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Weak spectral synthesis in commutative Banach algebras

Eberhard Kaniuth

Institut für Mathematik, Universität Paderborn, D-33095 Paderborn, Germany

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Abstract

Let A be a semisimple and regular commutative Banach algebra with structure space $\Delta(A)$. Generalizing the notion of spectral sets in $\Delta(A)$, the considerably larger class of weak spectral sets was introduced and studied in [C.R. Warner, Weak spectral synthesis, Proc. Amer. Math. Soc. 99 (1987) 244–248]. We prove injection theorems for weak spectral sets and weak Ditkin sets and a Ditkin–Shilov type theorem, which applies to projective tensor products. In addition, we show that weak spectral synthesis holds for the Fourier algebra $A(G)$ of a locally compact group G if and only if G is discrete.
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Introduction

Let A be a regular and semisimple commutative Banach algebra with structure space $\Delta(A)$ and Gelfand transform $a \to \widehat{a}$. For any subset M of A, the hull $h(M)$ of M is defined by $h(M) = \{\varphi \in \Delta(A): \varphi(M) = \{0\}\}$. Associated to each closed subset E of $\Delta(A)$ are two distinguished ideals with hull equal to E, namely

$$k(E) = \left\{a \in A: \widehat{a}(\varphi) = 0 \quad \text{for all } \varphi \in E\right\}$$

E-mail address: kaniuth@math.uni-paderborn.de.

and

$$j(E) = \{a \in A: \widehat{a} \text{ has compact support disjoint from } E\}.$$

Then $k(E)$ is the largest ideal with hull E and $j(E)$ is the smallest such ideal. Recall that E is a *spectral set* (or *set of synthesis*) if $k(E) = \overline{j(E)}$ (equivalently, $k(E)$ is the only closed ideal with hull equal to E). One says that *spectral synthesis holds* for A if every closed subset of $\Delta(A)$ is a spectral set. Moreover, E is a *Ditkin set* if $a \in \overline{aj(E)}$ for every $a \in k(E)$.

In connection with the union problem, that is, the question of when the union of two sets of synthesis is a set of synthesis, Warner [19] introduced and studied the class of weak spectral sets. Motivation certainly also arose from the fact that such sets appeared earlier in the work of Varopoulos [17,18] and others. The definition, although this is not the original one, can be given as follows. A closed subset E of $\Delta(A)$ is called a *weak spectral set* if there exists $n \in \mathbb{N}$ such that $a^n \in \overline{j(E)}$ for every $a \in k(E)$. Adopting the notation of [19], we let $\xi(E)$ denote the smallest such number n. When this happens for each E, we say that *weak spectral synthesis holds* for A. Generalizing the notion of a Ditkin set, we call E a *weak Ditkin set* if there exists $n \in \mathbb{N}$ such that $a^n \in \overline{a^n j(E)}$ for all $a \in k(E)$, and $\eta(E)$ will then stand for the minimal such n. So E is a spectral set (Ditkin set) if and only if $\xi(E) = 1$ ($\eta(E) = 1$). Subsequent to [19], the study of weak spectral sets and of the weak synthesis problem gained considerable attention [7,11,12,20], the more so because there are many commutative Banach algebras for which weak spectral synthesis holds, whereas spectral synthesis fails (compare Section 1).

The purpose of this paper is to investigate weak spectral sets and weak Ditkin sets under various aspects. We start in Section 1 by mentioning some examples of algebras for which spectral synthesis fails, but weak synthesis holds. These examples are followed by some preliminary results concerning countable unions, localness and a sufficient condition for a weak spectral set to be weak Ditkin. In Section 2 the setting is that of a closed ideal I of A together with the embedding i of $\Delta(A/I)$ into $\Delta(A)$. So-called injection theorems for spectral sets and Ditkin sets relate either of these properties of closed subsets E of $\Delta(A/I)$ to the corresponding property of their images $i(E)$ in $\Delta(A)$. We prove injection theorems for weak spectral sets (Theorem 2.2) and weak Ditkin sets (Theorem 2.5), at the same time providing estimates for the values of ξ and η.

The classical Ditkin–Shilov (or Helson–Reiter) theorem states that if singletons in $\Delta(A)$ are Ditkin sets, then every closed subset of $\Delta(A)$ with scattered boundary is a spectral set. A remarkable extension was obtained in [1, Theorem 1.2]. Here we establish, under somewhat weaker hypotheses, an analogous result for weak spectral sets (Theorem 3.1). Like [1, Theorem 1.2], Theorem 3.1 admits an application to projective tensor products (Theorem 3.5). Again, there are upper bounds for $\xi(E)$. Modifying Varopoulos' proof [18] of Malliavin's celebrated theorem [9], it was shown in [12, Theorem 3.1] that weak spectral synthesis fails for the Fourier algebra $A(G)$ of every non-discrete abelian locally compact group G. We conclude the paper by extending this result to arbitrary locally compact groups (Theorem 4.3).

1. Preliminaries, examples and some basic properties

Let A be a semisimple and regular commutative Banach algebra. In [19] a closed subset E of $\Delta(A)$ was defined to be a weak spectral set if every element of the quotient algebra $k(E)/\overline{j(E)}$ is nilpotent. Then, as shown in [19, Theorem 1.2] and [3, footnote 7, p. 885], there exists $n \in \mathbb{N}$ such that $x^n \in \overline{j(E)}$ for all $x \in k(E)$. So this latter property can equally well be taken as the definition. One of the important features of the class of weak spectral sets is that it is closed under

the formation of finite unions. Actually, for any two weak spectral sets E_1 and E_2, $\xi(E_1 \cup E_2) \leqslant \xi(E_1) + \xi(E_2)$ [19, Theorem 2.2] (see [11, Corollary 3.11] for a different approach). Although the class of weak spectral sets shares this finite union property with the class of Ditkin sets, in contrast to the latter a closed countable union of weak spectral sets need not be a spectral set [19, Theorem 2.6].

We point out that, throughout this paper, the most important fundamental tool is the *local membership principle* which we briefly recall for the readers convenience. Let I be a closed ideal in A. An element $x \in A$ is said to belong locally to I at $\varphi \in \Delta(A)$ (at infinity) if there exist a neighbourhood V of φ in $\Delta(A)$ (a compact subset K of $\Delta(A)$) and an element y of I such that $\widehat{x}(\psi) = \widehat{y}(\psi)$ for all $\psi \in V$ ($\psi \in \Delta(A) \setminus K$). If x belongs locally to I at every point of $\Delta(A)$ and at infinity, then $x \in I$. As general references to spectral synthesis we mention [4,13,14].

In this section, we first present three illustrative examples and then give some basic results on weak spectral sets and weak Ditkin sets. Concerning notation, we make the following convention. If $E \subseteq \Delta(A)$ is a singleton, say $\{\varphi\}$, we write $k(\varphi)$ and $j(\varphi)$ in place of $k(\{\varphi\})$ and $j(\{\varphi\})$, respectively.

Example 1.1. (1) For $n \in \mathbb{N}$, the unit sphere S^{n-1} in $\mathbb{R}^n = \Delta(L^1(\mathbb{R}^n))$ is a weak spectral set with $\xi(S^{n-1}) = \lfloor \frac{n+1}{2} \rfloor$ [18, Theorem 3].

(2) For each $n \in \mathbb{N}$, $\mathbb{T}^\infty = \Delta(L^1(\widehat{\mathbb{T}^\infty}))$ contains a weak spectral set E with $\xi(E) = n$ [20, Corollary 2.5(d)].

(3) Let $C^n[0, 1]$ be the algebra of all n-times continuously differentiable functions on $[0, 1]$ and identify $\Delta(C^n[0, 1])$ with $[0, 1]$. Then $\xi(E) = n + 1$ for every non-empty proper closed subset E of $[0, 1]$ [7, Example 2.4(i)].

Example 1.2. Let X be a compact metric space with metric d and let $0 < \alpha \leqslant 1$. Then $\mathrm{Lip}_\alpha X$ is the space of all complex-valued functions f on X such that

$$p_\alpha(f) = \sup\left\{ \frac{|f(x) - f(y)|}{d(x, y)^\alpha} : x, y \in X, \ x \neq y \right\}$$

is finite. With pointwise multiplication and the norm $\|f\| = \|f\|_\infty + p_\alpha(f)$, $\mathrm{Lip}_\alpha X$ is a commutative Banach algebra. These Lipschitz algebras were first extensively studied by Sherbert [15]. All the facts relevant to us, can now also be found in Section 4.4 of the monograph [5]. The map $x \to \varphi_x$, where $\varphi_x(f) = f(x)$ for $f \in \mathrm{Lip}_\alpha X$, is a homeomorphism from X onto $\Delta(\mathrm{Lip}_\alpha X)$. With this identification of X and $\Delta(\mathrm{Lip}_\alpha X)$, we have $k(E)^2 = \overline{j(E)}$ for every closed subset E of X, whereas E is a spectral set if and only if E is open in X. Consequently, if X is not discrete, then spectral synthesis fails for $\mathrm{Lip}_\alpha X$, but weak spectral synthesis holds. Actually, inspection of the proof of Theorem 4.4.31 of [5] shows that $f^2 \in \overline{f^2 j(E)}$ for every $f \in k(E)$, so that $\eta(E) \leqslant 2$. Moreover, if E is open and closed in X, then $1_E \in \mathrm{Lip}_\alpha X$, and hence E is a Ditkin set. Thus $\mathrm{Lip}_\alpha X$ provides an example of a semisimple and regular commutative Banach algebra A for which $\xi(E) = \eta(E) \leqslant 2$ for each closed subset E of $\Delta(A)$.

Example 1.3. Let M be the Mirkil algebra which was introduced in [10] as a counterexample to discrete spectral synthesis. Identifying $[-\pi, \pi)$ with the torus $\mathbb{T}$, M is defined to be the space of all functions $f \in L^2(\mathbb{T})$ such that f is continuous on $[-\pi/2, \pi/2]$. With convolution and the norm $\|f\| = \|f\|_2 + \|f|_{[-\pi/2,\pi/2]}\|_\infty$, M is a regular and semisimple commutative Banach algebra, and $\Delta(M)$ can be identified with $\mathbb{Z}$. Every subset E of $\mathbb{Z}$ is a weak Ditkin set with $\eta(E) \leqslant 2$

[20, Theorem 1.4]. On the other hand, no finite subset of $\mathbb{Z}$ is a Ditkin set [10, Section 5], [20, Corollary 2.2]. The Mirkil algebra also serves as a counterexample to the union question. In fact, the sets $4\mathbb{Z}$ and $4\mathbb{Z} + 2$ are spectral sets, whereas their union, $2\mathbb{Z}$, is not [2, Theorem] (see also the exposition in Section 4.5 of [5]). Since translates of Ditkin sets in $\mathbb{Z}$ are Ditkin sets [20, Theorem 2.1], it follows that

$$\xi(4\mathbb{Z}) = \xi(4\mathbb{Z} + 2) = 1, \qquad \xi(2\mathbb{Z}) = 2 \quad \text{and} \quad \eta(4\mathbb{Z}) = \eta(4\mathbb{Z} + 2) = \eta(2\mathbb{Z}) = 2.$$

This in particular shows that the inequality $\xi(E_1 \cup E_2) \leqslant \xi(E_1) + \xi(E_2)$ mentioned above cannot be improved.

As is shown by the sets $4\mathbb{Z}$ and $4\mathbb{Z} + 2$ in Example 1.3, in general spectral sets need not be Ditkin sets. However, if $E \subseteq \Delta(A)$ is a spectral set satisfying some additional hypothesis, which might be termed bounded regularity, then E is a Ditkin set. The analogous conclusion (Lemma 1.4), which will be used in Section 3, holds for weak spectral sets. We include the simple proof.

Lemma 1.4. *Let $E \subseteq \Delta(A)$ be a weak spectral set. Suppose that there exists a constant $C > 0$ such that for every compact subset K of $\Delta(A)$ which is disjoint from E, there exists $a \in j(E)$ such that $\|a\| \leqslant C$ and $\widehat{a} = 1$ on K. Then E is a weak Ditkin set and $\eta(E) = \xi(E)$.*

Proof. Let $n = \xi(E)$ and let $x \in k(E)$ and $\epsilon > 0$. There exists $y \in j(E)$ with $\|x^n - y\| \leqslant \epsilon$. By hypothesis, there exists $a \in j(E)$ such that $\|a\| \leqslant C$ and $\widehat{a} = 1$ on supp $\widehat{y}$. Then $y = ya$ since A is semisimple and $\widehat{ya} = \widehat{y}$. It follows that

$$\left\| x^n - x^n a \right\| \leqslant \left\| x^n - y \right\| + \left\| ya - x^n a \right\| \leqslant (1 + C)\epsilon.$$

This shows that $x^n \in \overline{x^n j(E)}$, and hence $\eta(E) \leqslant \xi(E)$. □

A closed countable union of Ditkin sets is a Ditkin set. Refined arguments allow to show the following version for weak Ditkin sets.

Proposition 1.5. *Suppose that $\emptyset$ is a weak Ditkin set. Let E be a closed subset of $\Delta(A)$ which is a union of closed subsets E_i, $i \in \mathbb{N}$. If $\eta(E_i) \leqslant N$ for some $N \in \mathbb{N}$ and all i, then $\eta(E) \leqslant N\eta(\emptyset)$.*

Proof. Let $m = \eta(\emptyset)$. Then there exists a sequence $(u_n)_n$ in A such that $x^m u_n \to x^m$ and each $\widehat{u_n}$ has compact support. It suffices to show that $(x^m u_n)^N \in \overline{(x^m u_n)^N j(E)}$ for all n. It therefore suffices to show that if $y \in k(E)$ is such that $\widehat{y}$ has compact support then $y^N \in \overline{y^N j(E)}$. Since A is semisimple and regular, this will follow once we have verified that y^N belongs locally to $\overline{y^N j(E)}$ at every point of $\Delta(A)$.

To that end, fix $\varphi \in \Delta(A)$ and a compact neighbourhood U of φ in $\Delta(A)$. There exists $u \in A$ such that $\widehat{u} = 1$ in a neighbourhood of φ and supp $\widehat{u} \subseteq U$. We claim that $y^N u \in \overline{y^N j(E)}$. Let $\epsilon > 0$ be given. Since all E_i are weak Ditkin sets with $\eta(E_i) \leqslant N$, we can construct by induction a sequence $(z_i)_i$ such that $z_i \in j(E_i)$, $\|y^N - y^N z_1\| \leqslant \frac{\epsilon}{2\|u\|}$ and

$$\left\| y^N - y^N z_i \right\| \leqslant \frac{\epsilon}{2^i \|u\| \cdot \prod_{l=1}^{i-1} \|y_l\|}$$

for $i \geqslant 2$. For each i, let V_i be an open set containing E_i such that $\widehat{z_i}$ vanishes on V_i. Now, since $E \cap U$ is compact, $E \cap U \subseteq \bigcup_{i=1}^{n} V_i$ for some $n \in \mathbb{N}$. Let $z = z_1 \cdot \ldots \cdot z_n$, then $z \in j(E \cap U)$ and hence $uz \in j(E)$ since $\widehat{u}$ vanishes on $\Delta(A) \setminus U$. Finally,

$$\|y^N u - y^N u z\| \leqslant \|y^N u - y^N u z_1\| + \sum_{i=2}^{n} \|y^N u z_1 \cdot \ldots \cdot z_{i-1} - y^N u z_1 \cdot \ldots \cdot z_i\|$$

$$\leqslant \|u\| \left(\|y^N - y^N z_1\| + \sum_{i=2}^{n} \|y^N - y^N z_i\| \prod_{l=1}^{i-1} \|z_l\| \right)$$

$$\leqslant \epsilon \sum_{i=1}^{n} \frac{1}{2^i} < \epsilon.$$

Since $y^N u z \in \overline{y^N j(E)}$ and $\epsilon > 0$ was arbitrary, it follows that $y^N u \in \overline{y^N j(E)}$. This shows that y^N belongs locally to $\overline{y^N j(E)}$ at φ because $\widehat{u} = 1$ near φ. $\square$

It is well known that the property of being a spectral set is local in the sense that if A satisfies Ditkin's condition at infinity and E is a closed subset of $\Delta(A)$ such that every point of E has a closed relative neighbourhood in E which is a spectral set for A, then E is a spectral set. The corresponding result for weak spectral sets does not hold (see Example 1.7 below). We have, however, the following

Proposition 1.6. *Let A be a regular and semisimple commutative Banach algebra satisfying Ditkin's condition at infinity and let E be a compact subset of $\Delta(A)$. Suppose that each point of E has a closed relative neighbourhood in E which is a weak spectral set for A. Then E is a weak spectral set for A.*

Proof. Let $x \in k(E)$. Since A is semisimple and x belongs locally to $\overline{j(E)}$ at every point of $\Delta(A) \setminus E$, it suffices to show that there exists $n \in \mathbb{N}$ such that x^n belongs locally to $\overline{j(E)}$ at every point φ of E.

By hypothesis, there exist a closed subset E_φ of E and an open neighbourhood U_φ of φ in $\Delta(A)$ such that $U_\varphi \cap E \subseteq E_\varphi$ and $x^{n_\varphi} \in \overline{j(E_\varphi)}$. A being regular, there exists $u_\varphi \in A$ such that $\operatorname{supp} \widehat{u_\varphi} \subseteq U_\varphi$ and $\widehat{u_\varphi} = 1$ in a neighbourhood of φ in $\Delta(A)$. Since E_φ is a weak spectral set, there exist $y_\varphi \in j(E_\varphi)$ and $n_\varphi \in \mathbb{N}$ such that $\|x^{n_\varphi} - y_\varphi\| < \epsilon/\|u_\varphi\|$. Then $\|u_\varphi x^{n_\varphi} - u_\varphi y_\varphi\| < \epsilon$ and $\widehat{u_\varphi y_\varphi}$ vanishes in a neighbourhood of E since $\widehat{y_\varphi} = 0$ in a neighbourhood of E_φ and $\widehat{u_\varphi} = 0$ in a neighbourhood of $\Delta(A) \setminus U_\varphi$ and $E \subseteq E_\varphi \cup (\Delta(A) \setminus U_\varphi)$. So $y_\varphi u_\varphi \in j(E)$. Since $\epsilon > 0$ was arbitrary, it follows that $x^{n_\varphi} u_\varphi \in \overline{j(E)}$. Finally, $\widehat{x^{n_\varphi} u_\varphi} = \widehat{x^{n_\varphi}}$ in a neighbourhood of φ and hence x^{n_φ} belongs locally to $\overline{j(E)}$ at φ.

Thus for each $\varphi \in E$, there exist an open neighbourhood V_φ of $\varphi \in \Delta(A)$ and $z_\varphi \in \overline{j(E)}$ such that $\widehat{x^{n_\varphi}} = \widehat{z_\varphi}$ on V_φ. Since E is compact, there exist $\varphi_1, \ldots, \varphi_m \in E$ such that $E \subseteq \bigcup_{i=1}^{m} V_{\varphi_i}$. Let $n = \max\{n_{\varphi_i} : 1 \leqslant i \leqslant m\}$. Then x^n belongs locally to $\overline{j(E)}$ at every point of E. $\square$

If the appropriate minor modifications are made, the proof of Proposition 1.6 applies to closed subsets E provided that the values $\xi(E_\varphi)$, $\varphi \in E$, are bounded.

Example 1.7. Let $(A_n)_n$ be a sequence of unital and semisimple commutative Banach algebras and A their c_0-direct sum. Then $\Delta(A)$ is the topological sum of the sets $\Delta(A_n)$ and A satisfies Ditkin's condition at infinity. Now suppose that for every $n \in \mathbb{N}$ there exists $\varphi_n \in \Delta(A_n)$ such that $\{\varphi_n\}$ is a weak spectral set with $\xi(\varphi_n) \geqslant n$, and let $E = \{\varphi_n : n \in \mathbb{N}\}$. Then each singleton $\{\varphi_n\}$ is open in E and there exists $x_n \in k(\varphi_n)$ such that $x_n^k \notin \overline{j(\varphi_n)}$ for all $k < n$. Viewing x_n as an element of A, we have $x_n \in k(E)$ and $x_n^k \notin \overline{j(E)}$ for all $k < n$. It follows that $\xi(E) \geqslant n$ for all $n \in \mathbb{N}$.

As an example of such sequences $(A_n)_n$ and $(\varphi_n)_n$, simply take $A_n = C^n[0, 1]$ and, after identifying $\Delta(A_n)$ with the interval $[0, 1]$, $\varphi_n = 0$. In fact, $\xi(t) = n + 1$ for each $t \in [0, 1]$ (Example 1.1(3)).

2. Injection theorems for weak spectral sets and weak Ditkin sets

Let G be a locally compact abelian group and H a closed subgroup of G. Then $L^1(G/H)$ is a quotient of $L^1(G)$ and $\widehat{G/H} = \Delta(L^1(G/H))$ embeds canonically into $\widehat{G} = \Delta(L^1(G))$. Then a closed subset of $\widehat{G/H}$ is a spectral set (Ditkin set) for $L^1(G/H)$ if and only if it is a spectral set (Ditkin set) for $L^1(G)$ (see [13, Theorems 7.3.15 and 7.4.13]). For the obvious reason, these results are referred to as *injection theorems*. The same problem naturally arises in the general context of a regular and semisimple commutative Banach algebra A and a closed ideal I of A, and it is worthwhile to consider weak spectral sets and weak Ditkin sets rather than just spectral sets and Ditkin sets. In this section, we establish such injection theorems. However, as the reader might expect, some additional hypotheses, which are automatically satisfied in the group algebra situation, have to be placed on A and I.

We start with a lemma which is needed to prove the injection theorem for weak spectral sets. In what follows, if I is a closed ideal of A, then $i : \Delta(A/I) \to \Delta(A)$ will denote the embedding defined by $i(\varphi)(x) = \varphi(x + I)$ for $\varphi \in \Delta(A/I)$ and $x \in A$.

Lemma 2.1. *Let A be a regular commutative Banach algebra and I a closed ideal of A such that $h(I) = \Delta(A)$. Let E be a closed subset of $\Delta(A/I)$ and suppose that $k(E)^n \subseteq \overline{j(E)}$ and $I^m \subseteq \overline{j(i(E))}$ for some $n, m \in \mathbb{N}$. Then*

$$k\big(i(E)\big)^{nm} \subseteq \overline{j\big(i(E)\big)}.$$

Proof. It suffices to show that given $x \in k(i(E))$ such that $\|x\| < 1$ and $0 < \epsilon < 1 - \|x\|$, there exists $y \in \overline{j(i(E))}$ with $\|x^{nm} - y\| < \epsilon m$. By hypothesis, there exists $y \in A$ such that $\widehat{y + I}$ has compact support disjoint from E and $\|(x^n + I) - (y + I)\| < \epsilon$. Since $\widehat{y + I}(\varphi) = \widehat{y}(i(\varphi))$ for all $\varphi \in \Delta(A/I)$ and since i is a homeomorphism from $\Delta(A/I)$ onto $h(I) = \Delta(A)$, we have $y \in j(i(E))$. Choose $z \in I$ so that $\|x^n - (y + z)\| < \epsilon$. Then $\|y + z\| \leqslant 1$ and

$$\big\|x^{nm} - (y + z)^m\big\| = \left\|\big(x^n - (y + z)\big) \sum_{j=0}^{m-1} x^{nj}(y + z)^{m-1-j}\right\|$$

$$< \epsilon \sum_{j=0}^{m-1} \|x\|^{nj} \|y + z\|^{m-1-j}$$

$$\leqslant \epsilon m.$$

Since both z^m and y are contained in $\overline{j(i(E))}$, it follows that

$$(y+z)^m = z^m + \sum_{j=1}^{m} \binom{m}{j} y^{j-1} z^{m-j} \in \overline{j(i(E))}.$$

So $u = (y+z)^m$ has the desired properties. $\square$

Theorem 2.2. *Let A be a regular and semisimple commutative Banach algebra, I a closed ideal of A and E a closed subset of $\Delta(A/I)$.*

(i) *If $i(E)$ is a weak spectral set for A, then E is a weak spectral set for A/I and $\xi(E) \leqslant \xi(i(E))$.*
(ii) *If E is a weak spectral set for A/I and $h(I)$ is a weak spectral set for A, then $i(E)$ is a weak spectral set for A and $\xi(i(E)) \leqslant \xi(E)\xi(h(I))$.*

Proof. (i) Let $n = \xi(i(E))$ and let $x \in A$ be such that $x + I \in k(E)$. Then $x \in k(i(E))$ and hence given $\epsilon > 0$, there exists $y \in j(i(E))$ such that $\|x^n - y\| < \epsilon$. It follows that $\|(y + I) - (x^n + I)\| < \epsilon$ and $\widehat{y+I}$ has compact support disjoint from E. Since $\epsilon > 0$ was arbitrary, $(x + I)^n \in \overline{j(E)}$.

(ii) Note first that $\overline{j(h(I))} \subseteq I$ since A is semisimple and regular. Let $A_1 = A/\overline{j(h(I))}$ and $I_1 = I/\overline{j(h(I))}$. Then $h(I_1) = \Delta(A_1)$, so that Lemma 2.1 applies to A_1 and its ideal I_1. Hence, with i_1 denoting the embedding $\Delta(A_1/I_1) \to \Delta(A_1)$, $k(i_1(E))^{nm} \subseteq \overline{j(i_1(E))}$. Let $q : A \to A_1$ denote the quotient homomorphism and $i_2 : \varphi \to \varphi \circ q$ the embedding of $\Delta(A_1)$ into $\Delta(A)$. Since $i(E) \subseteq h(I)$, we have $\overline{j(h(I))} \subseteq \overline{j(i(E))}$, and since $i = i_2 \circ i_1$, it follows that

$$\begin{aligned}
\overline{j(i(E))} = q^{-1}\big(q\big(\overline{j(i(E))}\big)\big) &\supseteq q^{-1}\big(\overline{j(i_1(E))}\big) \\
&\supseteq q^{-1}\big(k(i_1(E))^{nm}\big) \supseteq \big(q^{-1}\big(k(i_1(E))\big)\big)^{nm} \\
&= k(i(E))^{nm},
\end{aligned}$$

as was to be shown. $\square$

We continue with two consequences of Theorem 2.2.

Corollary 2.3. *Let A be a semisimple and regular commutative Banach algebra and suppose that $\emptyset$ is a Ditkin set. Let $E \subseteq \Delta(A)$ be a weak spectral set for A and F an open and closed subset of E. Then F is a weak spectral set and $\xi(F) \leqslant \xi(E)$.*

Proof. Let $I = k(E)$ and let F' and E' denote the sets in $\Delta(A/I)$ corresponding to F and E', respectively. Then $\Delta(A/I) = E'$ and F' is open and closed in $\Delta(A/I)$. Since A/I is semisimple and regular and satisfies Ditkin's condition at infinity, it follows that F' is a spectral set for A/I. Theorem 2.2(ii) now implies that $F = i(F')$ is a weak spectral set and

$$\xi(F) = \xi\big(i(F')\big) \leqslant \xi(F')\xi\big(h(I)\big) = \xi(E),$$

since $\xi(F') = 1$ and $E = h(I)$. $\square$

For a locally compact group G, let $A(G)$ denote the Fourier algebra of G as introduced and studied extensively by Eymard [6]. $A(G)$ is a regular and semisimple commutative Banach algebra whose spectrum can be identified with G. In fact, the map $t \to \varphi_t$, where $\varphi_t(u) = u(t)$ for $u \in A(G)$, is a homeomorphism from G onto $\Delta(A(G))$ [6, Théorème 3.34]. Recall that when G is abelian, $A(G)$ is isometrically isomorphic (by means of the Fourier transform) to $L^1(\widehat{G})$, the L^1-algebra of the dual group of G. An injection theorem for spectral sets of Fourier algebras was shown in [8, Theorem 3.4]. The following corollary generalizes [19, Corollary 2.5(c)].

Corollary 2.4. *Let G be a locally compact group, H a closed subgroup of G and $i : H = \Delta(A(H)) \to G = \Delta(A(G))$. Then $\xi(E) \leqslant \xi(i(E))$ for any closed subset E of H.*

Proof. Let $I = \{u \in A(G): u|_H = 0\}$. Then the map $u + I \to u|_H$ is an isometric isomorphism between $A(G)/I$ and $A(H)$. The statement now follows from Theorem 2.2 and the fact that H is a set of synthesis for $A(G)$ [16, Theorem 3]. $\quad\square$

We now proceed with the injection theorem for weak Ditkin sets. Note that the hypothesis in part (ii) of the following theorem implies that $h(I)$ is a weak Ditkin set with $\eta(h(I)) \leqslant m$. When $A = L^1(G)$ and I is the kernel of the quotient homomorphism $L^1(G) \to L^1(G/H)$, this condition, with $m = 1$, is always satisfied [13, Lemma 7.4.14].

Theorem 2.5. *Let A be a regular and semisimple commutative Banach algebra, I a closed ideal of A and E a closed subset of $\Delta(A/I)$.*

 (i) *If $i(E)$ is a weak Ditkin set for A, then E is a weak Ditkin set for A/I and $\eta(E) \leqslant \eta(i(E))$.*
(ii) *Suppose that there exist $m \in \mathbb{N}$ and a constant $C > 0$ with the following property: For every $a \in A$ and $\epsilon > 0$, there exists $b \in A$ such that $\|a^m - a^m b\| \leqslant C \|a^m + I\| + \epsilon$ and $\widehat{b}$ vanishes in a neighbourhood of $h(I)$. If E is a weak Ditkin set for A/I, then $i(E)$ is a weak Ditkin set for A and $\eta(i(E)) \leqslant m^2 \eta(E)^2$.*

Proof. In the following, q denotes the quotient homomorphism from A onto A/I.

(i) Let $n = \eta(i(E))$ and let $x \in A$ be such that $q(x) \in k(E)$. Then, given $\epsilon > 0$, there exists $y \in j(i(E))$ such that $\|x^n - x^n y\| \leqslant \epsilon$. It follows that $\|q(x)^n - q(x)^n q(y)\| \leqslant \epsilon$ and $\widehat{q(y)}$ has compact support and vanishes in a neighbourhood of E in $\Delta(A/I)$. Since $\epsilon > 0$ was arbitrary, $q(x)^n \in \overline{q(x)^n j(E)}$.

(ii) Let $n = \eta(E)$ and let $x \in k(i(E))$, $x \neq 0$, and $\epsilon > 0$ be given. Choose $\delta > 0$ such that

$$\delta \cdot \sum_{j=0}^{nm-1} \|x\|^{n(nm-j)} \big(\|x\|^n + \delta\big)^j \leqslant \epsilon.$$

Then $q(x) \in k(E)$ and hence there exists $u \in A$ such that

$$\big\| q\big(x^n - x^n u\big) \big\| = \big\| q(x)^n - q(x)^n q(u) \big\| \leqslant \delta$$

and $\widehat{q(u)}$ vanishes in a neighbourhood of E in $\Delta(A/I)$. For arbitrary Banach algebra elements s and t and $k \in \mathbb{N}$, we have

$$\left\| s^k - t^k \right\| = \left\| (s - t) \sum_{j=0}^{k-1} s^{k-j} t^j \right\| \leqslant \| s - t \| \cdot \sum_{j=0}^{k-1} \| s \|^{k-j} \big(\| s \| + \| t - s \| \big)^j.$$

Setting $s = q(x^n)$, $t = q(x^n u)$, and $k = nm$ and using that $\| q(x^n - x^n u) \| \leqslant \delta$, it follows from the choice of δ that

$$\left\| q\big(x^{n^2 m} - x^{n^2 m} u^{nm} \big) \right\| \leqslant \delta \cdot \sum_{j=0}^{nm-1} \| x \|^{n(nm-j)} \big(\| x \|^n + \delta \big)^j \leqslant \epsilon.$$

By Theorem 2.2(ii), $i(E)$ is a weak spectral set for A and

$$\xi\big(i(E) \big) \leqslant \xi(E) \xi\big(h(I) \big) \leqslant \eta(E) \eta\big(h(I) \big) = nm.$$

Since $u \in k(i(E))$, there exists $v \in A$ such that $\| u^{nm} - v \| \leqslant \epsilon / \| x \|^{n^2 m}$ and $\widehat{v}$ has compact support and vanishes in a neighbourhood of $i(E)$ in $\Delta(A)$. Then

$$\begin{aligned}
\left\| q\big(x^{n^2 m} - x^{n^2 m} v \big) \right\| &\leqslant \left\| q\big(x^{n^2 m} - x^{n^2 m} u^{nm} \big) \right\| + \left\| q\big(x^{n^2 m} \big) q\big(u^{nm} - v \big) \right\| \\
&\leqslant \left\| q\big(x^{n^2 m} - x^{n^2 m} u^{nm} \big) \right\| + \| x \|^{n^2 m} \left\| u^{nm} - v \right\| \\
&\leqslant 2\epsilon.
\end{aligned}$$

Now let $a = x^{n^2 m} - x^{n^2 m} v \in A$. By hypothesis, there exists $w \in j(h(I))$ such that $\| a^m - a^m w \| \leqslant C \| q(a^m) \| + \epsilon$. Write

$$\begin{aligned}
a^m - a^m w &= \sum_{j=0}^{m} \binom{m}{j} (-1)^j x^{n^2 m^2} \big(v^j - v^j w \big) \\
&= x^{n^2 m^2} - x^{n^2 m^2} \left(w - \sum_{j=1}^{m} \binom{m}{j} (-1)^{j-1} \big(v^j - v^j w \big) \right).
\end{aligned}$$

Finally, let

$$y = w - \sum_{j=1}^{m} \binom{m}{j} (-1)^{j-1} \big(v^j - v^j w \big) \in A.$$

Then $\widehat{y}$ has compact support since both $\widehat{v}$ and $\widehat{w}$ have compact support. Moreover, $\widehat{y}$ vanishes in a neighbourhood of $i(E)$ in $\Delta(A)$ since $\widehat{v}$ does so and $\widehat{w}$ vanishes in a neighbourhood of $h(I)$. So $y \in j(i(E))$ and

$$\left\| x^{n^2 m^2} - x^{n^2 m^2} y \right\| \leqslant C \| q(a^m) \| + \epsilon \leqslant C(2\epsilon)^m + \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we conclude that $x^{n^2 m^2} \in \overline{x^{n^2 m^2} j(i(E))}$. This finishes the proof of the theorem. $\quad\square$

3. A Ditkin–Shilov type theorem and an application to projective tensor products

The classical Ditkin–Shilov (or Wiener–Ditkin) theorem asserts that if A is a semisimple and regular commutative Banach algebra such that singletons in $\Delta(A)$ are Ditkin sets, then every closed subset of $\Delta(A)$ with scattered boundary is a set of synthesis.

We remind the reader that a topological space X is called scattered if every non-empty closed subset of X has an isolated point in the relative topology. Clearly, a countable locally compact Hausdorff space is scattered. Conversely, if X is a second countable locally compact Hausdorff space and X is scattered, then X is countable.

In [1, Theorem 1.2] Atzmon established, for unital A, a generalization which admits applications to projective tensor products. The first purpose of this section is to extend Theorem 1.2 of [1] to weak spectral sets and not necessarily unital A. In the sequel, for a closed subset E of a topological space, $\partial(E)$ will denote the boundary of E.

Theorem 3.1. *Let A be a regular and semisimple commutative Banach algebra, T a locally compact Hausdorff space and $f : \Delta(A) \to T$ a continuous, surjective and proper mapping. Suppose that for each $t \in T$, every closed subset of $f^{-1}(t)$ is a weak Ditkin set and that*

$$N = \sup\{\eta(F) : F \subseteq f^{-1}(t), \ F \ closed, \ t \in T\}$$

is finite. Let E be a closed subset of $\Delta(A)$ such that $f(\partial(E))$ is scattered. Then E is a weak spectral set and $\xi(E) \leqslant N$.

Proof. Let $a \in k(E)$ and let S denote the set of all $t \in T$ with the property that a^N does not at all points of $f^{-1}(t)$ belong locally to $\overline{j(E)}$. Since a belongs locally to $\overline{j(E)}$ at every point of $\Delta(A) \setminus \partial(E)$, we have $f^{-1}(t) \cap \partial(E) \neq \emptyset$ for every $t \in S$ and hence $S \subseteq f(\partial(E))$.

We first observe that S is closed in T. To see this, let $(s_\alpha)_\alpha$ be a net in S converging to some $t \in T$ and, towards a contradiction, suppose that $t \notin S$. For each α, choose $\varphi_\alpha \in f^{-1}(s_\alpha)$ such that a^N does not belong locally to $\overline{j(E)}$ at φ_α. Fix a compact neighbourhood U of t. Since $f^{-1}(U)$ is compact, after passing to a subnet if necessary, we can assume that $\varphi_\alpha \in f^{-1}(U)$ for all α and $\varphi_\alpha \to \varphi$ for some $\varphi \in \Delta(A)$. Then $\varphi \in f^{-1}(t)$ since $f(\varphi_\alpha) \to t$. Hence a^N belongs locally to $\overline{j(E)}$ at φ and since $\varphi_\alpha \to \varphi$, the same is true at φ_α for large α. This contradiction shows that S is closed in T.

Suppose that $S \neq \emptyset$. Then S has an isolated point s since $f(\partial(E))$ is scattered and S is closed. Choose an open subset V of T such that $V \cap S = \{s\}$. Since $C = \partial(E) \cap f^{-1}(s)$ is compact, we find an open neighbourhood W of C such that $\overline{W} \subseteq f^{-1}(V)$. Moreover, A being regular, there exists $u \in A$ such that $\widehat{u} = 1$ in a neighbourhood of C and $\widehat{u} = 0$ on $\Delta(A) \setminus \overline{W}$.

By hypothesis, C is a weak Ditkin set and $\eta(C) \leqslant N$. Therefore, there exists a sequence $(u_n)_n$ in $j(C)$ such that $\|a^N - a^N u_n\| \to 0$ and hence $\|a^N u - a^N u u_n\| \to 0$. Now, a^N belongs locally to $\overline{j(E)}$ at every point of $\Delta(A) \setminus f^{-1}(S)$ and of $\Delta(A) \setminus \partial(E)$ and $\widehat{u u_n}$ has compact support and vanishes on open sets containing C and $\Delta(A) \setminus f^{-1}(V)$, respectively. Since

$$\partial(E) \cap f^{-1}(S) \cap f^{-1}(V) = C,$$

this means that $a^N u u_n$ belongs locally to $\overline{j(E)}$ at every point of $\Delta(A)$ and at infinity. Semisimplicity of A implies that $a^N u u_n \in \overline{j(E)}$ and therefore $a^N u \in \overline{j(E)}$. Since $\hat{u} = 1$ in a neighbourhood of C, it follows that a^N belongs locally to $\overline{j(E)}$ at every point of $f^{-1}(s)$. This contradiction shows that $S = \emptyset$ and finishes the proof of the theorem. $\square$

One situation in which Theorem 3.1 applies is as follows. Let A be a unital, regular and semisimple commutative Banach algebra. Let I be a closed ideal of A and let $X = \Delta(I) \cup \{\omega\}$, the one-point compactification of $\Delta(I)$. Define $f : \Delta(A) \to X$ by $f(\varphi) = \varphi|_I$ for $\varphi \in \Delta(A) \setminus h(I)$ and $f(\varphi) = \omega$ for $\varphi \in h(I)$. Then f is continuous. In fact, every compact subset C of $\Delta(I)$ is closed in $\Delta(A)$ since A is regular, and hence $f^{-1}(X \setminus C) = \Delta(A) \setminus C$ is open in $\Delta(A)$. We leave the reformulation of Theorem 3.1 in this situation to the reader. A more important application concerns projective tensor products. In preparation for this, we need the following two lemmas.

In passing we remind the reader that the structure space of the projective tensor product $A \widehat{\otimes} B$ of two commutative Banach algebras A and B identifies naturally with the product space $\Delta(A) \times \Delta(B)$. More precisely, given $\varphi \in \Delta(A)$ and $\psi \in \Delta(B)$, there is a unique homomorphism $\varphi \widehat{\otimes} \psi : A \widehat{\otimes} B \to \mathbb{C}$ such that $(\varphi \widehat{\otimes} \psi)(a \otimes b) = \varphi(a)\psi(b)$ for all $a \in A$ and $b \in B$, and the map $(\varphi, \psi) \to \varphi \widehat{\otimes} \psi$ is a homeomorphism from $\Delta(A) \times \Delta(B)$ onto $\Delta(A \widehat{\otimes} B)$.

Lemma 3.2. *Let F be a closed subset of $\Delta(B)$ and $\varphi \in \Delta(A)$. Then*

$$k(\{\varphi\} \times F) = k(\varphi) \widehat{\otimes} B + A \widehat{\otimes} k(F).$$

Proof. Since obviously $k(\varphi) \widehat{\otimes} B + A \widehat{\otimes} k(F) \subseteq k(\{\varphi\} \times F)$, we only have to show the reverse inclusion. Let $u = \sum_{i=1}^\infty a_i \otimes b_i \in k(\{\varphi\} \times F)$, $a_i \in A$, $b_i \in B$, $\sum_{i=1}^\infty \|a_i\| \cdot \|b_i\| < \infty$. Choose $e \in A$ such that $\varphi(e) = 1$ and write $a_i = x_i + \lambda_i e$, where $x_i \in k(\varphi)$ and $\lambda_i \in \mathbb{C}$. Then, since $|\lambda_i| = |\varphi(a_i)| \leqslant \|a_i\|$ and $\|x_i\| = \|a_i - \lambda_i e\| \leqslant \|a_i\|(1 + \|e\|)$,

$$u = \sum_{i=1}^\infty x_i \otimes b_i + e \otimes \sum_{i=1}^\infty \lambda_i b_i.$$

Now, for each $\psi \in F$,

$$\psi\left(\sum_{i=1}^\infty \lambda_i b_i\right) = \psi\left(\sum_{i=1}^\infty \varphi(a_i) b_i\right) = (\varphi \otimes \psi)(u) = 0.$$

Thus $\sum_{i=1}^\infty \lambda_i b_i \in k(F)$ and hence, since $x_i \in k(\varphi)$, $u \in k(\varphi) \widehat{\otimes} B + A \widehat{\otimes} k(F)$. $\square$

Lemma 3.3. *Let F and φ be as in Lemma* 3.2. *Then*

$$\xi(\{\varphi\} \times F) \leqslant \xi(\varphi) + \xi(F) - 1.$$

Proof. It suffices to show that if $n = \xi(\varphi) < \infty$ and $m = \xi(F) < \infty$, then $u^{n+m-1} \in \overline{j(\{\varphi\} \times F)}$ for every $u \in k(\{\varphi\} \times F)$. First, let u be of the form $u = \sum_{i=1}^\infty x_i \otimes y_i$, $x_i \in k(\varphi)$, $y_i \in B$. Since

$x^n \in \overline{j(\varphi)}$ for every $x \in k(\varphi)$ and every n-fold product of elements of $k(\varphi)$ is a linear combination of nth powers of elements in $k(\varphi)$, it follows that

$$u^n = \sum_{i_1,\ldots,i_n=1}^{\infty} (x_{i_1} \cdot \ldots \cdot x_{i_n}) \otimes (y_{i_1} \cdot \ldots \cdot y_{i_n}) \in \overline{j(\varphi)} \,\widehat{\otimes}\, B.$$

Secondly, let $z \in k(F)$ and $v = e \otimes z$, where $e \in A$ is such that $\varphi(e) = 1$. Then $v^m = e^m \otimes z^m \in A \,\widehat{\otimes}\, \overline{j(F)}$. By the proof of Lemma 3.2, each element of $k(\{\varphi\} \times F)$ is of the form $u + v$, where u and v are as above. Then

$$(u + v)^{n+m-1} = \sum_{i=0}^{n+m-1} \binom{n+m-1}{i} u^i v^{n+m-1}$$

$$= u^n \sum_{i=n}^{n+m-1} \binom{n+m-1}{i} u^{i-n} v^{n+m-1} + v^m \sum_{i=0}^{n-1} \binom{n+m-1}{i} u^i v^{n-1-i}$$

$$\subseteq \overline{j(F)} \,\widehat{\otimes}\, B + A \,\widehat{\otimes}\, \overline{j(F)}.$$

Now, by Lemma 1.5 of [7], $\overline{j(\varphi)} \,\widehat{\otimes}\, B + A \,\widehat{\otimes}\, \overline{j(F)} \subseteq \overline{j(\{\varphi\} \times F)}$, and this finishes the proof. $\square$

By simply taking spectral sets for $\{\varphi\}$ and F, it is clear that the estimate in Lemma 3.3 cannot be improved.

We say that a closed subset E of $\Delta(A)$ satisfies *condition* (D) (D referring to Ditkin) if there exists a constant $C > 0$ such that for every neighbourhood U of E, there exists $b \in A$ such that $\|b\| \leqslant C$, supp $\widehat{b} \subseteq U$ and $\widehat{b} = 1$ in a neighbourhood of E. The relevance of this notion is due to the fact that for unital A, (D) is equivalent to the condition appearing in Lemma 1.4, as we point out next.

Remark 3.4. Suppose that A has an identity e. For a closed subset E of $\Delta(A)$, the following are equivalent.

(i) E satisfies condition (D).
(ii) There exists a constant $c > 0$ such that for every compact subset K of $\Delta(A)$ which is disjoint from E, there exists $a \in j(E)$ such that $\|a\| \leqslant c$ and $\widehat{a} = 1$ on K.

To see this, suppose first that (ii) holds and let U be an open set containing E. Choose an open set V such that $E \subseteq V$ and $\overline{V} \subseteq U$ and let $K = \Delta(A) \setminus V$. By (i), there exists $b \in j(E)$ so that $\widehat{b} = 1$ on K and $\|b\| \leqslant c$ (with c only depending on E). Let $a = e - b$, then $\|a\| \leqslant 1 + c$, supp $\widehat{a} \subseteq \overline{V} \subseteq U$ and $\widehat{a} = 1$ in a neighbourhood of E.

(i) $\Rightarrow$ (ii) is even easier.

Theorem 3.5. *Let A and B be regular commutative Banach algebras such that both are unital and $A \,\widehat{\otimes}\, B$ is semisimple. Suppose that the following conditions (1) and (2) are satisfied.*

(1) *Every closed subset of $\Delta(B)$ is a weak spectral set and satisfies condition* (D) *and*

$$N = \sup\{\xi(F)\colon F \subseteq \Delta(B) \text{ closed}\} < \infty.$$

(2) *Each singleton* $\{\varphi\}$, $\varphi \in \Delta(A)$, *is a weak spectral set and satisfies condition* (D) *and*

$$M = \sup\{\xi(\varphi) \colon \varphi \in \Delta(A)\} < \infty.$$

Let E be a closed subset of $\Delta(A) \times \Delta(B)$ *such that the set*

$$\{\varphi \in \Delta(A) \colon \partial(E) \cap (\{\varphi\} \times \Delta(B)) \neq \emptyset\}$$

is scattered. Then E is a weak spectral set and $\xi(E) \leqslant N + M - 1$.

Proof. We apply Theorem 3.1 to $A \widehat{\otimes} B$, $T = \Delta(A)$ and the map

$$f : \Delta(A \widehat{\otimes} B) = \Delta(A) \times \Delta(B) \to \Delta(A), \quad (\varphi, \psi) \to \varphi.$$

By hypothesis, $f(\partial(E))$ is scattered. It follows from conditions (1) and (2) and Lemma 3.3 that for every closed subset F of $\Delta(B)$ and $\varphi \in \Delta(A)$, $\{\varphi\} \times F$ is a weak spectral set and

$$\xi(\{\varphi\} \times F) \leqslant \xi(\varphi) + \xi(F) - 1 \leqslant N + M - 1.$$

Now, let c and d denote the constants occuring in condition (D) for $\{\varphi\}$ and F, respectively, and let U be an open neighbourhood of φ in $\Delta(A)$ and V an open neighbourhood of F in $\Delta(B)$. There exist $a \in A$ and $b \in B$ with the following properties: $\|a\| \leqslant c$, $\|b\| \leqslant d$, $\operatorname{supp} \widehat{a} \subseteq U$, $\operatorname{supp} \widehat{b} \subseteq V$, $\widehat{a} = 1$ near φ and $\widehat{b} = 1$ in a neighbourhood of F. Then the element $x = a \otimes b$ of $A \widehat{\otimes} B$ satisfies $\|x\| \leqslant cd$, $\operatorname{supp} \widehat{x} \subseteq U \times V$ and $\widehat{x} = 1$ in a neighbourhood of $\{\varphi\} \times F$. It follows from Remark 3.4 and Lemma 1.4 that $\{\varphi\} \times F$ is a weak Ditkin set and $\eta(\{\varphi\} \times F) = \xi(\{\varphi\} \times F) \leqslant N + M - 1$. Thus Theorem 3.1 applies and yields that E is a weak spectral set and $\xi(E) \leqslant N + M - 1$. $\quad\square$

Theorem 3.5 applies, for instance, to the projective tensor product of any two of the Banach algebras $C(X)$, X a compact Hausdorff space, $A(G)$ for a compact group G, $C^n[0, 1]$ and $\operatorname{Lip}_\alpha X$ (see Section 1).

4. Weak spectral synthesis for Fourier algebras

It was shown in [12, Theorem 3.1] that weak spectral synthesis fails in $A(G) = L^1(\widehat{G})$ for every non-discrete locally compact abelian group G. Employing this result as well as a deep theorem due to Zelmanov [21], we settle in this final section the weak synthesis problem for the Fourier algebras of arbitrary locally compact groups.

It is easy to see that weak spectral synthesis holds for $A(G)$ when G is discrete. In fact, $A(G)$ is Tauberian [6, Corollaire 3.38] and hence by [19, Corollary 2.4] (see also [11, Corollary 3.10]),

$$\xi(E) \leqslant 1 + \xi(\partial(E)) = 1 + \xi(\emptyset) = 2$$

for every subset E of G. Alternatively, one can appeal to the following simple observation.

Lemma 4.1. *Let A be a semisimple and regular commutative Banach algebra. Suppose that A is Tauberian, and let E be an open and closed subset of* $\Delta(A)$. *Then E is a weak spectral set and* $\xi(E) \leqslant 2$.

Proof. Let $x \in k(E)$. Since A is Tauberian, $x^2 \in xA \subseteq \overline{xj(\emptyset)}$. It therefore suffices to show that $xy \in \overline{j(E)}$ for every $y \in j(\emptyset)$. Now, xy belongs locally to $\overline{j(E)}$ at infinity, at every point of E since E is open, and at every point of $\Delta(A) \setminus E$ anyway. Thus $xy \in \overline{j(E)}$ since A is semisimple and regular. □

The next lemma will be needed to carry out a projective limit argument in the proof of Theorem 4.3 below.

Lemma 4.2. *Let K be a compact normal subgroup of G and identify G and G/K with $\Delta(A(G))$ and $\Delta(A(G/K))$, respectively. Let $q : G \to G/K$ denote the quotient homomorphism. Then $\xi(E) \leqslant \xi(q^{-1}(E))$ for every closed subset E of G/K.*

Proof. We can assume that $n = \xi(q^{-1}(E)) < \infty$. Let $u \in A(G/K)$ such that $u|_E = 0$ and $\epsilon > 0$. Then $u \circ q \in A(G)$ and $u \circ q|_{q^{-1}(E)} = 0$. By hypothesis, there exists $v \in A(G)$ such that $\|(u \circ q)^n - v\|_{A(G)} < \epsilon$ and v vanishes on some open set U which contains $q^{-1}(E)$ and is such that $G \setminus U$ is compact. Since $q^{-1}(E) = q^{-1}(E)K$ and K is compact, a simple topological argument shows that we can assume that $U = UK$. For $v \in A(G)$ and $k \in K$, let $R_k v \in A(G)$ be defined by $R_k v(x) = v(xk), x \in G$. Then the element $\int_K R_k v \, dk$ of $A(G)$ vanishes on U. Define $w \in A(G/K)$ by

$$w(xK) = \int_K R_k v(x) \, dk, \quad x \in G.$$

Then w vanishes on the neighbourhood $q(U)$ of E, $G/K \setminus q(U)$ is compact and

$$\|u^n - w\|_{A(G/K)} = \|u^n \circ q - w \circ q\|_{A(G)} = \|(u \circ q)^n - w \circ q\|_{A(G)}$$

$$= \left\| \int_K R_k (u \circ q)^n \, dk - \int_K R_k v \, dk \right\|_{A(G)}$$

$$\leqslant \int_K \|R_k ((u \circ q)^n - v)\|_{A(G)} \, dk$$

$$= \|(u \circ q)^n - v\|_{A(G)}.$$

So $w \in j(E)$ and $\|u^n - w\|_{A(G/K)} < \epsilon$. □

Theorem 4.3. *Let G be an arbitrary locally compact group. Then weak spectral synthesis holds for $A(G)$ if and only if G is discrete.*

Proof. Since $A(G)$ is Tauberian, by Lemma 4.1 we only have to show that G is discrete whenever weak synthesis holds for $A(G)$. As mentioned above, this conclusion is true if G is abelian [12, Theorem 3.1], and our proof is a reduction to this case.

In the first instance, assume that G is a connected Lie group and that the radical R of G, the maximal connected solvable closed normal subgroup of G, is non-trivial. Then R contains a non-trivial connected abelian closed normal subgroup H, namely the last non-trivial member of

the commutator series of R. By Corollary 2.4, weak synthesis holds for $A(H)$ and hence H has to be discrete. This contradiction shows that $R = \{e\}$. So G is semisimple. If G is non-trivial, it contains an infinite compact subgroup K, and by a theorem of Zelmanov [21, Theorem 2], K in turn contains an infinite (closed) abelian subgroup H. As above, this leads to a contradiction. Consequently, G is trivial.

Now, drop the hypothesis that G be a Lie group. The connected group G is a projective limit of Lie groups G/K_α. Since K_α is compact, by Lemma 4.2 weak spectral synthesis holds for G/K_α. By the preceding paragraph, $K_\alpha = G$ for all α and hence $G = \{e\}$.

Turning to the general case, let G_0 denote the connected component of the identity of G. Then weak synthesis holds for G_0 and then $G_0 = \{e\}$ by what we have already shown. So G is totally disconnected. Finally, fix a compact open subgroup K of G and suppose that K is infinite. Applying Zelmanov's theorem again leads to a contradiction. Thus G is discrete. $\square$

The reader should compare the preceding theorem with Proposition 2.2 of [8], where it was shown that:

(i) local spectral synthesis holds for $A(G)$ if and only if G is discrete;

(ii) spectral synthesis holds for $A(G)$ if and only if G is discrete and $u \in \overline{uA(G)}$ for every $u \in A(G)$.

So weak spectral synthesis and local spectral synthesis are equivalent for Fourier algebras. It is not unlikely that these properties already force spectral synthesis to hold for $A(G)$ since no example seems to be known of a discrete group G for which the condition $u \in \overline{uA(G)}$ (the existence of an approximate identity in $A(G)$ in the weakest possible sense) is not satisfied.

Acknowledgment

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