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All wheels with two missing consecutive spokes are chromatically unique

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Abstract

This paper shows that every wheel of order n, $n \geqslant 6$, with two missing consecutive spokes is chromatically unique.

1. Introduction

For a given graph G, let $P(G,\lambda)$ be its chromatic polynomial. Two graphs G and H are called *chromatically equivalent* if $P(G,\lambda) = P(H,\lambda)$, and a graph G is called *chromatically unique* if $P(H,\lambda) = P(G,\lambda)$ implies that H is isomorphic to G for any graph H.

Let W_n be the wheel of order n and let $W(n,k)$ be the graph obtained from W_n by deleting all but k consecutive spokes, where $n \geqslant 4$ and $1 \leqslant k \leqslant n-1$. The chromaticity of $W(n,k)$, where $1 \leqslant k \leqslant n-1$, is still not clear. In [8], Koh and Teo proposed the following two problems.

Problem 1. For $n \geqslant 9$, which graphs of the form $W(n,n-2)$ are chromatically unique?

Problem 2. Study the chromaticity of the graphs of the form $W(n,n-t)$, for $t = 3,4$, etc.

Chia [2] showed that $W(n,n-2)$ is chromatically unique for any even integer $n \geqslant 6$. In [1], $W(5,3)$ was proved to be chromatically unique. Very recently, the authors proved [6] that for any odd integer $n \geqslant 9$, $W(n,n-2)$ is chromatically unique, and

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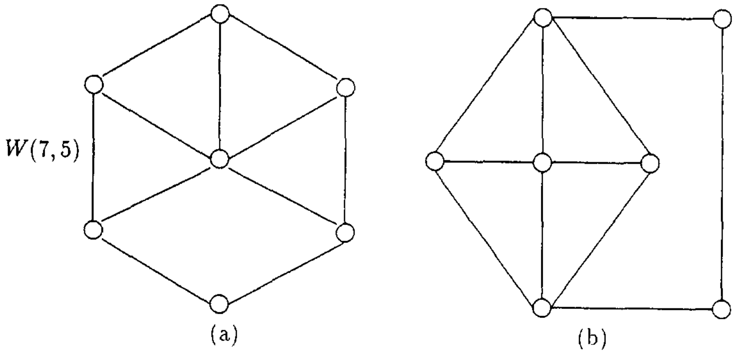


Fig. 1.

just one graph, shown in Fig. 1(b), is chromatically equivalent to $W(7,5)$, but not isomorphic to it. Thus Problem 1 is completely solved.

It is easy to check that $W(4,1)$ and $W(5,2)$ are chromatically unique. In this paper we show that $W(n,n-3)$ is chromatically unique for $n\geqslant 6$. Thus Problem 2 is solved for $t=3$.

2. Some known results

In this section, we introduce some known results, which will be used in the following sections.

Lemma 2.1. *Let G and H be two chromatically equivalent graphs. Then,*

(a) $|V(G)|=|V(H)|$, $|E(G)|=|E(H)|$ *and* $t_1(G)=t_1(H)$, *where* $t_1(G)$ *is the number of triangles in the graph G;*

(b) $t_2(G)-2t_3(G)=t_2(H)-2t_3(H)$, *where* $t_2(G)$ *and* $t_3(G)$ *are the number of cycles of order 4 without chords in G and the number of* K_4*'s in G, respectively;*

(c) *G and H have the same number of components, and if G and H are connected, then G and H contain the same number of blocks; and*

(d) $\chi(G)=\chi(H)$.

Let G be a graph. For a subset S of $V(G)$, let $G[S]$ or $[S]$ denote the subgraph of G induced by S. $G[S]$ is called an *induced* subgraph of G. If S is a cut-set of G and $G[S]$ is complete, then S is called a *complete* cut-set of G.

Lemma 2.2. *Let G and H be two chromatically equivalent graphs. If* $\chi(G)=3$ *and* $t_2(G)\leqslant 1$, *then*

$$3t_4(G)-t_5(G)=3t_4(H)-t_5(H),$$

where $t_4(G)$ and $t_5(G)$ are the number of induced wheel W_5's in G and the number of induced pentagons in G, respectively.

This result can be obtained directly from Theorem 2 in [7].

For a cycle C of G, if $|C|\geqslant 4$ and $G[C]\cong C$, i.e., every two non-consecutive vertices of C are not adjacent, then C is called a *chordless* cycle of G. G is called a *chordal* graph if G contains no chordless cycles. For a vertex x of G, x is called a *simplicial* vertex of G if $d(x)=0$ or $G[N_x]$ is complete.

Lemma 2.3 (Dirac [3]). *Let G be a chordal graph, but not complete. Then G contains two non-adjacent simplicial vertices.*

Lemma 2.4 (Dong [4]). *Let G be a connected graph of order n and x be a vertex of G with $d(x)<n-1$. Let $S_1,S_2,\dots,S_m$, where $m\geqslant 1$, be the vertex sets of the all components of $G-x-N_x$. Then, no chordless cycle of G contains x if and only if $G[N_x\cap N_{S_i}]$ is complete for every integer i with $1\leqslant i\leqslant m$, where N_{S_i} is the set of all vertices of G adjacent to some vertices in S_i.*

Corollary. *Let G be a connected graph of order n. If G contains no complete cut-sets, then, for any vertex $x\in V(G)$ with $d(x)<n-1$, there exists a chordless cycle C of G such that $x\in V(C)$.*

A graph G, is called a *forest* if G contains no cycles, and G is called a *tree* if G is connected and contains no cycles. Obviously, a tree is also a forest.

In what follows in this section, let G be a graph with chromatic number 3, and let A_1,A_2 and A_3 be the colour classes of a 3-colouring of G. Define $G_{i,j}$ to be $G[A_i\cup A_j]$ for all i and j with $1\leqslant i<j\leqslant 3$.

Lemma 2.5 (Dong et al. [6]). *If $G_{1,2},G_{1,3}$ and $G_{2,3}$ are forests, and $d(x)=0$ or $G[N_x]$ is a tree for every $x\in A_3$, then G is a chordal graph.*

Lemma 2.6 (Dong et al. [6]). *If G satisfies the following conditions, then G contains only one chordless cycle, i.e., the cycle C:*

(i) *$G_{1,2}$ contains just one cycle, denoted by C, and $G_{1,3}$ and $G_{2,3}$ are forests;*
(ii) *a vertex x in A_3 is adjacent to each vertex in C; and*
(iii) *for any y in A_3, $d(y)=0$ or $G[N_y]$ is connected.*

Lemma 2.7 (Dong et al. [6]). *If G satisfies the following conditions, then G contains only one chordless cycle, i.e., the cycle C, and A_3 contains a vertex which is adjacent to each vertex in C:*

(i) *$G_{1,2}$ has just one cycle, denoted by C, and $G_{1,3}$ and $G_{2,3}$ are forests; and*
(ii) *for any vertex y in A_1, $d(y)=0$ or $[N_y]$ is connected.*

3. The chromatic uniqueness of $W(n, n-3)$ when n is even

From now on, let G be a graph such that $P(G,\lambda) = P(W(n,n-3),\lambda)$, where $n \geqslant 6$. Obviously, $\chi(G) = 3$. Let A_1, A_2, A_3 be the colour classes of a 3-colouring of G and define $G_{i,j}$ to be $G[A_i \cup A_j]$ for all integers i and j with $1 \leqslant i < j \leqslant 3$.

It is found that

$$P(G,\lambda) = \lambda(\lambda-1)^2(\lambda-2)^{n-3} + \lambda(\lambda-2)[(\lambda-2)^{n-4} - (-1)^{n-4}].$$

By Lemmas 2.1 and 2.2, the following result is obtained.

Lemma 3.1. *G has the following properties:*

(a) *G has n vertices, $2n-4$ edges, and $n-4$ triangles;*

(b) *$\chi(G) = 3$ and G is 2-connected;*

(c) *$P(G,\lambda)$ is divisible by neither $(\lambda-1)^2$ nor $(\lambda-2)^2$;*

(d) *$t_2(G) = 0$, $t_3(G) = 0$, $t_4(G) = 0$ and $t_5(G) = 1$;*

(e) *there are exactly two ways when n is even and three ways when n is odd to separate the vertex set of G into three independent subsets, since*

$$s(G) = \frac{P(G,3)}{3!} = \frac{12 + 3[1-(-1)^{n-4}]}{6} = \begin{cases} 2, & \text{when } n \text{ is even;} \\ 3, & \text{otherwise.} \end{cases}$$

Corollary. *Let S be a complete cut-set of G, and G_1 and G_2 be two induced proper subgraphs of G such that $G_1 \cup G_2 = G$ and $V(G_1) \cap V(G_2) = S$. Then G_1 and G_2 are not chordal graphs.*

Proof. Suppose G_1 is a chordal graph. By Lemma 2.3, there is a vertex y in G_1 such that y is a simplicial vertex of G. Thus

$$P(G,\lambda) = (\lambda - d(y))P(G-y,\lambda).$$

Since $\chi(G) = 3$ and G is 2-connected, $d(y) = 2$. Since $t_1(G) > 1$, $\chi(G-y) = 3$. So $P(G,\lambda)$ can be divisible by $(\lambda-2)^2$, contrary to Lemma 3.1. Thus G_1 is not a chordal graph, and similarly, G_2 is also a non-chordal graph. □

In this section, let n be an even integer with $n \geqslant 6$.

Lemma 3.2. *Among the three subgraphs $G_{1,2}$, $G_{1,3}$ and $G_{2,3}$, one is a forest with two components and the other two are trees. Without loss of generality, suppose that $G_{1,2}$ is a forest with two components and thus $G_{1,3}$ and $G_{2,3}$ are trees.*

Proof. Since $s(G) = 2$, there is at most one subgraph $G_{i,j}$ which is not connected, where $1 \leqslant i < j \leqslant 3$. Without loss of generality, suppose that $G_{1,3}$ and $G_{2,3}$ are connected.

Thus $G_{1,2}$ is possibly not connected. So we have

$$\begin{cases} |E(G_{1,2})| \geqslant |V(G_{1,2})| - 2; \\ |E(G_{1,3})| \geqslant |V(G_{1,3})| - 1; \\ |E(G_{2,3})| \geqslant |V(G_{2,3})| - 1, \end{cases}$$

and further

$$|E(G)| = \sum_{1 \leqslant i < j \leqslant 3} |E(G_{i,j})| \geqslant 2|V(G)| - 4 = 2n - 4.$$

Since $|E(G)| = 2n - 4$. We have the following result:

$$\begin{cases} |E(G_{1,2})| = |V(G_{1,2})| - 2; \\ |E(G_{1,3})| = |V(G_{1,3})| - 1; \\ |E(G_{2,3})| = |V(G_{2,3})| - 1, \end{cases} \tag{1}$$

and thus $G_{1,2}$ is a forest with two components, and $G_{1,3}$ and $G_{2,3}$ are trees. □

For a graph H, let $1 + \rho(H)$ be the number of components of H, and let $\rho(x) = \rho(H[N_x])$ for any $x \in V(H)$. Then $\rho(x) \geqslant -1$ for any $x \in V(H)$. Furthermore, $\rho(x) = -1$ iff x is an isolated vertex, and $\rho(x) = 0$ iff $[N_x]$ is connected. Since G is 2-connected, $\rho(x) \geqslant 0$ for any $x \in V(G)$. $G[N_x]$ is a subgraph of $G - A_i$ if x is in A_i, $1 \leqslant i \leqslant 3$. Thus, by Proposition 3.2, for any vertex $x \in V(G)$, $G[N_x]$ is a forest, and $G[N_x]$ is a tree if and only if $\rho(x) = 0$. We calculate the number of triangles of G in the following way:

$$\begin{aligned} t_1(G) &= \sum_{x \in A_3} (d(x) - 1 - \rho(x)) = \sum_{x \in A_3} d(x) - |A_3| - \sum_{x \in A_3} \rho(x) \\ &= |E(G)| - |E(G_{1,2})| - |A_3| - \sum_{x \in A_3} \rho(x) \\ &= |E(G)| - |V(G)| + 2 - \sum_{x \in A_3} \rho(x) \\ &= n - 2 - \sum_{x \in A_3} \rho(x). \end{aligned}$$

Since $t_1(G) = n - 4$,

$$\sum_{x \in A_3} \rho(x) = 2. \tag{2}$$

Similarly, we have

$$\sum_{x \in A_1} \rho(x) = 1; \qquad \sum_{x \in A_2} \rho(x) = 1. \tag{3}$$

Lemma 3.3. *There exist four different vertices $x_1 \in A_1$, $x_2 \in A_2$ and x_3^1, $x_3^2 \in A_3$ such that for any x in G,*

$$\rho(x) = \begin{cases} 1, & \text{when } x = x_1, x_2, x_3^1, x_3^2; \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Proof. By (3), (4) is obtained for $x \in A_1 \cup A_2$.

If there is a vertex $x_3 \in A_3$ such that $\rho(x_3) \geqslant 2$, then by (2), $\rho(x_3) = 2$ and $\rho(x) = 0$ for any $x \in A_3 \backslash \{x_3\}$. Thus $G - x_3$ is not connected, i.e., x_3 is a cut vertex of G, contrary to Lemma 3.1(b). So there exist two different vertex x_3^1 and x_3^2 such that $\rho(x_3^1) = 1$, $\rho(x_3^2) = 1$ and $\rho(x) = 0$ for any $x \in A_3 \backslash \{x_3^1, x_3^2\}$. □

Lemma 3.4. *Any chordless cycle of G contains the vertices* x_1, x_2, x_3^1 *and* x_3^2.

Proof. By Lemma 2.5 and Lemma 3.3, $G - x_1$ is a chordal graph. So any chordless cycle of G contains x_1. Similarly, any chordless cycle of G contains x_2.

Since $G_{1,2}$ is not connected with two components and $\rho(x)=0$ for any $x \in A_3 \backslash \{x_3^1, x_3^2\}$, $G - x_3^1 - x_3^2$ is not connected to two components. Let S_1 and S_2 denote the vertex sets of the two components, and $G_i = G[S_i \cup \{x_3^1, x_3^2\}]$, $i = 1, 2$. For any vertex $x \in V(G_1) \cap A_3$, $G_1[N_x]$ is a tree. By Lemma 2.5, G_1 is a chordal graph. Similarly, G_2 is a chordal graph too. Therefore, any chordless cycle of G contains x_3^1 and x_3^2. □

Lemma 3.5. *G contains no complete cut-sets.*

Proof. Assume that A is a complete cut-set of G. Let G_1 and G_2 be two induced proper subgraphs of G such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = G[A]$. By the corollary to Lemma 3.1, G_1 and G_2 are not chordal graphs. Since $G[A]$ is complete, any chordless cycle of G is in G_1 or in G_2. Then, by Lemma 3.4, the four vertices x_1, x_2, x_3^1 and x_3^2 must be in A, contrary to $G[A]$ being complete. □

Lemma 3.6. *G contains just one vertex induced pentagon, and* $x_3^1 x_1 x_2 x_3^2$ *or* $x_3^1 x_2 x_1 x_3^2$ *or* $x_1 x_3^1 x_2 x_3^2$ *or* $x_2 x_3^1 x_1 x_3^2$ *or* $x_3^1 x_1 x_3^2 x_2$ *or* $x_3^1 x_2 x_3^2 x_1$ *is a path in the pentagon.*

Proof. By Lemma 3.1, there is exactly one induced pentagon in G. By Lemma 3.4, x_1, x_2, x_3^1 and x_3^2 are in the pentagon. Then the four vertices induce a path. Since x_3^1 and x_3^2 are in A_3, the two vertices are not consecutive in the path. This path can be expressed in six different ways: $x_3^1 x_1 x_2 x_3^2$ or $x_3^1 x_2 x_1 x_3^2$ or $x_1 x_3^1 x_2 x_3^2$ or $x_2 x_3^1 x_1 x_3^2$ or $x_3^1 x_1 x_3^2 x_2$ or $x_3^1 x_2 x_3^2 x_1$. □

Lemma 3.7. *If* $x_3^1 x_1 x_2 x_3^2$ *or* $x_3^1 x_2 x_1 x_3^2$ *is a path in G, then G is isomorphic to* $W(n, n-3)$.

Proof. Without loss of generality, let $x_3^1 x_1 x_2 x_3^2$ be a path in G. Let $x_4 x_3^1 x_1 x_2 x_3^2 x_4$ be the pentagon in G, shown in Fig. 2(a), where $x_4 \in A_1 \cup A_2$. Without loss of generality, suppose $x_4 \in A_1$. By Lemma 3.3, $\rho(x_4) = 0$, i.e., $G[N_{x_4}]$ is a tree. So there exists the unique path $P(x_3^1, x_3^2)$ in $G_{2,3}$ connecting x_3^1 and x_3^2, and x_4 is adjacent to each vertex in the path. Thus $P(x_3^1, x_3^2)$ does not contain x_1 or x_2. Let G_0 be the subgraph $G[V(P(x_3^1, x_3^2)) \cup \{x_4, x_1, x_2\}]$. By Lemma 3.2, G_0 is isomorphic to $W(m, m-3)$ for some even integer m, shown in Fig. 2.

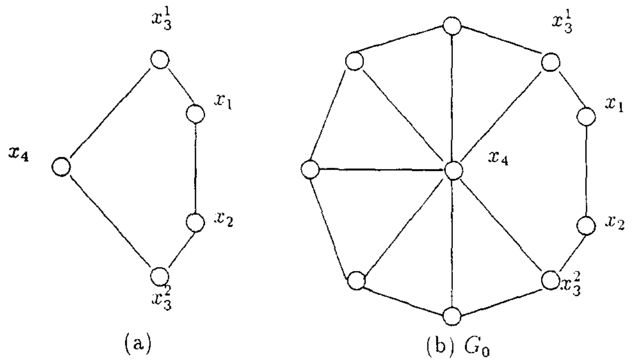


Fig. 2.

Suppose C is a chordless cycle of G and C is not in G_0. By Lemma 3.4, C contains x_1, x_2, x_3^1 and x_3^2. If x_4 is not adjacent to some vertices on the path $C - x_1 - x_2$, then there exists a chordless cycle in the subgraph induced by $\{x_4\} \cup V(C - x_1 - x_2)$, which does not contain x_1 and x_2, contrary to Lemma 3.4. If x_4 is adjacent to each vertex in $C - x_1 - x_2$, then the subgraph of G induced by $V(C - x_1 - x_2) \cup V(P(x_3^1, x_3^2)$ is a subgraph of $G_{2,3}$, which contains cycles, contrary to $G_{2,3}$ being a tree. Therefore, any chordless cycle of G is in the subgraph G_0.

Let x be any vertex of G. If $d(x) = n - 1$ and $x \in A_i$, then $|A_i| = 1$. But, from the subgraph G_0, we know $|A_i| \geqslant 2$ for any $i = 1, 2, 3$. Thus $d(x) < n - 1$ for any $x \in V(G)$. By Lemma 3.5 and the corollary to Lemma 2.4, x is contained in some chordless cycle of G, and thus $x \in V(G_0)$, which implies that $V(G) = V(G_0)$. So $G \cong G_0$, i.e., $G \cong W(m, m-3)$ for some even integer m. Hence $m = n$ and $G \cong W(n, n-3)$. □

Lemma 3.8. *G does not contain any one of the four paths* $x_1 x_3^1 x_2 x_3^2$, $x_2 x_3^1 x_1 x_3^2$, $x_3^1 x_1 x_3^2 x_2$, *and* $x_3^1 x_2 x_3^2 x_1$.

Proof. If G contains one of the four paths, we can show, in the same way as Lemma 3.7, that $G \cong W(m, m-3)$ for some odd integer m, a contradiction. □

By Lemmas 3.1–3.8, we have

Theorem 1. *For any even integer* $n \geqslant 6$, *the graph* $W(n, n-3)$ *is chromatically unique.*

4. The chromatic uniqueness of $W(n, n-3)$ when n is odd

Let us first introduce one result, which can be proved easily.

Lemma 4.0. *Let H be a graph with* $\chi(H) = m$, *and let* $B_1, \ldots, B_m$ *be the colour classes of an m-colouring of H. Define* $H_{i,j}$ *to be the graph* $H[B_i \cup B_j]$ *for all i and* *j with* $1 \leqslant i < j \leqslant m$. *Then*

$$s(H) \geqslant 1 + \sum_{1 \leqslant i < j \leqslant m} (2^{\rho(H_{i,j})} - 1),$$

where $s(H)$ *is the number of ways to separate* $V(H)$ *into* m *independent sets, i.e.,*

$$s(H) = \frac{P(H, \chi(H))}{\chi(H)!}.$$

In this section, let n be an odd integer with $n \geqslant 7$.

By Lemma 3.1, we have $s(G) = 3$. By Lemma 4.0, $\rho(G_{i,j}) \leqslant 1$ for all i and j with $1 \leqslant i < j \leqslant 3$, and there exists at least one $G_{i,j}$ such that $\rho(G_{i,j}) = 0$. Without loss of generality, suppose $\rho(G_{2,3}) = 0$. Then $G_{1,2}$ and $G_{1,3}$ possibly contain two components. So,

$$\begin{cases} |E(G_{1,2})| \geqslant |V(G_{1,2})| - 2; \\ |E(G_{1,3})| \geqslant |V(G_{1,3})| - 2; \\ |E(G_{2,3})| \geqslant |V(G_{2,3})| - 1 \end{cases}$$

and further

$$|E(G)| = \sum_{1 \leqslant i < j \leqslant 3} |E(G_{i,j})| \geqslant 2|V(G)| - 5 = 2n - 5.$$

Since $|E(G)| = 2n - 4$,

$$\begin{cases} |E(G_{1,2})| = |V(G_{1,2})| - 1 \\ |E(G_{1,3})| = |V(G_{1,3})| - 2 \\ |E(G_{2,3})| = |V(G_{2,3})| - 1, \end{cases}$$

or

$$\begin{cases} |E(G_{1,2})| = |V(G_{1,2})| - 2 \\ |E(G_{1,3})| = |V(G_{1,3})| - 2 \\ |E(G_{2,3})| = |V(G_{2,3})|, \end{cases}$$

or

$$\begin{cases} |E(G_{1,2})| = |V(G_{1,2})| - 2 \\ |E(G_{1,3})| = |V(G_{1,3})| - 1 \\ |E(G_{2,3})| = |V(G_{2,3})| - 1. \end{cases}$$

Thus we have five cases:

Case a: $G_{1,2}$ contains two components with just one cycle C_1, $G_{1,3}$ is a forest with two components, and $G_{2,3}$ is a tree.

Case b: $G_{1,3}$ is a forest with two components, and $G_{1,2}$ and $G_{2,3}$ are trees.

Case c: $G_{1,2}$ and $G_{1,3}$ are forests with two components, respectively, and $G_{2,3}$ is connected with just one cycle, denoted by C_2.

Case d: $G_{1,3}$ has two components with just one cycle, $G_{1,2}$ is a forest with two components, and $G_{2,3}$ is a tree.

Case e: $G_{1,2}$ is a forest with two components, and $G_{1,3}$ and $G_{2,3}$ are trees.

We need only consider cases a, b and c; cases d and e are treated in the same way as cases a and b.

4.1. *Case* a

Lemma 4.1. *In case* a, *we have*

$$\sum_{y\in A_2} \rho(y) = 2; \qquad \sum_{y\in A_1} \rho(y) = 1. \tag{5}$$

Proof. Since $G_{1,3}$ is a forest with two components,

$$\begin{aligned} t_1(G) &= \sum_{y\in A_2} [d(y) - 1 - \rho(y)] = \sum_{y\in A_2} d(y) - |A_2| - \sum_{y\in A_2} \rho(y) \\ &= |E(G)| - |E(G_{1,3})| - |A_2| - \sum_{x\in A_2} \rho(y) \\ &= |E(G)| - |V(G)| + 2 - \sum_{y\in A_2} \rho(y) \\ &= n - 2 - \sum_{y\in A_2} \rho(y). \end{aligned}$$

Since $t_1(G) = n - 4$, we have $\sum_{y\in A_2} \rho(y) = 2$. Similarly, $\sum_{y\in A_1} \rho(y) = 1$. □

Lemma 4.2. *In case* a, *there exists a vertex* y_3 *in* A_3 *which is adjacent to each vertex in the cycle* C_1, *and*

$$\sum_{y\in A_3} \rho(y) = 2. \tag{6}$$

Proof. If no vertex y in A_3 is adjacent to each vertex in C_1, then, similar to the proof of Lemma 4.1, we have

$$\sum_{y\in A_3} \rho(y) = 1.$$

This means that there is only one vertex y' in A_3 such that $\rho(y') = 1$. Since $G_{1,2}$ is not connected, y' is a cut vertex of G, contrary to Lemma 3.1 (b).

Let y_3 be the vertex in A_3 which is adjacent to each vertex in C_1. Then, in the same way as the proof of Lemma 4.1, we have $\sum_{y\in A_3} \rho(y) = 2$. □

Lemma 4.3. *In case* a, *there exist five different vertices* $y_1 \in A_1$, $y_2^1, y_2^2 \in A_2$, *and* $y_3^1, y_3^2 \in A_3$ *such that for any* y *in* G,

$$\rho(y) = \begin{cases} 1, & \text{when } y = y_1, y_2^1, y_2^2, y_3^1, y_3^2; \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

Proof. Similar to the proof of Lemma 3.3, the result is obtained by Lemmas 4.1 and 4.2. □

Lemma 4.4. *In case* a, *any chordless cycle of* G, *except for* C_1, *contains* y_1, y_2^1, y_2^2, y_3^1 *and* y_3^2.

Proof. Consider the graph $G' = G - y_1$ and let $B_1 = A_1 \backslash \{y_1\}$, $B_2 = A_2$ and $B_3 = A_3$. $G'[B_1 \cup B_3]$ and $G'[B_2 \cup B_3]$ are forest, and $G'[B_1 \cup B_2]$ contains at most one chordless cycle, i.e., C_1. Since $G'[N_y]$ is a tree for all $y \in B_1$, by Lemmas 2.5 and 2.7, G' contains at most one chordless cycle, i.e., C_1. Thus any chordless cycle of G, except for C_1, contains y_1.

Since $G_{1,2}$ is not connected with two components, by Lemma 4.3, $G - y_3^1 - y_3^2$ is disconnected with two components. Let S_1 and S_2 denote the vertex sets of the two components, and let $G_i = G[S_i \cup \{y_3^1, y_3^2\}]$, $i = 1, 2$. So C_1 is in G_1 or in G_2. Without loss of generality, assume C_1 is in G_1. By Lemma 2.6, G_1 contains just one chordless cycle, i.e., C_1, and by Lemma 2.5, G_2 is a chordal graph. Therefore, any chordless cycle of G, except for C_1, contains y_3^1 and y_3^2.

Similarly, any chordless cycle of G except for C_1 contains y_2^1 and y_2^2. □

Lemma 4.5. *In case* a, *there are exactly two chordless cycles in* G. *One is* C_1 *and the other is a pentagon, denoted by* C', *with vertex set* $\{y_1, y_2^1, y_2^2, y_3^1, y_3^2\}$.

Proof. By Lemma 3.1, G contains a chordless cycle with five vertices, denoted by C'. Obviously, $C' \neq C_1$. By Lemma 4.4, we know that C' contains the five vertices y_1, y_2^1, y_2^2, y_3^1 and y_3^2. So the five vertices induce a chordless cycle, C'. By Lemma 4.4, G contains exactly two chordless cycles C_1 and C'. □

Lemma 4.6. *In case* a, G *contains no complete cut-sets.*

Proof. Suppose S is a complete cut-set.

Let G_1' and G_2' be two induced proper subgraphs of G such that $G_1' \cup G_2' = G$ and $G_1' \cap G_2' = G[S]$. Since $G[S]$ is complete, any chordless cycle of G is contained in G_1' or in G_2'.

By the corollary to Lemma 3.1, G_1' and G_2' are not chordal graphs. By Lemma 4.5, G_1' and G_2' contain only one chordless cycle, respectively. Without loss of generality, suppose C_1 and C' are in G_1' and G_2', respectively.

Since G is 2-connected and $G[S]$ is complete, both G_1' and G_2 are 2-connected. By Lemma 4.2, there exists a vertex in A_3 which is adjacent to each vertex of C_1. Since

G'_1 contains only one chordless cycle, we have

$$P(G'_1,\lambda) = \lambda[(\lambda-2)^m + (\lambda-2)](\lambda-2)^{l_1},$$

where $m = |C_1|$ is even and $l_1 \geqslant 0$. Since G'_2 contains only one chordless cycle C' and no vertex of G is adjacent to each vertex of C',

$$P(G'_2,\lambda) = (\lambda-2)^{l_2}((\lambda-1)^5 - (\lambda-1)),$$

where $l_2 \geqslant 0$. So we have

$$P(G,\lambda) = \frac{P(G'_1,\lambda)P(G'_2,\lambda)}{P(G[S],\lambda)} = \frac{(\lambda-2)^{l_1+l_2}((\lambda-2)^m + (\lambda-2))((\lambda-1)^4-1)}{(\lambda-2)^i},$$

where $i = |S|-2 = 0$ or 1. Since $P(G,\lambda)$ is not divisible by $(\lambda-2)^2$, $l_1 = l_2 = 0$ and $i = 1$. Thus

$$P(G,\lambda) = ((\lambda-2)^{m-1}+1)((\lambda-1)^4-1).$$

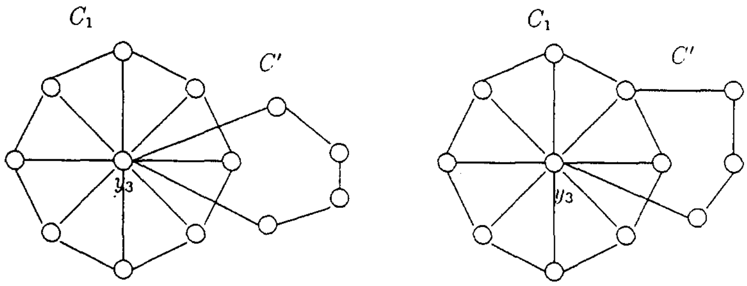
So $n = m+3$. Since G'_2 contains C' and $G[S]$ is complete, $|V(G'_2)| - |S| \geqslant 3$. Since G' contains the cycle C_1 and y_3, $|V(G'_1)| \geqslant m+1$. Thus $n \geqslant m+4$, a contradiction. □

Lemma 4.7. *Case* a *leads to a contradiction.*

Proof. If $d(x) = n-1$ for some vertex $x \in A_i$, then $|A_i| = 1$. Since $|A_1| \geqslant |A_1 \cap V(C_1)| \geqslant 2$, $|A_2| \geqslant |A_2 \cap V(C_1)| \geqslant 2$ and A_3 contains y_3^1 and y_3^2, $d(x) < n-1$ for all $x \in V(G)$. Since G contains no complete cut-sets, by the corollary to Lemma 2.4, any vertex x of G is contained in some chordless cycle and thus, by Lemma 4.5, $V(G) = V(C') \cup V(C_1)$. Since y_3 is adjacent to each vertex in C_1, y_3 is not in C_1 and thus $y_3 \in \{y_3^1 . y_3^2\}$. Then $|V(C_1) \cap V(C')| = k$, where $0 \leqslant k \leqslant 2$. So $n = m+5-k$, where $m = |V(C_1)|$, and $e(G) \geqslant 2m + (5-k)$. Since $e(G) = 2n-4$, we have $k = 0$ or 1. See Fig 3. When $k = 1$, G is the graph in Fig. 3(b), and G contains a cut-set, a contradiction. When $k = 0$, G is a graph obtained from the graph in Fig. 3(a) by adding one edge with end-vertices respectively in C_1 and C', contrary to the fact that G contains no complete cut-sets. □

4.2. *Case* c

In case c, $G_{1,2}$ and $G_{1,3}$ are forests with two components, respectively, and $G_{2,3}$ is connected with just one cycle, denoted by C_2. In the same way, as Lemmas 4.1 and 4.3, the following result is obtained.



(a) $k = 0$ (b) $k = 1$

Fig. 3.

Lemma 4.8. *In case* c, *we have*

$$\sum_{x \in A_2} \rho(x) = 2; \qquad \sum_{x \in A_3} \rho(x) = 2, \tag{8}$$

and there exist four different vertices $y_2^1, y_2^2 \in A_2$ *and* $y_3^1, y_3^2 \in A_3$ *such that for* $y \in A_2 \cup A_3$,

$$\rho(y) = \begin{cases} 1, & y = y_2^1, y_2^2, y_3^1, y_3^2; \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 4.9. *In case* c, *any chordless cycle of* G, *except for* C_2, *contains* y_2^1, y_2^2, y_3^1 *and* y_3^2.

Proof. Since $G_{1,3}$ is a forest with two components, by Lemma 4.8, $G - y_2^1 - y_2^2$ is disconnected with two components. Let S_1 and S_2 be the vertex sets of the two components, and let $G_i'' = G[S_i \cup \{y_2^1, y_2^2\}]$, $i = 1,2$. Consider the graph G_1''. Let $B_j = A_j \cap V(G_1'')$, $j = 1,2,3$. $G_1''[B_1 \cup B_2]$ and $G_1''[B_1 \cup B_3]$ are forests, and $G_1''[B_2 \cup B_3]$ contains at most one cycle, i.e., C_2. For any vertex $x \in B_2$, $G_1''[N_x]$ is connected. By Lemmas 2.5 and 2.7, G_1'' contains at most one chordless cycle, i.e., C_2. Similarly, G_2'' contains at most one chordless cycle, i.e., C_2. Thus any chordless cycle of G, except for C_2, contains y_2^1 and y_2^2. Similarly, any chordless cycle of G, except for C_2, contains y_3^1 and y_3^2. □

Lemma 4.10. *In case* c, *G contains just one induced pentagon, denoted by* C_3, *which can be written as* $y_4 y_2^1 y_3^1 y_2^2 y_3^2 y_4$ *or* $y_4 y_2^1 y_3^2 y_2^2 y_3^1 y_4$, *where* $y_4 \in A_1$.

Lemma 4.11. *In case* c, *G contains no complete cut-sets.*

Proof. Suppose S is a complete cut-set of G.

Let G_1 and G_2 be two induced proper subgraphs of G such that $G_1 \cup G_2 = G$ and $V(G_1) \cap V(G_2) = S$. By the corollary to Lemma 3.1, G_1 and G_2 are not chordal graphs.

We claim that there is a vertex $y' \in A_1$ such that y' is adjacent to each vertex of C_2. Suppose that no vertex in A_1 is adjacent to each vertex of C_2. In the same way as Lemma 4.1, it can be shown that $\sum_{x\in A_1} \rho(x) = 0$, i.e., $\rho(x) = 0$ for all vertex $x \in A_1$. Then, by Lemma 2.6, G has only one chordless cycle (the cycle C_2'). This implies that G_1 or G_2 is a chordal graph, a contradiction.

Suppose y' is adjacent to each vertex of C_2. Then $\sum_{x\in A_1} \rho(x) = 1$, i.e., there exists a vertex y'' such that for $y \in A_1$,

$$\rho(y) = \begin{cases} 1, & y = y''; \\ 0, & y \neq y''. \end{cases}$$

Without loss of generality, suppose C_2 is in G_1. Then y' is contained in G_1. If y'' is not in G_2, then by Lemma 2.5, G_2 is a chordal graph, a contradiction. If the subgraph of G induced by $V(G_2) \cap N_{y''}$ is connected, then by Lemma 2.5, G_2 is a chordal graph, a contradiction. Thus y'' is in G_2 and $G[V(G_2) \cap N_{y''}]$ is not connected.

Since $G[N_y \cap V(G_1)]$ is connected for any vertex $y \in V(G_1) \cap A_1$, by Lemma 2.6, G_1 contains just one chordless cycle, i.e., C_2. By Lemma 2.5, $G_2 - y''$ is a chordal graph, which implies that any chordless cycle of G_2 contains y''. Thus, any chordless cycle of G, except for C_2, contains y''. By Lemmas 4.9 and 4.10, $y'' = y_4$ and G_2 contains only one chordless cycle, i.e., C_3.

Since G_1 and G_2 contain only one chordless cycle, respectively, in the same way as in the proof of Lemma 4.6, a contradiction happens. Thus G contains no complete cut-sets. □

Lemma 4.12. *In case* c, *no vertex in* A_1 *is adjacent to each vertex in* C_2, *and* $\rho(x) = 0$ *for any* $x \in A_1$.

Proof. If there is a vertex y' in A_1 which is adjacent to each vertex in C_2, then in the same way as Lemma 4.1, we have

$$\sum_{x\in A_1} \rho(x) = 1.$$

So there is a vertex y'' in A_1 such that for $y \in A_1$,

$$\rho(y) = \begin{cases} 1, & y = y''; \\ 0, & \text{otherwise}. \end{cases}$$

If $y'' \neq y'$, by Lemma 2.6, $G - y''$ contains just one chordless cycle, i.e., the cycle C_2. If $y'' = y'$, let G' be the graph obtained from G by deleting all edges (y', y), where $y \in (A_2 \cup A_3) \backslash V(C_2)$. By Lemma 2.6, G' contains only one chordless cycle, i.e., C_2. So, if $y'' = y'$, $G - y''$ contains just one chordless cycle, i.e., C_2. Thus any chordless cycle of G except for C_2 contains y''.

By Lemma 4.10, $y'' = y_4$. By Lemmas 4.9 and 4.10, G contains just two chordless cycles, i.e., the cycles C_2 and C_3.

If $|A_1| = 1$, then $y' = y'' = y_4$. Since $G_{2,3}$ is connected with just one cycle C_2 and y_4 is adjacent to each vertex in the cycle, G is a wheel or contains complete cut-sets, a contradiction.

If $|A_1| \geqslant 2$, then $|A_i| \geqslant 2$ for all i with $1 \leqslant i \leqslant 3$, and thus $d(x) < n-1$ for all vertex $x \in V(G)$. Since G contains no complete cut-sets, by the corollary to Lemma 2.4, $V(G) = V(C_2) \cup V(C_3)$, which implies that $|A_1| = 1$, a contradiction.

Since no vertex in A_1 is adjacent to each vertex in C_2, in the same way as Lemma 4.1, we have $\rho(x) = 0$ for any $x \in A_1$. □

Lemma 4.13. *In case* c, *G is isomorphic to* $W(n, n-3)$.

Proof. By Lemma 4.10, without loss of generality, let $y_4 y_2^1 y_3^1 y_2^2 y_3^2 y_4$ be the cycle C_3, shown in Fig. 4(a). By Lemma 4.12, $\rho(y_4) = 0$. So there is a path $P(y_2^1, y_3^2)$ in $G_{2,3}$ connecting y_2^1 and y_3^2 and y_4 is adjacent to each vertex on the path. Thus y_3^1 and y_2^2 are not on $P(y_2^1, y_3^2)$, shown in Fig. 4(b). The joint of the two paths $P(y_2^1, y_3^2)$ and $y_2^1 y_3^1 y_2^2 y_3^2$ is a cycle in $G_{2,3}$, and thus it is the cycle C_2. Let G_0 be the graph shown in Fig. 4(b). Since $G_{2,3}$ contains only one cycle, G_0 is an induced subgraph of G.

Similar to the proof of Lemma 3.7, it can be shown that any chordless cycle of G is contained in G_0. For any vertex x in G, if $d(x) = n-1$, then x is in A_1 and thus $x = y_4$, contrary to the fact that $d(y_4) < n-1$. So $d(x) < n-1$ for any vertex x in G. Since G contains no complete cut-sets, by the corollary to Lemma 2.4, every vertex of G is contained in some chordless cycle. Thus, $V(G) = V(G_0)$. So, $G \cong G_0$ and hence $G \cong W(n, n-3)$. □

4.3. *Case* b

In case b, $G_{1,3}$ is a forest with two components, and $G_{1,2}$ and $G_{2,3}$ are trees, G having the same properties as in Lemma 3.2. By the same way as in Section 3, we can obtain a series of results, and finally we establish the following result.

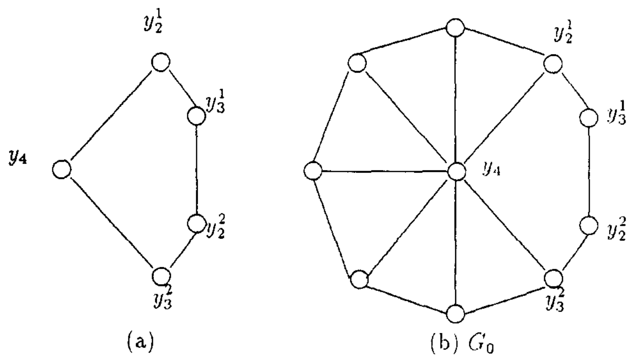


Fig. 4.

Lemma 4.14. *In case* b, *G is isomorphic to* $W(n, n-3)$.

By Lemmas 4.7, 4.13 and 4.14, we have

Theorem 2. *For any odd integer* $n \geqslant 7$, *the graph* $W(n, n-3)$ *is chromatically unique.*

Thus we have shown that the graph $W(n, n-3)$ is chromatically unique for any integer $n \geqslant 6$.

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