Discrete Mathematics 72 (1988) 361–366
North-Holland

P_3-FACTORIZATION OF COMPLETE BIPARTITE GRAPHS

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Received 25 July 1986
Revised 3 August 1987

In this paper, it is shown that a necessary and sufficient condition for the existence of a P_3-factorization of $K_{m,n}$ is (i) $m + n \equiv 0 \pmod 3$, (ii) $m \leqslant 2n$, (iii) $n \leqslant 2m$ and (iv) $3mn/2(m+n)$ is an integer.

1. Introduction

Let P_3 be a *path* on 3 points and $K_{m,n}$ be a *complete bipartite graph* with partite sets V_1 and V_2, where $|V_1| = m$ and $|V_2| = n$. A spanning subgraph F of $K_{m,n}$ is called a *P_3-factor* if each component of F is isomorphic to P_3. If $K_{m,n}$ is expressed as a line-disjoint sum of P_3-factors, then this sum is called a *P_3-factorization* of $K_{m,n}$.

In this paper, a necessary and sufficient condition for the existence of a P_3-factorization of $K_{m,n}$ will be given.

2. P_3-factor of $K_{m,n}$

The following theorem is on the existence of P_3-factors of $K_{m,n}$.

Theorem 1. *$K_{m,n}$ has a P_3-factor if and only if* (i) $m + n \equiv 0$ (*mod* 3), (ii) $m \leqslant 2n$ *and* (iii) $n \leqslant 2m$.

Proof. Suppose that $K_{m,n}$ has a P_3-factor F. Let t be the number of components of F. Then $t = \frac{1}{3}(m - n)$. Hence, Condition (i) is necessary. Among these t components, let x and y be the number of components whose endpoints are in V_2 and V_1, respectively. Then, since F is a spanning subgraph of $K_{m,n}$, we have $x + 2y = m$ and $2x + y = n$. Hence $x = \frac{1}{3}(2n - m)$ and $y = \frac{1}{3}(2m - n)$. From $0 \leqslant x \leqslant m$ and $0 \leqslant y \leqslant n$, we must have $m \leqslant 2n$ and $n \leqslant 2m$. Conditions (ii) and (iii) are, therefore, necessary.

For those parameters m and n satisfying (i)–(iii), let $x = \frac{1}{3}(2n - m)$ and $y = \frac{1}{3}(2m - n)$. Then x and y are integers such that $0 \leqslant x \leqslant m$ and $0 \leqslant y \leqslant n$. Hence $x + 2y = m$ and $2x + y = n$. Using x points in V_1 and $2x$ points in V_2, consider x P_3's whose endpoints are in V_2. Using the remaining $2y$ points in V_1 and the remaining y points in V_2, consider y P_3's whose endpoints are in V_1. Then these $x+y$ P_3's are line-disjoint and they form a P_3-factor of $K_{m,n}$. □

Corollary 1. *$K_{n,n}$ has a P_3-factor if and only if $n \equiv 0 \pmod 3$.*

3. P_3-factorization of $K_{m,n}$

Our main theorem is on the existence of P_3-factorizations of $K_{m,n}$.

Theorem 2. *$K_{m,n}$ has a P_3-factorization if and only if* (i) $m+n \equiv 0$ $(mod\ 3)$, (ii) $m \leqslant 2n$, (iii) $n \leqslant 2m$ *and* (iv) $3mn/2(m+n)$ *is an integer.*

Proof. Suppose that $K_{m,n}$ is factorized into r P_3-factors. By Theorem 1, Conditions (i)–(iii) are obviously necessary. Let t be the number of components of each P_3-factors. Then $t=\frac{1}{3}(m+n)$ and $r=3mn/2(m+n)$. Hence, condition (iv) is necessary. The proof of sufficiency will be given in Subsection 3.2.

3.1. *Extension theorem of P_3-factorization of $K_{m,n}$*

We prove the following extension theorem, which we use later in the paper.

Theorem 3. *If $K_{m,n}$ has a P_3-factorization, then $K_{sm,sn}$ has a P_3-factorization for every positive integer s.*

Proof. Let V_1, V_2 be the independent sets of $K_{sm,sn}$ where $|V_1|=sm$ and $|V_2|=sn$. Divide V_1 and V_2 into s subsets of m and n points each, respectively. Construct a new graph G with a point set consisting of the subsets which were just constructed. In this graph, two points are adjacent if and only if the subsets come from disjoint independent sets of $K_{sm,sn}$. G is a complete bipartite graph $K_{s,s}$. Noting that the cardinality of each subset identified with a point set of G is m or n and that $K_{s,s}$ has a 1-factorization, we see that the desired result is obtained. 1-factorizations of $K_{s,s}$ are discussed in [1, 2]. □

3.2. *The proof of the sufficiency of Theorem* 2

There are three cases to consider.

Case (1) $m=2n$: In this case, from Theorem 3, $K_{2n,n}$ has a P_3-factorization since $K_{2,1}$ is just P_3.

Case (2) $n=2m$: Obviously, $K_{m,2m}$ has a P_3-factorization.

Case (3) $m<2n$ and $n<2m$: In this case, let $x=\frac{1}{3}(2n-m)$, $y=\frac{1}{3}(2m-n)$, $t=\frac{1}{3}(m+n)$ and $r=3mn/2(m+n)$. Then from Conditions (i)–(iv), x, y, t, r

are integers and $0<x<m$ and $0<y<n$. We have $x+2y=m$ and $2x+y=n$. Hence $r=(x+y)+xy/2(x+y)$. Let $z=xy/2(x+y)$, which is a positive integer. And let $(x, 2y)=d$, $x=dp$, $2y=dq$, where $(p, q)=1$. Then dp is even and $z=dpq/2(2p+q)$. The following lemmas can be verified.

Lemma 1. $(p, q)=1 \Rightarrow (pq, p+q)=1$.

Lemma 2. $(p, q)=1 \Rightarrow (pq, 2p+q)=1$ *(q: odd) or* 2 *(q: even).*

Using these p, q, d the parameters m and n satisfying Conditions (i)–(iv) are expressed as follows:

Lemma 3. $(p, q)=1$ *and* $dpq/2(2p+q)$ *is an integer*

$\Rightarrow$
(I) $m=2(p+q)(2p+q)s$, $n=(4p+q)(2p+q)s$ *when* q *is odd,*
(II) $m=2(p+2q')(p+q')s$, $n=2(2p+q')(p+q')s$ *when* $q=2q'$ *and* q' *is odd,*
(III) $m=(p+4q'')(p+2q'')s$, $n=2(p+q'')(p+2q'')s$ *when* $q=4q''$, *where* s *is a positive integer.*

We use the following notations for sequences.

Notation. Let A and B be two sequences of the same size such as

A: $a_1, a_2, \ldots, a_u$
B: $b_1, b_2, \ldots, b_u$.

If $b_i=a_i+c$ $(i=1, 2, \ldots, u)$, then we write $B=A+c$. If $b_i=((a_i+c) \bmod w)$ $(i=1, 2, \ldots, u)$, then we write $B=A+c \bmod w$, where the residuals $a_i+c \bmod w$ are integers in the set $\{1, 2, \ldots, w\}$.

For the parameters m and n in (I)–(III) when $s=1$, we can construct a P_3-factorization of $K_{m,n}$.

Lemma 4. $(p, q)=1$ *and* q *is odd*

$$m=2(p+q)(2p+q),\ n=(4p+q)(2p+q)$$

$\Rightarrow$ $K_{m,n}$ *has a* P_3*-factorization.*

Proof. The proof is by construction (Algorithm I). Let $x=\frac{1}{3}(2n-m)$, $y=\frac{1}{3}(2m-n)$, $t=\frac{1}{3}(m+n)$, $r=3mn/2(m+n)$. Then we have $x=2p(2p+q)$, $y=q(2p+q)$, $t=(2p+q)^2$, $r=(p+q)(4p+q)$. Let $r_1=p+q$, $r_2=4p+q$, $m_0=m/r_1=2(2p+q)$, $n_0=n/r_2=2p+q$. Consider two sequences R and C of the same size $4(2p+q)$.

R: $1, 1, 2, 2, \ldots, 2(2p+q), 2(2p+q)$

C: $1, 2, \ldots, 4(2p+q)-1, 4(2p+q)$.

Construct p sequences R_i such that $R_i = R + 2(i-1)(2p+q)$ $(i = 1, 2, \ldots, p)$. Construct p sequences C_i such that $C_i = (C + 2(i-1) \bmod 4(2p+q)) + 4(i-1)(2p+q)$ $(i = 1, 2, \ldots, p)$. Consider two sequences R' and C' of the same size $2(2p+q)$.

R': $1, 2, \ldots, 2(2p+q)-1, 2(2p+q)$

C': $1, 3, \ldots, 2p+q, 2, 4, \ldots, 2p+q-1, 1, 3, \ldots, 2p+q, 2, 4, \ldots, 2p+q-1$.

Construct q sequences R'_i such that $R'_i = R' + 2(i-1)(2p+q) + 2p(2p+q)$ $(i = 1, 2, \ldots, q)$. Construct q sequences C'_i such that $C'_i = (C' + (i-1) + 2p \bmod 2p+q) + (i-1)(2p+q) + 4p(2p+q)$ $(i = 1, 2, \ldots, q)$. Consider two sequences I and J of the same size.

I: $R_1, R_2, \ldots, R_p, R'_1, R'_2, \ldots, R'_q$

J: $C_1, C_2, \ldots, C_p, C'_1, C'_2, \ldots, C'_q$.

Then the size of I or J is $2t$. Let i_k and j_k be the kth element of I and J, respectively $(k = 1, 2, \ldots, 2t)$. Join two points i_k in V_1 and j_k in V_2 with a line (i_k, j_k) $(k = 1, 2, \ldots, 2t)$. Construct a graph F with two point sets $\{i_k\}$ and $\{j_k\}$ and a line set $\{(i_k, j_k)\}$. Then F is a P_3-factor of $K_{m,n}$. This graph is called a *P_3-factor constructed with two sequences I and J.*
Construct r_1 sequences I_i such that $I_i = I + (i-1)m_0 \bmod m$ $(i = 1, 2, \ldots, r_1)$. Construct r_2 sequences J_j such that $J_j = J + (j-1)n_0 \bmod n$ $(j = 1, 2, \ldots, r_2)$. Construct r_1r_2 P_3-factors F_{ij} with I_i and J_j $(i = 1, 2, \ldots, r_1; j = 1, 2, \ldots, r_2)$. Then it is easy to show that F_{ij} are line-disjoint and that their sum is a P_3-factorization of $K_{m,n}$. $\square$

Lemma 5. *$(p, q) = 1$ and $q = 2q'$ (q': odd)*

$$m = 2(p+2q')(p+q'),\ n = 2(2p+q')(p+q')$$

$\Rightarrow$ *$K_{m,n}$ has a P_3-factorization.*

Proof. The proof is by construction (Algorithm II). Let $x = \frac{1}{3}(2n-m)$, $y = \frac{1}{3}(2m-n)$, $t = \frac{1}{3}(m+n)$, $r = 3mn/2(m+n)$. Then we have $x = 2p(p+q')$, $y = 2q'(p+q')$, $t = 2(p+q')^2$, $r = (p+2q')(2p+q')$. Let $r_1 = p+2q'$, $r_2 = 2p+q'$, $m_0 = m/r_1 = 2(p+q')$, $n_0 = n/r_2 = 2(p+q')$. Consider two sequences R and C of the same size $4(p+q')$.

R: $1, 1, 2, 2, \ldots, 2(p+q'), 2(p+q')$

C: $1, 2, \ldots, 4(p+q')-1, 4(p+q')$.

Construct p sequences R_i such that $R_i = R + 2(i-1)(p+q')$ $(i = 1, 2, \ldots, p)$. Construct p sequences C_i such that $C_i = (C + 2(i-1) \bmod 4(p+q')) + 4(i-

$1)(p+q')$ $(i=1,2,\ldots,p)$. Consider two sequences R' and C' of the same size $4(p+q')$.

R': $1, 2, \ldots, 4(p+q')-1, 4(p+q')$

C': $1, 3, \ldots, 2(p+q')-1, 1, 3, \ldots, 2(p+q')-1, 2, 4, \ldots, 2(p+q'),$
$2, 4, \ldots, 2(p+q')$.

Construct q' sequences R_i' such that $R_i' = R' + 4(i-1)(p+q') + 2p(p+q')$ $(i=1,2,\ldots,q')$. Construct q' sequences C_i' such that $C_i' = (C' + 2(i-1) + 2p \bmod 2(p+q')) + 2(i-1)(p+q') + 4p(p+q')$ $(i=1,2,\ldots,q')$.
Consider two sequences I and J of the same size $2t$.

I: $R_1, R_2, \ldots, R_p, R_1', R_2', \ldots, R_{q'}'$

J: $C_1, C_2, \ldots, C_p, C_1', C_2', \ldots, C_{q'}'$.

Construct r_1 sequences I_i such that $I_j = I + (i-1)m_0 \bmod m$ $(i=1,2,\ldots,r_1)$. Construct r_2 sequences J_j such that $J_j = J + (j-1)n_0 \bmod n$ $(j=1,2,\ldots,r_2)$. Construct r_1r_2 P_3-factors F_{ij} with I_i and J_j $(i=1,2,\ldots,r_1; j=1,2,\ldots,r_2)$. Then it is easy to show that F_{ij} are line-disjoint and that their sum is a P_3-factorization of $K_{m,n}$. □

Lemma 6. $(p,q)=1$ *and* $q=4q''$

$$m=(p+4q'')(p+2q''),\ n=2(p+q'')(p+2q'')$$

$\Rightarrow$ $K_{m,n}$ *has a* P_3*-factorization.*

Proof. The proof is by construction (Algorithm III). Let $x=\frac{1}{3}(2n-m)$, $y=\frac{1}{3}(2m-n)$, $t=\frac{1}{3}(m+n)$, $r=3mn/2(m+n)$. Then we have $x=p(p+2q'')$, $y=2q''(p+2q'')$, $t=(p+2q'')^2$, $r=(p+4q'')(p+q'')$. Let $r_1=p+4q''$, $r_2=p+q''$, $m_0=m/r_1=p+2q''$, $n_0=n/r_2=2(p+2q'')$. Consider two sequences R and C of the same size $2(p+2q'')$.

R: $1, 1, 2, 2, \ldots, p+2q'', p+2q''$

C: $1, 2, \ldots, 2(p+2q'')-1, 2(p+2q'')$.

Construct p sequences R_i such that $R_i = R + (i-1)(p+2q'')$ $(i=1,2,\ldots,p)$. Construct p sequences C_i such that $C_i = (C + 2(i-1) \bmod 2(p+2q'')) + 2(i-1)(p+2p'')$ $(i=1,2,\ldots,p)$. Consider two sequences R' and C' of the same size $4(p+2q'')$.

R': $1, 2, \ldots, 4(p+2q'')-1, 4(p+2q'')$

C': $1, 3, \ldots, 2(p+2q'')-1, 2, 4, \ldots, 2(p+2q''), 3, 5, \ldots, 2(p+2q'')-1, 1,$
$4, 6, \ldots, 2(p+2q''), 2$.

Construct q'' sequences R_i' such that $R_i' = R' + 4(i-1)(p+2q'') + p(p+2q'')$ $(i=1,2,\ldots,q'')$. Construct q'' sequences C_i' such that $C_i' = (C' + 4(i-1) +$

$2p \bmod 2(p+2q'')) + 2(i-1)(p+2q'') + 2p(p+2q'')$ $(i = 1, 2, \ldots, q'')$. Consider two sequences I and J of the same size $2t$.

I: $R_1, R_2, \ldots, R_p, R'_1, R'_2, \ldots, R'_{q''}$

J: $C_1, C_2, \ldots, C_p, C'_1, C'_2, \ldots, C'_{q''}$.

Construct r_1 sequences I_i such that $I_i = I + (i-1)m_0 \bmod m$ $(i = 1, 2, \ldots, r_1)$. Construct r_2 sequences J_j such that $J_j = J + (j-1)n_0 \bmod n$ $(j = 1, 2, \ldots, r_2)$. Construct $r_1 r_2$ P_3-factors F_{ij} with I_i and J_j $(i = 1, 2, \ldots, r_1; j = 1, 2, \ldots, r_2)$. Then it is easy to show that F_{ij} are line-disjoint and that their sum is a P_3-factorization of $K_{m,n}$. □

Applying Theorem 3 with Lemmas 4 to 6, it can be seen that for the parameters m and n satisfying Conditions (i)–(iv), $K_{m,n}$ has a P_3-factorization. This completes the proof of Theorem 2.

Corollary 2. *$K_{n,n}$ has a P_3-factorization if and only if* $n \equiv 0 \pmod{12}$.

References

[1] G. Chartrand and L. Lesniak, Graphs & digraphs, 2nd ed. (Wadsworth, California, 1986).
[2] F. Harary, Graph theory (Addison-Wesley, Massachusetts, 1972).