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An inequality between Jordan–von Neumann constant and James constant

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ABSTRACT

Let X be a non-trivial Banach space. L. Maligranda conjectured $C_{NJ}(X) \leq 1 + J(X)^2/4$ for James constant $J(X)$ and von Neumann–Jordan constant $C_{NJ}(X)$ for X. Recently, J. Alonso et al. gave a proof of it and conjectured that $C_{NJ}(X) \leq J(X)$ is also valid. In this paper, we show that this conjecture is true.

1. Introduction

We shall assume throughout this paper that X stands for a non-trivial Banach space, and use B_X and S_X to denote the unit ball and unit sphere of X, respectively. Many recent studies have focused on the von Neumann–Jordan (NJ) constant and James constant (cf. [1–16]). The constant

$$J(X) = \sup\{\min(\|x+y\|, \|x-y\|) : x, y \in S_X\}$$

is called the non-square or James constant of X. It is well known that [5,6]

(i) $\sqrt{2} \leq J(X) \leq 2$.
(ii) $J(X) = \sup\{\min(\|x+y\|, \|x-y\|) : x, y \in B_X\}$.
(iii) If $1 \leq p \leq \infty$ and *dim* $L_p(\mu) \geq 2$, then $J(L_p(\mu)) = \max\{2^{1/p}, 2^{1-1/p}\}$.

The von Neumann–Jordan constant of a Banach space X was introduced by Clarkson [3] as the smallest constant C for which

$$\frac{1}{C} \leq \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \leq C$$

holds for all $x, y \in X$ with $(x, y) \neq (0, 0)$. An equivalent definition of the NJ constant is found in [8] as the following form:

$$C_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x \in S_X, y \in B_X\right\}.$$

Recently, J. Alonso et al. defined the constant

$$C'_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{4} : x, y \in S_X\right\}.$$

Now let us collect some properties of these constants in [1,3,8,9]:

(1) $C_{NJ}(X) = C_{NJ}(X^*)$.

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(2) $1 \leq C_{NJ}(X) \leq 2$; X is a Hilbert space if and only if $C_{NJ}(X) = 1$.
(3) X is uniformly non-square if and only if $C_{NJ}(X) < 2$.
(4) For any non-trivial Banach space X,

$$\frac{J(X)^2}{2} \leq C_{NJ}(X) \leq 1 + \frac{J(X)^2}{4}. \tag{1.1}$$

(5) For any Banach space X,

$$C_{NJ}(X) \leq 2\left[1 + C'_{NJ}(X) - \sqrt{2C'_{NJ}(X)}\right]. \tag{1.2}$$

(6) $C'_{NJ}(X) \leq J(X)$.
(7) If $1 \leq p \leq \infty$ and $\dim L_p(\mu) \geq 2$, then $C_{NJ}(L_p(\mu)) = \max\{2^{\frac{2}{p}-1}, 2^{1-\frac{2}{p}}\}$.

In [11,12], L. Maligranda conjectured $C_{NJ}(X) \leq 1 + J(X)^2/4$ for any Banach space X. Recently, J. Alonso et al. gave a proof of it in [1], and another proof can also be found in [16]. In 2009 Wang and Pang [14] obtained the following inequality which improves this conjecture:

$$C_{NJ}(X) \leq J(X) + \sqrt{J(X) - 1}\left\{\sqrt{1 + (1 - \sqrt{J(X) - 1})^2} - 1\right\}.$$

In this paper, we also consider the constant

$$A_2(X) = \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} : x, y \in S_X\right\}$$

introduced by Baronti et al. in [2]. The aim of our main results is to prove $A_2(X) \leq \frac{3J(X)}{2} - \frac{J(X)^2}{4}$ and $C_{NJ}(X) \leq J(X)$.

2. Proof of the main results

First, we recall that the modulus of convexity of a Banach space X is defined for $\varepsilon \in [0, 2]$ as

$$\delta_X(\varepsilon) = \inf\{1 - \|x+y\|/2 : x, y \in S_X, \|x-y\| \geq \varepsilon\},$$

where "S_X" and "$\geq$" can be replaced by "B_X" and "$=$" respectively. Obviously, the modulus of convexity is a nondecreasing function in $[0, 2]$. Moreover, the function $\frac{\delta_X(\varepsilon)}{\varepsilon}$ is also nondecreasing on $(0, 2]$. It is worth noting that $J(X) = 2(1 - \delta(J(X)))$ is valid for any uniformly non-square space X.

Theorem 2.1. *For any Banach space X, we have*

$$2A_2(X) \leq 3J(X) - \frac{J(X)^2}{2}. \tag{2.1}$$

Proof. For simplicity we shall denote $J(X)$ by J, and we can assume that $J < 2$. If $\max\{\|x+y\|, \|x-y\|\} \leq J$, then $\|x+y\| + \|x-y\| \leq 2J \leq 3J - \frac{J^2}{2}$. So we may also assume that $\varepsilon := \|x-y\| \geq J$, otherwise, we may extract $\varepsilon := \|x+y\| \geq J$.

(1) If $J \leq \varepsilon \leq 2J - \frac{J^2}{2}$. Since $J = 2(1 - \delta_X(J))$, we have

$$\begin{aligned}\|x+y\| + \|x-y\| &\leq \varepsilon + 2 - 2\delta_X(\varepsilon) \\ &\leq 2J - \frac{J^2}{2} + 2 - 2\delta_X(J) \\ &= 3J - \frac{J^2}{2}.\end{aligned}$$

(2) If $2J - \frac{J^2}{2} \leq \varepsilon \leq 2$. By $\frac{2-J}{2J} = \frac{\delta_X(J)}{J} \leq \frac{\delta_X(2J - \frac{J^2}{2})}{2J - \frac{J^2}{2}}$, we have

$$\begin{aligned}\|x+y\| + \|x-y\| &\leq \varepsilon + 2 - 2\delta_X\left(2J - \frac{J^2}{2}\right) \\ &\leq 4 - \frac{2-J}{J}\left[2J - \frac{J^2}{2}\right] \\ &= 3J - \frac{J^2}{2}.\end{aligned}$$

Hence (2.1) is valid.

Let X^* be the dual space of X. Recently, J. Alonso et al. deduced an estimate

$$|J(X^*) - J(X)| \leq \frac{2-\sqrt{2}}{2}$$

by using the inequalities (see [8])

$$2J(X) - 2 \leq J(X^*) \leq \frac{J(X)}{2} + 1. \tag{2.2}$$

Here, we can obtain a better estimate by using Theorem 2.1. □

Corollary 2.1. *For any non-trivial Banach space X. Then*

$$|J(X^*) - J(X)| \leq \max\left\{\frac{2J(X)-J(X)^2}{4}, \frac{2J(X^*)-J(X^*)^2}{4}\right\} \leq \frac{\sqrt{2}-1}{2}.$$

Proof. Note that $A_2(X) = A_2(X^*)$ (see [2]). Since for any Banach space X, $J(X) \leq A_2(X)$, from Theorem 2.1 we have $J(X^*) - J(X) \leq A_2(X^*) - J(X) = A_2(X) - J(X) \leq \frac{J(X)}{2} - \frac{J(X)^2}{4}$, and a similar one for $J(X) - J(X^*)$. □

Now, in order to give a simple proof of $C_{NJ}(X) \leq J(X)$, first we have the following lemma.

Lemma 2.1. *Let X be a Banach space. Then*

$$\|x+y\|^2 + \|x-y\|^2 \leq 2J(X) + 4\sqrt{J(X)-1} \tag{2.3}$$

for any $x, y \in S_X$.

Proof. Let $J = J(X)$. We can assume that $J < 2$. If $\max\{\|x+y\|, \|x-y\|\} \leq 1+\sqrt{J-1}$, then $\|x+y\|^2+\|x-y\|^2 \leq 2J+4\sqrt{J-1}$. On the other hand, we may assume that $\varepsilon := \|x-y\| \geq 1+\sqrt{J-1}$. Since $\frac{2-J}{2J} = \frac{\delta_X(J)}{J} \leq \frac{\delta_X(\varepsilon)}{\varepsilon}$, then

$$\|x+y\|^2 + \|x-y\|^2 \leq \varepsilon^2 + [2-2\delta_X(\varepsilon)]^2 \leq \varepsilon^2 + \left[2 - \frac{2-J}{J}\varepsilon\right]^2.$$

Let $h(t) = t^2 + [2 - \frac{2-J}{J}t]^2$. Since $h(t)$ is increasing for $t \geq \frac{J(2-J)}{J^2-2J+2}$, and $\frac{J(2-J)}{J^2-2J+2} \leq 1+\sqrt{J-1} \leq \varepsilon \leq 2$, we have that

$$\|x+y\|^2 + \|x-y\|^2 \leq h(\varepsilon) \leq h(2).$$

To complete the proof of (2.3) we only need to see that $h(2) \leq 2J+4\sqrt{J-1}$. Now, $h(2)-2J-4\sqrt{J-1} = -\frac{2f(J)}{J^2}$, where $f(J) = -8+16J+2(\sqrt{J-1}-5)J^2+J^3$. To see that $f(J) \geq 0$, take $\alpha = \sqrt{J-1}$. Then $f(J) = (\alpha^4+4\alpha^3-1)(\alpha-1)^2$, with $\sqrt{\sqrt{2}-1} \leq \alpha \leq 1$. Finally, since $g(\alpha) = \alpha^4+4\alpha^3-1$ is increasing for $\sqrt{\sqrt{2}-1} \leq \alpha \leq 1$, we have $g(\alpha) \geq g(\sqrt{\sqrt{2}-1}) = 2-2\sqrt{2}+4(\sqrt{2}-1)^{3/2} > 0$. □

Theorem 2.2. *For any non-trivial Banach space X we have*

$$C_{NJ}(X) \leq J(X). \tag{2.4}$$

Proof. By Lemma 2.1, we have

$$C'_{NJ}(X) \leq \frac{J(X)}{2} + \sqrt{J(X)-1}. \tag{2.5}$$

From (1.2) we know that $C_{NJ}(X) \leq 2(1+C'_{NJ}(X)-\sqrt{2C'_{NJ}(X)})$. The function $g(t) := 2(1+t-\sqrt{2t})$ is increasing in [1,2], so (2.4) follows from (2.5). □

3. An example

In this section we shall compute the value of $C_{NJ}(X), J(X)$ for some space X.

Lemma 3.1 ([8]). *Let $X = (X, \|\cdot\|)$ be a non-trivial Banach space, and $X_1 = (X_1, \|\cdot\|_1)$, where $\|\cdot\|_1$ is an equivalent norm on X satisfying, for $\alpha, \beta > 0$ and $x \in X$,*

$$\alpha\|x\| \leq \|x\|_1 \leq \beta\|x\|.$$

Then

$$\frac{\alpha}{\beta}J(X) \leq J(X_1) \leq \frac{\beta}{\alpha}J(X)$$

and

$$\frac{\alpha^2}{\beta^2}C_{NJ}(X) \le C_{NJ}(X_1) \le \frac{\beta^2}{\alpha^2}C_{NJ}(X).$$

Example. Let $\lambda > 0$, $X_\lambda = \mathbb{R}^2$ endowed with the norm

$$|x|_\lambda = (\|x\|_p^2 + \lambda\|x\|_q^2)^{\frac{1}{2}}.$$

(i) If $2 \le p \le q \le \infty$, then $J(X_\lambda) = 2\sqrt{\frac{\lambda+1}{2^{\frac{2}{p}}+\lambda 2^{\frac{2}{q}}}}$, and $C_{NJ}(X_\lambda) = C'_{NJ}(X_\lambda) = \frac{2(\lambda+1)}{2^{\frac{2}{p}}+\lambda 2^{\frac{2}{q}}}$.

(ii) If $1 \le p \le q \le 2$, then $J(X_\lambda) = \sqrt{\frac{2^{\frac{2}{p}}+\lambda 2^{\frac{2}{q}}}{\lambda+1}}$, and $C_{NJ}(X_\lambda) = C'_{NJ}(X_\lambda) = \frac{2^{\frac{2}{p}}+\lambda 2^{\frac{2}{q}}}{2(\lambda+1)}$.

Proof. (i) First, we show that the following inequality is valid

$$\|x\|_p^2(1 + \lambda 2^{\frac{2}{q}-\frac{2}{p}}) \le |x|_\lambda^2 \le (\lambda + 1)\|x\|_p^2. \tag{3.1}$$

In fact, the right is obvious, and by Hölder's inequality we have:

$$\|x\|_p^2 \le 2^{\frac{2}{p}-\frac{2}{q}}\|x\|_q^2.$$

Therefore, (3.1) is valid. From Lemma 3.1, $J((\mathbb{R}^2, \|\cdot\|_p)) = 2^{1-\frac{1}{p}}$ and $C_{NJ}((\mathbb{R}^2, \|\cdot\|_p)) = 2^{1-\frac{2}{p}}$, we have

$$C_{NJ}(X_\lambda) \le \frac{2(\lambda+1)}{2^{\frac{2}{p}} + \lambda 2^{\frac{2}{q}}},$$

and

$$J(X_\lambda) \le 2\sqrt{\frac{\lambda+1}{2^{\frac{2}{p}} + \lambda 2^{\frac{2}{q}}}}.$$

Now letting $x = (\frac{a}{2^{\frac{2}{p}}}, \frac{a}{2^{\frac{2}{p}}})$, and $y = (\frac{a}{2^{\frac{2}{p}}}, -\frac{a}{2^{\frac{2}{p}}})$, where $a = \frac{2^{\frac{2}{p}}}{\sqrt{2^{\frac{2}{p}}+\lambda 2^{\frac{2}{q}}}}$. Then $\|x\|_\lambda = \|y\|_\lambda = 1$, $x + y = (\frac{2a}{2^{\frac{2}{p}}}, 0)$, and $x - y = (0, \frac{2a}{2^{\frac{2}{p}}})$. Hence,

$$C_{NJ}(X_\lambda) \ge C'_{NJ}(X_\lambda) \ge \frac{2(\lambda+1)}{2^{\frac{2}{p}} + \lambda 2^{\frac{2}{q}}}$$

and

$$J(X_\lambda) \ge 2\sqrt{\frac{\lambda+1}{2^{\frac{2}{p}} + \lambda 2^{\frac{2}{q}}}}.$$

(ii) By using Hölder's inequality, we have

$$\|x\|_q^2(\lambda + 1) \le |x|_\lambda^2 \le (\lambda + 2^{\frac{2}{p}-\frac{2}{q}})\|x\|_q^2.$$

Applying Lemma 3.1, $J((\mathbb{R}^2, \|\cdot\|_q)) = 2^{\frac{1}{q}}$ and $C_{NJ}((\mathbb{R}^2, \|\cdot\|_q)) = 2^{\frac{2}{q}-1}$, we have the following inequalities

$$C_{NJ}(X_\lambda) \le \frac{2^{\frac{2}{p}} + \lambda 2^{\frac{2}{q}}}{2(\lambda+1)},$$

and

$$J(X_\lambda) \le \sqrt{\frac{2^{\frac{2}{p}} + \lambda 2^{\frac{2}{q}}}{\lambda+1}}.$$

Now letting $x = (\frac{a}{2^{\frac{2}{p}}}, 0)$, and $y = (0, -\frac{a}{2^{\frac{2}{p}}})$, where $a = \frac{2^{\frac{2}{p}}}{\sqrt{\lambda+1}}$. Then $\|x\|_\lambda = \|y\|_\lambda = 1$, $x + y = (\frac{a}{2^{\frac{2}{p}}}, -\frac{a}{2^{\frac{2}{p}}})$, and $x - y = (\frac{a}{2^{\frac{2}{p}}}, \frac{a}{2^{\frac{2}{p}}})$. Hence

$$C_{NJ}(X_\lambda) \ge C'_{NJ}(X_\lambda) \ge \frac{2^{\frac{2}{p}} + \lambda 2^{\frac{2}{q}}}{2(\lambda+1)},$$

and

$$J(X_\lambda) \geq \sqrt{\frac{2^{\frac{2}{p}} + \lambda 2^{\frac{2}{q}}}{\lambda + 1}}.$$

Therefore, the proof of the example is complete. □

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References

[1] J. Alosnso, P. Matrín, P.L. Papini, wheeling around von Neumann–Jordan constant in Banach spaces, Studia Math. 188 (2008) 135–152.
[2] M. Baronti, E. Casini, P.L. Papini, Trangles inscribed in a semicricle, in Minkowski planes and in normed spaces, J. Math. Anal. Appl. 252 (2000) 124–146.
[3] J.A. Clarkson, The von Neumann–Jordan constant for the Lebesgue space, Ann. of Math. 38 (1937) 114–115.
[4] S. Dhompongsa, P. Piraisangjun, S. Saejung, Generalized Jordan–von Neumann constants and uniform normal structure, Bull. Austral. Math. Soc. 67 (2003) 225–240.
[5] J. Gao, K.S. Lau, On the geometry of spheres in normed linear spaces, J. Aust. Math. Soc. Ser. A 48 (1990) 101–112.
[6] J. Gao, K.S. Lau, On two classes of Banach spaces with uniform normal structure, Studia Math. 99 (1991) 41–56.
[7] P. Jordan, J. von Neumann, On inner products in linear metric spaces, Ann. of Math. 36 (1935) 719–723.
[8] M. Kato, L. Maligranda, Y. Takahashi, On James and Jordan–von Neumann constants and the normal structure coefficient of Banach spaces, Studia Math. 144 (2001) 275–292.
[9] M. Kato, Y. Takahashi, Von Neumann–Jordan constant for Lebesgue–Bochner spaces, J. Inequal. Appl. 2 (1998) 89–97.
[10] M. Kato, L. Maligranda, On James and Jordan–von Neumann constants of Lorentz sequence spaces, J. Math. Anal. Appl. 258 (2001) 457–465.
[11] L. Maligranda, L. Nikolova, L.E. Persson, T. Zachariades, On n-th James and Khintchine constants of Banach spaces, Math.Inequalities and Appl. 11 (2008) 1–22.
[12] L. Maligranda, On James nonsquare and related constants of Banach spaces, preprint.
[13] S. Saejung, On James and von Neumann–Jordan constants and sufficient conditions for the fixed point property, J. Math. Anal. Appl. 323 (2006) 1018–1024.
[14] F. Wang, B. Pang, Some inequalities concerning the James constant in Banach spaces, J. Math. Anal. Appl. 353 (1) (2009) 305–310.
[15] C. Yang, F. Wang, On a new geometric constant related to the von Neumann–Jordan constant, J. Math. Anal. Appl. 324 (2006) 555–565.
[16] C. Yang, A note of Jordan–von Neumann constant and James constant, J. Math. Anal. Appl. 357 (1) (2009) 98–102.