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q-Analogs of the inclusion-exclusion principle and permutations with restricted position

William Y.C. Chen

C-3, Mail Stop B265, Los Alamos National Laboratory, Los Alamos, NM 87545, USA

Gian-Carlo Rota

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

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Abstract

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We derive a q-analog of the principle of inclusion-exclusion, and use it to derive a q-analog of the Kaplansky–Riordan theory of permutations with restricted position. Some analogies with the theory of Mahonian statistics are pointed out at the end, leading to a conjectured relationship between the two.

1. Introduction

Phillip Hall was first to remark that the principle of inclusion-exclusion has a q-analog; he derived the basic properties of such a q-analog, and made use of it in his theory of p-groups. Soon afterwards, however, the discovery of Möbius inversion on a general partially ordered set, and the manifold applications they were eventually made of such a principle, distracted everyone's attention from the q-analog, which came to appear as a special case without particular significance.

We wish presently to return to the q-analog of the inclusion-exclusion principle, to derive some of its finer properties and to display some of its further applications. The q-analog of the Boolean algebra of subsets of a set is the lattice of subspaces of a vector space; for purposes of finite enumeration, one is forced to choose a finite-dimensional vector space over a finite field with q elements (we

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feel that this restriction will some day be lifted, but do not at present see a way out of it). The fact that joins and meets in the lattice of subspaces does not satisfy the distributive law that holds for union and intersection of sets was once seen as a fatal drawback. Recent work on q-analogs of some deep enumerative properties of finite sets has shown such fears to be groundless: so far, every combinatorial property pertaining to subsets of a finite set has eventually been ensconced in the more rarefied, but more refreshing, air of vector spaces. The proofs of the q-analogs of combinatorial properties of finite sets are often far more difficult, but they are more rewarding, in that they often display features of the problem that were invisible in dealing with sets. Thus, the failure of the distributive law of set theory has proved to be a purely psychological obstacle, at least as far as enumeration goes.

Our motivation is threefold. First, we derive the q-analog of inclusion-exclusion by direct reasoning on vector spaces, leading to identities which are notably stronger than those that could be obtained by straightforward application of Möbius inversion. We thus derive q-analogs of some variants of the inclusion-exclusion principle which have proved useful in the case of sets, but which have been missed in the case of vector spaces. Secondly, we dispose of the long-conjectured q-analog of the Kaplansky–Riordan theory of permutations with restricted position. An early proposal by Joni and Rota [18], though leading to a q-analog of Laguerre polynomials, cannot be pronounced a success. We give a complete answer to this problem, by showing that an honest q-analog of the theory of permutations with restricted position (namely, automorphisms with prescribed behavior) allows for only two possibilities, namely, a diagonal board and a rectangular board. The case of a diagonal board is the q-analog of the classical derangements problem. Lastly, we relate the q-analog of inclusion-exclusion to the enumeration problems arising in Mahonian statistics, showing the unimodality of q-derangement numbers for the major index, as well as remarking some analogies between vector space enumeration and Mahonian statistics enumeration, which we hope someone will be able to further explain.

This paper is self-contained, except for some elementary properties of the Möbius function.

2. The q-analog of inclusion-exclusion principle

We shall always assume that V is an n-dimensional vector space over the finite field of q elements, and we shall use $\{n\}!$ to denote the number of automorphisms on V. It is easy to see that

$$\{n\}! = (q^n - 1)(q^n - q) \cdots (q^n - q^{n-1}).$$

Define $\{0\}! = 1$. We set

$$\{n\} = q^{n-1}(q^n - 1), \quad \text{for } n > 0, \quad \text{and} \quad \{0\} = 1.$$

$$\langle n \rangle = q^n - 1, \qquad \langle n \rangle ! = \langle n \rangle \langle n-1 \rangle \cdots \langle 1 \rangle, \qquad \langle 0 \rangle ! = 1.$$

Therefore, we have

$$\{n\}! = \{n\}\{n-1\} \cdots \{1\} = q^{\binom{n}{2}} \langle n \rangle !.$$

Let $\left[{n \atop k}\right]$ denote the number of k-dimensional subspaces of V. It is well known that

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{\langle n \rangle !}{\langle k \rangle ! \, \langle n-k \rangle !}.$$

• Let $L(V)$ be the lattice of subspaces of V, and let X be a set of N elements. Suppose $\mathcal{P}$ is a set of properties on X indexed by V, denoted

$$\mathcal{P} = \{P_v \mid v \in V\}.$$

In other words, P_v is the set of elements in X which satisfy the property P_v. For any $x \in X$, define

$$V(x) = \{v \in V \mid x \in P_v\}.$$

The property set $\mathcal{P}$ is called *q-consistent* if $V(x)$ is a subspace of V for every $x \in X$.

For any q-consistent property set and a subspace T of V, we define $S(T)$ to be the number of elements x in X such that T is a subspace of $V(x)$, and for any $i \geqslant 0$, we define

$$S_i = \sum_{T \leqslant V, \dim T = i} S(T),$$

writing $T \leqslant V$ whenever T is a subspace of V (and not just a subset). Let $p(k)$, for $k = 0, 1, \ldots, n$, denote the number of elements x in X such that $\dim V(x) = k$.

The q-analog of the principle of inclusion-exclusion can now be stated as follows:

Theorem 2.1.

$$p(k) = \sum_{i=k}^{n} (-1)^{i-k} q^{\binom{i-k}{2}} \begin{bmatrix} i \\ k \end{bmatrix} S_i. \tag{2.1}$$

Proof. Let L be a subspace of V and $P(L)$ denote the number of elements x in X such that $V(x) = L$. Since $V(x)$ is a subspace of V for every $x \in X$, we have

$$S(L) = \sum_{L \leqslant T} P(T).$$

Denote by t and l the dimensions of T and L respectively. Since for $L \leq T$ the Möbius function $\mu(L, T)$ on $L(V)$ equals $(-1)^{t-l}q^{\binom{t-l}{2}}$, we have

$$\begin{aligned} p(k) &= \sum_{L \leq V, \dim L=k} P(L) = \sum_{L \leq V, \dim L=k} \sum_{L \leq T} (-1)^{t-l} q^{\binom{t-l}{2}} S(T) \\ &= \sum_{T \leq V, \dim T \geq k} \sum_{L \leq T, \dim L=k} (-1)^{t-k} q^{\binom{t-k}{2}} S(T) \\ &= \sum_{T \leq V, \dim T \geq k} \begin{bmatrix} t \\ k \end{bmatrix} (-1)^{t-k} q^{\binom{t-k}{2}} S(T) \\ &= \sum_{i \geq k} \begin{bmatrix} i \\ k \end{bmatrix} (-1)^{i-k} q^{\binom{i-k}{2}} \sum_{T \leq V, \dim T=i} S(T) \\ &= \sum_{i \geq k} \begin{bmatrix} i \\ k \end{bmatrix} (-1)^{i-k} q^{\binom{i-k}{2}} S_i. \end{aligned}$$

This completes the proof. □

In $L(V)$, it was proved by Crapo [3] and independently by Kovacs [21] that the number of complements of a subspace of dimension k is $q^{k(n-k)}$. We give a slightly stronger result below, which may be obtained by the method used in [15].

Proposition 2.2. *Let Z be an m-dimensional subspace of V. The number of l-dimensional subspaces W of V such that $Z \cap W = 0$ is $\left[\begin{smallmatrix} n-m \\ l \end{smallmatrix}\right] q^{lm}$.*

Proof. Let $V = Z \oplus Y$. Suppose W is an l-dimensional subspace of V such that $Z \cap W = 0$. Any $w \in W$ can be uniquely expressed as $w = z + y$ for some $z \in Z$ and $y \in Y$. Let W_Y be the projection of W on Y, i.e., let

$$W_Y = \{ y \mid w = z + y, w \in W, z \in Z, y \in Y \}.$$

W_Y is a subspace of Y. It is clear that $Z + W_Y = Z + W$ and $W_Y \cap Z = 0$. Hence $\dim W_Y = \dim W = l$. Since W_Y is a subspace of Y, there are $\left[\begin{smallmatrix} n-m \\ l \end{smallmatrix}\right]$ ways to choose W_Y. Suppose $y_1, y_2, \ldots, y_l$ is a basis of W_Y, and $z_1, z_2, \ldots, z_l$ are vectors in Z. Then the space spanned by $y_1 + z_1, y_2 + z_2, \ldots, y_l + z_l$, denoted $L(y_1 + z_1, y_2 + z_2, \ldots, y_l + z_l)$, satisfies

$$Z \cap L(y_1 + z_1, y_2 + z_2, \ldots, y_l + z_l) = 0.$$

Let $(z_1', z_2', \ldots, z_l')$ is a sequence of vectors in Z which are not identical to $(z_1, z_2, \ldots, z_l)$. Suppose $z_i \neq z_i'$. It is easy to see that $y_i + z_i' \notin L(y_1 + z_1, \ldots, y_l + z_l)$. Suppose

$$y_i + z_i' = \sum_{1 \leq j \leq l} \alpha_j (y_j + z_j).$$

Since $W_Y \cap Z = 0$, we have

$$y_i = \sum_{1 \leq j \leq l} \alpha_j y_j.$$

That is $\alpha_i = 1$ and $\alpha_j = 0$ for $j \neq i$. Thus, $z_i = z_i'$, a contradiction. We conclude that the subspaces $L(y_1 + z_1, y_2 + z_2, \ldots, y_l + z_l)$ are different for different choices of $(z_1, z_2, \ldots, z_l)$. Since there are q^m vectors in Z, it follows that there are q^{lm} choices for $(z_1, z_2, \ldots, z_l)$. This completes the proof. □

Using Theorem 2.1 we can obtain another expression for the number in the preceding Proposition.

Proposition 2.3.

$$q^{lm}\begin{bmatrix} n-m \\ l \end{bmatrix} = \sum_{k=0}^{m} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} n-k \\ l-k \end{bmatrix} \tag{2.2}$$

Proof. Let X be the set of all l-dimensional subspaces W of V. Define the property P_z for $z \in Z$ as

$$P_z = \{W \in X \mid z \in Z \cap W\},$$

therefore, $Z(W) = Z \cap W$ for $W \in X$. For any subspace T of Z, define $S(T)$ to be the number of subspaces W in X such that $T \subseteq Z \cap W$. It can be seen that

$$S(T) = \begin{bmatrix} n-k \\ l-k \end{bmatrix},$$

where $k = \dim T$. Hence

$$S_k = \sum_{T \subseteq Z, \dim T = k} S(T) = \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} n-k \\ l-k \end{bmatrix}.$$

Therefore, (2.2) follows from Theorem 2.1. □

The preceding identity becomes more transparent after introducing the q-backward difference operator of order m acting on n as

$$\nabla_q^m f(n) = \sum_{k=0}^{m} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} m \\ k \end{bmatrix} f(n-k).$$

Setting $t = n - l$, we may rewrite (2.2) as follows:

$$\nabla_q^m \begin{bmatrix} n \\ t \end{bmatrix} = q^{m(n-t)} \begin{bmatrix} n-m \\ t-m \end{bmatrix}. \tag{2.3}$$

This can be viewed as a q-analog of the identity $\nabla^m \binom{n}{t} = \binom{n-m}{t-m}$ for ordinary backward differences.

Example 2.4. *Let U and W be an l-dimensional and an m-dimensional subspace of V respectively. Let $X_n(l, m)$ denote the number of endomorphisms f of V such* *that for any* $0 \neq u \in U$, $f(u) \notin W$. *Then we have*

$$X_n(l, m) = \sum_{k \geq 0} (-1)^k q^{\binom{k}{2}} q^{n(n-k)} q^{mk} \begin{bmatrix} l \\ k \end{bmatrix}. \tag{2.4}$$

Proof. For any k-dimensional subspace T of U, define $S(T)$ to be the number of endomorphisms f of V such that for any $u \in T$, $f(u) \in W$. Then

$$S(T) = q^{km} q^{n(n-k)}.$$

Therefore, (2.4) follows from Theorem 2.1. □

3. The q-analog of premutations with restricted position

We are now ready to give the main definition of this paper, and we begin by recalling the classical definition of permutations with restricted position. Given an n by n matrix $B = (b_{ij})$ of zeros and ones (the board), the problem is to enumerate all n by n permutation matrices $P = (p_{ij})$ such that $p_{ij} b_{ij} = 0$ for all i and j. Such permutations are said to be restricted by the board B.

The problem is solved by Möbius inversion on the partially ordered set $\Sigma(B)$ (which turns out to be a simplicial complex) of partial violations matrices $Q = (q_{ij})$ such that $q_{ij} = 1$ only if $b_{ij} = 1$. The ordering is by inclusion of the set of ones. The actual enumeration leads to the rook polynomials which have been widely studied [4, 9–13, 20, 23–24]. Several attempts have been made to obtain a q-analog of this problem. It is our present objective to show why these attempts failed, and why only very few q-analogs of permutations with restricted position are possible. To this end, we begin with what we believe to be a clear statement of the problem. The crucial step is a proper definition of the q-analog of a board. We define a q-board to be a subspace S of the Cartesian product $V \times V$. We consider the problem of enumerating the set of automorphisms f of V which are restricted by the board S in the following sense: An automorphism f is said to be restricted by a board S whenever $(v, f(v)) \notin S$ for any $v \neq 0$.

The q-analog $\Sigma_q(B)$ of the simplicial complex of partial violations is defined as follows. Let $\Sigma_q(S)$ be the partially ordered set whose elements are all partial isomorphisms of V, namely, of all isomorphisms g from a subspace U of V to a subspace W of V such that $(u, g(u)) \in S$ for any $u \in U$. If $g' : U' \to W'$ is another such partial isomorphism, we define a partial order $g \leq g'$ whenever $U \leq U'$ and $W \leq W'$, and furthermore, g is a the restriction of g' to U.

The partially ordered set $\Sigma_q(S)$ is a q-analog of a simplicial complex, and an element of $\Sigma_q(S)$ may be called a face. Let $W_k(S)$, or W_k for short, denote the number of faces in $\Sigma_q(S)$ whose domain has dimension k.

Again in analogy with the classical case, for any automorphism f of V we let $\beta(f)$ be the subspace of V consisting of all violations, that is, of all elements v

such that $(v, f(v)) \in S$. Let N_k denote the number of automorphisms f such that $\dim \beta(f) = k$.

Again in analogy with the classical case, we have the following.

Theorem 3.1. *Let S be a q-board on V, and r_S be the number of automorphisms restricted by S. Then*

$$r_S = q^{\binom{n}{2}} \sum_{k=0}^{n} (-1)^k \langle n-k \rangle!\, W_k. \tag{3.1}$$

Proof. Let X be the set of all automorphisms of V, and P_v be the property

$$P_v = \{f \in X \mid (v, f(v)) \in S\}, \quad v \in V.$$

Hence,

$$V(f) = \{v \mid (v, f(v)) \in S\}, \quad f \in X.$$

Since S is a subspace of $V \times V$, for any $u, v \in \beta(f)$, we have $(u+v, f(u+v)) = (u+v, f(u)+f(v)) = (u, f(u)) + (v, f(v)) \in S$. Hence $V(f)$ is a subspace of V for any $f \in X$. For any subspace T of V, we use $S(T)$ to denote the number of automorphisms f such that T is a subspace of $V(f)$. Set

$$S_i = \sum_{T \leq V, \dim T = i} S(T).$$

For any automorphism f, it is clear that an i-dimensional subspace T is a subspace of $V(f)$ if and only if the induced isomorphism of f from T to $f(T)$ is an i-face in $\Sigma_q(S)$. However, for any i-face $g \in \Sigma_q(S)$, there are

$$(q^n - q^i)(q^n - q^{i+1}) \cdots (q^n - q^{n-1}) = q^{\binom{n}{2}} q^{-\binom{i}{2}} \langle n-i \rangle!$$

ways to extend it to an automorphism of V. Therefore,

$$S_i = q^{\binom{n}{2}} q^{-\binom{i}{2}} \langle n-i \rangle!\, W_i.$$

By Theorem 2.1, we have

$$N_k = q^{\binom{n}{2}} \sum_{i=k}^{n} (-)^{i-k} q^{\binom{i-k}{2}} q^{-\binom{i}{2}} \langle n-i \rangle! \begin{bmatrix} i \\ k \end{bmatrix} W_i. \tag{3.2}$$

Since $r_S = N_0$, this completes the proof. □

As our first example, we consider the q-analog of the classical derangements problem (the problem of computing the number of permutations without fixpoints).

Kung [22] considered the following q-analog of the derangement numbers. Let $H_n(q)$ be the number of automorphisms of V which have no one-dimensional invariant subspace, or equivalently, which have no nonzero eigenvectors. Kung obtained the following formula for $H_n(q)$ (see also [29]):

$$H_n(q) = \{n\}! \sum_{j_1+\cdots+j_{q-1}\le n} \prod_{k=1}^{q-1} \frac{(-1)^{j_k} q^{\binom{j_k}{2}}}{\{j_k\}!}.$$

Our q-analog of the derangement number is different. We consider the number of automorphisms of V which do not fix any vector in V except for 0. We call such automorphisms *q-derangements* of V. Let $G_n(q)$ or simply G_n denote the number of q-derangements on V. We shall give a formula for G_n which is analogous to the formula $D_n = n! \sum_{k=0}^{n} (-1)^k/k!$ for classical derangements.

Example 3.2. *Let G_n denote the number of q-derangements on V. Then we have*

$$G_n(q) = \{n\}! \sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}}/\{k\}!. \tag{3.3}$$

Proof. Let $I = \{(v, v) \mid v \in V\}$. Clearly, I is a q-board, and we have $W_i = \left[{n \atop i}\right]$. Then (3.3) follows from Theorem 3.1. $\square$

Let $F_n(k)$ denote the number automorphisms of V which fix exactly a k-dimensional subspace. From (3.2), we may have

$$\begin{aligned} F_n(k) &= \sum_{i=k}^{n} (-1)^{i-k} q^{\binom{i-k}{2}} \begin{bmatrix} i \\ k \end{bmatrix} \frac{\{n\}!}{\{i\}!}, \qquad (3.4)\\ &= \begin{bmatrix} n \\ k \end{bmatrix} \{n-k\}! \sum_{j=0}^{n-k} (-1)^j q^{\binom{j}{2}} q^{k(n-k-j)}/\{j\}!. \end{aligned}$$

It should be noted that $F_n(k) \neq \left[{n \atop k}\right] G_{n-k}(q)$ unlike the case of sets. However, we have the following identity for the $F_n(k)$:

$$\frac{\{n\}!}{\{t\}!} = \sum_{k=t}^{n} F_n(k) \begin{bmatrix} k \\ t \end{bmatrix}, \quad (t = 0, 1, \ldots, n). \tag{3.5}$$

Proof. By formula (3.4), the right-hand side of (3.5) equals

$$\begin{aligned} &\sum_{k=t}^{n} \begin{bmatrix} k \\ t \end{bmatrix} \sum_{i=k}^{n} (-1)^{i-k} q^{\binom{i-k}{2}} \begin{bmatrix} i \\ k \end{bmatrix} \frac{\{n\}!}{\{i\}!} \\ &= \sum_{k=0}^{n} \begin{bmatrix} k \\ t \end{bmatrix} \sum_{i=k}^{n} (-1)^{i-k} q^{\binom{i-k}{2}} \begin{bmatrix} i \\ k \end{bmatrix} \frac{\{n\}!}{\{i\}!} \\ &= \sum_{i=0}^{n} \frac{\{n\}!}{\{i\}!} \sum_{k=0}^{i} \begin{bmatrix} k \\ t \end{bmatrix} (-1)^{i-k} q^{\binom{i-k}{2}} \begin{bmatrix} i \\ k \end{bmatrix} \\ &= \sum_{i=0}^{n} \frac{\{n\}!}{\{i\}!} \sum_{k=0}^{i} \begin{bmatrix} i-k \\ t \end{bmatrix} (-1)^{k} q^{\binom{k}{2}} \begin{bmatrix} i \\ k \end{bmatrix} \end{aligned}$$

$$= \sum_{i=0}^{n} \frac{\{n\}!}{\{i\}!} \sum_{k=0}^{i} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} i \\ k \end{bmatrix} \begin{bmatrix} i-k \\ t \end{bmatrix}$$

$$= \sum_{i=0}^{n} \frac{\{n\}!}{\{i\}!} \nabla_q^i \begin{bmatrix} i \\ t \end{bmatrix} \qquad (\nabla_q^i \text{ acts on } i)$$

$$= \sum_{i=0}^{n} \frac{\{n\}!}{\{i\}!} q^{i(i-t)} \begin{bmatrix} 0 \\ t-i \end{bmatrix} \qquad \text{(by identity (2.3))}.$$

Since $[{}_{t-i}^{\ 0}] = \delta_{i,t}$, the above summation reduces to one term $\{n\}!/\{t\}!$. □

We note that implicit in (3.5) is the following recursion for $F_n(k)$:

$$F_n(t) = \sum_{k=t+1}^{n} F_n(k)\left(q^t(q^{t+1}-1)\begin{bmatrix} k \\ t+1 \end{bmatrix} - \begin{bmatrix} k \\ t \end{bmatrix}\right), \quad (t<n).$$

Recall the function $E_q(x)$ defined in [15]:

$$E_q(x) = \sum_{k\geqslant 0} \frac{x^k}{(1-q)(1-q^2)\cdots(1-q^k)}.$$

Then we have the following generating function for $G_n(q)$:

$$\sum_{n\geqslant 0} G_n(q)\frac{x^n}{\{n\}!} = \frac{E_q(x)}{1-x}.$$

When $n \to \infty$, we have

$$\frac{G_n(q)}{\{n\}!} \to E_q(1).$$

The derangement numbers D_n satisfy the recursive formulae $D_n = nD_{n-1} + (-1)^n$ and $D_n = (n-1)(D_{n-1} + D_{n-2})$. If we define $G_0(q) = 1$, we have the analogous formulae for $G_n(q)$:

$$G_n = \{n\}G_{n-1} + (-1)^n q^{\binom{n}{2}},$$

$$G_n = q^{n-1}((\langle n\rangle - 1)G_{n-1} + \{n-1\}G_{n-2}).$$

The next example we consider is the q-analog of a rectangular board. We shall see shortly that the q-rectangular board and the q-derangement board exhaust all possibilities for q-boards.

Example 3.3. *Let U and W be an l-dimensional and an m-dimensional subspace of V respectively. We may use $Y_n(l, m)$ denote the number of automorphisms of V restricted by the board $S = U \times W$ which is called the q-rectangular board of size $l \times m$. Then we have*

$$Y_n(l, m) = \{n\}! \sum_{k\geqslant 0} (-1)^k q^{\binom{k}{2}} \frac{\begin{bmatrix} l \\ k \end{bmatrix}\begin{bmatrix} m \\ k \end{bmatrix}}{\begin{bmatrix} n \\ k \end{bmatrix}}. \tag{3.6}$$

Proof. If $U_1 \leqslant U$, $W_1 \leqslant W$ and $\dim U_1 = \dim W_1 = k$, then there are $\{k\}!$ isomorphisms between U_1 and W_1. Thus, the number of k-faces in $\Sigma_q(S)$ is

$$W_k = \begin{bmatrix} l \\ k \end{bmatrix}\begin{bmatrix} m \\ k \end{bmatrix}\{k\}!, \tag{3.7}$$

Thus, (3.6) follows from Theorem 3.1. □

Another expression for $Y_n(l, m)$ can be obtained by the following argument. Let $V = U \oplus Y$ and $y_1, y_2, \ldots, y_{n-l}$ be a basis of Y. Then, r_S equals the number of automorphisms f of V such that $W \cap f(U) = 0$. By Proposition 2.2, we have $[{}^{n-m}_{\;\;l}]q^{lm}$ choices for $f(U)$. There are $\{l\}!$ isomorphisms from U to $f(U)$ and there are $(q^n - q^l)(q^n - q^{l+1}) \cdots (q^n - q^{n-1})$ ways to choose the $n - l$ independent vectors $z_1, z_2, \ldots, z_{n-l}$ which are the images of y_i's. Combining these counts, we obtain

$$\begin{aligned} Y_n(l, m) &= \begin{bmatrix} n-m \\ l \end{bmatrix} q^{lm}\{l\}!\,(q^n - q^l)(q^n - q^{l+1}) \cdots (q^n - q^{n-1}), \\ &= q^{2lm}\frac{\{n-l\}!\,\{n-m\}!}{\{n-l-m\}!}. \end{aligned} \tag{3.8}$$

We note that expression (3.6) can be obtained from (3.8) by using identity (2.2).

We shall next prove that the two preceding examples are essentially the only possible examples of q-boards. Thus, the q-analog of the theory of permutations with restricted position is necessarily more limited than the set-theoretic version.

Proposition 3.4. *Let S be a q-board of a vector space V. Then S can be expressed as*

$$S = (X \times Y) \oplus \{(u, f(u)) \mid u \in U\},$$

where X, Y, U and W are subspaces of V and f is an isomorphism from U to W satisfying $X \cap U = 0$ and $Y \cap W = 0$.

Proof. Let

$$X = \{x \in V \mid (x, 0) \in S\} \quad \text{and} \quad Y = \{y \mid (0, y) \in S\},$$

and let

$$\begin{aligned} A &= \{u \in V \mid \exists v \in V \text{ such that } (u, v) \in S\}, \\ B &= \{v \in V \mid \exists u \in V \text{ such that } (u, v) \in S\}. \end{aligned}$$

Clearly, X, Y, A and B are subspaces of V, $X \leqslant A$ and $Y \leqslant B$. Now write

$$A = X \oplus U \quad \text{and} \quad B = Y \oplus W.$$

Let u be a vector in U. Suppose there exist w_1 and w_2 in W such that $(u, w_1) \in S$ and $(u, w_2) \in S$. It follows that $(0, w_1 - w_2) \in S$, that is $w_1 - w_2 \in Y$. Since W is a subspace of V, we have $w_1 - w_2 \in W$. Thus, $w_1 - w_2 \in Y \cap W = 0$. So $w_1 = w_2$. In other words, for any $u \in U$, there is exactly one vector $w \in W$ such that $(u, w) \in S$. Hence we may define the mapping $f: U \to W$ as $f(u) = w$, where $w \in W$ is the only vector satisfying $(u, w) \in S$. It is obvious that f is linear. A similar argument shows that f is invertible, so that f is an isomorphism between U and W.

Let $(a, b) \in S$. Since $A = X \oplus U$, a can be uniquely written in the form $a = x + u$ for some $x \in X$ and $u \in U$. Similarly, b can be uniquely written in the form $b = y + w$ for some $y \in Y$ and $w \in W$. Clearly, $(x, y) = (x, 0) + (0, y) \in S$. Thus, $(u, w) = (a, b) - (x, y) \in S$. It follows that $(a, b) = (x, y) + (u, w)$, that is

$$S = (X \times Y) + \{(u, f(u)) \mid u \in U\}.$$

Since $A = X \oplus U$ and $B = Y \oplus W$, we have

$$X \times Y \cap \{(u, f(u)) \mid u \in U\} = 0.$$

This completes the proof. □

4. The Mahonian statistic q-analog

There are some connections between the q-counting of permutations with restricted position, as developed above, and the theory of Mahonian statistics. Let S_n be the set of all permutations on $\{1, 2, \ldots, n\}$. A Mahonian statistic on S_n is a function $\text{stat}(\pi)$ defined on every permutation π and taking nonnegative integer values, such that

$$\sum_{\pi \in S_n} q^{\text{stat}(\pi)} = [n]!,$$

where $[n] = 1 + q + q^2 + \cdots + q^{n-1}$ and $[n]! = [1][2] \cdots [n]$.

Two important examples of Mahonian statistics are the inversion number $\text{inv}(\pi)$ and the major index $\text{maj}(\pi)$, which are defined as follows. Given a permutation $\pi = a_1 a_2 \cdots a_n$ on $\{1, 2, \ldots, n\}$, the number of inversions of π, denoted by $\text{inv}(\pi)$, is the number of pairs (a_i, a_j) such that $i < j$ and $a_i > a_j$, and the major index of π, denoted by $\text{maj}(\pi)$, is defined by the sum of indices k $(k < n)$ such that $a_k > a_{k+1}$. For example, let $\pi = 3142756$, then $\text{inv}(\pi) = 5$ and $\text{maj}(\pi) = 1 + 3 + 5 = 9$.

In this section, we propose a Mahonian statistic q-analog of the theory of permutations with restricted positions. Let $A = (a_{ij})$ be an $n \times n$ (0, 1)-matrix. We define

$$I_q(A) = \sum_{\pi = i_1 i_2 \cdots i_n \in S_n} q^{\text{inv}(\pi)} a_{1i_1} a_{2i_2} \cdots a_{ni_n},$$

$$M_q(A) = \sum_{\pi = i_1 i_2 \cdots i_n \in S_n} q^{\text{maj}(\pi)} a_{1i_1} a_{2i_2} \cdots a_{ni_n}.$$

Proposition 4.1. *Let $A = (a_{ij})$ and A_{ij} be the matrix obtained from A by deleting the ith row and the jth column. Then we have*

$$I_q(A) = \sum_{k=1}^{n} q^{k-1} a_{1k} I_q(A_{1k}) \tag{4.1}$$

Proof. Let $i_1 i_2 \cdots i_n$ be a permutation on $\{1, 2, \ldots, n\}$. It is clear that

$$\mathrm{inv}(i_1 i_2 \cdots i_n) = (i_1 - 1) + \mathrm{inv}(i_2 \cdots i_n) = (i_1 - 1) + \mathrm{inv}(i_2' \cdots i_n').$$

where $i_2' \cdots i_n'$ is the permutation of $\{1, 2, \ldots, n-1\}$ obtained from $i_2 \cdots i_n$ by replacing every element i_t by $i_t - 1$ for $i_t > i_1$. Therefore, (4.1) follows immediately. □

In the following example, we shall use J to denote the $n \times n$ matrix with every element equal to 1, and use $K_m = (a_{ij})$ to denote the $n \times n$ triangular (0, 1)-*matrix* in which $a_{ij} = 1$ if and only if $i + j \leqslant m + 1$.

Example 4.2. *Let $T_{n,m}(q) = I_q(J - K_m)$. Then we have*

$$T_{n,m}(q) = q^{\binom{m+1}{2}}[n-m]^m [n-m]! \tag{4.2}$$

Proof. From Proposition 4.1, we have

$$T_{n,m}(q) = \sum_{k=m+1}^{n} q^{k-1} T_{n-1,m-1}(q).$$

Hence the conclusion. □

Let B be an $n \times n$ (0, 1)-matrix. We would like to treat B as the q-analog of a board in the sense of the Kaplansky–Riordan theory of permutations with restricted position, and to define a q-analog of the rook polynomials. We surmise that such a q-analog is to be related to either of the polynomials $I_q(J - B)$ or $M_q(J - B)$. We shall derive some identities (of a very preliminary nature, to be sure) which seem to support this surmise.

Let $J_{l,m}$ be the rectangular board of size $l \times m$, i.e., $J_{l,m}$ is an $n \times n$ matrix with entry 1 at the positions (i, j) for $1 \leqslant i \leqslant l$, $1 \leqslant j \leqslant m$, and 0 at other positions.

Example 4.3. *Let $R_n(l, m)$ denote $I_q(J - J_{l,m})$. Then we have*

$$R_n(l, m) = q^{lm} \frac{[n-l]!\,[n-m]!}{[n-l-m]!}. \tag{4.3}$$

Proof. From Proposition 4.1 we have

$$R_n(l, m) = \sum_{i=1}^{n-m} q^{m+i-1} R_{n-1}(l-1, m) = q^m [n-m] R_{n-1}(l-1, m).$$

Therefore, (4.3) follows immediately. □

By identity (2.2) or by comparing (3.6) and (3.8), we obtain another expression for $R_n(l, m)$, namely

$$R_n(l, m) = [n]! \sum_{l \geq 0} (-1)^l q^{\binom{l}{2}} \frac{\left[{k \atop l}\right]\left[{m \atop l}\right]}{\left[{n \atop l}\right]}. \tag{4.4}$$

We have not been able to derive the above expression by Möbius inversion, despite its suggestive look. Comparison with (3.6) suggests a connection between the vector space q-analog developed in this paper and the Mahonian statistic q-analog proposed in this section. Following this analogy, one may conjecture a Mahonian statistic analog of derangements. In fact, a simple computation shows that the inversion number is not the right choice for such an analog, so that we are led to consider the major index, following a suggestion of I. Gessel and M. Wachs. Let I be the identity matrix of order n, and let $D_n(q)$ denote $M_q(J - I) = \sum_{\pi \in \boldsymbol{D}_n} q^{\mathrm{maj}(\pi)}$, where $\boldsymbol{D}_n$ is the set of all derangements on $\{1, \ldots, n\}$. $D_n(q)$ has the following expression analogous to that of the q-derangement number $G_n(q)$ [7, 30]:

$$D_n(q) = [n]! \sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} / [k]!,$$

We establish the following property of the above q-derangement numbers, which we believe to be new.

Proposition 4.4. *The q-derangement number $D_n(q)$ is unimodal for all n.*

Proof. The following recursion for $D_n(q)$ is easily established:

$$D_n(q) = [n]D_{n-1}(q) + (-1)^n q^{\binom{n}{2}}. \tag{4.5}$$

We show the following: (1). the coefficient of $q^{\binom{n}{2}-1}$ is nonzero. (2). $D_n(q)$ contains the term $q^{\binom{n}{2}}$ if and only if n is even. These assertions are clearly true for $n = 1, 2$. Suppose they hold for $n - 1$. Since $\binom{n-1}{2} + n - 1 = \binom{n}{2}$, from the recursion (4.5) it follows that the coefficient of $q^{\binom{n}{2}-1}$ in $D_n(q)$ is nonzero. It is also clear that $D_n(q)$ has the term $q^{\binom{n}{2}}$ if and only if n is even. Hence, statements (1) and (2) follow for any n by induction.

For $n = 1, 2$, $D_n(q)$ is clearly unimodal. Suppose $D_{n-1}(q)$ is unimodal. Then we have that $[n]D_{n-1}$ is also unimodal because $[n]$ is log-concave without internal zeros [28]. When n is even, we know that $[n]D_{n-1}(q)$ does not contain the term $q^{\binom{n}{2}}$ and that the coefficient of $q^{\binom{n}{2}-1}$ in $[n]D_{n-1}(q)$ is nonzero. Therefore, $D_n(q) = [n]D_{n-1} + q^{\binom{n}{2}}$ is unimodal when n is even. Similarly, we may show that $D_n(q)$ is unimodal when n is odd. Hence, it follows by induction that $D_n(q)$ is unimodal for all n. $\square$

In general, the coefficients of $D_n(q)$ are not symmetric around the middle as in other q-analogs, however, the maximum coefficient seems to appear exactly at the middle. We have verified this conjecture for $n \leq 20$.

Conjecture 4.5. The maximum coefficient appearing in $D_n(q)$ is that of $q^{\lceil n(n-1)/4 \rceil}$, where $\lceil x \rceil$ is the usual notation for the smallest integer not less than x.

By way of evidence for the above conjecture, we list $D_n(q)$ for $n \leq 8$.

$$\begin{aligned}
D_1(q) &= 0,\\
D_2(q) &= q,\\
D_3(q) &= q + q^2,\\
D_4(q) &= q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6,\\
D_5(q) &= q + 3q^2 + 5q^3 + 7q^4 + 8q^5 + 8q^6 + 6q^7 + 4q^8 + 2q^9,\\
D_6(q) &= q + 4q^2 + 9q^3 + 16q^4 + 24q^5 + 32q^6 + 37q^7 + 38q^8 + 35q^9 + 28q^{10}\\
&\quad + 20q^{11} + 12q^{12} + 6q^{13} + 2q^{14} + q^{15},\\
D_7(q) &= q + 5q^2 + 14q^3 + 30q^4 + 54q^5 + 86q^6 + 123q^7 + 160q^8 + 191q^9\\
&\quad + 210q^{10} + 214q^{11} + 202q^{12} + 176q^{13} + 141q^{14} + 104q^{15} + 69q^{16}\\
&\quad + 41q^{17} + 21q^{18} + 9q^{19} + 3q^{20},\\
D_8(q) &= q + 6q^2 + 20q^3 + 50q^4 + 104q^5 + 190q^6 + 313q^7 + 473q^8 + 663q^9\\
&\quad + 868q^{10} + 1068q^{11} + 1240q^{12} + 1362q^{13} + 1417q^{14} + 1398q^{15}\\
&\quad + 1307q^{16} + 1157q^{17} + 968q^{18} + 763q^{19} + 564q^{20} + 388q^{21}\\
&\quad + 247q^{22} + 143q^{23} + 74q^{24} + 33q^{25} + 12q^{26} + 3q^{27} + q^{28}.
\end{aligned}$$

A possible q-analog of ménage numbers could be the following. Let M_n be the set of all ménage permutations, i.e., the set of all permutations $\pi = a_1 a_2 \cdots a_n$ on $\{1, 2, \ldots, n\}$ such that $a_i \neq i$ or $i+1 \pmod{n}$. Define

$$P_n(q) = \sum_{\pi \in M_n} q^{\mathrm{inv}(\pi)}, \qquad Q_n(q) = \sum_{\pi \in M_n} q^{\mathrm{maj}(\pi)}.$$

We conjecture that $P_n(q)$ and $Q_n(q)$ are unimodal.

Some new statistic may be needed to get a q-analog of the formula for ménage numbers. As a partial result in this direction, we shall derive an analog of the lemma proved by Kaplansky in his solution of the ménage problem [19].

Let $[1, n]$ denote the set $\{1, 2, \ldots, n\}$. For any subset S of $[1, n]$, we associate a sequence $w(S) = w_1 w_2 \cdots w_n$ of 0's and 1's defined as

$$w_i = \begin{cases} 1 & \text{if } i \in S,\\ 0 & \text{if } i \notin S. \end{cases}$$

We define a statistic on a sequence of 0's and 1's as follows. Let $w = w_1 w_2 \cdots w_n$ be a $(0, 1)$-sequence. We shall use w' to denote the sequence obtained from w by deleting w_i if $w_i = 0$ and $w_{i-1} = 1$. For example, $w = 0110100100010$, then $w' = 011101001$. Define

$$s(w) = |\{(i, j) \mid (w_i, w_j) = (0, 1),\ 1 \leq i < j \leq n\}|,$$

and set $t(w) = s(w')$. Then the statistic $t(w)$ gives a q-analog of Kaplansky's lemma. Let $L_{n,k}$ be the set of all k-subsets S of $[1, n]$ such that no consecutive numbers are contained in S, and let $C_{n,k}$ be the set of all k-subsets S of $[1, n]$ such that no consecutive numbers (mod n) are contained in S. Then one can prove the following proposition by induction.

Proposition 4.6.

$$\sum_{S \in L_{n,k}} q^{t(w(S))} = \begin{bmatrix} n-k+1 \\ k \end{bmatrix},$$

$$\sum_{S \in C_{n,k}} q^{t(w(S))} = \frac{[n]}{[n-k]} \begin{bmatrix} n-k \\ k \end{bmatrix}.$$

Compare these identities with the Mahonian definition of q-binomial coefficients: $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ can be explained as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{S \subseteq M, |S|=k} q^{\mathrm{inv}(w(S))} = \sum_{S \subseteq M, |S|=k} q^{\mathrm{maj}(w(S))}.$$

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