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Singular Integral Operators with Non-necessarily Bounded Kernels on Spaces of Homogeneous Type*

ROBERTO A. MACÍAS

PEMA, CONICET, and Universidad Nacional del Litoral, 3000 Sante Fe, Argentina

CARLOS SEGOVIA

IAM, CONICET, and Universidad de Buenos Aires (FCEyN), Argentina

AND

JOSÉ L. TORREA

Universidad Autónoma de Madrid, 28049 Madrid, Spain

INTRODUCTION

The purpose of this paper is twofold. First, we intend to clarify the relevance of conditions of the type considered in [A, DJS, MT] on the measure of coronas in the study of singular integral operators. The main result in this direction is given in Theorem (1.19), where we show that for a space of homogeneous type satisfying condition (H_α), see (1.5), a normalization can be given to satisfy condition (L_α), see (1.3). This result allows us to interpret (H_α) as a quantitative property ensuring that the order of the normalized space is at least equal to α. Examples show that, in general, α cannot be improved. An approximation of the identity of R. Coifman's type is obtained for normalized spaces of order α without restrictions on the measure of the whole space X or the existence of atoms for the measure. This allows us to get rid of the condition (H_α) in the results of Chapter II.

Second, in Chapter II we study singular integral operators with conditions on the associated kernel which generalize those of [A, DJS, MT], allowing the kernel to be unbounded, see [KW].

The conditions we assume on the kernel are stated in (2.3), (2.4), (2.5),

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and (2.6). They are inspired in the L^r-Dini condition of [KW]. The main result of the paper is to show that T is weakly bounded if and only if $T\psi$ is a function given by an explicit formula involving the kernel associated to T and $T1 = g$, see Theorem (2.27). By a systematic use of this formula we obtain the following results:

If T is a weakly bounded singular integral operator and $T1$ belongs to B.M.O., then

(a) The kernel associated to T is equal to zero if and only if there exist $h(x) \in L^\infty$ and $Tf(x) = h(x) f(x)$ (see (2.31).

(b) T maps Lipschitz functions into bounded Lipschitz functions if and only if $T1 = 0$ (see (2.32)). For related results see [L].

(c) If T^*1 also belongs to B.M.O., then T satisfies estimates of the type given in Lemma 2.3 of [DJS], which allow the L^2 theory to develop (see (2.34)).

Finally, we give an application to operators defined by principal value integrals, see (2.37), obtaining a priori Lipschitz estimates for some parabolic partial differential equations.

I. Geometry of Spaces of Homogeneous Type

We say that a real valued function $d(x, y)$ defined on $X \times X$ is a quasi-distance on X if

$$\begin{aligned} &\text{(i)} \quad d(x, y) \geqslant 0 \text{ and } d(x, y) = 0 \text{ if and only if } x = y, \\ &\text{(ii)} \quad d(x, y) = d(y, x), \text{ and} \\ &\text{(iii)} \quad d(x, y) \leqslant K[d(x, z) + d(z, y)], \end{aligned} \tag{1.1}$$

hold for every x, y, and z in X and K a finite constant. The set $\{y : d(x, y) \leqslant r\}$ is denoted by $B_d(x, r)$. This quasi-distance defined a uniform structure on X, the family $\{(x, y) : d(x, y) < \varepsilon\}$ being a basis of the uniformity. Let μ be a positive measure on a σ-algebra of subsets of X which contains the open sets and the balls $B_d(x, r)$. We say that (X, d, μ) is a space of homogeneous type if there exists a finite constant A such that

$$\mu(B_d(x, 2Kr)) \leqslant A\mu(B_d(x, r)) \tag{1.2}$$

holds for every $x \in X$ and $r > 0$. It is known [MS1] that it is always possible to find a quasi-distance $d'(x, y)$ equivalent to $d(x, y)$ and $0 < \beta \leqslant 1$, such that

$$(L_\beta) \mid d'(x, z) - d'(y, z)| \leqslant Cr^{1-\beta} d(x, y)^\beta \tag{1.3}$$

holds for whenever $d'(x, z)$ and $d'(y, z)$ are smaller than or equal to r, with C a finite constant. Thus we can assume that $d(x, y)$ satisfies condition (L_β) for some $0<\beta\leqslant 1$.

We say that a triple (X, d, μ) is a normalized space if there exist constants K_1, K_2, A_1, and A_2 such that

$$\begin{aligned}
&\text{(i)} \quad \text{if } K_1\mu(\{x\})\leqslant r\leqslant K_2\mu(X), \text{ then } A_1 r\leqslant \mu(B_d(x, r))\leqslant A_2 r,\\
&\text{(ii)} \quad \text{if } r<K_1\mu(\{x\}), \text{ then } B_d(x, r)=\{x\}, \text{ and} \qquad (1.4)\\
&\text{(iii)} \quad \text{if } r>K_2\mu(X), \text{ then } B_d(x, r)=X.
\end{aligned}$$

These there conditions imply that (X, d, μ) is a space of homogeneous type.

Let (X, d, μ) be a space of homogeneous type, with its quasi-distance satisfying condition (L_β). Then we shall say that this space satisfies the condition (H_α), $0<\alpha\leqslant 1$, if

$$\begin{aligned}
&\mu(B_d(x, r+r^{1-\beta}s^\beta))-\mu(B_d(x, r-r^{1-\beta}s^\beta))\\
&\qquad \leqslant C\mu(B_d(x, r))^{1-\alpha}\,\mu(B_d(x, s))^\alpha \qquad (1.5)
\end{aligned}$$

holds for $0\leqslant s\leqslant r$ and $x\in X$, with C a finite constant.

The main purpose of this chapter is to prove that in a space of homogeneous type satisfying condition (H_α), (1.5), a normalization can be found such that its quasi-distance satisfies condition (L_α), (1.4). Also, an approximation of the identity, made of Lipschitz functions of order α, of the type introduced by R. Coifman is given.

(1.6) LEMMA. *Let (X, d, μ) satisfy condition (H_α). Then either $\mu(\{x\})=0$ for every $x\in X$ or $\mu(\{x\})>0$ for every $x\in X$.*

This result is proved in [MT]. We give a proof here for the sake of completeness.

Proof. Let us assume that there is a point $x\in X$ such that $\mu(\{x\})=0$. Let $y\in X$, $y\neq x$. Then y belongs to $B_d(x, d(x, y)+d(x, y)^{1-\beta}s^\beta)\sim B_d(x, d(x, y)-d(x, y)^{1-\beta}s^\beta)$, for every $s\leqslant d(x, y)$. By condition (H_α), we have

$$\mu((\{y\}))\leqslant C\mu(B_d(x, d(x, y)))^{1-\alpha}\,\mu(B_d(x, s))^\alpha.$$

Since $\lim_{s\to 0}\mu(B_d(x, s))=\mu(\{x\})=0$, we get $\mu(\{y\})=0$.

Let (X, d, μ) be a space of homogeneous type and define

$$\delta(x, x)=0 \qquad \text{and} \qquad \text{if } x\neq y,\ \delta(x, y)=\mu(B_d(x, d(x, y))). \qquad (1.7)$$

(1.8) PROPOSITION. *The function $\delta(x, y)$ satisfies*

$$\begin{aligned} &\text{(i)} && \delta(x, y) \geqslant 0 \text{ and } \delta(x, y) = 0 \quad \text{if and only if } x = y, \\ &\text{(ii)}' && \delta(x, y) \leqslant A\delta(y, x), \quad \text{and} \\ &\text{(iii)} && \delta(x, y) \leqslant A^2 \, |\delta(x, z) + \delta(y, z)|, \end{aligned}$$

for every x, y, and z in X.

Proof. Part (i) is obvious. Let us consider (ii)′. If $v \in B_d(x, d(x, y))$, we have $d(v, y) \leqslant K\,|d(v, x) + d(x, y)| \leqslant 2Kd(x, y)$; then $\delta(x, y) = \mu(B_d(x, d(x, y))) \leqslant A(B_d(y, d(x, y))) = A\delta(y, x)$. Let us consider (iii). If $d(x, z) \leqslant d(z, y)$, we have that $u \in B_d(x, d(x, y))$ implies $d(u, y) \leqslant K\,|d(u, x) + d(x, y)| \leqslant 2Kd(x, y)$ and since $d(x, y) \leqslant K\,|d(x, z) + d(z, y)| \leqslant 2Kd(z, y)$, it follows that $d(u, y) \leqslant (2K)^2\, d(z, y)$. Thus,

$$\delta(x, y) \leqslant \mu(B_d(x, d(x, y)) \leqslant A^2 \mu(B_d(y, d(y, z))) = A^2 \delta(y, z).$$

Analogously, if $d(z, y) \leqslant d(x, z)$ it turns out that $\delta(x, y) \leqslant A^2 \delta(x, z)$. This proves part (iii).

We observe that $\delta(x, y)$ does not necessarily satisfy condition (ii) of (1.1), but it does satisfy (ii)′ of (1.8). We shall call this $\delta(x, y)$ the non-necessarily symmetric quasi-distance associated to (X, d, μ). We denote by $B_\delta(x, r)$ the set $\{y : \delta(x, y) \leqslant r\}$.

(1.9) PROPOSITION. *Let (X, d, μ) be a space of homogeneous type and $\delta(x, y)$ the non-necessarily symmetric quasi-distance associated to (X, d, μ). Then the following properties hold:*

$$\begin{aligned} &\text{(i)} && \text{if } 0 < r < \mu(\{x\}), \text{ then } B_\delta(x, r) = \{x\}, \\ &\text{(ii)} && \text{if } \mu(\{x\}) \leqslant r, \text{ then } \mu(B_\delta(x, r)) \leqslant r, \\ &\text{(iii)} && \text{if } \mu(X) \leqslant r, \text{ then } B_\delta(x, r) = X, \quad \text{and} \\ &\text{(iv)} && \text{if } r < \mu(X), \text{ then } A^{-2} r \leqslant \mu(B_\delta(x, r)). \end{aligned}$$

Proof. Part (i): if $y \in B_\delta(x, r)$ and $y \neq x$, then $r < \mu(\{x\}) \leqslant \mu(B(x, d(x, y))) = \delta(x, y) \leqslant r$, which is a contradiction. Then $B_\delta(x, r) = (\{x\})$. Part (ii): if $\mu(\{x\}) \leqslant r$, since

$$B_\delta(x, r) = \bigcup \{B_d(x, d(x, y)) : y \in B_\delta(x, r)\},$$

it turns out that $\mu(B_\delta(x, r))) \leqslant r$. Part (iii): let $y \in X$; since $\mu(B_d(x, d(x, y)) \leqslant \mu(X) \leqslant r$, it follows that $y \in B_\delta(x, r)$. Part (iv): assume that $B_\delta(x, r) = \{x\}$. This implies that for every $y \neq x$, $\mu(B_d(x, d(x, y))) > r$. Let $\{y_n\}$ be a sequence of points of X such that

$$m = \lim d(x, y_n) = \inf\{d(x, y) : y \in X, y \neq x\}.$$

If this limit m is equal to zero, we have $\mu(\{x\}) = \lim \mu(B(x, d(x, y_n))) \geqslant r$ and therefore $\mu(B_\delta(x, r)) = \mu(\{x\}) \geqslant r > A^{-2}r$. If m is positive, then $B_\delta(x, 3m/4) = \{x\}$ and $\mu(B_d(x, 2k3m/4)) > r$. Thus,

$$r < A\mu(B_d(x, 3m/4)) = A\mu(\{x\}) = A\mu(B_\delta(x, r)),$$

verifying (iv). Let us assume now that $B_\delta(x, r) \neq \{x\}$. Let $s = \sup\{d(x, y) : x \neq y,\ y \in B_\delta(x, r)\}$. Then $s > 0$, and moreover s is finite, since otherwise $B_\delta(x, r) = X$ and then $r < \mu(X) = \mu(B_\delta(x, r)) \leqslant r$, which is a contradiction. Let $t < s < 2t$. If $A^{-2}r > \mu(B_\delta(x, r))$, we shall show that for every positive integer n, $B_d(x, (2K)^n t) = B_d(x, s)$ holds. For $n = 1$, we have

$$\mu(B_d(x, 2Kt)) \leqslant A\mu(B_d(x, t)) \leqslant A\mu(B_\delta(x, r)) \leqslant A^{-1}r < r.$$

If there were $y \in B_d(x, 2Kt) \sim B_d(x, s)$, there would exist $y \in B_\delta(x, r)$ and $d(x, y) > s$, contradicting the definition of s. For $n + 1$, we have

$$\begin{aligned} \mu(B_d(x, (2K)^{n+1} t)) &\leqslant A\mu(B_d(X, (2K)^n t)) = A\mu(B_d(x, s)) \\ &\leqslant A\mu(B_d(x, 2Kt)) \leqslant A^2\mu(B_d(x, t)) \\ &\leqslant A^2\mu(B_\delta(x, r)) < r. \end{aligned}$$

Again, since $(2K)^{n+1}t > s$, it follows that $B_d(x, (2K)^{n+1} t) = B_d(x, s)$. Therefore, we have $B_d(x, s) = X$. From

$$\begin{aligned} r < \mu(X) = \mu(B_d(x, s)) = \mu(B_d(x, 2Kt)) \leqslant A\mu(B_d(x, t)) \\ \leqslant A\mu(B_\delta(x, r)), \end{aligned}$$

it follows that

$$A^{-2}r < A^{-1}r \leqslant \mu(B_\delta(x, r)),$$

which is a contradiction and (iv) is proved.

(1.10) Lemma. *Let $K' = (C + K)^{2/\beta}$, where C is the constant in condition (L_β) of* (1.3). *Then, if (X, d, μ) saatisfies conditions (L_β) and (H_α) of* (1.3) *and* (1.5), *respectively, we have*

$$\begin{aligned} &|\mu(B_d(x, d(x', y))) - \mu(B_d(x', d(x', y)))| \\ &\qquad \leqslant C''\mu(B_d(x, d(x', y)))^{1-\alpha}\, \mu(B_d(x, d(x, x')))^\alpha, \end{aligned}$$

provided that $K'd(x, x') \leqslant d(x', y)$.

Proof. Let us assume first that $\mu(B_d(x, d(x', y)))$ is larger than $\mu(B_d(x', d(x', y)))$. If $z \in B(x, d(x', y))$, we have

$$d(z, x') \leqslant K\,|d(z, x) + d(x, x')| \leqslant 2Kd(x', y).$$

Then, by condition (L_β) of (1.3),

$$d(z, x') \leqslant d(z, x) + C(2K)^{1-\beta}\, d(x', y)^{1-\beta}\, d(x, x')^{\beta},$$

or

$$d(z, x') \leqslant d(x', y) + d(x', y)^{1-\beta}\, (C^{1/\beta}(2K)^{(1-\beta)/\beta}\, d(x, x'))^{\beta}.$$

Since $C^{1/\beta}(2K)^{(1-\beta)/\beta}\, d(x, x') \leqslant K'd(x, x') \leqslant d(x', y)$, condition (H_α) implies

$$\begin{aligned}
&\mu(B_d(x, d(x', y))) - \mu(B_d(x', d(x', y)))\\
&\quad \leqslant C''\mu(B_d(x', d(x', y)))^{1-\alpha}\, \mu(B_d(x', d(x, x')))^{\alpha}\\
&\quad \leqslant C''\mu(B_d(x, d(x', y)))^{1-\beta}\, \mu(B_d(x, d(x, x')))^{\alpha}.
\end{aligned}$$

The case $\mu(B_d(x, d(x', y))) \leqslant \mu(B_d(x', d(x', y)))$ is similar and even simpler.

(1.11) PROPOSITION. *Let (X, d, μ) be a space of homogeneous type satisfying conditions (L_β) and (H_α). Then, the non-necessarily symmetric quasi-distance $\delta(x, y)$ associated to the space satisfies*

(i) $|\delta(x, y) - \delta(x', y)| \leqslant Cr^{1-\alpha}\delta(x, x')^{\alpha}$, *whenever* $\delta(x, y)$ *and* $\delta(x', y)$ *are less than or equal to r, and*

(ii) *for every $x \in X$, $\delta(x, y)$ is a continuous function of y.*

Proof. We can assume that $d(x, y) \geqslant d(x', y)$. Let $r = [d(x, y) + d(x', y)]/2$ and $s = [d(x, y) - d(x', y)]^{1/\beta}.\ [d(x, y) + d(x, y)]^{1-1/\beta}/2$. It is easy to see that

$$(s/r)^{\beta} = [d(x, y) - d(x', y)]/[d(x, y) - d(x', y)] \leqslant 1,$$

that is to say, $s \leqslant r$. Moreover,

$$r + r^{1-\beta}s^{\beta} = d(x, y) \qquad \text{and} \qquad r - r^{1-\beta}s^{\beta} = d(x', y).$$

By condition (L_β), we have

$$d(x, y) - d(x', y) \leqslant Cd(x, y)^{1-\beta}\, d(x, x')^{\beta};$$

therefore, $s \leqslant Cd(x, x')$. It is also evident that $r \leqslant d(x, y)$. Applying condition (H_α) with the given r,

$$\mu(B_d(x, d(x, y))) - \mu(B_d(x, d(x', y)) \\ \leqslant C\mu(B_d(x, d(x, y)))^{1-\alpha}\, \mu(B_d(x, d(x, x')))^{\alpha}.$$

On the other hand, by Lemma (1.10), it follows that if $K'd(x, x')^{\beta} < d(x', y)^{\beta}$,

$$|\mu(B_d(x, d(x', y))) - \mu(B_d(x', d(x', y)))| \\ \leqslant C\mu(B_d(x, d(x', y)))^{1-\alpha}\, \mu(B_d(x, d(x, x')))^{\alpha}.$$

If we assume that $K'd(x, x')^{\beta} \geqslant d(x', y)^{\beta}$, we have

$$\mu(B_d(x, d(x', y))) \\ = \mu(B_d(x, d(x', y)))^{1-\alpha}\, \mu(B_d(x, d(x', y)))^{\alpha} \\ \leqslant C\mu(B_d(x, d(x, y)))^{1-\alpha}\, \mu(B_d(x, d(x, x')))^{\alpha}.$$

On the other hand,

$$\mu(B_d(x', d(x', y))) \\ = \mu(B_d(x', d(x', y)))^{1-\alpha}\, \mu(B_d(x', d(x', y)))^{\alpha} \\ \leqslant C\mu(B_d(x', d(x', y)))^{1-\alpha}\, \mu(B_d(x', d(x, x')))^{\alpha}.$$

Let $u \in B_d(x', d(x', y))$; we have

$$d(u, y) \leqslant K[d(u, x') + d(x', y)] \leqslant 2Kd(x', y) \leqslant 2Kd(x, y),$$

showing that $B_d(x', d(x', y) \subset B_d(y, 2Kd(x, y))$. Therefore,

$$\mu(B_d(x', d(x', y))) \leqslant A\mu(B_d(y, d(x, y))) \leqslant C'\mu(B_d(x, d(x, y)))^{\alpha}.$$

Thus, we have

$$\mu(B_d(x', d(x', y)) \leqslant C''\mu(B_d(x, d(x, y)))^{1-\alpha}\, \mu(B_d(x, d(x, x')))^{\alpha}.$$

Collecting results, it follows that

$$|\delta(x, y) - \delta(x', y)| \leqslant |\mu(B_d(x, d(x, y))) - \mu(B_d(x, d(x', y)))| \\ + |\mu(B_d(x, d(x', y))) - \mu(B_d(x', d(x', y)))| \\ \leqslant C\mu(B_d(x, d(x', y)))^{1-\alpha}\, \mu(B_d(x, d(x, x')))^{\alpha} \\ = C\delta(x, y)^{1-\alpha}\, \delta(x, x')^{\alpha},$$

which implies (i).

As for part (ii), by virtue of Lemma (1.6) we have two possible cases. First, for every $x \in X$, $\mu(\{x\}) > 0$. In this case X is a discrete space for both d and δ and therefore, every function on X is continuous. The second case is when $\mu(\{x\}) = 0$. Then, if $d(x, y) > d(x, y')$, choosing r and s as

$$r + r^{1-\beta}s^{\beta} = d(x, y), \qquad \text{and} \qquad r - r^{1-\beta}s^{\beta} = d(x, y'),$$

we get

$$r = [d(x, y) + d(x, y')]/2$$
$$s = \{([d(x, y) - d(x, y')]/2([d(x, y) + d(x, y')]/2)^{1-\beta}\}^{1/\beta},$$

$s \leqslant r$, and $r \leqslant d(x, y)$. Thus, by condition (H_α), it follows that

$$|\delta(x, y) - \delta(x, y')| \leqslant C\mu(B_d(x, d(x, y)))^{1-\alpha}\, \mu(B_d(x, s))^{\alpha}.$$

Since y' tending to y implies that s tends to zero and $\lim \mu(B_d(x, s)) = \mu(\{x\}) = 0$, the continuity of $\delta(x, y)$ is proved.

In the rest of this chapter, (X, δ, μ) will be a triple satisfying the following conditions:

$$\begin{aligned}
&\text{(i)} && 0 \leqslant \delta(x, y) < \infty \text{ and } \delta(x, y) = 0 \quad \text{if and only if } x = y\\
&\text{(ii)} && \delta(x, y) \leqslant K\delta(y, x),\\
&\text{(iii)} && \delta(x, y) \leqslant K[\delta(x, z) + \delta(z, y)],\\
&\text{(iv)} && \text{if } K_1\mu(\{x\}) \leqslant r \leqslant K_2\mu(X), \quad \text{then}\\
& && rA_1 \leqslant \mu(B_\delta(x, r)) \leqslant rA_2,\\
&\text{(v)} && \text{if } r < K_1\mu(\{x\}), \quad \text{then } B_\delta(x, r) = \{x\} \text{ and}\\
&\text{(vi)} && \text{if } r > K_2\mu(X), \quad \text{then } B_\delta(x, r) = X,
\end{aligned} \tag{1.12}$$

where K, K_1, K_2, A_1, and A_2 are constants. These conditions imply the existence of a constant satisfying (1.2), i.e., $\mu(B_\delta(x, 2Kr)) \leqslant A\mu(B_\delta(x, r))$. We shall call a triple (X, δ, μ) satisfying conditions (1.12) a non-necessarily symmetric normalized space. The only difference between this and a normalized space is that instead of assuming δ to be symmetric, we assume that (ii) of (1.12) holds with K non-necessarily equal to one.

(1.13) THEOREM (Approximation of the Identity). *Let (X, δ, μ) be a non-necessarily symmetric normalized space of order α, that is to say*

$$|\delta(x, y) - \delta(x', y)| \leqslant Cr^{1-\alpha}\delta(x, x')^{\alpha} \tag{1.14}$$

holds for an α, $0 < \alpha \leqslant 1$, whenever $\delta(x, y) < r$ and $\delta(x', y) < r$. If $\delta(x, y)$ is *non-symmetric, we assume that $\delta(x, y)$ is a continuous function of y. Then, for every t, $0 < t < C\mu(X)$, there is a function $s_t(x, y)$ satisfying*

(i) $0 \leqslant s_t(x, y) \leqslant C[\mu(B_\delta(x, t))^{-1} + \mu(B_\delta(y, t))^{-1}]$,

(ii) *if* $\delta(x, y) < C^{-1}t$,
then $s_t(x, y) \geqslant C^{-1}[\mu(B_\delta(x, t))^{-1} + \mu(B_\delta(y, t))^{-1}]$,

(iii) $s_t(x, y) = s_t(y, x)$

(iv) $\operatorname{supp} s_t \subset \{(x, y) : \delta(x, y) < Ct\}$

(v) $|s_t(x, y) - s_t(x', y)|$
$\leqslant C\delta(x, x')^\alpha\, [\mu(B_\delta(x, t))^{-1} + \mu(B_\delta(x', t))^{-1}]^{1+\alpha}$

(vi) $\int s_t(x, y)\, d\mu(y) = 1$,

where C is a finite constant. If necessary, C can be chosen as large as desired.

In order to prove this theorem, we shall need some lemmas.

Let $h(t)$ be a C^∞ function defined on $|0, \infty)$ that satisfies $h(t) = 1$ if $0 \leqslant t \leqslant 1$, $h(t) = 0$ if $t \geqslant A$, and $0 \leqslant h(t) \leqslant 1$ for every $t \geqslant 0$.

(1.15) LEMMA. *If $u_t(x, y) = h(\delta(x, y)/t)$, then*

$$|u_t(x, y) - u_t(x', y)| \leqslant C\delta(x, x')^\alpha\, [\mu(B_\delta(x, t))^{-1} + (B_\delta(x', t))^{-1}\}^\alpha.$$

Proof. Let $\delta(x, y) \leqslant 2KAt$ and $\delta(x', y) \leqslant 2KAt$. Then, by (1.14), we have

$$|u_t(x, y) - u_t(x', y)| \leqslant \|h'\|_\infty\, |\delta(x, y) - \delta(x', y)|/t \leqslant C(\delta(x, x')/t)^\alpha.$$

If $\delta(x, y) > 2KAt$ and $\delta(x', y) \leqslant At$, then

$$2KAt < \delta(x, y) \leqslant K(\delta(x, x') + \delta(x', y)) \leqslant K\delta(x, x') + KAt;$$

thus, $t \leqslant At \leqslant \delta(x, x')$. Therefore

$$|u_t(x, y) - u_t(x', y)| = 1 \leqslant (\delta(x, x')/t)^\alpha.$$

The other possible cases are trivial. Now, if $K_2\mu(X) \geqslant t \geqslant \min(K_1 A^{-1}\mu(\{x\}), K_1 A^{-1}\mu(\{x'\}))$ then

$$|u_t(x, y) - u_t(x', y)| \leqslant C'\delta(x, x')^\alpha\, [\mu(B_\delta(x, t))^{-1} + \mu(B_\delta(x', t))^{-1}]^\alpha.$$

If $t < \min(K_1 A^{-1}\mu(\{x\}), K_1 A^{-1}\mu(\{x'\}))$, then $B_\delta(x, t) = \{x\}$, $B_\delta(x', t) = \{x'\}$, and

$$u_t(x, y) = 1 \text{ if } x = y \quad \text{and} \quad u_t(x, y) = 0 \text{ if } x \neq y,$$
$$u_t(x', y) = 1 \text{ if } x' = y \quad \text{and} \quad u_t(x', y) = 0 \text{ if } x \neq y.$$

Assume $x \neq x'$. Then $K_1 \mu(\{x\}) \leqslant \delta(x, x')$ and $K_1 \mu(\{x\}) \leqslant \delta(x', x) < K\delta(x, x')$, yielding

$$|u_t(x, y) - u_t(x', y)| \leqslant 1 \leqslant C\delta(x, x')^\alpha [\mu(\{x\})^{-1} + \mu(\{x'\})^{-1}]^\alpha$$
$$\leqslant C'\delta(x, x')^\alpha [\mu(B_\delta(x, t))^{-1} + \mu(B_\delta(x', t))^{-1}]^\alpha.$$

(1.16) LEMMA. *Let*

$$m_t(x) = \int u_t(x, y)\, d\mu(y).$$

Then $m_t(x)$ *is well defined and*

(i) $|m_t(x) - m_t(x')| \leqslant C\delta(x, x')^\alpha [\mu(B_\delta(x, t))^{-1} + \mu(B_\delta(x', t))^{-1}]^\alpha \cdot [\mu(B_\delta(x, t)) + \mu(B_\delta(x', t)];$

moreover,

(ii) $\mu(B_\delta(x, t)) \leqslant m_t(x) \leqslant \mu(B_\delta(x, At))$.

Proof. The function $m_t(x)$ is well defined since we assume that $\delta(x, y)$ is a continuous function of y. On the other hand, by Lemma (1.15), we have

$$|m_t(x) - m_t(x')| \leqslant \int |u_t(x, y) - u_t(x', y)|\, d\mu(y)$$
$$\leqslant C'\delta(x, x')^\alpha [\mu(B_\delta(x, t))^{-1} + (\mu(B_\delta(x', t))^{-1}]^\alpha$$
$$\times [\mu(B_\delta(x, t)) + \mu(B_\delta(x', t))].$$

As for (ii), since $u_t(x, y) = 1$ if $y \in B_\delta(x, t)$ and $u_t(x, y) = 0$ if $y \notin B(x, t)$, (ii) follows.

(1.17) LEMMA. *Let*

$$v_t(x, y) = m_t(x)^{-1}\, u_t(x, y).$$

Then,

(i) $|v_t(x, y) - v_t(x', y)|$
$\leqslant C\delta(x, x')^\alpha [\mu(B_\delta(x, t))^{-1} + \mu(B_\delta(x', t))^{-1}]^{1+\alpha}$,

(ii) $\int v_t(x, y)\, d\mu(y) = 1,$ *and*

(iii) $C^{-1} \leqslant \int v_t(x, y)\, d\mu(x) \leqslant C,$

where C *is a finite constant.*

Proof. We can assume that $m_t(x') \leqslant m_t(x)$. Then

$$v_t(x, y) - v_t(x', y) = m_t(x)^{-1}\,[u_t(x, y) - u_t(x', y)]$$
$$+ u_t(x', y)[m_t(x') - m_t(x)]\, m_t(x)^{-1}\, m_t(x')^{-1}.$$

By Lemmas (1.15) and (1.16), it follows that

$$|v_t(x, y) - v_t(x', y)| \leqslant C'\delta(x, x')^\alpha\, [\mu(B_\delta(x, t))^{-1} + \mu(B_\delta(x', t))^{-1}]^{1+\alpha}.$$

As for (ii), it is apparent from the definition of $v_t(x, y)$. In order to prove (iii), we observe that

$$C^{-1} m_t(y) \leqslant m_t(x) \leqslant C m_t(y),$$

for $x \in B_\delta(y, At)$. This implies (iii).

Proof of Theorem (1.13). Let

$$w(z) = \left(\int v_k(x, z)\, d\mu(x)\right)^{-1}.$$

We define

$$s_t(x, y) = \int v_t(x, z)\, w(z)\, v_t(y, z)\, d\mu(z).$$

Part (i) By definition of v_t and from part (iii) of Lemma (1.17), we get

$$0 \leqslant s_t(x, y) \leqslant (m_t(x)^{-1}\, m_t(y)^{-1}\, [\mu(B_\delta(x, t)) + \mu(B_\delta(y, t))]$$
$$\leqslant C[\mu(B(x, t))^{-1} + \mu(B_\delta(y, t))^{-1}].$$

Part (ii). If $\delta(x, z) < C^{-1}t$ and $\delta(x, y) < C^{-1}t$, then $\delta(y, z) \leqslant K(\delta(y, x) + \delta(x, z)) \leqslant 2KAC^{-1}t < t$, if C is chosen to be $2KA < C$. Then

$$s_t(x, y) \geqslant C' m_t(x)^{-1}\, m_t((y)[\mu(B_\delta(x, t)) + \mu(B_\delta(y, t))]$$
$$\geqslant C^{-1}[\mu(B_\delta(x, t))^{-1} + \mu(B_\delta(y, t))^{-1}].$$

Part (iii). follows from the definition of $s_t(x, y)$.

Part (iv). If $s_t(x, y) > 0$, there exists z such that $\delta(x, z) < At$ and $\delta(y, z) < At$, therefore $\delta(x, y) \leqslant Ct$.

Part (v). By Lemma (1.17) we have

$$|s_t(x, y) - s_t(x', y)| \leqslant \int |v_t(x, z) - v_t(x', z)|\, w(z)\, v_t(y, z)\, d\mu(z)$$
$$\leqslant C\delta(x, x')^\alpha\, [\mu(B_\delta(x, t))^{-1} + \mu(B_\delta(x', t))^{-1}]^{1+\alpha}.$$

Part (vi). By Lemma (1.7) we have

$$\int s_t(x, y)\, d\mu(y) = \int v_t(x, z)\, w(z) \left(\int v_t(y, z)\, d\mu(y) \right) d\mu(z)$$

$$= \int v_t(x, z)\, d\mu(z) = 1.$$

(1.18) THEOREM. *If (X, δ, μ) is a non-necessarily symmetric normalized space of order α, then there exists δ', symmetric and equivalent to δ, such that (X, δ', μ) is a normalized space of order α, that is to say, it satisfies conditions* (1.4) *and* (L_α).

Proof. Let C be the constant of Theorem (1.13). If $x \neq y$, let i be the integer such that $cA^{-i-1} < \delta(x, y) \leqslant CA^{-i}$. Let p be the integer satisfying

$$C^{-1}A^{-p-2} < K_2\mu(X) \leqslant C^{-1}A^{-p-1},$$

and let n be the positive integer satisfying

$$C^2A^{-n} < 1 \leqslant C^2A^{-n+1}.$$

Then, if $k \leqslant i$, we have

$$CA^{-k} \geqslant CA^{-i} \geqslant \delta(x, y) \geqslant K_1\mu(\{x\});$$

thus, $\mu(B_\delta(x, A^{-i})) \approx \mu(B_\delta(x, CA^{-i})) \approx A^{-i}$. On the other hand, we have

$$CA^{-i-1} < \delta(x, y) \leqslant K_2\mu(X) \leqslant C^{-1}A^{-p-1},$$

therefore,

$$1 < C^2A^{-n+1} < A^{i-p-n},$$

thus, $i \geqslant p + n$.

Moreover,

$$\delta(x, y) \leqslant CA^{-i} = C^2A^{-n}C^{-1}A^{-i-n} < C^{-1}A^{-(i-n)}$$

and if $k \geqslant i + 1$, then

$$\delta(x, y) > CA^{-i-1} \geqslant CA^{-k}.$$

We have that

$$s(x, y) = \sum_{k=p}^{s} s_{A^{-k}}(x, y)$$

satisfies

$$s(x, y) = \sum_{k=p}^{i} s_{A^{-k}}(x, y) \leqslant C' \sum_{k=p}^{i} A^k \leqslant C'' A^i \leqslant C''' \delta(x, y)^{-1}$$

and

$$s(x, y) \geqslant s_{A^{-(i-n)}}(x, y) \geqslant C A^i \geqslant C \delta(x, y)^{-1}.$$

Next, we estimate $|s(x, y) - s(x', y)|$. We can assume that $0 < \delta(x, y) \leqslant \delta(x', y)$. Let m be an integer satisfying $A^m \geqslant 2K$. Then, if $A^m \delta(x, y) \leqslant \delta(x', y)$, we have

$$A^m \delta(x, y) \leqslant \delta(x', y) \leqslant K\delta(x', x) + K\delta(x, y),$$

which implies $\delta(x', y)/2 \leqslant K\delta(x, x')$. Then

$$|s(x, y) - s(x', y)| \leqslant C'\delta(x, y)^{-1} \leqslant C'' \frac{\delta(x', x)}{\delta(x, y)\, \delta(x', y)^{\alpha}}.$$

If $\delta(x, y) \leqslant \delta(x', y) \leqslant A^m \delta(x, y) \leqslant C A^{m-i+1}$, and since for $k > i$, $CA^{-k} \leqslant CA^{-i-1} < \delta(x, y) \leqslant \delta(x', y)$, we have $s_{A^{-k}}(x, y) = s_{A^{-k}}(x', y) = 0$; thus

$$|s(x, y) - s(x', y)| \leqslant \sum_{k=p}^{i} |s_{A^{-k}}(x, y) - s_{A^{-k}}(x', y)|,$$

and by Theorem (1.13), we get that

$$\begin{aligned} |s(x, y) - s(x', y)| &\leqslant C'\delta(x, x')^{\alpha} \sum_{k=p}^{i} A^{k(1+\alpha)} \\ &\leqslant C'' A^{i(1+\alpha)} \delta(x, y)^{\alpha} \leqslant C''' \delta(x', y)^{-(1+\alpha)}\, \delta(x, x')^{\alpha}. \end{aligned}$$

Now, let us define

$$\delta'(x, x) = 0 \qquad \text{and}$$
$$\delta(x, y) = s(x, y)^{-1} \qquad \text{for} \quad x \neq y.$$

We have already shown that there exists a constant $C > 0$ such that

$$C^{-1}\delta(x, y) < \delta'(x, y) \leqslant C\delta(x, y).$$

Let us estimate $|\delta'(x, y) - \delta'(x', y)|$. If $x = y$, then

$$|\delta'(x, x) - \delta'(x', x)| \leqslant C r^{1-\alpha} \delta(x, x')^{\alpha}$$

if $\delta(x, x') < r$. Analogously for $x' = y$. Thus, we can assume that $x \neq x'$, $y \neq x$, and $y \neq x'$. Then

$$|\delta'(x, y) - \delta'(x', y)| \leqslant C' \, |s(x, y) - s(x', y)| \, \delta'(x, y) \, \delta'(x', y),$$

which, by previous estimates on $s(x, y)$, is smaller than or equal to

$$C'' \frac{\delta(x, x')}{\delta(x, y) \, \delta(x', y)^{\alpha}} \, \delta'(x, y) \, \delta(x', y) \leqslant C''' r^{1-\alpha} \delta(x, x')^{\alpha},$$

if $\delta(x, y) \leqslant \delta(x', y) \leqslant r$. This ends the proof of the theorem.

(1.19) THEOREM. *Let (X, d, μ) be a space of homogeneous type satisfying conditions (L_β) and (H_α), Then a normalization of order α can be found for this space.*

Proof. The normalization is given by the quasi-distance $\delta'(x, y)$ of Theorem (1.18), where $\delta(x, y)$ is the non-necessarily symmetric quasi-distance associated to (X, d, μ) in (1.7). Propositions (1.9) and (1.11) and Theorem (1.18) show that (X, δ', μ) is a normalized space of order α.

(1.20) PROPOSITION. *Let f be a Lipschitz function of order $\eta \leqslant \alpha$, with respect to the quasi-distance δ, supported in $B_\delta(x_0, r)$, and (X, δ, μ) a normalized space of order α. Then if $0 < \eta' < \eta$, we have that the functions*

$$f_t(x) = \int S_t(x, y) \, f/y) \, d\mu(y),$$

for $t < K_2 \mu(X)$, satisfy

(i) $\operatorname{supp} f_t \subset B(x_0, r + C'' r^{1-\alpha}, t^\alpha)$, *if* $t < r$,

(ii) $|f_t(x) - f_t(x')| \leqslant C'' t^{-(1+\alpha)} \mu(B(x_0, r))^{1+\eta} \, \delta(x, x')$.

(iii) $|(f_t(x) - f(x)) - (f_t(x') - f(x'))| \leqslant C(t) \, \delta(x, x')^{\eta}$, *where* $\lim_{t \to 0} C(t) = 0$.

Proof. The support of $f_t(x)$ is contained in the set of point x such that there exists y satisfying $\delta(x, y) < Ct$ and $\delta(x_0, y) < r$. Then $|\delta(x_0, x) - \delta(x_0, y)| \leqslant C'(t + r)^{1-\alpha} \, \delta(x, y)^\alpha \leqslant C'(t + r)^{1-\alpha} \, t^\alpha \leqslant C'' r^{1-\alpha} t^\alpha$.

Let us consider part (ii). We have

$$|f_t(x) - f_t(x')| \leqslant \int |s_t(x, y) - s_t(x', y)| \, |f(y)| \, d\mu(y).$$

By Theorem (1.13), this is smaller than or equal to

$$C\delta(x, x')^{\alpha} [\mu(B_{\delta}(x, t))^{-1} + \mu(B_{\delta}(x', t))^{-1}]^{1+\alpha} \int |f(y)| \, d\mu(y)$$
$$\leqslant C'\delta(x, x')^{\alpha} t^{-(1+\alpha)} C\mu(B(x_0, r))^{\eta+1}.$$

As for part (iii), given $\varepsilon > 0$, assume that $t < \varepsilon$; then

$$|f_t(x) - f(x)| \leqslant \int s_t(x, y) f(y) - f(x)| \, d\mu(y)$$
$$\leqslant C \int s_t(x, y) \, \delta(x, y)^{\eta} \, d\mu(y) \leqslant Ct^{\eta}.$$

If $\delta(x, x') \geqslant t$, we get

$$|f_t(x) - f(x)| \leqslant C\varepsilon^{\eta - \eta'} \delta(x, x')^{\eta'}.$$

Analogously for $f_t(x') - f(x')$. If $\delta(x, x') < t$, we have

$$|(f_t(x) - f(x)) - (f_t(x') - f(x'))|$$
$$\leqslant |f_t(x) - f_t(x')| + |f(x) - f(x')| = I_1 + I_2.$$

For I_1, we have

$$|f_t(x) - f_t(x')| = \left| \int |s_t(x, y) - s_t(x', y)| \, f(y) \, d\mu(y) \right|$$
$$\leqslant \int |s_t(x, y) - s_t(x', y)| \, |f(y) - f(x)| \, d\mu(y)$$
$$\leqslant C\delta(x, x')^{\alpha} t^{-1-\alpha} \int_{B_{\delta}(x, At) \cup B_{\delta}(x', At)} \delta(x, y)^{\eta} \cdot d\mu(y)$$
$$\leqslant C'\delta(x, x')^{\alpha} t^{-1-\alpha} t^{\eta+1} \leqslant C''\delta(x, x')^{\eta'} \varepsilon^{\eta - \eta'}.$$

The same estimate holds for $|f(x) - f(x')| = I_2$. This ends the proof of the proposition.

II. Singular Integral Operators

In this chapter (X, δ, μ) will be a triple satisfying the following conditions:

(i) $0 \leqslant \delta(x, y) < \infty$ and $\delta(x, y) = 0$ if and only if $x = y$,

(ii) $\delta(x, y) = \delta(y, x)$,

$$\begin{aligned}
&\text{(iii)} \quad \delta(x, y) \leqslant K(\delta(x, z) + \delta(z, y)),\\
&\text{(iv)} \quad \text{if } k_1\mu(\{x\}) \leqslant r \leqslant k_2\mu(X) \text{ then } rA_1 \leqslant \mu(B_\delta(x, r)) \leqslant rA_2,\\
&\text{(v)} \quad \text{if } r < k_1\mu(\{x\}) \text{ then } B_\delta(x, r) = \{x\}, \qquad (2.1)\\
&\text{(vi)} \quad \text{if } r > k_2\mu(X) \text{ then } B_\delta(x, r) = X, \text{ and}\\
&\text{(vii)} \quad \text{there exists } \alpha,\ 0 < \alpha \leqslant 1, \text{ such that}\\
&\qquad |\delta(x, y) - \delta(x', y)| \leqslant C r^{1-\alpha}\delta(x, x')^\alpha\\
&\qquad \text{holds, whenever } \delta(x, y) < r \text{ and } \delta(x', y) < r,
\end{aligned}$$

where K, k_1, k_2, A_1, A_2, and C are constants. These conditions imply the existence of a constant A satisfying (1.2). For the sake of simplicity we shall assume that $A = K$.

Given a ball B and a number γ, $0 < \gamma \leqslant \alpha$, we denote by $\Lambda(B)$ the Banach space of complex-valued functions supported on B, such that

$$|\psi(x) - \psi(y)| \leqslant C\delta(x, y). \tag{2.2}$$

Given $\psi \in \Lambda(B)$ we shall denote by $|||\psi|||_\gamma$ the infimum of the constants C appearing in (2.2).

We say that ψ belongs to Λ_0^γ if $\psi \in \Lambda^\gamma(B)$ for some ball B. On Λ_0^γ we define the topology which is the inductive limit of the spaces $\Lambda^\gamma(B)$, see [MS2], and $(\Lambda_0^\gamma)'$ denotes the space of all continuous linear functions on Λ_0^γ. By $\{\Lambda_0^\gamma\}_0$ we denote the subspace of all functions ψ in Λ_0^γ such that $\int \psi(x)\, d\mu(x) = 0$. Λ_b^γ stands for the space of bounded functions ψ satisfying (2.2). As usual, B.M.O. is the space of all the locally integrable functions g on X such that

$$\mu(B)^{-1} \int_B |g(x) - m_B g|\, d\mu(x) \leqslant C,$$

where B is any ball and $m_B g = \mu(B)^{-1} \int_B g(x)\, d\mu(x)$.

We consider a continuous linear operator T from Λ_0^γ into $(\Lambda_0^\gamma)'$ for some γ, $0 < \gamma \leqslant \alpha$, associated to a kernel $k(x, y)$, that is to say, for any x not in the support of f

$$Tf(x) = \int k(x, y) f(y)\, d\mu(y).$$

Let $\tilde{k}(x, y)$ be the function defined by

$$\sup\{\mu(B_\delta(x, \varepsilon))^{-1}\, \mu(B_\delta(y, \varepsilon))^{-1} \cdot \iint_{\substack{\delta(u, x) < \varepsilon\\ \delta(v, y) < \varepsilon}} |k(u, v)|\, d\mu(u)\, ds(v) : \delta(x, y) > \varepsilon 4A^2\}. \tag{2.3}$$

We say that k satisfies an L^r-Dini condition $1 \leqslant r \leqslant \infty$, if the following conditions hold:

for any $R > 0$,

$$\left(\int_{R < \delta(x, y) \leqslant AR} (|\tilde{k}(x, y)|^r + |\tilde{k}(y, x)|^r) \, d\mu(y) \right)^{1/r} \leqslant CR^{-1/r'}, \tag{2.4}$$

there exists η, $0 < \eta \leqslant \alpha$, such that if $A\delta(y, z) \leqslant R$, then

$$\left(\int_{R < \delta(y, x) \leqslant AR} |k(y, x) - k(z, x)|^r \, d\mu(x) \right)^{1/r} \leqslant CR^{-1/r'} \left(\frac{\delta(y, z)}{R} \right)^{\eta}, \tag{2.5}$$

and

$$\left(\int_{R < \delta(y, x) \leqslant AR} |k(x, y) - k(x, z)|^r \, d\mu(x) \right)^{1/r} \leqslant CR^{-1/r'} \left(\frac{\delta(y, z)}{R} \right)^{\eta}. \tag{2.6}$$

(2.7) Lemma. *Let $k(x, y)$ be a kernel satisfying* (2.4), *and* η, $0 < \eta \leqslant \alpha$, *then*

$$\int_{B_\delta(x, s)} \delta(x, y)^\eta \, \tilde{k}(x, y) \, d\mu(y) \leqslant C \min(s^\eta, \mu(B_\delta(x, s))^\eta)$$

Proof. If $s < k_1 \mu(\{x\})$, then the integral is equal to zero. It is enough to assume $s < k_2 \mu(X)$. Then

$$\begin{aligned} &\int_{B_\delta(x, s)} \delta(x, y)^\eta \, \tilde{k}(x, y) \, d\mu(y) \\ &\quad \leqslant \sum_{j=0}^{\infty} \int_{A^{-j}s < \delta(x, y) A^{-j+1}s} \delta(x, y)^\eta \, \tilde{k}(x, y) \, d\mu(y) \\ &\quad \leqslant \sum_{j=0}^{\infty} \left(\int_{A^{-j}s < \delta(x, y) \leqslant A^{-j+1}} |\tilde{k}(x, y)|^r \, d\mu(y) \right)^{1/r} \\ &\quad\quad \times \left(\int_{A^{-j}s < \delta(x, y) \leqslant A^{-j+1}s} \delta(x, y)^{\eta r'} \, d\mu(y) \right)^{1/r'} \\ &\quad \leqslant C \sum_{j=0}^{\infty} (A^{-j}s)^{-1/r'} (A^{-j}s)^\eta (A^{-j}s)^{1/r'} \leqslant Cs^\eta. \end{aligned}$$

(2.8) Definition. We say that T is weakly bounded of order γ, $0 < \leqslant \alpha$, if T is a linear operator from Λ_0^γ into $(\Lambda_0^\gamma)'$ and

$$|\langle Tf, g \rangle| \leqslant C\mu(B)^{1+2\gamma} \, |||f|||_\gamma \, |||g|||_\gamma \tag{2.9}$$

holds for any ball B and functions f and g with their supports contained in B.

(2.10) LEMMA. *Let T be a continuous linear operator from Λ_0^γ into $(\Lambda_0^\gamma)'$ for some γ, $0<\gamma<\alpha$, associated to a kernel satisfying* (2.4) *and* (2.5). *Let us assume that T is weakly bounded of order η, for some η, $\gamma \leqslant \eta$. Then, for any f, g, and h in $\Lambda_0^{\gamma'}$, $\gamma'>\gamma$,*

$$\langle Tgh, f\rangle = \langle Th, fg\rangle + \iint f(x)[g(y)-g(x)] \times k(x, y)\, h(y)\, d\mu(x)\, d\mu(y) \tag{2.11}$$

holds.

Proof. It is clear that (2.11) holds if T is defined by integration against a locally bounded kernel.

In the general case let T_t be defined from $\Lambda_0^{\gamma'}$ into $(\Lambda_0^{\gamma'})'$ by

$$\langle T_t f, g\rangle = \langle Tf_t, g_t\rangle.$$

f_t and g_t are introduced in Proposition (1.20). Let $B=B_\delta(x_0, r)$ be a ball containing the support of f; then for $z\in B$, the support of $s_t(\cdot, z)$ is contained in the ball $B_\delta(x_0, C_t r)$. Thus, the application

$$z \to s_t(\cdot, z), \qquad z\in B,$$

is a $\Lambda^{\gamma'}(B_\delta(x_0, C_t r))$-valued Bochner integrable function with respect to the measure $|f(z)|\, d\mu(z)$. Therefore,

$$T_t(z, y) = \langle Ts_t(\cdot, z), s_t(\cdot, y)\rangle$$

is the kernel associated to T_t.

Since by Theorem (1.13) $s_t(\cdot, z)\in \Lambda_0^\eta$, then, by (2.9) (weak boundedness) and (2.4), if $t<k_2\mu(X)$, we get

$$|T_t(z, y)| \leqslant C\, |\mu(B_\delta(z, t)) + \mu(B_\delta(y, t))|^{-1}.$$

Then (2.11) holds for T_t. On the other hand, by Proposition (1.20), f_t converges to f in Λ_0^γ for f in $\Lambda_0^{\gamma'}$ when t goes to zero. Therefore, $\langle T_t f, g\rangle$ converges to $\langle Tf, g\rangle$ for f and g in $\Lambda_0^{\gamma'}$. Moreover, $T_t(x, y)$ converges pointwise to $k(x, y)$. Using again (2.4) and weak boundedness, it follows that for t sufficiently small, $|T_t(x, y)| \leqslant C\tilde{k}(x, y)$. Then, by the Lebesgue dominated convergence theorem, the right hand side of (2.11) is equal to the limit of

$$\iint f(x)\, |g(y)-g(x)|\, T_t(x, y)\, h(y)\, d\mu(x)\, d\mu(y).$$

Given a ball $B=B_\delta(z,s)$ we define

$$h_B(y)=h(\delta(z,y)/4A^2s), \tag{2.12}$$

where h is the function considered in (1.15).

(2.13) LEMMA. *Let $k(x,y)$ be a kernel satisfying* (2.5) *and $B=B_\delta(z,s)$. Then for any $x_1, x_2 \in B$*

$$\left|\int (k(x_1,y)-k(x_2,y))(1-h_B(y))\,d\mu(y)\right| \leqslant C\left(\frac{\delta(x_1,x_2)}{A\mu(B)}\right)^\eta \leqslant C.$$

Proof. It is enough to prove the lemma for $k_1\mu(\{z\})\leqslant s\leqslant k_2\mu(X)$. Then

$$\begin{aligned}
&\int_{4A^2s<\delta(z,y)} |k(x_1,y)-k(x_2,y)|\,d\mu(y)\\
&\quad\leqslant \int_{3As<\delta(x_1,y)} |k(x_1,y)-k(x_2,y)|\,d\mu(y)\\
&\quad\leqslant \sum_{j=0}^{\infty}\int_{A^j3As<\delta(x_1,y)\leqslant A^{j+1}3As} |k(x_1,y)-k(x_2,y)|\,d\mu(y).
\end{aligned}$$

Therefore, by (2.5), this is less than

$$\begin{aligned}
&\sum_{j=0}^{\infty} (A^{j+1}3As)^{1/r'}(A^j3As)^{-1/r'}(\delta(x_1,x_2))^\eta (A^j3As)^{-\eta}\\
&\quad\leqslant C\left(\frac{\delta(x_1,x_2)}{As}\right)^\eta \sum_{j=0}^{\infty}\frac{1}{A^{j\eta}} = C\left(\frac{\delta(x_1,x_2)}{As}\right)^\eta\\
&\quad\leqslant C\left(\frac{\delta(x_1,x_2)}{A\mu(B)}\right)^\eta.
\end{aligned}$$

(2.14) LEMMA. *Let $k(x,y)$ be a kernel satisfying* (2.5), *$B=B_\delta(z,s)$, and $\phi\in\Lambda_b^\gamma$, $0<\gamma\leqslant\alpha$. Then*

$$I_B\phi(x)=\int (k(x,y)-k(z,y))\,\phi(y)(1-h_B(y))\,d\mu(y)$$

is well defined for any $x\in B$. Moreover, $I_B\phi\in\Lambda^\gamma(B)'$ and I_B satisfies (2.9) *for functions supported on B.*

Proof. We can assume $s \leqslant k_2 \mu(X)$, since otherwise $I_B \phi = 0$. Let $\psi \in \Lambda^\gamma(B)$. By Lemma (2.13) we get

$$\left| \int I_B\phi(x)\, \psi(x)\, d\mu(x) \right| \leqslant C\, \|\psi\|_\infty\, \|\phi\|_\infty \int_{\delta(x,z)<s} \left(\frac{\delta(x,z)}{s}\right)^\eta d\mu(x)$$

$$\leqslant C\, \|\psi\|_\infty\, \|\phi\|_\infty\, \mu(B) \leqslant C\mu(B)^{1+\gamma}\, \|\phi\|_\infty\, |||\psi|||_\gamma.$$

If $\phi \in \Lambda^\gamma(B)$ then

$$\left| \int I_B\phi(x)\, \psi(x)\, d\mu(x) \right| \leqslant C\mu(B)^{1+2\gamma}\, |||\phi|||_\gamma\, |||\psi|||_\gamma.$$

(2.15) Definition. Let T be a linear operator from Λ_0^γ into $(\Lambda_0^\gamma)'$. Given $B = B_\delta(z, r)$ we define T_B from Λ_b^γ into $\Lambda^\gamma(B)'$ as

$$T_B\phi = T(\phi h_B) + I_B\phi.$$

(2.16) Lemma. *Let T be a continuous linear operator from Λ_0^γ into $(\Lambda_0^\gamma)'$ associated to a kernel satisfying* (2.5). *Then for any pair of balls $B_1 = B_\delta(z_1, r_1) \subset B_2 = B_\delta(z_2, r_2)$,*

$$\langle T_{B_1}\phi, \psi \rangle = \langle T_{B_2}\phi, \psi \rangle$$

holds for any $\psi \in \{\Lambda^\gamma(B_1)\}_0$, the set of functions in $\Lambda^\gamma(B_1)$ with integral equal to zero, and $\phi \in \Lambda_b^\gamma$.

Proof. We have

$$\begin{aligned}\langle T_{B_2}\phi, \psi \rangle &= \langle T(\phi h_{B_2}), \psi \rangle + \langle I_{B_2}\phi, \psi \rangle \\ &= \langle T(\phi h_{B_1}), \psi \rangle + \langle T\phi(h_{B_2} - h_{B_1}), \psi \rangle \\ &\quad + \int I_{B_2}\phi(x)\, \psi(x)\, d\mu(x) \\ &= \langle T(\phi h_{B_1}), \psi \rangle + \int \psi(x) \int k(x, y)[h_{B_2}(y) - h_{B_1}(y)] \\ &\quad \times \phi(y)\, d\mu(y)\, d\mu(x) + \int I_{B_2}\phi(x)\, \psi(x)\, d\mu(x).\end{aligned}$$

Clearly,

$$T\phi(h_{B_2} - h_{B_1})(z_1) = \int k(z_1, y)\, \phi(y)[h_{B_2} - h_{B_1}(y)]\, dy,$$

and

$$-I_{B_2}\phi(z_1)=\int [k(z_2, y)-k(z_1, y)][1-h_{B_2}(y)]\,\phi(y)\,dy.$$

Then, since $\int \psi = 0$, we get

$$\begin{aligned}\langle T_{B_2}\phi, \psi\rangle &= \langle T(\phi h_{B_1}), \psi\rangle \\ &\quad + \int \psi(x) \int [k(x, y)-k(z_1, y)]\,\phi(y)[1-h_{B_1}(y)]\,d\mu(y)\,d\mu(x) \\ &= \langle T(\phi h_{B_1}), \psi\rangle + \langle I_{B_1}\phi, \psi\rangle = \langle T_{B_1}\phi, \psi\rangle.\end{aligned}$$

It is clear that

$$\langle T_B\phi, \psi\rangle = \langle T\phi, \psi\rangle,$$

whenever $\operatorname{supp}(\phi)\subset B_1$. Then Lemma (2.16) allows us to introduce the following extension of T.

(2.17) Definition. Let T be a continuous linear operator from Λ_0^γ into $(\Lambda_0^\gamma)'$ associated to a kernel satisfying (2.5). For any $\phi \in \Lambda_b^\gamma$ and $\psi \in \{\Lambda_0^\gamma\}_0$ with $\operatorname{supp}\psi \subset B$, we define

$$\langle T\phi, \psi\rangle = \langle T_B\phi, \psi\rangle.$$

(2.18) Lemma. *Let T be a continuous linear operator from Λ_0^γ into $(\Lambda_0^\gamma)'$ associated to a kernel $k(x, y)$ satisfying (2.5), and such that T is weakly bounded of order γ. Assume that $T1 = g$ with $g \in$ B.M.O. Then, given a ball $B = B_\delta(z, r)$, there exists a constant c_B such that for any $\phi \in \Lambda^\gamma(B)$*

$$\begin{aligned}\langle Th_B, \phi\rangle &= \int (g(x)-m_B(g))\,\phi(x)\,d\mu(x) + c_B \int \phi(x)\,d\mu(x) \\ &\quad - \int I_B 1(x)\,\phi(x)\,d\mu(x).\end{aligned}$$

Moreover, $\sup_B |c_B| \leqslant C$, where C is an absolute constant depending on the constants appearing in (2.5), (2.9), *and* $\|g\|_{\mathrm{BMO}}$.

Proof. Given the ball $B = B_\delta(z, r)$, consider the function

$$h'_B(y) = h(A^2\delta(z, y)/r),$$

where h is the function considered in (1.15). This function is supported in $B_\delta(z, r/A)$. Therefore the function

$$l_B(y) = \left(\int h'_B(y)\, d\mu(y)\right)^{-1} h'_B(y)$$

is supported in $B_\delta(z, r/A)$ and its integral is equal to one.

Then, given $\phi \in \Lambda^\gamma(B)$, we have

$$\begin{aligned}
&\langle Th_B + I_B 1, \phi \rangle \\
&= \left\langle Th_B + I_B 1, \phi - \left(\int \phi\right) l_B \right\rangle + \left\langle Th_B + I_B 1, \left(\int \phi\right) l_B \right\rangle \\
&= \left\langle g, \phi - \left(\int \phi\right) l_B \right\rangle + \langle Th_B + I_B 1, l_B \rangle \int \phi(x)\, d\mu(x) \\
&= \int (g(x) - m_B g)\, \phi(x)\, d\mu(x) + m_B g \int \phi(x)\, d\mu(x) \\
&\quad - \langle g, l_B \rangle \int \phi(x)\, d\mu(x) + \langle Th_B + I_B 1, l_B \rangle \int \phi(x)\, d\mu(x) \\
&= \int (g(x) - m_B g)\, \phi(x)\, d\mu(x) + c_B \int \phi(x)\, d\mu(x),
\end{aligned}$$

where

$$c_B = \langle Th_B + I_B 1 - (g - m_B(g)), l_B \rangle.$$

It is easy to check that

$$|||h'_B|||_\gamma \leqslant C\mu(B)^{-\gamma} \qquad \text{and} \qquad |||l_B|||_\gamma \leqslant C\mu(B)^{-(1+\gamma)};$$

then, by weak boundedness (2.9),

$$|\langle Th_B, l_B \rangle| \leqslant C\mu(B)^{1+2\gamma}\, |||h_B|||_\gamma\, |||l_B|||_\gamma \leqslant C,$$

and, by Lemma (2.13),

$$|\langle I_B 1, l_B \rangle| \leqslant C\mu(B)^{1+\gamma}\, |||l_B|||_\gamma \leqslant C.$$

Finally, it is clear that

$$|\langle g - m_B g, l_B \rangle| \leqslant C\, \|g\|_{\mathrm{BMO}}.$$

These estimates show that $|c_B|$ is bounded by a constant C not depending on B.

(2.19) COROLLARY. *Let T be an operator satisfying all the conditions of Lemma* (2.18). *Then $g \in L^\infty$ if and only if $|\langle Th_B, \phi\rangle| \leqslant C\,\|\phi\|_1$ for any $\phi \in \Lambda^\gamma(B)$, where C is an absolute constant not depending on B.*

(2.20) DEFINITION. Let T be an operator satisfying the conditions of Lemma (18.1). Given $\phi \in \Lambda^\gamma(B)$ and $x \in B$, we define

$$T^B\phi(x) = (g(x) - m_B\, g)\,\phi(x) + c_B\phi(x) - I_B 1(x)\,\phi(x) + \int [\phi(y) - \phi(x)]\,k(x, y)\,h_B(y)\,d\mu(y).$$

(2.21) LEMMA. *Let $B_1 = B_\delta(z_1, r_1) \subset B_2 = B_\delta(z_2, r_2)$ and $\phi \in \Lambda^\gamma(B_1)$. Then*

$$T^{B_2}\phi(x) = T^{B_1}\phi(x), \qquad \textit{for} \quad x \in B_1,$$

Proof. First observe that

$$\begin{aligned} c_{B_2} - c_{B_1} &= \langle Th_{B_2} + I_{B_2}1 - (g - m_2\, g), l_{B_2} - l_{B_1}\rangle \\ &\quad + \langle Th_{B_2} + I_{B_2}1 - (g - m_{B_2} g), l_{B_1}\rangle \\ &\quad - \langle Th_{B_1} + I_{B_1}1 - (g - m_{B_1} g), l_{B_1}\rangle \\ &= \langle T(h_{B_2} - h_{B_1}) + I_{B_2}1 - I_{B_1}1, l_{B_1}\rangle + m_{B_2}\, g - m_{B_1}\, g. \end{aligned} \tag{2.22}$$

On the other hand

$$\begin{aligned} &I_{B_1}1(x) - I_{B_2}1(x) \\ &\quad = \int k(x, y)(h_{B_2}(y) - h_{B_1}(y))\,d\mu(y) \\ &\quad\quad + \int (k(z_2, y) - k(z_1, y))(1 - h_{B_2}(y))\,d\mu(y) \\ &\quad\quad - \int k(z_1, y)(h_{B_2}(y) - h_{B_1}(y))\,d\mu(y) \\ &\quad = T(h_{B_2} - h_{B_1})(x) - I_{B_2}1(z_1) - T(h_{B_2} - h_{B_1})(z_1); \end{aligned} \tag{2.23}$$

consequently,

$$\begin{aligned} \langle T(h_{B_2} - h_{B_1}) - I_{B_2}1 - I_{B_1}1, l_B\rangle &= \langle I_{B_2}1(z_1) + T(h_{B_2} - h_{B_1})(z_1), l_B\rangle \\ &= I_{B_2}1(z_1) + T(h_{B_2} - h_{B_1})(z_1). \end{aligned} \tag{2.24}$$

Moreover,

$$\int |\phi(y)-\phi(x)|\, k(x, y)(h_{B_2}(y)-h_{B_1}(y))\, d\mu(y)$$

$$= -\phi(x) \int k(x, y)(h_{B_2}(y)-h_{B_1}(y))\, d\mu(y)$$

$$= -\phi(x)\, T(h_{B_2}-h_{B_1})(x). \tag{2.25}$$

Then passing up together (2.22), (2.23), (2.24), and (2.25), we obtain the result sought.

Given $\phi \in \Lambda_0^\gamma$, Lemma (2.21) allows us to define $\tilde{T}\phi$ as the function

$$\tilde{T}\phi(x) = T^B\phi(x), \tag{2.26}$$

where B is a ball containing the support of ϕ and $x \in B$.

Now we can prove the main result.

(2.27) THEOREM. *Let T be a continuous linear operator from Λ_0^γ into $(\Lambda_0^\gamma)'$, for every $0<\gamma\leqslant\alpha$, with an associated kernel satisfying* (2.4) *and* (2.5), *and such that $T1=g$, $g\in$ BMO. Then for any η, $0<\eta\leqslant\alpha$, the following conditions are equivalent*:

$$T \text{ is weakly bounded of order } \eta. \tag{2.28}$$

$$\text{For any } \phi\in\Lambda_0^\eta,\ T\phi=\tilde{T}\phi. \tag{2.29}$$

Proof. Let us show that (2.28) implies (2.29). Let ψ, $\phi\in\Lambda^\eta(B)$. Then, by Lemma (2.10),

$$\langle T\phi, \psi\rangle = \langle Th_B, \phi\psi\rangle + \iint \psi(x)[\phi(y)-\phi(x)]\, k(x, y)\, h_B(y)\, d\mu(x)\, d\mu(y),$$

and (2.29) follows by applying Lemma (2.18). Let us prove the converse. Given $B=B_\delta(z, s)$, we apply Lemma (2.7), getting

$$\left|\int [\phi(y)-\phi(x)]\, k(x, y)\, h_B(y)\, d\mu(y)\right|$$

$$\leqslant C\, |||\phi|||_\eta \int_{B_\delta(z,\, As)} \delta(x, y)^\eta\, \tilde{k}(x, y)\, d\mu(y)$$

$$\leqslant C\, |||\phi|||_\eta \int_{B_\delta(x,\, 2A^2 s)} \delta(x, y)^\eta\, \tilde{k}(x, y)\, d\mu(y)$$

$$\leqslant C\, |||\phi|||_\eta\, \mu(B)^\eta;$$

therefore, for $\phi, \psi \in \Lambda^{\gamma}(B)$,

$$|\langle T\phi, \psi\rangle| \leqslant \left|\int (g(x)-m_B g)\,\phi(x)\,\psi(x)\,d\mu(x)\right| + C\int |\phi(x)\,\psi(x)|\,d\mu(x)$$

$$+\int |I_B 1(x)|\,|\phi(x)\,\psi(x)|\,d\mu(x) + C\,|||\phi|||_\eta\,\mu(B)^\eta \int |\psi(x)|\,d\mu(x)$$

$$\leqslant (\|g\|_{\mathrm{BMO}}+C)\,\|\phi\|_\infty\,\|\psi\|_\infty\,\mu(B) + |||\phi|||_\eta\,\mu(B)^{1+2\eta}\,|||\psi|||_\eta$$

$$\leqslant (\|g\|_{\mathrm{BMO}}+C)\,\mu(B)^{1+2\eta}\,|||\phi|||_\eta\,|||\psi|||_\eta.$$

(2.30) *Remark.* Consider the operator

$$T\phi(x) = g(x)\,\phi(x).$$

If T is weakly bounded of order γ then, for every ball B,

$$|\langle Th_B, l_B\rangle| \leqslant C\mu(B)^{1+2\gamma}\,|||h_B|||_\gamma\,|||l_B|||_\gamma \leqslant C.$$

This means that for every B,

$$\left|\int g(x)\,l_B(x)\,dx\right| \leqslant C,$$

and by differentiation (assuming that it holds) we get $|g(x)| \leqslant C$.

(2.31) Corollary. *Let T be an operator satisfying the hypotheses and conclusions of Theorem* (2.27). *Then the kernel $k(x, y)$ is zero if and only if $T\phi(x) = h(x)\,\phi(x)$, with $h \in L^\infty$.*

Proof. Assume that the kernel is zero. Then

$$T\phi(x) = (g(x)-m_B g)\,\phi(x) + c_B\phi(x) = (g(x)-m_B g + c_B)\,\phi(x).$$

Therefore, by Remark (2.30), $g(x)-m_B g+c_B$ must be bounded, but since c_B is bounded this tells us that g must be bounded. In other words, $h(x) = g(x) - m_B g + c_B$.

(2.32) Theorem. *Let T be a continuous linear operator defined from Λ_0^γ into $(\Lambda_0^\gamma)'$ for every γ, $0<\gamma\leqslant\alpha$, weakly bounded of order η for some η, $0<\eta\leqslant\alpha$, and with an associated kernel satisfying* (2.4) *and* (2.5) *for $\eta+\varepsilon$ with $\varepsilon>0$. Assume that $T1 = g$ belongs to B.M.O. Then T satisfies*

$$|||T\phi|||_\eta \leqslant C\,|||\phi|||_\eta \qquad \textit{and} \qquad T\phi \textit{ is a bounded function,}$$

if and only if $T1 = 0$.

Proof. Assume first that $T1=0$. Given $x_1, x_2 \in X$, $\phi \in \Lambda_0^\eta$, and $B_1 = B_\delta(x_1, \delta(x_1, x_2))$, we consider $B = B_\delta(x_1, s)$ such that x_1, $x_2 \in B$, supp $\phi \subset B$, and $A\delta(x_1, x_2) < s$.

We want to show that $T^B\phi$ is a Lipschitz function. Let us estimate the difference

$$\begin{aligned}
&|T^B\phi(x_1) - T^B\phi(x_2)| \\
&\quad \leqslant c_B\,|\phi(x_1) - \phi(x_2)| \\
&\qquad + |I_B 1(x_1)\,\phi(x_1) - I_B 1(x_2)\,\phi(x_2)| \\
&\qquad + \left| \int [\phi(y) - \phi(x_1)]\, k(x_2, y)\, h_B(y)\, d\mu(y) \right. \\
&\qquad \left. - \int [\phi(y) - \phi(x_2)]\, k(x_2, y)\, h_B(y)\, d\mu(y) \right| \\
&\quad = \sigma_1 + \sigma_2 + \sigma_3.
\end{aligned}$$

We have

$$\sigma_1 \leqslant \sup_B |c_B|\ |||\phi|||_\eta\, \delta(x_1, x_2)^\eta.$$

On the other hand, since $I_B 1(x_1) = 0$, by Lemma (2.13) we have

$$\sigma_2 \leqslant C\, \|\phi\|_\infty \left(\frac{\delta(x_1, x_2)}{A\mu(B)} \right)^\eta \leqslant C\, |||\phi|||_\eta\, \delta(x_1, x_2)^\eta.$$

As for σ_3, we have

$$\begin{aligned}
\sigma_3 &\leqslant \left| \int [\phi(y) - \phi(x_1)]\, k(x_1, y)\, h_B(y)\, h_{B_1}(y)\, d\mu(y) \right| \\
&\quad + \left| \int [\phi(y) - \phi(x_2)]\, k(x_2, y)\, h_B(y)\, h_{B_1}(y)\, d\mu(y) \right| \\
&\quad + \left| \int \{[\phi(y) - \phi(x_1)]\, k(x_1, y) \right. \\
&\quad \left. - [\phi(y) - \phi(x_2)]\, k(x_2, y)\}\, h_B(y)(1 - h_{B_1}(y))\, d\mu(y) \right| \\
&= \sigma_{31} + \sigma_{31} + \sigma_{33}.
\end{aligned}$$

By Lemma (2.7) we have

$$\sigma_{31} \leqslant C\, |||\phi|||_\eta \int \delta(x_1, y)^\eta \tilde{k}(x_1, y)\, h_B(y)\, h_{B_1}(y)\, d\mu(y)$$

$$\leqslant C\, |||\phi|||_\eta \int_{\delta(x_1, y) < A^2\delta(x_1, x_2)} \delta(x_1, y)^\eta \tilde{k}(x_1, y)\, d\mu(y)$$

$$\leqslant C\, |||\phi|||_\eta\, \delta(x_1, x_2)^\eta.$$

Analogously,

$$\sigma_{32} \leqslant C\, |||\phi|||_\eta \int_{\delta(x_1, y) < A^2\delta(x_1, x_2)} \delta(x_2, y)^\eta \tilde{k}(x_2, y)\, d\mu(y)$$

$$\leqslant C\, |||\phi|||_\eta \int_{\delta(x_2, y) < A^3\delta(x_1, x_2)} \delta(x_2, y)^\eta \tilde{k}(x_2, y)\, d\mu(y)$$

$$\leqslant C\, |||\phi|||_\eta\, \delta(x_1, x_2)^\eta.$$

It is clear that

$$\sigma_{33} \leqslant |\phi(x_2) - \phi(x_1)| \left| \int K(x_1, y)\, h_B(y)(1 - h_{B_1}(y))\, d\mu(y) \right|$$

$$+ \int |\phi(y) - \phi(x_2)|$$

$$\times |K(x_1, y) - K(x_2, y)|\, h_B(y)(1 - h_{B_1}(y))\, d\mu(y)$$

$$= \sigma_{331} + \sigma_{332}.$$

By the definition of the associated kernel and Corollary (2.19),

$$\sigma_{331} \leqslant C\, |||\phi|||_\eta\, \delta(x_1, x_2)^\eta\, (|Th_B(x_1)| + |Th_{B_1}(x_1)|)$$
$$\leqslant C\, |||\phi|||_\eta\, \delta(x_1, x_2)^\eta.$$

On the other hand, by (2.5)

$$\sigma_{332} \leqslant |||\phi|||_\eta \int_{A\delta(x_1, x_2) < \delta(x_1, y)} \delta(x_2, y)^\eta\, |k(x_1, y) - k(x_2, y)|\, d\mu(y)$$

$$\leqslant |||\phi|||_\eta \sum_{j=0}^{\infty} \left(\int_{A^jA\delta(x_1, x_2) < \delta(x_1, y) < A^{j+1}A\delta(x_1, x_2)} |k(x_1, y) - k(x_2, y)|^r\, d\mu(y) \right)^{1/r}$$

$$\cdot \left(\int_{A^jA\delta(x_1, x_2) < \delta(x_1, y) < A^{j+1}A\delta(x_1, x_2)} \delta(x_2, y)^{\eta r'}\, d\mu(y) \right)^{1/r'}$$

$$\leqslant C\, |||\phi|||_\eta \sum_{j=0}^{\infty} (A^j\delta(x_1, x_2))^{-1/r'} \left(\frac{\delta(x_1, x_2)}{A^j\delta(x_1, x_2)}\right)^{\eta+\varepsilon}$$
$$\cdot (A^j\delta(x_1, x_2))^\eta \cdot (A^j\delta(x_1, x_2))^{1/r'}$$
$$\leqslant C\, |||\phi|||_\eta\, \delta(x_1, x_2)^\eta \sum_{j=0}^{\infty} A^{-j\varepsilon} \leqslant C\, |||\phi|||_\eta\, \delta(x_1, x_2)^\eta.$$

Finally, we shall prove that if $\operatorname{supp} \phi \subset B_0$,

$$\|T\phi(x)\|_\infty \leqslant C\, |||\phi|||_\eta\, \mu(B_0)^\eta.$$

It is enough to show that

$$\left|\int [\phi(y) - \phi(x)]\, k(x, y)\, h_B(y)\, d\mu(y)\right| \leqslant C\, |||\phi|||_\eta\, (\operatorname{diam}(\operatorname{supp} \phi))^\eta,$$

for any sufficiently large B.

Let $B_0 = B_\delta(z, r_0)$, $B_1 = B_\delta(z, A^2 r_0)$, and $B = B_\delta(z, r)$ be such that $\operatorname{supp} \phi \subset B_0$ and $A^3 r_0 < r$.

Assume first that $x \notin B_\delta(z, A^2 r_0)$. Then

$$\left|\int [\phi(y) - \phi(x)]\, k(x, y)\, h_B(y)\, d\mu(y)\right| = \left|\int \phi(y)\, k(x, y)\, h_B(y)\, d\mu(y)\right|$$
$$= \left|\int \phi(y)\, k(x, y)\, d\mu(y)\right|.$$

In this integral the relevant points y satisfy $\delta(z, y) < r_0$, since $y \in \operatorname{supp} \phi$, and $\delta(x, z) > A^2 r_0$.

Then, if $A^j r_0 < \delta(x, z) \leqslant A^{j+1} r_0$, $j \geqslant 2$, we have $A^{j-2}(A-1)\, r_0 < \delta(x, y) \leqslant 2A^{j+2} r_0$.

Therefore, for $x \in B(z, A^{j+1} r_0) \backslash B(z, A^j r_0)$, $j \geqslant 2$, we have

$$\left|\int \phi(y)\, k(x, y)\, d\mu(y)\right|$$
$$= \left|\int_{A^{j-2}(A-1)\, r_0 < \delta(x, y) < 2A^{j+2} r_0} \phi(y)\, k(x, y)\, d\mu(y)\right|$$
$$\leqslant \|\phi\|_\infty \int_{A^{j-2}(A-1)\, r_0 < \delta(x, y) < 2A^{j+2} r_0} \tilde{k}(x, y)\, d\mu(y)$$
$$\leqslant C\, \|\phi\|_\infty \left(\int_{A^{j-2} r_0 < \delta(x, y) < 2A^{j+2} r_0} \tilde{k}(x, y)^r\, d\mu(y)\right)^{1/r} \left(\mu(B_\delta(x, 2A^{j+2} r_0)\right)^{1/r'}$$
$$\leqslant C\, \|\phi\|_\infty \leqslant C\, |||\phi|||_\eta\, \mu(B_0)^\eta.$$

If $x \in B(z, A^2 r_0)$, using (2.4), (2.19), and (2.7), we get

$$\left| \int [\phi(y) - \phi(x)]\, k(x, y)\, h_B(y)\, d\mu(y) \right|$$
$$\leqslant \left| \int [\phi(y) - \phi(x)]\, k(x, y)\, h_B(y)\, h_{B_1}(y)\, d\mu(y) \right|$$
$$+ \left| \int [\phi(y) - \phi(x)]\, k(x, y)\, h_B(y)(1 - h_{B_1}(y))\, d\mu(y) \right|$$
$$\leqslant \left| C \int_{\delta(x, y) \leqslant 2A^3 r_0} |||\phi|||_\eta\, \delta(x, y)^\eta\, \tilde{k}(x, y)\, d\mu(y) \right|$$
$$+ \left| \phi(x) \int k(x, y)(h_B(y) - h_{B_1}(y))\, d\mu(y) \right|$$
$$\leqslant C\, |||\phi|||_\eta\, \mu(B_0)^\eta + C\, \|\phi\|_\infty \leqslant C\, |||\phi|||_\eta\, \mu(B_0)^\eta.$$

In order to prove the converse, assume that T is continuous from Λ_0^η into Λ_b^η. Then, by the computations above, this implies that the function defined for $x \in B$ as

$$(g(x) - m_B\, g)\phi(x)$$

is a Lipschitz function for any $\phi \in \Lambda_0^\eta$; moreover

$$|||(g(\cdot) - m_B\, g)\, \phi(\cdot)|||_\eta \leqslant C\, |||\phi|||_\eta. \tag{2.33}$$

Now take x_1, x_2, and $B = B_\delta(z, r)$ such that $x_1, x_2 \in B$; then by (2.33),

$$\begin{aligned} |g(x_1) - g(x_2)| &= |(g(x_1) - m_B\, g) - (g(x_2) - m_B\, g)| \\ &= |(g(x_1) - m_B\, g)\, h_B(x_1) - (g(x_2) - m_B\, g)\, h_B(x_2)| \\ &\leqslant C\, |||h_B|||_\eta \leqslant C r^{-\eta}. \end{aligned}$$

Now letting $r \to \infty$ we obtain $g(x_1) = g(x_2)$. In other words, $g(x)$ is constant and $T1 = 0$.

Let us define

$$t_j(x, y) = s_{A^{-j}}(x, y) - s_{A^{-j-1}}(x, y),$$

where $s_t(x, y)$ is the approximation of the identity introduced in Theorem (1.13). We define

$$k_{j_1, j_2}(x, y) = \langle t_{j_1}(x, \cdot), T t_{j_2}(y, \cdot\ \cdot) \rangle.$$

(2.34) THEOREM. *Let T be a continuous linear operator defined from Λ_0^γ into $(\Lambda_0^\gamma)'$ for every γ, $0<\gamma\leqslant\alpha$, weakly bounded of order η, for some η, $0<\eta\leqslant\alpha$, and with an associated kernel satisfying* (13.1) *and* (13.2) *with $1/r'+\eta>1$. Assume that $T1=0$. Then the following inequality holds for $j_1\geqslant j_2$:*

$$|k_{j_1,j_2}(x,y)|\leqslant\frac{A^{\eta(j_2-j_1)}A^{j_2}A^{-j_2(1/r'+\eta)}}{\delta(x,y)^{1/r'+\eta}+A^{-j_2(1/r'+\eta)}}.$$

Proof. Let B be a ball with radius bigger than A^{-j_2} and such that

$$\{z:\delta(x,z)<CA^{-j_1}\}\cup\{z:\delta(y,z)<CA^{-j_2}\}\subset B.$$

Theorem (2.27) tells us that

$$\begin{aligned}k_{j_1,j_2}(x,y)&=\langle t_{j_1}(x,\cdot),T^B t_{j_2}(y,\cdot)\rangle\\&=c_B\int t_{j_1}(x,z)\,t_{j_2}(y,z)\,d\mu(z)\\&\quad-\int t_{j_1}(x,z)\,I_B1(z)\,t_{j_2}(y,z)\,d\mu(z)\\&\quad+\int t_{j_1}(x,z)\left(\int(t_{j_2}(y,u)-t_{j_2}(y,z))\,k(z,u)\,d\mu(u)\right)d\mu(z).\end{aligned}\tag{2.35}$$

Assume first that $\delta(x,y)\leqslant A(A+1)A^{-j_2}$. Then, by Theorem (1.13), we have

$$\begin{aligned}&\left|\int t_{j_1}(x,z)\,t_{j_2}(y,z)\,d\mu(z)\right|\\&=\left|\int t_{j_1}(x,z)(t_{j_2}(y,z)-t_{j_2}(y,x))\,d\mu(z)\right|\\&\leqslant C\int t_{j_1}(x,z)\,A^{j_2(1+\eta)}\delta(x,z)^\eta\,dz\\&<CA^{-j_1\eta}A^{j_2(1+\eta)}\leqslant C\,\frac{A^{-j_1\eta}}{\delta(x,y)^{1+\eta}+A^{-j_2(1+\eta)}}\\&=CA^{\eta(j_2-j_1)}\,\frac{A^{-j_2(1+\eta)}A^{j_2}}{\delta(x,y)^{1+\eta}+A^{-j_2(1+\eta)}}.\end{aligned}$$

Analogously, by Lemma (2.13), we have

$$\left|\int t_{j_1}(x, z)\, I_B 1(z)\, t_{j_2}(y, z)\, d\mu(z)\right|$$

$$= \left|\int t_{j_1}(x, z)[I_B 1(z)\, t_{j_2}(y, z) - I_B 1(x)\, t_{j_2}(y, x)]\, d\mu(z)\right|$$

$$\leqslant C \int t_{j_1}(x, z)\, A^{j_2(1+\eta)} \delta(x, z)^\eta\, d\mu(z)$$

$$\leqslant CA^{\eta(j_2 - j_1)} \frac{A^{-j_2(1+\eta)}}{\delta(x, y)^{1+\eta} + A^{-j_2(1+\eta)}}.$$

Analogously, by Theorem (2.32), we have

$$\left|\int t_{j_1}(x, z) \left(\int (t_{j_2}(y, u) - t_{j_2}(y, z))\, k(z, u)\, d\mu(u)\right) d\mu(z)\right|$$

$$= \left|\int t_{j_1}(x, z) \left(\int (t_{j_2}(y, u) - t_{j_2}(y, z))\, k(z, u)\, d\mu(u)\right.\right.$$

$$\left.\left. - \int (t_{j_2}(y, u) - t_{j_2}(y, x))\, k(x, u)\, d\mu(u)\right) d\mu(z)\right|$$

$$\leqslant \int t_{j_1}(x, z)\, A^{j_2(1+\eta)} \delta(x, z)^\eta\, d\mu(z)$$

$$\leqslant CA^{\eta(j_2 - j_1)} \frac{A^{-j_2(1+\eta)}}{\delta(x, y)^{1+\eta} + A^{-j_2(1+\eta)}}.$$

Let us assume now that $\delta(x, y) > A(A+1)\, A^{-j_2}$. If $t_{j_2}(y, z) \neq 0$, then

$$A(A+1)\, A^{j_2} < \delta(x, y) \leqslant A(\delta(x, z) + \delta(z, y)) \leqslant A(\delta(x, z) + A^{-j_2}).$$

In other words,

$$\delta(x, z) > AA^{-j_2} > A^{-j_2} \geqslant A^{-j_1}.$$

This tells us that $t_{j_1}(x, z) = 0$ and therefore the first two integrals in (2.35) are zero.

We estimate now

$$\int t_{j_1}(x, z) \left(\int (t_{j_2}(y, u) - t_{j_2}(y, z))\, k(z, u)\, d\mu(u)\right) d\mu(z).$$

As we have seen before, if $t_{j_2}(y, z) \neq 0$, then $t_{j_1}(x, z) = 0$. Then it is enough to estimate

$$\int t_{j_1}(x, z)\left(\int t_{j_2}(y, u)\, k(z, u)\, d\mu(u)\right) d\mu(z)$$

$$= \int t_{j_1}(x, z)\left(\int t_{j_2}(y, u)(k(z, u) - k(x, u))\, d\mu(u)\right) d\mu(z).$$

Observe that

$$\delta(x, y) \leqslant A(\delta(x, u) + \delta(u, y)) < A(\delta(x, u) + A^{-j_2})$$

$$\leqslant A\delta(x, u) + \frac{1}{A+1}\, \delta(x, y);$$

then $\delta(x, u)(A+1) \geqslant \delta(x, y)$, and moreover

$$\delta(x, z) < A^{-j_1} \leqslant A^{-j_2} < \frac{1}{A(A+1)}\, \delta(x, y). \tag{2.36}$$

Therefore, if we define

$$E = \{u : \delta(x, y) < (A+1)\, \delta(x, u);\ A(A+1)\, \delta(x, z) < \delta(x, y)\}$$

and

$$E_h = \left\{u : \frac{A^h}{A+1}\, \delta(x, y) < \delta(x, u) \leqslant \frac{A^{h+1}}{A+1}\, \delta(x, y),\right.$$

$$\left.\delta(x, z) < \frac{1}{A(A+1)}\, \delta(x, y)\right\},$$

we obtain by Hölder's inequality that the last integral is less than or equal to

$$\int t_{j_1}(x, z)\left\{\left(\int |t_{j_2}(y, u)|^{r'}\, d\mu(\mu)\right)^{1/r'}\right.$$

$$\left.\times\left(\int_E |k(z, u) - k(x, u)|^r\, d\mu(u)\right)^{1/r}\right\} d\mu(z)$$

$$\leqslant C \int t_{j_1}(x, z)\, A^{j_2} A^{-j_2(1/r')}$$

$$\times\left(\sum_h \int_{E_h} |k(z, u) - k(x, u)|^r\, d\mu(u)\right)^{1/r} d\mu(z).$$

By (2.5), this is less than

$$C \int t_{j_1}(x, z)\, A^{j_2} A^{-j_2(1/r')} \left(\sum_h \left(A^h \delta(x, y) \right)^{-r/r'} \left(\frac{\delta(x, z)}{A^h \delta(x, y)} \right)^{\eta r} \right)^{1/r} d\mu(z)$$

$$\leqslant C \int t_{j_1}(x, z)\, A^{j_2} A^{-j_2(1/r')} \delta(x, y)^{-(1/r' + \eta)} A^{-j_1 \eta}$$

$$\times \left(\sum_h A^{-h(r/r' + \eta r)} \right)^{1/r} d\mu(z)$$

$$\leqslant C \frac{A^{j_2} A^{-j_2(1/r')} A^{-j_1 \eta}}{\delta(x, y)^{1/r' + \eta}} \leqslant C \frac{A^{\eta(j_2 - j_1)} A^{-j_2(1/r' + \eta)}}{\delta(x, y)^{1/r' + \eta} + A^{-j_2(1/r' + \eta)}}.$$

(2.37) Corollary. *Under the conditions of Theorem* (2.34), *if we define*

$$T_{j_1, j_2} f(x) = \int k_{j_1, j_2}(x, y)\, f(y)\, dy,$$

then T_{j_1, j_2} *is a bounded operator from* $L^2(X, d\mu)$ *into* $L^2(X, d\mu)$ *with norm less than or equal to* $A^{\eta(j_2 - j_1)}$.

(2.38) Application. *Assume that* $k(x, y)$ *is a singular integral kernel* $k(x, y)$ *satisfying* (2.4), (2.5) *for* $\eta + \varepsilon$ *with* $\varepsilon > 0$ *and the following cancellation property*:

$$\text{let } 0 < r < R < \infty, \text{ then}$$
$$\int_{r < \delta(x, y) \leqslant R} k(x, y)\, d\mu(y) = 0, \qquad \text{for every } x \in X. \tag{2.39}$$

Under these conditions we define for $\phi \in \Lambda_0^\eta$

$$Tf(x) = \lim_{r \to 0} \int_{r < \delta(x, y)} k(x, y)\, \phi(y)\, dy. \tag{2.40}$$

Then the operator T *is well defined and maps* Λ_0^η *into* Λ_b^η.

In order to prove this result we show that T satisfies the hypotheses of Theorem (2.32) and in addition, $T1 = 0$.

Let x be a fixed point in X and $\phi \in \Lambda_0^\eta$ such that $\operatorname{supp} \phi \subset B(z, s)$, $s \leqslant k_2 \mu(z)$. Then. by (2.39), we have

$$\begin{aligned}
T\phi(x) &= \lim_{r\to 0} \int_{r<\delta(x,y)} k(x,y)\,\phi(y)\,dy \\
&= \lim_{r\to 0} \int_{r<\delta(x,y)\leqslant A(\delta(x,z)+s)} k(x,y)\,\phi(y)\,dy \\
&= \lim_{r\to 0} \int_{r<\delta(x,y)\leqslant A(\delta(x,z)+s)} k(x,y)(\phi(y)-\phi(x))\,dy \\
&= \int_{\delta(x,y)\leqslant A(\delta(x,z)+s)} k(x,y)(\phi(y)-\phi(x))\,dy.
\end{aligned}$$

The last integral converges since, by Lemma (2.7),

$$\begin{aligned}
&\int_{\delta(x,y)\leqslant A(\delta(x,z)+s)} |k(x,y)(\phi(y)-\phi(x))|\,dy \\
&\quad\leqslant |||\phi|||_\eta \int_{\delta(x,y)\leqslant A(\delta(x,z)+s)} \tilde{k}(x,y)\,\delta(x,y)^\eta\,dy \\
&\quad\leqslant C\,|||\phi|||_\eta\, A(\delta(x,z)+s)^\eta.
\end{aligned}$$

Therefore, (2.40) is well defined. Using the same kind of argument, if $(\operatorname{supp}\phi)\cup(\operatorname{supp}\phi)\subset B_\delta(z,s)$, we have

$$\begin{aligned}
|\langle T\phi,\psi\rangle| &= \left|\int\left(\lim_{r\to 0}\int_{r<\delta(x,y)} k(x,y)\,\phi(y)\,dy\right)\psi(x)\,dx\right| \\
&\leqslant C\,|||\phi|||_\eta \int (\delta(x,z)+s)^\eta\,|\psi(x)|\,dx \\
&\leqslant Cs^\eta\,|||\phi|||_\eta \int |\psi(x)|\,dx \\
&\leqslant C\mu(B_\delta(z,s))^{1+2\eta}\,|||\phi|||_\eta\,|||\psi|||_\eta.
\end{aligned}$$

Finally, let us compute $T1$. Assume that $\psi\in\{\Lambda_0^\eta\}_0$ with $\operatorname{supp}\psi\subset B=B_\delta(z,s)$. Then

$$\begin{aligned}
&\langle Th_B,\psi\rangle+\langle I_B1,\psi\rangle \\
&\quad=\int\left(\lim_{r\to 0}\int_{r<\delta(x,y)} k(x,y)\,h_B(y)\,dy\right)\psi(x)\,dx \\
&\qquad+\int\left(\int (k(x,y)-k(z,y))\,(1-h_B(y))\,dy\right)\psi(x)\,dx \\
&\quad=\int\left[\lim_{r\to 0}\int_{r<\delta(x,y)} k(x,y)\,h_B(y)\,dy\right. \\
&\qquad\left.+\int (k(x,y)-k(z,y))(1-h_B(y))\,dy\right]\psi(x)\,dx.
\end{aligned}$$

By (2.39), this integral is equal to

$$\int \left| \lim_{\substack{r \to 0 \\ R \to \infty}} \int_{r < \delta(x, y) \leq R} k(z, y)(1 - h_B(y))\, dy \right| \psi(x)\, dx$$

$$= \int \left| \lim_{\substack{r \to 0 \\ R \to \infty}} \int_{r < \delta(x, y) \leq R} k(z, y)(h_B(z) - h_B(y))\, dy \right| \psi(x)\, dx$$

$$= \int \left| \int k(z, y)(h_B(z) - h_B(y))\, dy \right| \psi(x)\, dx = 0,$$

since the innermost integral does not depend on x and $\psi \in \{\Lambda_0^\eta\}_0$.

A particular case of this application is the following:

Given a homogeneous polynomial $P(x)$ of even degree m, defined on $\mathbb{C}^n$ with negative real part for real x, we consider the parabolic differential equation

$$L\,|u| = \frac{\partial}{\partial t} u - (-1)^{m/2} P(D) u = f.$$

In [J] the following expression was considered in order to obtain a priori estimates:

$$D_x^\rho u(x, t) = \lim_{\varepsilon \to 0} \int_0^{t-\varepsilon} \int_{\mathbb{R}^n} s(x - y, t - s)\, f(y, s)\, dy\, ds,$$

where ρ is a multi-index, $|\rho| = \rho_1 + \cdots + \rho_n = m$, and $s(x, t)$ is the ρth spatial derivative of a fundamental solution of the homogeneous equation $L(U) = 0$.

It has been observed in [RT] that a priori estimates can be obtained from

$$\lim_{\varepsilon \to 0} \int_{|x - y| + t - s)^{1/m} > \varepsilon} s(x - y, t - s)\, f(y, s)\, dy\, ds.$$

This limit is viewed as defining a singular integral operator associated to the kernel $k(\bar{x}, \bar{y}) = s(x - y, t - s)$, on the space of homogeneous type (X, d, μ) given by

$$X = \mathbb{R}^n x \mid 0, \infty),$$

$$d(\bar{x}, \bar{y}) = d((x, t), (y, s)) = |x - y| + |t - s|^{1/m},$$

and μ the Lebesgue measure on $\mathbb{R}^n x \mid 0, \infty)$.

In [MT] it is proved that the kernel satisfies (2.4), (2.5) for $\gamma = (m+n)^{-1}$, and (2.35); therefore the a priori estimate

$$|||D_x^{p_u}|||_\eta \leqslant C\, |||L(u)|||_\eta$$

holds for any $0 < \eta < (m+n)^{-1}$.

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