Characterizations of Lie derivations of $B(X)$

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ABSTRACT

Let $X$ be a Banach space of dimension greater than 2. We prove that if $\delta : B(X) \rightarrow B(X)$ is a linear map satisfying

$$\delta([A, B]) = [\delta(A), B] + [A, \delta(B)]$$

for any $A, B \in B(X)$ with $AB = 0$ (resp. $AB = P$, where $P$ is a fixed nontrivial idempotent), then $\delta = d + \tau$, where $d$ is a derivation of $B(X)$ and $\tau : B(X) \rightarrow \mathbb{C}$ is a linear map vanishing at commutators $[A, B]$ with $AB = 0$ (resp. $AB = P$).

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1. Introduction

In general there are two directions in the study of the local actions of derivations of operator algebras. One is the well known local derivation problem (for example, see [5,11–13,18] and references therein). The other is to study conditions under which derivations of operator algebras can be completely determined by the action on some sets of operators (for example, see [1,3,4,7–9,19] and references therein).

Recall that a linear map $\delta$ from an algebra $A$ into itself is called a Lie derivation if $\delta([A, B]) = [\delta(A), B] + [A, \delta(B)]$ for any $A, B \in A$, where $[A, B] = AB - BA$ is the usual Lie product. In recent years there has been a great interest in the study of Lie derivations of operator algebras (see, for example,
[2,10,14–17]). But, so far, there have been no papers on the study of local actions of Lie derivations of operator algebras. Note that, in most cases, every Lie derivation of operator algebras can be expressed as the sum of a derivation and a linear map with image in the center vanishing at commutators (see, for example, [2,10,14–17]). Therefore the study of local Lie derivations is related to that of local derivations. However, it is not the case for the study of conditions under which Lie derivation can be completely determined by their actions on some sets of operators. It is the aim of this paper to initiate the study of Lie derivations of operator algebras in this direction.

Let $X$ be a Banach space of dimension greater than 2. The purpose of this paper is to show that if $\delta : B(X) \to B(X)$ is a linear map satisfying

$$\delta([A,B]) = [\delta(A), B] + [A, \delta(B)]$$

for any $A, B \in B(X)$ with $AB = 0$ (resp. $AB = P$, where $P$ is a fixed nontrivial idempotent), then $\delta = d + \tau$, where $d$ is a derivation of $B(X)$ and $\tau : B(X) \to \mathbb{C}I$ is a linear map vanishing at commutators $[A,B]$ with $AB = 0$ (resp. $AB = P$). We want to mention here that there is no continuity assumed.

Before proceeding let us fix some notations. Let $X$ be a Banach space and $B(X)$ the algebra of all bounded linear operators on $X$. An operator $P$ is an idempotent provided that $P^2 = P$. For any $x \in X$ and $f \in X^*$, where $X^*$ is the topological dual space of $X$, we denote $x \otimes f$ the rank one operator defined by $(x \otimes f)(z) = f(z)x$ for any $z \in X$. Note that every rank one operator can be written in this form. The operator $x \otimes f$ is an idempotent if and only if $f(x) = 1$.

2. Result and proof

One of our main results is the following.

**Theorem 2.1.** Let $X$ be a Banach space of dimension greater than 2, and $\delta : B(X) \to B(X)$ be a linear map satisfying

$$\delta([A,B]) = [\delta(A), B] + [A, \delta(B)]$$

for any $A, B \in B(X)$ with $AB = 0$. Then there exists an operator $R \in B(X)$ and a linear map $\tau : B(X) \to \mathbb{C}I$ vanishing at commutators $[A,B]$ when $AB = 0$ such that

$$\delta(A) = RA - AR + \tau(A), \quad A \in B(X).$$

**Proof.** Fix an idempotent $P_1 \in B(X)$ and let $P_2 = I - P_1$. In what follows, we write $A_{ij} = P_i B(X) P_j$ for $i, j = 1, 2$. Then every operator $A \in B(X)$ can be written as $A = \sum_{i,j=1}^2 A_{ij}$. Note that notation $A_{ij}$ denotes an arbitrary element of $A_{ij}$.

Now we organize the proof in a series of claims.

**Claim 1.** $\delta(I) = \lambda I$ for some $\lambda \in \mathbb{C}$.

For any idempotent $P \in B(X)$, it is obvious that $P(I - P) = (I - P)P = 0$. Then we have

$$0 = \delta([P, I - P]) = [\delta(P), I - P] + [P, \delta(I - P)] = \delta(P) - P\delta(P) - P\delta(P) + P\delta(P) + P\delta(I - P) - P\delta(P) - \delta(I) P + \delta(P) P = [P, \delta(I)].$$

Now for any non-zero vector $x$ in $X$, we take $f \in X^*$ such that $f(x) = 1$. Then we have $\delta(I)x \otimes f = x \otimes f \delta(I)$. So $\delta(I)x = \lambda(x)x$ for some $\lambda(x) \in \mathbb{C}$. Hence $\delta(I) = \lambda I$ for some $\lambda \in \mathbb{C}$.

**Claim 2.** $P_1 \delta(P_1) P_1 + P_2 \delta(P_2) P_2 = \mu I$ for some $\mu \in \mathbb{C}$.

Indeed, for any rank one operator $x \otimes f \in B(X)$, since $(P_1 x \otimes f P_2) P_1 = 0$, we have

$$\delta([P_1 x \otimes f P_2, P_1]) = [\delta(P_1 x \otimes f P_2), P_1] + [P_1 x \otimes f P_2, \delta(P_1)].$$
Consequently,

\[-\delta(P_1 x \otimes fP_2) = \delta(P_1 x \otimes fP_2)P_1 - P_1 \delta(P_1 x \otimes fP_2) + P_1 x \otimes fP_2 \delta(P_1) - \delta(P_1)P_1 x \otimes fP_2.\]

Multiplying the above equality from the left by $P_1$ and from the right by $P_2$, we arrive at

\[P_1 x \otimes fP_2 \delta(P_1)P_2 = P_1 \delta(P_1)P_1 x \otimes fP_2.\]

Equivalently,

\[P_1 x \otimes (P_2 \delta(P_1)P_2)^* f = P_1 \delta(P_1)P_1 x \otimes P_2^* f.\]

It follows that $P_1 \delta(P_1)P_1 x = \mu P_1 x$ and $(P_2 \delta(P_1)P_2)^* f = \mu P_2^* f$ for some $\mu \in \mathbb{C}$. Since $x$ and $f$ are arbitrary, we can assert that $P_1 \delta(P_1)P_1 = \mu P_1$ and $P_2 \delta(P_1)P_2 = \mu P_2$. Hence $P_1 \delta(P_1)P_1 + P_2 \delta(P_1)P_2 = \mu I$, proving the claim.

In the sequel, we define

\[\Delta(A) = \delta(A) + \delta(P_1 \delta(P_1)P_2 - P_2 \delta(P_1)P_1) (A),\]

where $\delta(P_1 \delta(P_1)P_2 - P_2 \delta(P_1)P_1)$ is the inner derivation implemented by $P_1 \delta(P_1)P_2 - P_2 \delta(P_1)P_1$, that is,

\[\delta(P_1 \delta(P_1)P_2 - P_2 \delta(P_1)P_1) (A) = (P_1 \delta(P_1)P_2 - P_2 \delta(P_1)P_1)A - A(P_1 \delta(P_1)P_2 - P_2 \delta(P_1)P_1)\]

for all $A \in B(X)$. One can verify that

\[\Delta([A, B]) = [\Delta(A), B] + [A, \Delta(B)]\]

for all $A, B \in B(X)$ with $AB = 0$. Moreover, by Claim 2, we have

\[\Delta(A_1) = \delta(A) - \delta(P_1 \delta(P_1)P_2 - P_2 \delta(P_1)P_1) = \delta(P_1)P_1 + \delta(P_1)P_2 - P_1 \delta(P_1)P_2 - P_2 \delta(P_1)P_1 = P_1 \delta(P_1)P_1 + P_2 \delta(P_1)P_2 = \mu I,\]

and then $\Delta(P_2) = \Delta(I) - \Delta(P_1) = \delta(I) - \Delta(P_1) = (\lambda - \mu)I$.

The excellent feature of the map $\Delta$ will be shown in Claims 3–6 which say that $\Delta$ leaves each $A_{ij}$ invariant up to scalar summands.

**Claim 3.** For arbitrary $A_{12} \in A_{12}$, we claim that $\Delta(A_{12}) \in A_{12}$.

Since $\Delta(A_1) = \mu I$ and $A_1 A_1 = 0$, we have

\[\Delta(A_{12}) = \Delta([A_1, A_{12}]) = [\Delta(A_1), A_{12}] + [A_1, \Delta(A_{12})] = P_1 \Delta(A_{12}) - \Delta(A_{12})P_1.\]

This leads to

\[P_1 \Delta(A_{12})P_1 = P_2 \Delta(A_{12})P_2 = P_2 \Delta(A_{12})P_1 = 0\]

and so $\Delta(A_{12}) \in A_{12}$.

**Claim 4.** There exists a linear functional $\tau_1$ on $A_{11}$ such that

\[\Delta(A_{11}) - \tau_1(A_{11}) I \in A_{11}\]

for any $A_{11} \in A_{11}$.

For $A_{11} \in A_{11}$, since $A_{11} P_2 = 0$ and $\Delta(P_2) = (\lambda - \mu)I$, it follows that

\[0 = \Delta([A_{11}, P_2]) = [\Delta(A_{11}), P_2] + [A_{11}, \Delta(P_2)] = [\Delta(A_{11}), P_2] = \Delta(A_{11})P_2 - P_2 \Delta(A_{11}).\]
From this, we get $P_1 \Delta(A_{11})P_2 = P_2 \Delta(A_{11})P_1 = 0$. Therefore, we can write
\[ \Delta(A_{11}) = X_{11} + Y_{22} \]
for some $X_{11} \in A_{11}$ and $Y_{22} \in A_{22}$.

Now, for any $B_{22} \in A_{22}$, since $A_{11}B_{22} = 0$, we have
\[ 0 = \Delta([A_{11}, B_{22}]) = [\Delta(A_{11}), B_{22}] + [A_{11}, \Delta(B_{22})]. \]
It follows that $P_2 [\Delta(A_{11}), B_{22}]P_2 = 0$. Accordingly, $[Y_{22}, B_{22}] = 0$ for any $B_{22} \in A_{22}$. Therefore, there exists a scalar $\tau_1(A_{11})$ such that $Y_{22} = \tau_1(A_{11})P_2$. Thus we get
\[ \Delta(A_{11}) = X_{11} + Y_{22} = X_{11} + \tau_1(A_{11})P_2 = X_{11} + \tau_1(A_{11})I - \tau_1(A_{11})P_1. \]
Consequently,
\[ \Delta(A_{11}) - \tau_1(A_{11})I = X_{11} - \tau_1(A_{11})P_1 \in A_{11}. \]
One can verify that $\tau_1$ is linear.

**Claim 5.** For any $A_{21} \in A_{21}$, we claim that $\Delta(A_{21}) \in A_{21}$.

With the similar argument in Claim 3, one can get the result by considering $\Delta([P_1, A_{21}])$.

**Claim 6.** There exists a linear functional $\tau_2$ on $A_{22}$ such that $\Delta(A_{22}) - \tau_2(A_{22})I \in A_{22}$ for any $A_{22} \in A_{22}$.

We omit the proof as it is similar to the proof of Claim 4.

Now for any $A = \sum_{i,j=1}^{2} A_{ij} \in B(X)$, we define two linear mappings $\tau : B(X) \to \mathbb{C}I$ and $d : B(X) \to B(X)$ by
\[ \tau(A) = (\tau_1(A_{11}) + \tau_2(A_{22}))I \quad \text{and} \quad d(A) = \Delta(A) - \tau(A). \]
Note that $d(A_{ij}) \subseteq A_{ij}$ for $1 \leq i, j \leq 2$ and $d(A_{ij}) = \Delta(A_{ij})$ for $1 \leq i \neq j \leq 2$.

**Claim 7.** $d$ is a derivation.

By Claims 3–6, it is sufficient to show that
\[ d(A_{ij}B_{jk}) = d(A_{ij})B_{jk} + A_{ij}d(B_{jk}) \]
for $1 \leq i, j, k \leq 2$.

We divide the proof into the following six steps.

**Step 1.** For any $A_{11} \in A_{11}$ and $B_{12} \in A_{12}$, since $B_{12}A_{11} = 0$, we have
\[ d(A_{11}B_{12}) = \Delta(A_{11}B_{12}) = \Delta([A_{11}, B_{12}]) \]
\[ = [\Delta(A_{11}), B_{12}] + [A_{11}, \Delta(B_{12})] = [\Delta(A_{11}), B_{12}] + [A_{11}, d(B_{12})] \]
\[ = [d(A_{11}) + \tau(A_{11}), B_{12}] + [A_{11}, d(B_{12})] = d(A_{11})B_{12} + A_{11}d(B_{12}). \]

Thus we have
\[ d(A_{11}B_{12}) = d(A_{11})B_{12} + A_{11}d(B_{12}). \]
Similarly, we can get
\[ d(A_{12}B_{22}) = d(A_{12})B_{22} + A_{12}d(B_{22}), \]
\[ d(A_{21}B_{11}) = d(A_{21})B_{11} + A_{21}d(B_{11}), \]
\[ d(A_{22}B_{21}) = d(A_{22})B_{21} + A_{22}d(B_{21}). \]
Step 2. Let $A_{11}, B_{11} \in A_{11}$. For any $C_{12} \in A_{12}$, on one hand, by Step 1, we have

$$d(A_{11}B_{11}C_{12}) = d(A_{11})B_{11}C_{12} + A_{11}d(B_{11}C_{12})$$
$$= d(A_{11})B_{11}C_{12} + A_{11}d(B_{11})C_{12} + A_{11}B_{11}d(C_{12}).$$

On the other hand,

$$d(A_{11}B_{11}C_{12}) = d(A_{11}B_{11})C_{12} + A_{11}B_{11}d(C_{12}).$$

Comparing these two equalities, we have

$$(d(A_{11}B_{11}) - d(A_{11})B_{11} - A_{11}d(B_{11}))C_{12} = 0$$

for all $C_{12} \in A_{12}$, which is equivalent to

$$(d(A_{11}B_{11}) - d(A_{11})B_{11} - A_{11}d(B_{11}))p_1B(X)p_2 = 0.$$

Since $B(X)$ is prime, we get

$$(d(A_{11}B_{11}) - d(A_{11})B_{11} - A_{11}d(B_{11}))p_1 = 0.$$

Note that $d(A_{11}) \subseteq A_{11}$. Then we have

$$d(A_{11}B_{11}) - d(A_{11})B_{11} - A_{11}d(B_{11}) = 0.$$

Similarly, by considering $d(A_{22}B_{22}C_{21})$, we can get

$$d(A_{22}B_{22}) = d(A_{22})B_{22} + A_{22}d(B_{22}).$$

Step 3. Let $A = A_{11} + A_{12} + A_{22}$ and $B = B_{11} + B_{12} + B_{22}$ be in $A_{11} \oplus A_{12} \oplus A_{22}$. On one hand, by step 1 and step 2, we have

$$d(AB) = d((A_{11} + A_{12} + A_{22})(B_{11} + B_{12} + B_{22}))$$
$$= d(A_{11}B_{11}) + d(A_{11}B_{12}) + d(A_{11}B_{22}) + d(A_{22}B_{22})$$
$$= d(A_{11})B_{11} + A_{11}d(B_{11}) + d(A_{11})B_{12} + A_{11}d(B_{12})$$
$$+ d(A_{12})B_{22} + A_{12}d(B_{22}) + d(A_{22})B_{22} + A_{22}d(B_{22}),$$

on the other hand, since $d(A_{ij}) \subseteq A_{ij}$ we have

$$d(A)B + Ad(B) = d(A_{11} + A_{12} + A_{22})(B_{11} + B_{12} + B_{22})$$
$$+ (A_{11} + A_{12} + A_{22})d(B_{11} + B_{12} + B_{22})$$
$$= d(A_{11})B_{11} + A_{11}d(B_{11}) + d(A_{11})B_{12} + A_{11}d(B_{12})$$
$$+ d(A_{12})B_{22} + A_{12}d(B_{22}) + d(A_{22})B_{22} + A_{22}d(B_{22}).$$

So, $d(AB) = d(A)B + Ad(B)$, and so the restriction of $d$ to $A_{11} \oplus A_{12} \oplus A_{22}$ is a derivation. Furthermore, since $A_{11} \oplus A_{12} \oplus A_{22}$ is the nest algebra associated to the nest $\{ 0, P(X), X \}$, by the main result in [6], there exists an operator $T \in B(X)$ such that

$$d(A) = TA - AT$$

(2.1)

for $A \in A_{11} \oplus A_{12} \oplus A_{22}$.

Similarly, we can conclude that the restriction of $d$ to $A_{11} \oplus A_{21} \oplus A_{22}$ is a derivation, and so there exists an operator $S \in B(X)$ such that

$$d(A) = SA - AS$$

(2.2)

for $A \in A_{11} \oplus A_{21} \oplus A_{22}$.

Step 4. Now, we write

$$T = \sum_{i,j=1}^{2} T_{ij} \quad \text{and} \quad S = \sum_{i,j=1}^{2} S_{ij}.$$
Then by (2.1) and (2.2) we have
\[ T(A_{11} + B_{22}) - (A_{11} + B_{22})T = S(A_{11} + B_{22}) - (A_{11} + B_{22})S \]
for all \( A_{11} \in \mathcal{A}_{11} \) and \( B_{22} \in \mathcal{A}_{22} \). This yields that
\[ (T - S)A_{11} = A_{11}(T - S) \] and \( (T - S)B_{22} = B_{22}(T - S) \).
Multiplying the above two equations by \( P_2 \) and \( P_1 \) from the left, respectively, we get that
\[ (T_{21} - S_{21})A_{11} = 0 \] and \( (T_{12} - S_{12})B_{22} = 0 \).
Similarly we may obtain
\[ (T_{11} - S_{11})A_{11} = A_{11}(T_{11} - S_{11}) \] and \( (T_{22} - S_{22})B_{22} = B_{22}(T_{22} - S_{22}) \).
These imply that
\[ T_{12} = S_{12}, \quad T_{21} = S_{21}, \quad T_{11} - S_{11} = \lambda_1 P_1, \quad T_{22} - S_{22} = \lambda_2 P_2 \] (2.3)
for some \( \lambda_1, \lambda_2 \in \mathbb{C} \).

**Step 5.** For any \( A_{12} \in \mathcal{A}_{12} \) and \( B_{21} \in \mathcal{A}_{21} \), using (2.1), (2.2) and (2.3), we have
\[
d(A_{12}B_{21}) - d(A_{12})B_{21} - A_{12}d(B_{21}) = TA_{12}B_{21} - A_{12}B_{21}T - (TA_{12} - A_{12}T)B_{21} - A_{12}(SB_{21} - B_{21}S) = -A_{12}B_{21}T + A_{12}TB_{21} - A_{12}SB_{21} + A_{12}B_{21}S = (\lambda_2 - \lambda_1)A_{12}B_{21} \] (2.4)
and
\[
d(B_{21}A_{12}) - d(B_{21})A_{12} - B_{21}d(A_{12}) = (SB_{21}A_{12} - B_{21}A_{12}S) - (SB_{21} - B_{21}S)A_{12} - B_{21}(TA_{12} - A_{12}T) = -B_{21}A_{12}S + B_{21}S_{11}A_{12} - B_{21}T_{11}A_{12} + B_{21}A_{12}T = (\lambda_2 - \lambda_1)B_{21}A_{12}. \] (2.5)

**Step 6.** Since at least one of \( P_1 \) or \( P_2 \) is of rank greater than or equal to 2, there must be \( A'_{12} \in \mathcal{A}_{12} \) and \( B'_{21} \in \mathcal{A}_{21} \) satisfying either
\[ A'_{12}B'_{21} = 0 \quad \text{and} \quad B'_{21}A'_{12} = 0, \]
or
\[ B'_{21}A'_{12} = 0 \quad \text{and} \quad A'_{12}B'_{21} = 0. \]
Without loss of generality, suppose that \( A'_{12}B'_{21} = 0 \) and \( B'_{21}A'_{12} = 0 \). Applying equalities (2.4) and (2.5), we get
\[
\tau ([A'_{12}, B'_{21}]) = \Delta([A'_{12}, B'_{21}]) - d([A'_{12}, B'_{21}]) = [\Delta(A'_{12}), B'_{21}] + [A'_{12}, \Delta(B'_{21})] - d([A'_{12}, B'_{21}]) = [d(A'_{12}), B'_{21}] + [A'_{12}, d(B'_{21})] - d([A'_{12}, B'_{21}]) = d(A'_{12})B'_{21} - B'_{21}d(A'_{12}) + A'_{12}(d(B'_{21}) - d(B'_{21})) - d(B'_{21})A'_{12} = (\lambda_1 - \lambda_2)A'_{12}B'_{21} + (\lambda_2 - \lambda_1)B'_{21}A'_{12} = (\lambda_2 - \lambda_1)B'_{21}A'_{12}. \]

Since \( \tau ([A'_{12}, B'_{21}]) \in \mathbb{C}I \) and \( B'_{21}A'_{12} \neq 0 \), it follows that \( \lambda_1 = \lambda_2 \).
Thus, equalities (2.4) and (2.5) imply that
\[
d(A_{12}B_{21}) = d(A_{12})B_{21} + A_{12}d(B_{21}),
\]
\[
d(B_{21}A_{12}) = d(B_{21})A_{12} + B_{21}d(A_{12}) \]
for any \( A_{12} \in \mathcal{A}_{12} \) and \( B_{21} \in \mathcal{A}_{21} \).
Now, by steps 1, 2 and 6, we can infer that \( d \) is a derivation. Since every derivation of \( B(X) \) is inner, there exists an operator \( R_1 \in B(X) \) such that \( d(A) = R_1A - AR_1 \) for every \( A \in B(X) \). Hence letting \( R = R_1 + P_1\delta(P_1)P_2 - P_2\delta(P_1)P_1 \), we have
\[
\delta(A) = RA - AR + \tau(A)
\]
for all \( A \in B(X) \).

**Claim 8.** \( \tau \) vanishes at commutators \([A, B]\) with \( AB = 0 \).

Suppose that \( AB = 0 \). We compute
\[
\tau([A, B]) = \Delta([A, B]) - d([A, B])
\]
\[
= [\Delta(A), B] + [A, \Delta(B)] - d(AB - BA)
\]
\[
= [d(A), B] + [A, d(B)] - d([A, B])
\]
\[
= 0.
\]
The proof is complete. \( \square \)

The proof of the following theorem shares the same outline as that of Theorem 2.1 but need different technique.

**Theorem 2.2.** Let \( X \) be a Banach space of dimension greater than 2, and \( \delta : B(X) \to B(X) \) be a linear map satisfying
\[
\delta([A, B]) = [\delta(A), B] + [A, \delta(B)]
\]
for all \( A, B \in B(X) \) with \( AB = P \), where \( P \in B(X) \) is a fixed non-trivial idempotent operator. Then \( \delta = d + \tau \), where \( d \) is a derivation of \( B(X) \) and \( \tau : B(X) \to \mathbb{C}I \) is a linear map vanishing at \([A, B]\) with \( AB = P \).

**Proof.** Let \( P_1 = P, P_2 = I - P_1 \), and \( A_{ij} = P_1B(X)P_3 \) for \( i, j = 1, 2 \).

**Claim 1.** \( P_1\delta(P_1)P_1 + P_2\delta(P_1)P_2 = \lambda I \) for some \( \lambda \in \mathbb{C} \).

For any \( A_{12} \in A_{12}, \) since \( (P_1 + A_{12})P_1 = P_1 = P \), we have
\[
\delta([P_1 + A_{12}, P_1]) = [\delta(P_1 + A_{12}), P_1] + [P_1 + A_{12}, \delta(P_1)].
\]
Multiplying the above equation by \( P_1 \) from the left and by \( P_2 \) from the right, we get
\[
A_{12}\delta(P_1)P_2 = P_1\delta(P_1)A_{12}.
\]
Now, for any rank one operator \( x \otimes f \in B(X) \), we have
\[
P_1x \otimes fP_2\delta(P_1)P_2 = P_1\delta(P_1)P_1x \otimes fP_2.
\]
With the same argument as Claim 2 in the proof of Theorem 2.1, we see that there exists \( \lambda \in \mathbb{C} \) such that \( P_1\delta(P_1)P_1 + P_2\delta(P_1)P_2 = \lambda I \).

Now, we define
\[
\Delta = \delta + \delta P_1\delta(P_1)P_2 - P_2\delta(P_1)P_1
\]
where \( \delta P_1\delta(P_1)P_2 - P_2\delta(P_1)P_1 \) is the inner derivation implemented by \( P_1\delta(P_1)P_2 - P_2\delta(P_1)P_1 \). Then we have \( \Delta(P_1) = \lambda I \) and
\[
\Delta([A, B]) = [\Delta(A), B] + [A, \Delta(B)]
\]
for all \( A, B \in B(X) \) with \( AB = P \).
Claim 2. For any $A_{12} \in A_{12}$ and $B_{21} \in A_{21}$, there hold

$\Delta(A_{12}) \in A_{12}$ and $\Delta(B_{21}) \in A_{21}$.

Noting that $(P_1 + A_{12})P_1 = P_1 = P$, we have

$$\Delta(A_{12}) = \Delta([P_1, P_1 + A_{12}])$$

$$= [\Delta(P_1), P_1 + A_{12}] + [P_1, \Delta(P_1 + A_{12})] = [P_1, \Delta(A_{12})],$$

which implies that $\Delta(A_{12}) \in A_{12}$.

Similarly, we can get $\Delta(B_{21}) \in A_{21}$ by considering $P_1(P_1 + B_{21}) = P_1$.

Claim 3. $\Delta(I) \in \mathbb{C}I$.

Since $IP_1 = P_1$, we have

$$0 = \Delta([I, P_1]) = [\Delta(I), P_1] + [I, \Delta(P_1)] = [\Delta(I), P_1].$$

This gives that $\Delta(I) = P_1 \Delta(I) P_1 + P_2 \Delta(I) P_2$.

Hence, for any $A_{12} \in A_{12}$, since $(P_1 - A_{12})(I + A_{12}) = P_1$, by Claim 2 we have

$$\Delta(A_{12}) = \Delta([P_1 - A_{12}, I + A_{12}])$$

$$= [\Delta(P_1 - A_{12}), I + A_{12}] + [P_1 - A_{12}, \Delta(I + A_{12})]$$

$$= [P_1, \Delta(I)] - [A_{12}, \Delta(I)] + \Delta(A_{12})$$

$$= -[A_{12}, \Delta(I)] + \Delta(A_{12}).$$

So $[A_{12}, \Delta(I)] = 0$ for all $A_{12} \in A_{12}$. This implies that $P_1 \Delta(I) P_1 + P_2 \Delta(I) P_2 \in \mathbb{C}I$. Consequently, $\Delta(I) \in \mathbb{C}I$.

Claim 4. $\Delta(A_{11}) \subseteq A_{11} \oplus A_{22}$.

Let $A_{11}$ be in $A_{11}$. First suppose that $A_{11}$ is invertible in $A_{11}$, i.e., there exists an operator $A_{11}^{-1} \in A_{11}$ such that $A_{11}A_{11}^{-1} = A_{11}^{-1}A_{11} = P_1$. From $A_{11}A_{11}^{-1} = P_1$ and $(A_{11}^{-1} + P_2)A_{11} = P_1$, we get

$$0 = \Delta([A_{11}, A_{11}^{-1}]) = [\Delta(A_{11}), A_{11}^{-1}] + [A_{11}, \Delta(A_{11}^{-1})]$$

and hence

$$0 = \Delta([A_{11}^{-1} + P_2, A_{11}]) = [\Delta(A_{11}^{-1} + P_2), A_{11}] + [A_{11}^{-1} + P_2, \Delta(A_{11})]$$

$$= [\Delta(A_{11}^{-1}), A_{11}] + [A_{11}^{-1}, \Delta(A_{11})] + [P_2, \Delta(A_{11})] = [P_2, \Delta(A_{11})].$$

(Note that in the third equality, we apply the fact that $\Delta(P_2) \in \mathbb{C}I$.) From this, we see $\Delta(A_{11}) \in A_{11} \oplus A_{22}$.

If $A_{11}$ is not invertible in $A_{11}$, we may find a sufficiently small number $\alpha$ such that $P_1 - \alpha A_{11}$ is invertible in $A_{11}$. It follows from the preceding case that $\Delta(P_1 - \alpha A_{11}) \in A_{11} \oplus A_{22}$. Therefore, we have

$$\Delta(A_{11}) = -\frac{1}{\alpha} \Delta(P_1 - \alpha A_{11}) + \frac{1}{\alpha} \Delta(P_1) \in A_{11} \oplus A_{22}.$$
For any $B_{22} \in \mathcal{A}_{22}$, from $(P_1 + B_{22})P_1 = P_1 = P$, we have

$$0 = \Delta((P_1 + B_{22}, P_1)) = [\Delta(P_1 + B_{22}), P_1] + [P_1 + B_{22}, \Delta(P_1)] = [\Delta(B_{22}), P_1].$$

This yields that $P_1 \Delta(B_{22})P_2 = P_2 \Delta(B_{22})P_1 = 0$. From this we see that $\Delta(B_{22}) \in \mathcal{A}_{11} \oplus \mathcal{A}_{22}$.

**Claim 6.** There exist a linear functional $\tau_1$ on $\mathcal{A}_{11}$ such that

$$\Delta(A_{11}) - \tau_1(A_{11}) I \in \mathcal{A}_{11}$$

for all $A_{11} \in \mathcal{A}_{11}$ and a linear functional $\tau_2$ on $\mathcal{A}_{22}$ such that

$$\Delta(A_{22}) - \tau_2(A_{22}) I \in \mathcal{A}_{22}$$

for all $A_{22} \in \mathcal{A}_{22}$.

Let $A_{11} \in \mathcal{A}_{11}$ and $B_{22} \in \mathcal{A}_{22}$ be arbitrary operators. By Claims 4 and 5, we can write $\Delta(A_{11}) = X_{11} + X_{22}$ and $\Delta(B_{22}) = Y_{11} + Y_{22}$. First, suppose that $A_{11}$ is invertible in $\mathcal{A}_{11}$ with inverse $A_{11}^{-1}$. Note that $A_{11}A_{11}^{-1} = P_1 = P$ and $(A_{11}^{-1} + B_{22})A_{11} = P_1 = P$. We get

$$0 = \Delta([A_{11}, A_{11}^{-1}]) = [\Delta(A_{11}), A_{11}^{-1}] + [A_{11}, \Delta(A_{11}^{-1})]$$

and hence

$$0 = \Delta([A_{11}^{-1} + B_{22}, A_{11}])$$

$$= [\Delta(A_{11}^{-1} + B_{22}), A_{11}] + [A_{11}^{-1} + B_{22}, \Delta(A_{11})]$$

$$= [\Delta(B_{22}), A_{11}] + [B_{22}, \Delta(A_{11})].$$

For any general $A_{11} \in \mathcal{A}_{11}$, letting $P_1 + \alpha A_{11}$ be invertible in $\mathcal{A}_{11}$ for some sufficiently small number $\alpha$, then

$$0 = \Delta(B_{22}), P_1 + \alpha A_{11}] + [B_{22}, \Delta(P_1 + \alpha A_{11})]$$

$$= \alpha[\Delta(B_{22}), A_{11}] + \alpha[B_{22}, \Delta(A_{11})]$$

$$= \alpha[Y_{11} + Y_{22}, A_{11}] + \alpha[B_{22}, X_{11} + X_{22}]$$

$$= \alpha[Y_{11}, A_{11}] + \alpha[B_{22}, X_{22}].$$

This implies that $[Y_{11}, A_{11}] = 0$ and $[B_{22}, X_{22}] = 0$. And hence $Y_{11} \in C P_1$ and $X_{22} \in C P_2$. The rest is similar to Claim 4 in the proof of Theorem 2.1.

Now, for any $A = \sum_{i,j=1}^2 A_{ij} \in B(X)$, we define

$$\tau(A) = (\tau_1(A_{11}) + \tau_2(A_{22})) I$$

and $d(A) = \Delta(A) - \tau(A)$.

One can verify that $d(A_{ij}) \subseteq A_{ij}$ for $i, j = 1, 2$, $d(A_{ij}) = \Delta(A_{ij})$ for $i \neq j$, and $\tau$ is linear.

**Claim 7.** $d(A_{11}B_{12}) = d(A_{11})B_{12} + A_{11}d(B_{12})$ for any $A_{11} \in \mathcal{A}_{11}$ and $B_{12} \in \mathcal{A}_{12}$.

Let $A_{11} \in \mathcal{A}_{11}$ and $C_{12} \in \mathcal{A}_{12}$ be arbitrary. If $A_{11}$ is invertible in $\mathcal{A}_{11}$ with inverse operator $A_{11}^{-1}$, then $(A_{11}^{-1} + A_{11}^{-1}C_{12})A_{11} = P_1 = P$. We have

$$d(C_{12}) = \Delta(C_{12}) = \Delta([A_{11}, A_{11}^{-1} + A_{11}^{-1}C_{12}])$$

$$= [\Delta(A_{11}), A_{11}^{-1} + A_{11}^{-1}C_{12}] + [A_{11}, \Delta(A_{11}^{-1} + A_{11}^{-1}C_{12})]$$

$$= [\Delta(A_{11}), A_{11}^{-1}] + [\Delta(A_{11}), A_{11}^{-1}C_{12}]$$

$$+ [A_{11}, \Delta(A_{11}^{-1})] + [A_{11}, \Delta(A_{11}^{-1}C_{12})]$$
\[ = [\Delta(A_{11}), A_{11}^{-1}C_{12}] + [A_{11}, \Delta(A_{11}^{-1}C_{12})] \]
\[ = [d(A_{11}), A_{11}^{-1}C_{12}] + [A_{11}, d(A_{11}^{-1}C_{12})] \]
\[ = d(A_{11})A_{11}^{-1}C_{12} + A_{11}d(A_{11}^{-1}C_{12}). \]

This is,
\[ d(C_{12}) = d(A_{11})A_{11}^{-1}C_{12} + A_{11}d(A_{11}^{-1}C_{12}). \]

Replacing \( C_{12} \) with \( A_{11}B_{12} \), we arrive at \( d(A_{11}B_{12}) = d(A_{11})B_{12} + A_{11}d(B_{12}). \)

Note that any \( A_{11} \) is a linear combination of \( P_1 \) and \( P_1 - \alpha A_{11} \) for some sufficiently small \( \alpha \) such that \( P_1 - \alpha A_{11} \) is invertible in \( A_{11} \). Then, for any \( A_{11} \in A_{11} \) and \( B_{12} \in A_{12} \), we have \( d(A_{11}B_{12}) = d(A_{11})B_{12} + A_{11}d(B_{12}). \)

Similarly, we have

Claim 8. \( d(A_{21}B_{11}) = d(A_{21})B_{11} + A_{21}d(B_{11}) \) for any \( A_{21} \in A_{21} \) and \( B_{11} \in A_{11} \).

With the same approach as in Step 2 in the proof of Theorem 2.1, we can get

Claim 9. \( d(A_{11}B_{11}) = d(A_{11})B_{11} + A_{11}d(B_{11}) \) for any \( A_{11}, B_{11} \in A_{11} \).

Claim 10. For any \( A_{22} \in A_{22}, B_{21} \in A_{21}, C_{12} \in A_{12}, \) and \( D_{22} \in A_{22} \), we have
\[ d(A_{22}B_{21}) = d(A_{22})B_{21} + A_{22}d(B_{21}) \]

and
\[ d(C_{12}D_{22}) = d(C_{12})B_{21} + C_{12}d(D_{22}). \]

We only show that \( d(A_{22}B_{21}) = d(A_{22})B_{21} + A_{22}d(B_{21}) \). Observe that \( (P_1 + A_{22} - A_{22}B_{21}) \times (P_1 + B_{21}) = P_1 = P \). We compute
\[ d(B_{21}) = \Delta(B_{21}) = \Delta([P_1 + B_{21}, P_1 + A_{22} - A_{22}B_{21}]) \]
\[ = [\Delta(P_1 + B_{21}), P_1 + A_{22} - A_{22}B_{21}] \]
\[ + [P_1 + B_{21}, \Delta(P_1 + A_{22} - A_{22}B_{21})] \]
\[ = [d(B_{21}), P_1 + A_{22} - A_{22}B_{21}] + [P_1 + B_{21}, d(A_{22} - A_{22}B_{21})] \]
\[ = d(B_{21}) - A_{22}d(B_{21}) = d(A_{22})B_{21} + A_{22}d(B_{21}). \]

Consequently, \( d(A_{22}B_{21}) = d(A_{22})B_{21} + A_{22}d(B_{21}) \).

Claim 11. \( d(A_{22}B_{22}) = d(A_{22})B_{22} + A_{22}d(B_{22}) \) holds for any \( A_{22}, B_{22} \in A_{22} \).

It follows similarly to Step 2 in the proof of Theorem 2.1.

Claim 12. For any \( A_{12} \in A_{12} \) and \( B_{21} \in A_{21} \), we have
\[ d(A_{12}B_{21}) = d(A_{12})B_{21} + A_{12}d(B_{21}) \]

and
\[ d(B_{21}A_{12}) = d(B_{21})A_{12} + B_{21}d(A_{12}). \]

Repeating the argument in Steps 3–5 in the proof of Theorem 2.1, we may find two numbers \( \lambda_1, \lambda_2 \in \mathbb{C} \) such that for any \( A_{12} \in A_{12} \) and \( B_{21} \in A_{21} \),
\[ d(A_{12}B_{21}) - d(A_{12})B_{21} - A_{12}d(B_{21}) = (\lambda_2 - \lambda_1)A_{12}B_{21} \]
and \[ d(B_{21}A_{12}) - d(B_{21})A_{12} - B_{21}d(A_{12}) = (\lambda_2 - \lambda_1)B_{21}A_{12}. \]

As discussed in Step 6 in the proof of Theorem 2.1, with no loss of generality, we may assume that there exist \( A_{12}' \in A_{12} \) and \( B_{21}' \in A_{21} \) such that \( A_{12}'B_{21}' = 0 \) and \( B_{21}'A_{12}' \neq 0 \). Then \((P_1 + A_{12}')(P_1 + B_{21}') = P_1\). Now we have

\[
\tau((P_1 + A_{12}', P_1 + B_{21}')) = \Delta((P_1 + A_{12}', P_1 + B_{21}')) - d((P_1 + A_{12}', P_1 + B_{21}')) \\
= \Delta(P_1 + A_{12}', P_1 + B_{21}') + [P_1 + A_{12}', \Delta(P_1 + B_{21}')] - d(A_{12}'B_{21}') + d(A_{12}') + d(B_{21}') + d(B_{21}'A_{12}') \\
= \Delta(A_{12}', P_1 + B_{21}') + [P_1 + A_{12}', \Delta(B_{21}') - d(A_{12}', P_1 + B_{21}')] + [P_1 + A_{12}', d(B_{21}') - d(A_{12}')P_1 + B_{21}'] + d(B_{21}'A_{12}') \\
= (\lambda_2 - \lambda_1)B_{21}'A_{12}'.
\]

Notice that \( \tau((P_1 + A_{12}', P_1 + B_{21}')) \in \mathbb{C}I \), which implies that \( \lambda_2 - \lambda_1 = 0 \). Hence we can get

\[ d(A_{12}B_{21}) - A_{12}d(B_{21}) - d(A_{12})B_{21} = d(B_{21}A_{12}) - B_{21}d(A_{12}) - d(B_{21})A_{12} = 0. \]

Now, we can conclude that \( d \) is a derivation. With the same argument as in the proof of Theorem 2.1, one can verify that \( \tau([A, B]) = 0 \) when \([A, B] = P\).

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**References**


