all these results depend upon the condition \( b_n \leq c_{n-1} \), or similar conditions.

A corollary similarly related to Theorem 2.2 is the following:

**Corollary 2.6.** If \( b_n \geq c_{n-1} \) and \( c_n/c_{n-1} \leq 1/4 \) for all sufficiently large \( n \), then (1.1) is non-oscillatory.
The proof is immediate. The following example shows that the constant \(1/4\) is best possible.

**Example 2.2.** Let \(b_n = c_{n-1} = 1/4^{n-1}\). Then (1.1) is non-oscillatory, since \(x = 2^n\) is a solution.

The above example also shows that the condition \(b_n < c_{n-1}\) is not in itself sufficient to imply oscillation.

Similarly, we note that the condition \(c_n/c_{n-1} < 1/4\) in Corollary 2.6 is not sufficient for non-oscillation. This can be seen by the example \(c_n = 4^{-n}, b_n = c_{n-1}/2\). Here \(c_n/c_{n-1} = 1/4\), but \(q_n = 1\) for all \(n\), hence (1.1) is oscillatory by Theorem 2.1.

### 3. Further Oscillation Criteria

In this section we prove an oscillation criterion for (1.1) with \(c_n \equiv 1\), and then go on to discuss some related criteria for the case \(c_n \not\equiv 1\), all involving the condition \(\sum_{n=1}^{\infty}(b_n - c_n - c_{n-1}) = -\infty\).

Assume \(c_n \equiv 1\) in (1.1). Then (1.1) can be written as

\[
x_{n+1} + x_{n-1} = (a_n + 2)x_n,
\]

where \(a_n = b_n - 2, n = 1, 2, ...,\) and the alternate form (1.2) becomes

\[
-A^2 x_{n-1} + a_n x_n = 0.
\]

By Theorem 1.1, Eq. (3.1) is non-oscillatory if and only if the related Riccati equation

\[
r_n + 1/r_{n-1} = a_n + 2
\]

has a solution \(r_n\) defined for all sufficiently large \(n\). In the lemma below we will compare solutions of (3.3) with solutions of an equation of the same form, in which the coefficients \(a_n\) are replaced by coefficients \(\alpha_n\) defined as follows. For any fixed integer \(M > 1\), let

\[
\alpha_n = a_n, \quad n \leq M - 1,
\]

\[
\alpha_M = a_M + a_{M+1},
\]

\[
\alpha_n = a_{n+1}, \quad n \geq M + 1.
\]

For such a sequence of coefficients, we consider the equation

\[
u_{n+1} + u_{n-1} = (\alpha_n + 2) u_n
\]
and the related Riccati equation

\[ \rho_n + 1/\rho_{n-1} = x_n + 2. \]  \hspace{1cm} (3.6)

**Lemma 3.1.** Suppose (3.1) is non-oscillatory, and let \( \{x_n\} \) be a solution of (3.1) such that \( x_n > 0 \) for \( n \geq N - 1 \), for some \( N \geq 1 \). For any fixed \( M > N \), define the sequence \( \{a_n\} \) as in (3.4). Then (3.5) is non-oscillatory. Moreover, if \( \{u_n\} \) is the solution of (3.5) satisfying the initial conditions \( u_{M-1} = x_{M-1} \) and \( u_M = x_M \), then \( u_n > 0 \) for \( n \geq N - 1 \), and the sequence \( \{\rho_n\} \) satisfying \( \rho_n = u_{n+1}/u_n \), \( n \geq N - 1 \), is a solution of (3.6) satisfying

\[ \begin{align*}
\rho_n &= r_n > 0, & N - 1 \leq n \leq M - 1, \\
\rho_n &\geq r_{n+1} > 0, & n \geq M,
\end{align*} \]  \hspace{1cm} (3.7)

where \( r_n = x_{n+1}/x_n \), \( n \geq N - 1 \).

**Proof.** Given a solution \( \{x_n\} \) of (3.1) such that \( x_n > 0 \) for \( n \geq N - 1 \), let \( r_n = x_{n+1}/x_n \), \( n \geq N - 1 \), so that \( \{r_n\} \) is a solution of (3.3). Let \( \{u_n\} \) be the solution of (3.5) as stated. Since \( u_{M-1} = x_{M-1} \), \( u_M = x_M \), and \( a_n = a_n \) for \( n \leq M - 1 \), it is clear from (3.1) and (3.5) that \( u_n = x_n \) for \( n \leq M \). Thus \( u_n > 0 \) for \( N - 1 \leq n \leq M \), so we may define

\[ \rho_n = u_{n+1}/u_n, \hspace{1cm} N - 1 \leq n \leq M. \]

Then \( \rho_n > 0 \) for \( N - 1 \leq n \leq M - 1 \). Also, dividing (3.5) by \( u_n \), we see that \( \rho_n \) satisfies (3.6) for \( N \leq n \leq M \). We need to show that \( \rho_M > 0 \), so that (3.6) can be used to define \( \rho_{M+1} \). To show this, we first write Eq. (3.3) for \( n = M \) and \( n = M + 1 \) and add the results to obtain

\[ r_{M+1} = a_M + a_{M+1} + 4 - (r_M + 1/r_M) - 1/r_{M-1}. \]  \hspace{1cm} (3.8)

Now \( u_n = x_n \) for \( n \leq M \), so, in particular \( r_{M-1} = \rho_{M-1} \). Also \( r_M + 1/r_M \geq 2 \). Thus (3.4) and (3.8) imply that

\[ r_{M+1} \leq a_M + 2 - 1/\rho_{M-1} = \rho_M. \]

Thus

\[ r_{M+1} \leq \rho_M, \]  \hspace{1cm} (3.9)

and since \( r_{M+1} > 0 \), we have \( \rho_M > 0 \).

We may therefore define \( \rho_{M+1} \) by (3.6), i.e.,

\[ \rho_{M+1} = a_{M+1} + 2 - 1/\rho_M. \]  \hspace{1cm} (3.10)
It follows from (3.5), (3.10), and the definition of $\rho_M$ that $\rho_{M+1} = u_{M+2}/u_{M+1}$. Also, (3.3), (3.4), and (3.9) imply that

$$r_{M+2} = a_{M+2} + 2 - 1/r_{M+1} \leq \alpha_{M+1} + 2 - 1/\rho_M = \rho_{M+1},$$

(3.11)

hence

$$0 < r_{M+2} \leq \rho_{M+1}.$$

(3.12)

Proceeding inductively as in steps (3.10) through (3.12), we conclude that $\rho_n$ is defined for all $n \geq N - 1$ and satisfies (3.7), which completes the proof.

**Theorem 3.2.** Let $\{a_n\}$, $n \geq 1$, be a sequence with the property that for any $N > 0$ there exist integers $M > N$ and $k \geq 1$ such that

$$a_M + a_{M+1} + \cdots + a_{M+k} \leq -2.$$  

(3.13)

Then (3.1) is oscillatory.

**Proof.** Let $\{a_n\}$ be such a sequence, and suppose that (3.1) is non-oscillatory. Let $\{x_n\}$ be a solution of (3.1) with $x_n > 0$ for $n \geq N - 1$, for some $N \geq 1$. Then the sequence $\{r_n\}$ defined by $r_n = x_{n+1}/x_n$, $n \geq N - 1$, satisfies the Riccati equation (3.3). By hypothesis, we may choose $M > N$ and $k \geq 1$ such that (3.13) holds.

For each $i = 0, 1, 2, \ldots, k$, we define a sequence $\{\alpha_n^i\}$, $n \geq 1$, by setting

$$\alpha_n^0 = a_n, \quad n \geq 1,$$

and for each $i = 1, 2, \ldots, k$, defining

$$\alpha_n^i = a_n, \quad n \leq M - 1,$$

(3.14a)

$$\alpha_n^i = \alpha_{n-M}^i + \alpha_{M+1}^{i-1},$$

(3.14b)

and

$$\alpha_n^i = \alpha_{n-M+1}^{i-1}, \quad n \geq M + 1.$$  

(3.14c)

We consider the difference equations (3.5) and (3.6) with the coefficients $\alpha_n$, replaced by $\alpha_n^i$, $i = 1, 2, \ldots, k$, as follows:

$$u_{n+1} + u_{n-1} = (\alpha_n^i + 2) u_n$$

(3.5i)

and

$$\rho_n + 1/\rho_{n-1} = \alpha_n^i + 2.$$  

(3.6i)

For each $i = 1, 2, \ldots, k$, we let $\{u_n^i\}$, $n \geq 0$, be the solution of (3.5i) satisfying the initial conditions $u_{M-1}^i - x_{M-1}$ and $u_M^i = x_M$. Then repeated application of Lemma 3.1 shows that for each $i = 1, 2, \ldots, k$, Eq. (3.5i) is
non-oscillatory and the sequence $\{\rho_i^j\} = \{u_{n+j}/u_n\}$ is defined for $n \geq N-1$ and is a solution of (3.6i) satisfying
\[ \rho_i^j = \rho_{i-1}^{j-1} > 0, \quad N-1 \leq n \leq M-1, \tag{3.15} \]
and
\[ \rho_i^j \geq \rho_{i+1}^{j-1} > 0, \quad n \geq M, \tag{3.16} \]
where $\rho_n^0 = r_n$, $n \geq N-1$.
It follows that the right-hand side of (3.6i) is positive, hence $\alpha_i^j > -2$ for $i = 1, 2, \ldots, k$ and $n \geq N-1$. However, repeated application of (3.14b) and (3.14c) yields
\[ a_M^r = a_M^0 + a_{M+1}^0 + \cdots + a_{M+k}^0 \]
\[ = a_M + a_{M+1} + \cdots + a_{M+k} \leq -2, \]
by hypothesis, which contradicts $a_M^r > -2$ and concludes the proof.

We note that if $\lim \inf a_k = -\infty$, then the hypotheses of Theorem 3.2 are satisfied. Thus we have as an immediate corollary the following result, which is a discrete analogue for the case $c_n = 1$ of the familiar Leighton-Wintner oscillation criterion for differential equations [11, Theorem 2.24], with the slightly weaker hypothesis "lim inf" instead of "lim."

**Corollary 3.3.** If $\lim \inf a_k = -\infty$, then (3.1) is oscillatory.

Turning to the general case (1.1), where $c_n \neq 1$, recall that (1.1) may be written in the form (1.2), with $a_n = b_n - c_n - c_{n-1}$. The analogue of the Leighton-Wintner criterion for (1.1) also requires $\Sigma 1/c_n = \infty$. This has been proved by Hinton and Lewis [4, Theorem 4], who present the following example to show that the condition $\Sigma a_n = -\infty$ by itself is not sufficient to imply oscillation of (1.1).

**Example 3.1.** Let $c_n = n^2$ and $b_n = n(2n^2 - 1)/(n+1)$, so that $a_n = -1/(n+1)$. Then $y_n = 1/n$ is a solution of (1.1).

It turns out that the behavior of the solution $y_n = 1/n$ in this example is typical for this case, as the following theorem indicates.

**Theorem 3.4.** Suppose $\Sigma \alpha = -\infty$ and (1.1) is non-oscillatory. Then not only must every solution be eventually of one sign but it must be decreasing in absolute value as well.

**Proof.** Suppose that $\Sigma a_n = -\infty$ and (1.1) is not oscillatory. Then any solution $\{x_n\}$ is eventually of one sign, say $x_n > 0$ for all $n \geq N$, or
$x_n < 0$ for all $n \geq N$, for some $N \geq 0$. Let $r_n = x_{n+1}/x_n$, $n \geq N$. Then the Riccati equation (1.3) may be written as

$$c_n(r_n - 1) + c_{n-1}(1/r_n - 1) - a_n, \quad n \geq N + 1.$$  \quad (3.18)

Summing (3.18) from $N + 1$ to $n$ and rearranging terms yields

$$c_n(1 - r_n) = c_N(1/r_N - 1) + \sum_{k=N+1}^{n-1} c_k(r_k - 1)^2/r_k - \sum_{k=N+1}^{n} a_k.$$  \quad (3.19)

Since $\sum_{k=1}^\infty a_k = -\infty$, the right-hand side of (3.19) is positive for all sufficiently large $n$, say $n > K > N + 1$. Therefore $1 - r_n > 0$ for all $n > K$, hence $0 < r_n < 1$, $n > K$. Thus

$$0 < x_{n+1}/x_n < 1, \quad n > K.$$  \quad (3.20)

It follows that $|x_{n+1}| < |x_n|$ for all $n > K$, which completes the proof.

We note, however, that although solutions must eventually be decreasing in absolute value under the hypotheses of Theorem 3.4, it is not true that all solutions tend to 0 asymptotically under these conditions. This can be shown by Example 3.1 above. This example was discussed further in [7] in connection with the notion of recessive and dominant solutions. (See [7] for an explanation of this terminology.) In addition to the solution $y_n = 1/n$ given above, another solution given in [7] is

$$u_n = \frac{1}{n(1 + 1/n)}.$$

The solution $\{y_n\}$ is recessive and $\{u_n\}$ is dominant, i.e., $y_n/u_n \to 0$ as $n \to \infty$. The solution $\{u_n\}$ is eventually decreasing and tends to 1 as $n \to \infty$.

It is proved in [7] that if (1.1) is non-oscillatory it must have a recessive solution and a dominant solution (which may both be taken to be positive). Theorem 3.4 shows that if, in addition, $\sum_{k=1}^\infty a_k = -\infty$, then both of these solutions must eventually be monotone decreasing.

The argument used in the preceding theorem also yields the following lemma, from which various oscillation criteria readily follow.

**Lemma 3.5.** Suppose $\sum_{k=1}^\infty a_k = -\infty$ and suppose for any $N$ there exists $K > N + 1$ such that

$$b_K - c_{K-1} + \sum_{k=N+1}^{K-1} a_k \leq 0.$$  \quad (3.21)

Then (1.1) is oscillatory.
Proof. Assume (1.1) is non-oscillatory and \( \sum a_n = -\infty \). We may use \( a_n = b_n - c_n - c_{n-1} \) to rewrite (3.19) as

\[
c_n - c_n r_n = c_N (1/r_N - 1) + \gamma(N, n) - b_n + c_n + c_{n-1} - \sum_{k=N+1}^{n-1} a_k
\]

where \( \gamma(N, n) > 0 \) is the sum from \( N+1 \) to \( n-1 \) of \( c_k (r_k - 1)^2/r_k \). By Theorem 3.4, we may also assume \( N \) is so large that \( r_N < 1 \), so that \( c_N (1/r_N - 1) > 0 \). Subtracting \( c_n \) from both sides we obtain

\[
-c_n r_n = c_N (1/r_N - 1) + \gamma(N, n) - b_n - c_n + c_{n-1} - \sum_{k=N+1}^{n-1} a_k.
\]

For \( n = K > N+1 \) such that (3.21) holds, the right side of (3.22) is positive, but the left is negative, a contradiction which completes the proof.

As immediate corollaries we have the following theorems.

**Theorem 3.6.** If \( \sum a_n = -\infty \) and \( b_n \leq c_{n-1} \) for all sufficiently large \( n \), then (1.1) is oscillatory.

**Theorem 3.7.** If \( \sum a_n = -\infty \) and \( \{b_n\} \) is a bounded sequence, then (1.1) is oscillatory.

For the case \( c_n \equiv 1 \), we have \( a_n = b_n - c_n - c_{n-1} = b_n - 2 \), so Theorem 3.7 has the immediate corollary:

**Corollary 3.8.** If \( c_n \equiv 1 \), \( \{b_n\} \) is bounded, and

\[
\sum_{n=0}^{\infty} (b_n - 2) = -\infty
\]

then (1.1) is oscillatory.

It is interesting to compare this result with Theorem 6 of Hinton and Lewis [4], where the conditions which imply oscillation of (1.1) are \( c_n \equiv 1 \), \( b_n \leq 2 \),

\[
\sum_{n=0}^{\infty} |b_n - 2| < \infty \quad \text{and} \quad \liminf_{k \to \infty} \sum_{n=k}^{\infty} (b_n - 2) < -1.
\]

Condition (3.24) requires absolute convergence of the series \( \sum (b_n - 2) \), while condition (3.23) calls for divergence to \(-\infty\). Yet in both results, (1.1) is oscillatory.

We now consider some examples in which we can make use of Theorem 3.7 and at the same time illustrate a technique of determining
oscillation or non-oscillation by transforming two distinct equations of form (1.1) to the same Riccati type equation.

**Example 3.2.** Consider (1.1) with $c_n = 1$ for $n$ odd, $c_n = 1/2$ for $n$ even, and $b_n = \sqrt{2}$ for all $n \geq 1$.

**Example 3.3.** Consider (1.1) with $c_n = b_n = 1$ for $n$ odd, $c_n = 1/2$ for $n$ even, and $b_n = 2$ for $n$ even.

In Example 3.2, $b_n$ is constant and $a_n = b_n - c_n - c_{n-1} = \sqrt{2} - 1 - 1/2$, so Theorem 3.7 implies that (1.1) is oscillatory in this case.

None of the direct oscillation criteria in this paper apply to Example 3.3. Consider, however, the substitution $s_n = (b_{n+1}x_{n+1})/(c_nx_n)$ which transforms (1.1) into the form (1.5), $q_ns_n + 1/s_{n-1} = 1$. If this transformation is applied to Example 3.2 and to Example 3.3, we obtain in each case the same equation of the form (1.5), since in both cases we have $q_n = 1/2$ for all odd $n$, and $q_n = 1/8$ for all even $n$. Thus, by Theorem 1.1, Examples 3.2 and 3.3 are both non-oscillatory if and only if the corresponding equation (1.5) has a positive solution defined for all sufficiently large $n$. But we know that Example 3.2 is oscillatory, hence Example 3.3 must be oscillatory also.

We note that Example 2.1 also leads to precisely the same transformed equation (1.5) as do Examples 3.2 and 3.3. Hence we have here a much briefer argument for the oscillation of Example 2.1 than we gave in Section 2.

### 4. Comparison Theorems

In addition to (1.1) and (1.5), consider the equations

$$C_nX_{n+1} + C_{n-1}X_{n-1} = B_nX_n, \quad (4.1)$$

and

$$Q_nS_n + 1/S_{n-1} = 1, \quad (4.2)$$

where $Q_n = C_n^2/(B_nB_{n+1})$. We have the following comparison theorem, which generalizes Theorem 9 of [6].

**Theorem 4.1.** Suppose $Q_n \geq q_n$ for all sufficiently large $n$. If (4.1) is non-oscillatory then (1.1) is non-oscillatory also.

**Proof.** If (4.1) is non-oscillatory, Theorem 1.1 implies that (4.2) has a positive solution $\{S_n\}$ defined for $n \geq N$ for some $N \geq 0$. We may assume
that $Q_n \geq q_n$ for all $n \geq N$, also. Note that $S_n > 1$ for all $n \geq N$, since (4.2) implies that $1/S_{n-1} < 1$ for all $n \geq N$. Choose $s_N$ such that $s_N \geq S_N > 1$, and define $s_{N+1}$ using (1.5). Thus, from (1.5) and (4.2),

$$q_{N+1}s_{N+1} = 1 - 1/s_N = Q_{N+1}S_{N+1} + 1/S_N - 1/s_N \geq Q_{N+1}S_{N+1},$$

hence,

$$s_{N+1} \geq (Q_{N+1}/q_{N+1}) S_{N+1} > S_{N+1} > 1.$$ 

By induction, we may thus obtain a solution $\{s_n\}$ of (1.5) for $n \geq N$, satisfying $s_n > 1$ for all $n \geq N$. Theorem 1.1 then implies that (1.1) is non-oscillatory, which completes the proof.

**Corollary 4.2** [6, Theorem 9]. Suppose $C_n \geq c_n$ and $B_n \leq b_n$ for all sufficiently large $n$. If (4.1) is non-oscillatory, then (1.1) is non-oscillatory also.

**Proof.** By hypothesis,

$$C_n^2/c_n^2 \geq 1 \geq (B_nB_{n+1})/b_nb_{n+1},$$

which implies that $Q_n \geq q_n$. The result follows from Theorem 4.1.

It is interesting to compare Corollary 4.2 with the difference equation analogue of the classical Sturm comparison theorem, which has been discussed by Fort [2, p. 153]. It follows immediately from Fort's theorem that if $C_n \leq c_n$ and $A_n \leq a_n$ for all sufficiently large $n$, then if the equation

$$-A(C_{n-1} \Delta X_{n-1}) + A_nX_n = 0, \quad n = 1, 2, \ldots, \quad (4.3)$$

is non-oscillatory, so is (1.2). As noted above in Section 1, (1.2) is equivalent to (1.1), with $a_n = b_n - c_n - c_{n-1}$, $n \geq 1$. Thus we have the following analogue of Sturm's comparison theorem.

**Theorem 4.2.** Suppose

$$C_n \leq c_n \quad \text{and} \quad B_n - C_n - C_{n-1} \leq b_n - c_n - c_{n-1} \quad (4.4)$$

for all sufficiently large $n$. Then if (4.1) is non-oscillatory, (1.1) is non-oscillatory also.

It is somewhat surprising to note that while Theorem 4.2 and Corollary 4.2 have the same conclusion, the inequality conditions on the coefficients $c_n$ in the hypotheses of these two results have the opposite sense. That is, Corollary 4.2 implies that if we increase $c_n$ and decrease $b_n$ in (1.1) we obtain "faster" oscillation, while the Sturmian theorem 4.2 implies that if we decrease both $c_n$ and $b_n - c_n - c_{n-1}$ we obtain the same effect.
Not surprisingly, the hypothesis $B_n - C_n - C_{n-1} \leq b_n - c_n - c_{n-1}$ in Theorem 4.2 cannot be replaced by $B_n \leq b_n$. For example, if $B_n = b_n = 2$, $C_n = 1$, and $c_n = 2$ for all $n$, then (4.1) is non-oscillatory since $x_n \equiv 1$ is a solution, but (1.1) is oscillatory, by Theorem 2.1.

We note also that, as in the case of the analogous differential equation, (1.2) must be non-oscillatory in case $a_n > 0$, hence (1.1) must be non-oscillatory if $b_n \geq c_n + c_{n-1}$. Indeed, in this case it is known (see [7, Theorem 2]) that (1.1) has a recessive solution $\{x_n\}$ such that $x_n > 0$ and $\Delta x_n \leq 0$ for all $n$. Thus the cases of interest in Theorem 4.2 occur when $b_n - c_n - c_{n-1} \leq 0$. For examples in which $b_n - c_n - c_{n-1} < 0$ and (1.1) is non-oscillatory, see Example 3.1 above, as well as [6, p. 454].

Finally, we consider a comparison example in which Theorem 4.1 is applicable, but Corollary 4.2 and Theorem 4.2 are not. Also, none of the direct oscillation criteria in this paper or our earlier papers are applicable to this example.

**Example 4.1.** Let $C_n = 1$, $n > 0$, and let

$$B_{2n} = 2/3^{n-1} \quad \text{and} \quad B_{2n-1} = 3^{n-1}, \quad n \geq 1. \quad (4.5)$$

We will apply Theorem 4.1 to show that (4.1) is oscillatory in this case. Let us compare the coefficients of Example 4.1 with the coefficients in Example 2.1. We have $C_n = c_n$, $B_{2n-1} \leq b_{2n-1}$, and $B_{2n} \geq b_{2n}$, $n \geq 1$, hence Corollary 4.2 and Theorem 4.2 are not applicable. However, $Q_{2n-1} = 1/2$ and $Q_{2n} = 1/6$ for $n \geq 1$ in Example 4.1, while $q_{2n-1} = 1/2$ and $q_{2n} = 1/8$ for $n \geq 1$, in Example 2.1.

Thus Theorem 4.1 is applicable, and since Example 2.1 was shown to be oscillatory, Example 4.1 is oscillatory also.

**REFERENCES**


