Abstract

Let $\mathcal{T}$ be a $\mathbb{Q}$-linear closed symmetric monoidal triangulated category in the sense of [M. Hovey, Model Categories, Math. Surveys Monogr., vol. 63, Amer. Math. Soc., Providence, RI, 1999]. We prove an additivity for evenly and oddly finite-dimensional vertices of distinguished triangles in $\mathcal{T}$ (Theorem 1). As a corollary, we get motivic finite dimensionality for quasi-projective curves over a field (Theorem 3). The last result has been independently obtained by C. Mazza, see [C. Mazza, Schur functors and motives, preprint, 2003, http://www.math.uiuc.edu/K-theory/0641/].

1. Introduction

Let $\mathcal{C}$ be a $\mathbb{Q}$-linear, pseudoabelian and symmetric monoidal category with a product $\otimes$. Let $\Sigma_n$ be the symmetric group of permutations of $n$ elements. For any object $X$ in $\mathcal{C}$ one can define its wedge $X^{(n)}$ and symmetric $X^{[n]}$ powers as the images of the idempotents in $\text{End}(X^{\otimes n})$ corresponding to the “vertical” and “horizontal” irreducible representations of $\Sigma_n$ over $\mathbb{Q}$. These powers generalize the usual notions of wedge and symmetric powers of vector spaces over a field of characteristic zero. Then $X$ is called evenly (respectively, oddly) finite-dimensional if $X^{(n)}$ (respectively, $X^{[n]}$) is zero for some $n$. In general, $X$ is finite-dimensional if $X \cong X_+ \oplus X_-$, where $X_+$ is evenly and $X_-$ is oddly finite-dimensional, see [7, Section 3] or [1, Section 9].
Let $k$ be a field and let $\text{CHM}$ be the category of Chow motives over $k$ with coefficients in $\mathbb{Q}$, see [14]. Let $X$ be a smooth projective curve of genus $g$ over $k$, and let $M(X)$ be the Chow motive of $X$ in $\text{CHM}$. Then $M(X)$ can be decomposed into a direct sum $M(X) = 1 \oplus M^1(X) \oplus L$, where 1 and $L$ are, respectively, the unit and the Lefschetz motive in $\text{CHM}$, and $M^1(X)$ is the middle part of $M(X)$ related to the Jacobian variety of $X$. The wedge squares of 1 and $L$ vanish. In [7, Theorem 4.2], Kimura has proved that the $(2g + 1)$th symmetric power of $M^1(X)$ vanishes as well, thus, $M(X)$ is finite-dimensional.

Let $\text{DM}$ be Voevodsky’s triangulated category $DM_{-}(k) \otimes \mathbb{Q}$ of motives over $k$ with coefficients in $\mathbb{Q}$, see [17]. The goal of the present paper is to generalize Kimura’s result to the motives of quasi-projective curves over $k$ considered in the category $\text{DM}$.

We start with a general $\mathbb{Q}$-linear pseudoabelian symmetric monoidal triangulated category and use the following two ideas. The first idea (suggested by U. Jannsen) is to associate a filtration with a given distinguished triangle $X \to Y \to Z \to \Sigma X$, which should be similar to the filtration for a short exact sequence of locally free sheaves of modules on a manifold, see [3, p. 127]. Without further assumptions, however, it seems difficult to show the required compatibilities in diagrams of distinguished triangles related to the above filtration. The second idea is to work in the homotopy category $T$ of a pointed simplicial model monoidal$^1$ category $C$, with the monoidal structure on $T$ induced by the monoidal structure on $C$ (see [4]). The category $T$ being simplicial, the suspension $\Sigma X = X \wedge S^1$ by the simplicial circle $S^1$ defines a shift functor in $T$. It turns out that it is possible to control powers of vertices in distinguished triangles in $T$ using cofiber sequences in the underlying category $C$. This second idea takes its roots in [8].

**Theorem 1.** Let $T$ be a triangulated category as above, and assume, furthermore, that $T$ is $\mathbb{Q}$-linear and pseudoabelian. Then, for any distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

in $T$, if $X$ and $Z$ are evenly (respectively, oddly) finite-dimensional, it follows that $Y$ is also evenly (respectively, oddly) finite-dimensional.

**Remark 2.** Equivalently Theorem 1 can be stated as follows: if $X$ is evenly (respectively, oddly) finite-dimensional and $Y$ is oddly (respectively, evenly) finite-dimensional, it follows that $Z$ is oddly (respectively, evenly) finite-dimensional. But one cannot make a similar statement if $X$ and $Y$ are both odd or both even.

In particular, Theorem 1 holds in the motivic stable homotopy category $\text{MSH}$ of Morel and Voevodsky (see [16], as well as [6,11], for the description of this category) considered with coefficients in $\mathbb{Q}$. Comparing $\text{MSH}$ with $\text{DM}$, and using some easy computations with canonical distinguished triangles in $\text{DM}$, we get the following result:

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$^1$ From now on, for short of notation, a monoidal category is always symmetric and closed monoidal, and one can also use the word “tensor” instead of the word “monoidal.”
Theorem 3. Let \( k \) be a field of characteristic zero, and let \( X \) be a quasi-projective curve over \( k \). Then its motive \( M(X) \), considered in Voevodsky’s category \( \text{DM} \), is finite-dimensional.

Remark 4. After publishing of the first version of this paper on the Internet I was informed that Theorem 3 has been independently obtained by C. Mazza [9, Theorem 5.8].

The paper is organized as follows. For the convenience of the reader, in the second section we recall the definitions and basic results about finite-dimensional objects and Hovey’s triangulated categories after [4,8]. We also recall that \( \text{MSH} \) is an example of such a category. In Section 3 we develop a homotopy technique to deal with finite dimensionality of vertices in distinguished triangles and show the existence of the above filtration on \( Y^{[n]} \) with graded pieces \( Z^{[p]} \wedge X^{[q]} \) where \( p + q = n \) (and the same for symmetric powers), and then prove Theorem 1. In Section 4 we prove Theorem 3.

2. Preliminary results

2.1. Wedge and symmetric powers

Let \( C \) be a monoidal category with a monoidal product \( \otimes \). For any \( n \) and any object \( X \) in \( C \), let \( X^{(n)} \) denote the \( n \)-fold product \( X \otimes^n \) in \( C \). If \( f : X \to Y \) is a morphism in \( C \), then let \( f^{(n)} : X^{(n)} \to X^{(n)} \) denote the \( n \)-fold product of \( f \).

Let \( \Sigma_n \) be the group of permutations of \( n \) letters, and let \( A = \mathbb{Q} \Sigma_n \) be the group algebra (over \( \mathbb{Q} \)) of \( \Sigma_n \). The set of all irreducible representations of \( \Sigma_n \) over \( \mathbb{Q} \) is in one-to-one correspondence with the set \( P_n \) of all partitions \( \lambda \) of \( n \), and there exists a finite collection \( \{ e_\lambda \} \) of pairwise orthogonal idempotents in \( A \), such that \( \sum_{\lambda \in P_n} e_\lambda = 1 \) in \( A \), and each \( e_\lambda \) induces the corresponding irreducible representation of \( \Sigma_n \) up to an isomorphism [2, Section 4].

Assume now that \( C \) is, in addition, \( \mathbb{Q} \)-linear and pseudoabelian. For any \( n \) and \( X \in C \) let \( \Gamma : A \to \text{End}(X^{(n)}) \) be the homomorphism sending any \( \sigma \in \Sigma_n \) to the endomorphism \( \Gamma_{\sigma} : X^{(n)} \to X^{(n)} \) permuting factors according to \( \sigma \) and the commutativity and associativity constraints in \( C \). For each \( \lambda \in P_n \) let \( d_\lambda = \Gamma_{e_\lambda} \). Since \( \sum_{\lambda \in P_n} e_\lambda = 1 \) in \( A \), it follows that \( \sum_{\lambda \in P_n} d_\lambda = 1 \) in \( \text{End}(X^{(n)}) \). The category \( C \) being pseudoabelian, it follows that \( X^{(n)} \) is a direct sum of images \( \text{im}(d_\lambda) \) of the idempotents \( d_\lambda \).

Let \( d_n^+ \) be the projector \( d_\lambda \) when \( \lambda \) is the partition \( (1, \ldots, 1) \) of \( n \), and let \( d_n^- \) be the projector \( d_\lambda \) when \( \lambda \) is the partition \( (n) \) of \( n \). In other words,

\[
d_n^+ = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \text{sgn}(\sigma) \Gamma_{\sigma} \quad \text{and} \quad d_n^- = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \Gamma_{\sigma}.
\]

The \( n \)th wedge and symmetric powers of \( X \) are defined as \( X^{[n]} = \text{im}(d_n^+) \) and \( X^{(n)} = \text{im}(d_n^-) \), respectively. We say that \( X \) is evenly (respectively, oddly) finite-dimensional if
$X^{(n)} = 0$ (respectively, $X_{-}^{(n)} = 0$) for some $n$. The object $X$ is finite-dimensional if it can be decomposed into a direct sum $X = X_+ \oplus X_-$, where $X_+$ is evenly and $X_-$ is oddly finite-dimensional.

Finite-dimensional objects have the following properties, see [7, Sections 5, 6] and [1, Section 9.1]. The tensor product of two finite-dimensional objects is finite-dimensional and a subobject\(^2\) of a finite-dimensional object is also finite-dimensional. If $X$ and $Y$ are evenly or oddly finite-dimensional and of the same parity, then $X \otimes Y$ is evenly finite-dimensional, and if $X$ and $Y$ have different parity, then $X \otimes Y$ is oddly finite-dimensional. If $X$ is finite-dimensional and $X \cong X_+ \oplus X_- \cong Y_+ \oplus Y_-$ be two decompositions of $X$ into even and odd parts, then $X_+ \cong Y_+$ and $X_- \cong Y_-.$

2.2. Homotopy category of a pointed model monoidal category

Let $C$ be a pointed model monoidal category with a monoidal product $\wedge : C \times C \to C$ and unit object $S$. The coproduct of two objects $X$ and $Y$ in $C$ will be denoted by $X \vee Y$. Let $f : X \to Y$ and $f' : X' \to Y'$ be two maps in $C$. Consider the coproduct $X \wedge Y' \coprod_{X \wedge X'} Y \wedge X'$, that is the colimit of the following diagram:

\[
\begin{array}{ccc}
X \wedge X' & \xrightarrow{f \wedge 1} & Y \wedge X' \\
\downarrow^{1 \wedge f'} & & \downarrow \\
X \wedge Y' & & 
\end{array}
\]

The pushout smash product of $f$ and $f'$ is the unique map

\[ f \Box f' : X \wedge Y' \coprod_{X \wedge X'} Y \wedge X' \longrightarrow Y \wedge Y' \]

determined by the above colimit. The connection between the model and monoidal structures can be expressed by the following axioms [4, 4.2]:

- If $f$ and $f'$ are cofibrations then $f \Box f'$ is also a cofibration. If, in addition, one of two maps $f$ and $f'$ is a weak equivalence, then so is $f \Box f'$.
- If $q : QS \to S$ is a cofibrant replacement for the unit object $S$, then the maps $q \wedge 1 : QS \wedge X \to S \wedge X$ and $1 \wedge q : X \wedge QS \to X \wedge S$ are weak equivalences for all cofibrant $X$.

Let $C$ be, in addition, a simplicial category. Then for any $X$ we have the suspension $\Sigma X = X \wedge S^1$ by the simplicial circle $S^1$, and the cone $CX = X \wedge I$, where $I$ is the

\(^2\) I.e. a kernel of an idempotent.
simplicial interval. For any cofibration \( f : X \to Y \) between cofibrant objects, the mapping cone \( C(f) \) is the colimit of the following diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & CX \\
\downarrow & & \downarrow \\
Y & &
\end{array}
\]

Assume that \( \Sigma \) is a Quillen equivalence with adjoint loop functor \( \Omega \). Then \( T = Ho(C) \) is a triangulated category with the shift endofunctor given by \( \Sigma \), see [4, 6.5, 6.6, 7.1]. To be more precise, \( T \) is a pre-triangulated category [4, 6.5], and the suspension functor \( \Sigma \) is an autoequivalence on \( T \). It can be shown that any pre-triangulated category is classically triangulated [4, Proposition 7.1.6].

The category \( T \) has the following useful properties: the localization functor \( C \to T \) is monoidal, the monoidal and triangulated structures are strongly compatible in \( T \), and distinguished triangles in \( T \) can be described in terms of cofiber sequences in \( C \) [4, Section 6.5], [8, Section 5]. If \( f : X \to Y \) is a map in \( C \), then, using the cofiber replacement in \( C \), one can assume that \( f \) is a cofibration between cofibrant objects \( X \) and \( Y \). Then we have the cofiber distinguished triangle

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
& & \xrightarrow{\partial} \Sigma X
\end{array}
\]

in \( T \). If \( Z = Y/X \) is the quotient of \( Y \) by \( X \), i.e., the colimit of the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
* & &
\end{array}
\]

where \( * \) is the initial–terminal object in \( C \), then \( Z \) is weakly equivalent to \( C(f) \), [8, Lemma 5.3]. The suspension \( \Sigma X \), being a cogroup object, coacts on \( Z \), [4, Theorem 6.2.1]. In particular, one can define the standard boundary map \( \partial : Z \to \Sigma X \) as the composition of the coaction \( Z \to Z \amalg \Sigma X \) with the evident map \( Z \amalg \Sigma X \to \Sigma X \) [4, 6.2]. Then we have the cofiber distinguished triangle

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
& & \xrightarrow{\partial} \Sigma X.
\end{array}
\]

Any distinguished triangle in \( T \) is isomorphic to a cofiber distinguished triangle of the above type, see [4, 6.2–7.1] and [8, Section 5].

**Lemma 5.** Let \( f : X \to Y \) and \( f' : X' \to Y' \) be two cofibrations of cofibrant objects in \( C \) with cofibers \( Z \) and \( Z' \), respectively. Let \( a : X \to X' \) and \( b : Y \to Y' \) be maps, such that
Let $bf = f'a$, and let $c : Z \to Z'$ be the induced map on the cones. Then $c$ is equivariant with respect to the cogroup homomorphism $\Sigma a$, so that the diagram

$$
\begin{array}{cccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{\partial} & \Sigma X \\
\downarrow{a} & & \downarrow{b} & & \downarrow{c} & & \downarrow{\Sigma a} \\
X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{\partial} & \Sigma X'
\end{array}
$$

is a map of distinguished triangles in $\mathcal{T}$.

**Proof.** See [4, Proposition 6.2.5]. $\square$

### 2.3. The motivic stable homotopy category

Let us consider now a particular case of the above abstract situation. Let $k$ be a field and let $\mathbf{Sm}$ be the category of all smooth schemes, separated and of finite type over $k$. Let $\mathbf{Spc}$ be the category of spaces, i.e., the category of simplicial sheaves for the Nisnevich topology on $\mathbf{Sm}$, and let $\mathbf{Spc}_*$ be the corresponding pointed category with the evident terminal–initial object $*$, [16]. The model structure on $\mathbf{Spc}_*$ is described in [6,11,12,16]. It is constructed on the base of $\mathbb{A}^1$-weak equivalences in $\mathbf{Spc}_*$. The homotopy categories of simplicial sheaves and presheaves are canonically isomorphic via the forgetful functor [6, Theorem 1.2(2)].

The composition of the Yoneda embedding with the functor from (pre)sheaves into simplicial (pre)sheaves allows one to identify a smooth scheme with the corresponding representable simplicial (pre)sheaf. Since $\mathbf{Spc}$ is cocomplete, one can consider colimits in $\mathbf{Spc}$, for example, quotients, contractions, gluings, etc. In particular, let

$$
T = \mathbb{A}^1 / (\mathbb{A}^1 - 0)
$$

be the quotient of $\mathbb{A}^1$ by $\mathbb{A}^1 - 0$, where $\mathbb{A}^1$ and $\mathbb{A}^1 - 0$ are pointed by 1. In the homotopy category $Ho(\mathbf{Spc}_*)$ one has

$$
T \cong \mathbb{P}^1 \cong S^1 \wedge (\mathbb{A}^1 - 0),
$$

where $S^1$ is the simplicial circle, viewed as a constant sheaf, and $\mathbb{P}^1$ is pointed at $\infty$.

A $T$-spectrum $X$ (or a motivic spectrum) is a sequence of objects $X^n \in \mathbf{Spc}_*$ and bonding maps $T \wedge X^n \to X^{n+1}$ for each $n$. A map of spectra $f : X \to Y$ consists of maps $f^n : X^n \to Y^n$ commuting with the bonding maps. A motivic symmetric spectrum $X$ is a motivic spectrum $X$ with an extra (left) action of the symmetric group $\Sigma_n$ on each $X^n$ and with $\Sigma_m \times \Sigma_n$-equivariant compositions of the bonding maps $T^{(m)} \wedge X^n \to X^{m+n}$. A map of motivic symmetric spectra is equivariant for the symmetric group action. All of this can be found in [6, Section 4] (the topological theory of symmetric spectra is developed in [5]).
Let $\text{Spc}_T^\Sigma$ be the category of motivic symmetric spectra. In [6] Jardine described the structure of a pointed simplicial model monoidal category on $\text{Spc}_T^\Sigma$ (arising from $\mathbb{A}^1$-weak equivalences in $\text{Spc}_+$, of course). As we have seen in Section 2.2, there exists a structure of a triangulated monoidal category on the corresponding homotopy category $\text{Ho}(\text{Spc}_T^\Sigma)$, such that its shift functor is the simplicial suspension and the localization functor is monoidal. Theorem 4.30 in [6] asserts that $\text{Ho}(\text{Spc}_T^\Sigma)$ is the desired motivic stable homotopy category $\text{MSH}$. So, $\text{MSH}$ is an example of a category satisfying the assumptions of Theorem 1.

In order to connect the category $\text{MSH}$ with the category $\text{DM}$, we need the following theorem and two lemmas:

**Theorem 6.** Let $k$ be a field, such that $\text{char}(k) = 0$ and $-1$ is a sum of squares in $k$. There is a monoidal and triangulated equivalence of $\mathbb{Q}$-localized categories $\text{MSH} \cong \text{DM}$.

**Proof.** See [12, Section 5.2] or [19]. □

**Lemma 7.** Let $\mathcal{T}$ and $\mathcal{T}'$ be $\mathbb{Q}$-linear pseudoabelian monoidal categories and let $F : \mathcal{T} \to \mathcal{T}'$ be an additive and monoidal functor. If $X$ is a finite-dimensional (respectively, evenly, oddly finite-dimensional) object in $\mathcal{T}$, then so is $F(X)$. If, moreover, $F$ is an equivalence of categories whose quasi-inverse functor is additive and monoidal, $X$ is finite-dimensional (respectively, evenly, oddly finite-dimensional) if and only if $F(X)$ is finite-dimensional (respectively, evenly, oddly finite-dimensional).

**Proof.** Since $F$ is monoidal, $F(X^{(n)}) = F(X)^{(n)}$ for any $n$ and any object $X$ in $\mathcal{T}$. Since $F$ is additive, it commutes with direct sums. Then, $F(X^{(n)}) = F(X)^{(n)}$ and the same holds for symmetric powers. Hence, given an object $X$ in $\mathcal{T}$, if $X$ is finite-dimensional (respectively, evenly, oddly finite-dimensional), then so is $F(X)$. Assume $F$ is an equivalence of categories with an additive and monoidal quasi-inverse $G$, and $F(X)$ is finite-dimensional in $\mathcal{T}'$. Since $X$ is isomorphic to the object $GF(X)$ and $GF(X)$ is finite-dimensional (respectively, evenly, oddly finite-dimensional) by the above argument, $X$ is also finite-dimensional (respectively, evenly, oddly finite-dimensional). □

**Lemma 8.** Let $k$ be a field of characteristic zero, let $L/k$ be a finite normal field extension and let $i^* : \text{DM}(k) \to \text{DM}(L)$ be the scalar extension functor induced by the morphism $i : \text{Spec}(L) \to \text{Spec}(k)$. Then, for any object $M \in \text{DM}(k)$, the motive $M$ is finite-dimensional if and only if the motive $i^* M$ is finite-dimensional.

**Proof.** The functor $i^*$ is additive and monoidal. By Lemma 7, it carries finite-dimensional objects to finite-dimensional objects. Hence we have only to show that finite dimensionality of $i^* M$ implies finite dimensionality of $M$. Let $d$ be the degree of $L$ over $k$ and let

$$M(i) : M(\text{Spec}(L)) \longrightarrow M(\text{Spec}(k))$$
be the morphism induced by the morphism $i$. Since $L$ is finite and normal over $k$, there exists the transfer morphism
\[ \text{tr}(i) : M(\text{Spec}(k)) \longrightarrow M(\text{Spec}(L)), \]
such that
\[ M(i) \circ \text{tr}(i) = d \cdot \text{id}_{M(\text{Spec}(k))} \]
[18, Proposition 4.1.4]. Let $M$ be any object in $\text{DM}(k)$. It is well known that $i^*$ has left adjoint
\[ i_\# : \text{DM}(L) \longrightarrow \text{DM}(k) \]
(it is induced by the corresponding left adjoint to the scalar extension functor on finite correspondences [10, Example 1.12]) and
\[ i_\# i^* M = M \otimes M(\text{Spec}(L)). \]
Let
\[ a : M \longrightarrow i_\# i^* M \quad \text{and} \quad b : i_\# i^* M \longrightarrow M \]
be the morphisms induced by the morphisms $\text{tr}(i)$ and $M(i)$, respectively. Note that $b$ is induced also by the adjunction morphism
\[ \Phi : i_\# i^* \longrightarrow \text{Id}_{\text{DM}(k)}, \]
i.e., $b = \Phi_M$. We also need the adjunction morphism
\[ \Psi : \text{Id}_{\text{DM}(L)} \longrightarrow i^* i_. \]
Since $b \circ a = d \cdot \text{id}_M$ and we work in the categories with coefficients in $\mathbb{Q}$, the morphism $b$ is left inverse to the morphism $a/d = \frac{1}{d} \cdot a$. In other words, $M$ is a subobject in $i_\# i^* M$. Assume that $i^* M$ is finite-dimensional and let
\[ i^* M = N_+ \oplus N_- \]
be the decomposition of $i^* M$ in its even and odd parts, i.e., $N_+^{[n]} = 0$ and $N_-^{[n]} = 0$ for some $n$. The category $\text{DM}(k)$ being pseudoabelian, the decomposition $i_\# i^* M = i_\# N_+ \oplus i_\# N_-$ induces the decomposition
\[ M = M_+ \oplus M_. \]
In particular,
\[ a/d = (a/d)_+ \oplus (a/d)_-, \]
where $(a/d)_\pm$ is a morphism from $M_\pm$ to $i_\# N_\pm$. 
The identity morphism

\[
i^* \Psi_{\circ i^*} \rightarrow i^* i_{\# i^*} \rightarrow i^* \Phi \rightarrow i^*
\]
gives rise to the commutative diagram:

\[
\begin{array}{ccc}
i^* M & \xrightarrow{(\Psi_{\circ i^*})_M} & i^* i_{\#} i^* M & \xrightarrow{(i^* \circ \Phi)_M} & i^* M \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
N_+ \oplus N_- & \xrightarrow{\Psi_{i^* N_+} \oplus \Psi_{i^* N_-}} & i^* i_{\#} i^* N_+ \oplus i^* i_{\#} i^* N_- & \xrightarrow{i^* (\Phi_M)} & i^* M_+ \oplus i^* M_-
\end{array}
\]

whose horizontal rows are the identity morphisms. Since \( \Phi_M = b, b \circ (a/d) = \text{id}_M \) and \( a/d = (a/d)_+ \oplus (a/d)_- \), we have that the composition \((i^* \circ \Phi)_M \circ (\Psi_{\circ i^*})_M\) induces two isomorphisms

\[
N_+ \cong i^* M_+ \quad \text{and} \quad N_- \cong i^* M_-.
\]

Since \( i^* \) is monoidal and additive,

\[
i^*(M_+^{(n)}) = (i^* M_+)^{(n)} \cong N_+^{(n)} = 0.
\]

Then, of course, \( i_{\#} i^* M_+^{(n)} = 0 \). But \( M_+^{(n)} \) is a subobject of \( i_{\#} i^* M_+^{(n)} \) (see the argument above), whence \( M_+^{(n)} = 0 \), i.e., \( M_+ \) is evenly finite-dimensional. Analogously, \( M_- \) is oddly finite-dimensional. \( \square \)

**Corollary 9.** Theorem 1 implies the same additivity for evenly (oddly) finite-dimensional objects in distinguished triangles in the category DM.

**Proof.** By Lemma 8 we may assume that \( \sqrt{-1} \) is in the ground field. Let \( F : \text{MSH} \rightarrow \text{DM} \) be the equivalence of categories from Theorem 6, and let \( G : \text{DM} \rightarrow \text{MSH} \) be its quasi-inverse. Let \( X \rightarrow Y \rightarrow Z \rightarrow X[1] \) be a distinguished triangle in DM, such that \( X \) and \( Y \) are evenly (respectively, oddly) finite-dimensional.\(^3\) \( G(X) \) and \( G(Z) \) are evenly (respectively, oddly) finite-dimensional by Lemma 7. Since \( G \) is an exact functor, one has the distinguished triangle \( G(X) \rightarrow G(Y) \rightarrow G(Z) \rightarrow \Sigma G(X) \) in MSH. The object \( G(Y) \) is evenly (respectively, oddly) finite-dimensional by Theorem 1. Then \( Y \) is evenly (respectively, oddly) finite-dimensional by Lemma 7. \( \square \)

\(^3\) Traditionally, we denote the shift functor in DM by means of square brackets.
3. Finite-dimensional objects in distinguished triangles

3.1. Cofiber sequences and combinatorics of powers

Let \( T \) be a homotopy category of a pointed simplicial model and monoidal category \( C \), as in Section 2.2 (but not necessary \( \mathbb{Q} \)-linear). We denote the monoidal product in \( C \) by \( \wedge \) and the coproduct by \( \vee \). The monoidal product in \( T \) will be denoted by \( \otimes \) and the direct sum by \( \oplus \). The canonical (localization) functor \( C \rightarrow T \) is monoidal, i.e., it carries an object \( X \otimes Y \) in \( C \) into the object \( X \otimes Y \) in \( T \). The category \( T \) is triangulated and the shift endofunctor in \( T \) is given by smashing with \( S^1 \) (see Section 2.2). Let us also recall that “monoidal” always means “closed and symmetric monoidal.” In particular, for any fibrant \( X \in C \) both functors \(- \wedge X \) and \( X \wedge - \) preserve colimits in \( C \).

Let \( X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{\partial} \Sigma X \) be a distinguished triangle in \( T \). Our goal is to study wedge and symmetric powers of the vertices in this triangle. Without loss of generality, applying cofibrant replacement, we can assume that both \( X \) and \( Y \) are cofibrant and the above distinguished triangle is a cofibration triangle, so that \( Z = Y/X \).

Let \( m \) be a natural number and let \( V_m \) be the collection of all ordered sets \( v = (v_1, \ldots, v_m) \), such that \( v_j \in \{0, 1\} \) for each \( 1 \leq j \leq m \). In particular, we have vectors \( 0 = (0, \ldots, 0) \) and \( 1 = (1, \ldots, 1) \). The elements of \( V_m \) can be considered as the vertices of the unit cube \( K_m \) in \( \mathbb{R}^m \). Let \( D_v \) be a smash-product \( A_1 \wedge \cdots \wedge A_m \) in \( C \) with \( A_j = X \) if \( v_j = 0 \) and \( A_j = Y \) if \( v_j = 1 \). Evidently, \( D_0 = X^{(m)} \) and \( D_1 = Y^{(m)} \). Place \( D_v \) on the vertex \( v \) and interpret morphisms between vertices induced by the cofibration \( f : X \rightarrow Y \) as oriented edges of the cube \( K_m \). Then \( K_m \) can be considered as a commutative diagram involving all the mixed powers of \( X \) and \( Y \) of degree \( m \). For example, \( K_2 \) is the commutative diagram:

\[
\begin{array}{ccc}
X \wedge X & \xrightarrow{1 \wedge f} & X \wedge Y \\
\downarrow{f \wedge 1} & & \downarrow{f \wedge 1} \\
Y \wedge X & \xrightarrow{1 \wedge f} & Y \wedge Y
\end{array}
\]

where the objects \( X \wedge Y \) and \( Y \wedge X \) correspond to the vertices \((0, 1)\) and \((1, 0)\), respectively.

For any \( 0 \leq i \leq m \) let \( V^i_m \) be the subset in \( V_m \) consisting of all the vertices \( v \), such that the number of units in \( v \) is less or equal to \( i \). Let \( K^i_m \) be the commutative subdiagram in \( K_m \) generated by the vertices from \( V^i_m \). We will show how the filtration

\[
K_m^0 \subset K_m^1 \subset \cdots \subset K_m^m = K_m
\]

leads us to the desired filtration on the wedge and the symmetric \( m \)-powers of the object \( Y \).
Let

\[ D^i_m = \colim K^i_m \]

be the colimit of the diagram \( K^i_m \) in the category \( C \). If \( U^i_m = V^i_m - V^{i-1}_m \) is the set of vertices in \( K_m \) containing exactly \( i \) units, then for each vertex \( v \in U^i_m \) let

\[ r_v : D^i_v \longrightarrow D^i_m \]

be the canonical morphism to the colimit. For any \( i \) the inclusion \( K^i_m \subset K^{i+1}_m \) induces the map

\[ w_{m,i} : D^i_m \longrightarrow D^{i+1}_m \]

on the colimits in \( C \). Clearly, \( D^0_m = X(m) \) and \( D^m_m = Y(m) \).

For each \( v \in V_m \) let \( E^i_v \) be the product \( A_1 \land \cdots \land A_m \) in \( C \), such that \( A_j = X \) if \( v_j = 0 \) and \( A_j = Z \) if \( v_j = 1 \), and let

\[ E^i_m = \bigvee_{v \in U^i_m} E^i_v. \]

Considering these objects in the category \( T \) we have that

\[ E^i_m = \bigoplus_{v \in U^i_m} E^i_v. \]

**Proposition 10.** The morphism \( w_{m,i} \) is a cofibration for any \( i \). Moreover, the corresponding quotient object \( D^{i+1}_m / D^i_m \) is canonically isomorphic to \( E^{i+1}_m \), so that we have the cofibration distinguished triangle

\[ D^i_m \xrightarrow{w_{m,i}} D^{i+1}_m \xrightarrow{} E^{i+1}_m \xrightarrow{} \Sigma D^i_m \]

in the category \( T \).

**Proof.** Since \( C \) is a closed monoidal model category, it follows that, for any cofibrant object \( B \) in \( C \), both endofunctors \( - \land B \) and \( B \land - \) on \( C \) are Quillen functors, see [4, 4.2]. In particular, they preserve cofibrations. By assumption, \( X \) and \( Y \) are cofibrant, so that all edges in the commutative diagram \( K_m \) are cofibrations. Then the \( w^i_m \)'s are cofibrations as well. The proof of this is similar to the proof of that cofibrations are closed under pushouts (see, for instance, [4, Corollary 1.1.11]). To be more precise, let

\[ \begin{array}{ccc}
D^i_m & \xrightarrow{w_{m,i}} & A \\
\downarrow & & \downarrow t \\
D^{i+1}_m & \xrightarrow{} & B
\end{array} \]
be a commutative diagram in $\mathcal{C}$ where $t$ is a trivial fibration. We have to show that there exists a map $h : D^{i+1}_{m} \to A$ preserving the commutativity of the diagram.

Let $u$ be any vertex from $U^{i+1}_{m}$. Let $v$ be a vertex from $U^{i}_{m}$, such that there is an edge $w_{v,u} : D_{v} \to D_{u}$ in the commutative diagram $K_{m}$. Then we have the commutative square

$$
\begin{array}{ccc}
D_{u} & \xrightarrow{r_{u}} & D^{i}_{m} \\
\downarrow w_{v,u} & & \downarrow w_{m,i} \\
D_{v} & \xrightarrow{r_{v}} & D^{i+1}_{m}
\end{array}
$$

in the category $\mathcal{C}$. Composing the last two commutative squares we see that, by the left lifting property for the cofibrations $w_{v,u}$, there exists a map $h_{u}$ making the diagram

$$
\begin{array}{ccc}
D_{v} & \xrightarrow{h_{u}} & A \\
\downarrow w_{v,u} & & \downarrow t \\
D_{u} & \xrightarrow{r_{v}} & B
\end{array}
$$

commutative. It is not hard to check that the system of morphisms $\{h_{u}\}_{u \in U^{i+1}_{m}}$ gives rise to a cone over the diagram $K^{i+1}_{m}$, so that we have a canonical map $h : D^{i+1}_{m} \to A$. Using the uniqueness of universal maps for colimits, one can easily show that this $h$ is the desired map.

To show the second assertion of the proposition we use induction on $m$. For $m = 1$ it is trivially true. Assume that it holds for the diagrams $K_{1}, \ldots, K_{m-1}$. For any $0 \leq i \leq m - 1$ and $1 \leq j \leq m$ let $V^{i,j}_{m}$ be the subset in $V^{i}_{m}$ consisting of all the vertices $v = (v_{1}, \ldots, v_{m})$, such that $v_{j} = 0$, and let $K^{i,j}_{m}$ be the commutative subdiagram in $K^{i}_{m}$ generated by the vertices from $V^{i,j}_{m}$. Then $K^{i,j}_{m}$ can be considered as the termwise smash product $K^{i}_{m-1} \land X$ with permuted factors in each term $D_{x} \land X$, $x \in V^{i}_{m-1}$, in order to put $X$ on $j$th place. Let

$$
D^{i,j}_{m} = \text{colim } K^{i,j}_{m}.
$$

If $U^{i,j}_{m} = U^{i}_{m} \cap V^{i,j}_{m}$, then for each vertex $v \in U^{i,j}_{m}$ we have the canonical morphism

$$
r_{v,j} : D_{v} \to D^{i,j}_{m}.
$$

Since $K^{i,j}_{m}$ is a subdiagram in $K^{i}_{m}$, we have the universal morphisms

$$
r_{i,j} : D^{i,j}_{m} \to D^{i}_{m}.
$$
Let $L_i^j$ be the diagram obtained by adding all the morphisms $r_{i,j}$ to the diagram $K_i^j$, where $1 \leq j \leq m$ and $v \in U_m^{i,j}$. Note that, in particular, $K_i^j$ is a subdiagram in $L_i^m$. Evidently, $D_i^j$ can be considered also as the colimit of this enriched diagram $L_i^j$:

\[ D_i^j = \text{colim } L_i^j. \]

Moreover, any canonical morphism $r_{i,j}, v \in U_i^{j,m}$, can be factored through $D_i^j$ for some $j$:

\[ r_{i,j} = r_{i,j} \circ r_{i,j}. \]

It means that, speaking informally, we “glue” the colimit $D_i^j$ out of the partial colimits $D_i^j$ along the whole diagram $K_i^j$.

In particular, $V_0^0 = \{0\}$, whence $D_m^0 = X(m)$, and $V_m^0 = V_m^{m-1,j}$ for any $j$. We also need the objects

\[ E_i^{j,j} = \bigvee_{v \in V_i^{j,j}} E_v. \]

In order to compute $D_i^{j+1} / D_m^i$ we will consider two cases separately: when $i$ is less or equal to $m - 2$ and when $i = m - 1$. By the induction hypothesis we have the cofiber distinguished triangle

\[ D_i^j \langle w_{m-1,i} \rangle \to D_i^{j+1} \to E_i^{j+1} \to \Sigma D_i^{j+1}. \]

Smashing with $X$ we get the cofiber distinguished triangle

\[ D_i^j \langle w_{m-1,i} \rangle \times X \to D_i^{j+1} \times X \to E_i^{j+1} \times X \to \Sigma D_i^{j+1} \times X. \]

Permuting factors in the smash-products in $K_i^{m-1} \times X$, $K_i^{m-1} \times X$ and $E_i^{j+1} \times X$ we obtain the distinguished triangle

\[ D_i^j \langle w_{m,j} \rangle \to D_i^{j+1} \to E_i^{j+1} \to \Sigma D_i^{j+1}. \]

For any object $A$ in the diagram $K_m^i$ let

\[ r_A : A \to D_i^j. \]
be the canonical morphism into the colimit $D^i_m$. If $f : A \to B$ is a morphism from the diagram $K^{i+1}_m$, such that $A$ belongs to the subdiagram $K^i_m$, then we have the commutative square

$$
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow r_A & & \downarrow r_B \\
D^i_m & \rightarrow & D^{i+1}_m
\end{array}
$$

The morphisms $r_A$ and $r_B$ induce the morphism on cones in the model category $C$:

$$r_f : \text{cone}(f) \rightarrow D^{i+1}_m / D^i_m.$$

If $r_B$ can be factored through $D^{i+1,j}_m$, the morphism $r_A$ can be factored through $D^{i,j}_m$, so that we have the commutative diagram

$$
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
D^{i,j}_m & \rightarrow & D^{i+1,j}_m \\
\downarrow & & \downarrow \\
D^i_m & \rightarrow & D^{i+1}_m
\end{array}
$$

Since we work in the model category $C$, the cone-construction is functorial (see Section 2.2). Consequently, one has the corresponding commutative diagram on cones:

$$
\begin{array}{ccc}
\text{cone}(f) & \rightarrow & E^{i+1,j}_m \\
\downarrow & & \downarrow \\
D^{i+1,j}_m / D^i_m
\end{array}
$$
Of course, if \( f^\prime : A^\prime \to B^\prime \) is another one morphism from the diagram \( K_m^{i+1} \), such that \( A^\prime \) is in \( K_m^i \), and if we are given a commutative square

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow t_A & & \downarrow t_B \\
A^\prime & \xrightarrow{f^\prime} & B^\prime
\end{array}
\]

then the pair \( t = (t_A, t_B) \) induces the morphism on cones:

\[
e_t : \text{cone}(f) \longrightarrow \text{cone}(f^\prime).
\]

Let us note again that, since cones in \( C \) are represented as colimits, the morphisms \( e_t \) are compatible with the morphisms \( e_{f,j} \). Let then \( F_m^{i+1} \) be the diagram generated by all possible morphisms \( e_{f,j} \) and \( e_t \). Then

\[
D_m^{i+1} / D_m^i = \text{colim} F_m^{i+1}
\]

with canonical morphisms given by \( e_f \) and \( e_{i+1,j} \). On the other hand,

\[
\text{colim} F_m^{i+1} = E_m^{i+1},
\]

whence

\[
D_m^{i+1} / D_m^i = E_m^{i+1}.
\]

Let us see what is going on when, for example, \( m = 2 \) and \( i = 0 \). The diagram \( K_2^{0,1} = K_2^{0,2} \) consists of only one object \( X \land X \), the diagram \( K_2^{1,1} \) is just the morphism \( 1_X \land f \), and the diagram \( K_2^{1,2} \) is the morphism \( f \land 1_X \). The diagram \( L_2^0 \) coincides with \( K_2^0 \), \( L_2^1 \) coincides with \( K_1^1 \), and \( D_2^1 \) is the colimit

\[
\text{colim} K_2^1 = (Y \land X) \coprod_{X \land X} (X \land Y).
\]

Since \( \text{cone}(1_X \land f) = X \land Z \) and \( \text{cone}(f \land 1_X) = Z \land X \), we have that \( E_2^{1,1} = X \land Z \) and \( E_2^{1,2} = Z \land X \). These two objects \( E_2^{1,1} \) and \( E_2^{1,2} \) are connected by the cone of the identity morphism \( X \land X \to X \land X \) only. In other words, \( D_2^1 / D_2^0 \) is the colimit of the diagram

\[
X \land Z = E_2^{1,1} \xrightarrow{e_{1_X \land X}^1} 0 \xrightarrow{e_{1_X \land X}^2} E_2^{1,2} = Z \land X
\]

So, we get

\[
D_2^1 / D_2^0 = (Z \land X) \lor (X \land Z) = E_2^1.
\]
If \( i = m - 1 \) we cannot use the inductive hypothesis fixing places of the object \( X \) in vertices of the diagrams \( K^{m-1}_m \) and \( K^m_m \) because the expression \( K^{m-1}_m \) does not make sense. But we can fix places of the object \( Y \). So, in the case \( i = m - 1 \) we use essentially the same arguments, but with slightly different diagrams. To be more precise, for any \( 1 \leq j \leq m \) let \( \tilde{D}^{m-1}_{m,j} \) be the colimit of the subdiagram in \( K^{m-1}_m \), generated by the vertices \( v \in V^{m-1}_m \) with \( v_j = 1 \). By the induction hypothesis one has the cofiber distinguished triangle

\[
D^{m-2}_{m-1} \xrightarrow{w_{m-1,m-2}} Y^{(m-1)} \xrightarrow{} Z^{(m-1)} \xrightarrow{} \Sigma D^{m-2}_{m-1}.
\]

After smashing it with \( Y \) and permutation of factors we get the triangle

\[
\tilde{D}^{m-1}_{m,j} \xrightarrow{w_{m,m-1,j}} Y^{(m)} \xrightarrow{} \tilde{E}^{m,j}_m \xrightarrow{} \Sigma \tilde{E}^{m-1,j}_m,
\]

where \( \tilde{E}^{m,j}_m \) is the product \( Z \wedge \cdots \wedge Z \wedge Y \wedge Z \wedge \cdots \wedge Z \) with \( Y \) on \( j \)th place.

Now, again, if \( f : A \to B \) is a morphism from the diagram \( K^m_m \), such that \( A \) belongs to the subdiagram \( K^{m-1}_m \), then we have the commutative square

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{r_A} & & \downarrow{r_B} \\
D^{m-1}_m & \xrightarrow{w_{m,m-1}} & Y^{(m)}
\end{array}
\]

The morphisms \( r_A \) and \( r_B \) induce the morphism on cones \( r_f : \text{cone}(f) \to Y^{(m)} / D^{m-1}_m \). If \( r_A \) can be factored through \( \tilde{D}^{m-1}_{m,j} \), we have the commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{\tilde{r}_{A,j}} & & \downarrow{r_B} \\
\tilde{D}^{m-1}_{m,j} & \xrightarrow{\tilde{w}_{m,m-1,j}} & Y^{(m)} \\
\downarrow{r_{A,j}} & & \downarrow{r_B} \\
D^{m-1}_m & \xrightarrow{w_{m,m-1}} & Y^{(m)}
\end{array}
\]
This diagram gives rise to the corresponding diagram on cones:

\[ \text{cone}(f) \]  
\[ \tilde{e}_{f,j} \]  
\[ e_f \]  
\[ \tilde{E}_{m,j} \]  
\[ \tilde{e}_{m,j} \]  
\[ Y^m / D^{m-1}_m \]

If \( f' : A' \to B' \) is another one morphism of the above type, and if we have a commutative square

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{t_A} & & \downarrow{t_B} \\
A' & \xrightarrow{f'} & B'
\end{array}
\]

then, as above, the pair \( t = (t_A, t_B) \) induces the morphism on cones \( e_t : \text{cone}(f) \to \text{cone}(f') \). Then,

\[
Y^m / D^{m-1}_m = \text{colim} \tilde{F}_m^m,
\]

where \( \tilde{F}_m^m \) is the diagram generated by morphisms \( \tilde{e}_{f,j} \) and \( e_t \). The canonical morphisms are given by \( e_f \) and \( \tilde{e}_{m,j} \). On the other hand, \( \text{colim} \tilde{F}_m^m = Z^m(m) \). Thus, we get:

\[
Y^m / D^{m-1}_m = Z^m(m).
\]

For example, when \( m = 2 \) and \( i = 1 \) we have that \( D^1_2 = (Y \wedge X) \coprod_{X \wedge X} (X \wedge Y) \), and the cone of the morphism \( w_{2,1} : D^1_2 \to Y^{(2)} \) is the colimit of the diagram

\[
Z \wedge Y = \tilde{E}^{2,2}_2 \xrightarrow{\tilde{e}_{f \wedge f,2}} \text{cone}(f \wedge f) \xrightarrow{\tilde{e}_{f \wedge f,1}} \tilde{E}^{2,1}_2 = Y \wedge Z.
\]

But this colimit is equal to \( Z^{(2)} \). □

3.2. Mixed idempotents

For any vertex \( \nu = (v_1, \ldots, v_m) \) in \( V_m \) and any permutation \( \sigma \in \Sigma_m \), let

\[
\sigma(\nu) = (v_{\sigma(1)}, \ldots, v_{\sigma(m)}),
\]
and let
\[ \Gamma_{\sigma, v}: D_v \longrightarrow D_{\sigma(v)} \]
be the isomorphism permuting factors according to \( \sigma \) and the commutativity and associativity constraints in \( C \). Let also
\[ \Gamma_{\sigma, i}: D_m^i \longrightarrow D_m^i \]
be the morphism on the colimits induced by all the maps \( \Gamma_{\sigma, v} \) with fixed \( \sigma \). Then, for any \( i \in \{1, \ldots, m\} \), we have the following commutative diagram:
\[
\begin{array}{ccc}
D_m^i & \xrightarrow{w_{m,i}} & D_m^{i+1} \\
\downarrow{\Gamma_{\sigma, i}} & & \downarrow{\Gamma_{\sigma, i+1}} \\
D_m^i & \xrightarrow{w_{m,i}} & D_m^{i+1}
\end{array}
\]
In the same fashion, any permutation \( \sigma \) induces the corresponding map on \( E_m^i \):
\[ \Xi_{\sigma, i}: E_m^i \longrightarrow E_m^i. \]
Then, for any \( i \in \{1, \ldots, m\} \), we have the morphism of cofibered sequences:
\[
\begin{array}{ccc}
D_m^i & \xrightarrow{w_{m,i}} & D_m^{i+1} \\
\downarrow{\Gamma_{\sigma, i}} & & \downarrow{\Gamma_{\sigma, i+1}} \\
D_m^i & \xrightarrow{w_{m,i}} & D_m^{i+1}
\end{array}
\]
Applying Lemma 5, we claim that for any \( i \) the triple \((\Gamma_{\sigma, i}, \Gamma_{\sigma, i+1}, \Xi_{\sigma, i+1})\) is, in fact, an automorphism of the distinguished triangle from Proposition 10.
From now on we assume that \( T \) is \( \mathbb{Q} \)-linear. Let
\[
d_{m, i}^+ = \frac{1}{m!} \sum_{\sigma \in \Sigma_m} \text{sgn}(\sigma) \Gamma_{\sigma, i}, \quad d_{m, i}^- = \frac{1}{m!} \sum_{\sigma \in \Sigma_m} \Gamma_{\sigma, i},
\]
\[
e_{m, i}^+ = \frac{1}{m!} \sum_{\sigma \in \Sigma_m} \text{sgn}(\sigma) \Xi_{\sigma, i} \quad \text{and} \quad e_{m, i}^- = \frac{1}{m!} \sum_{\sigma \in \Sigma_m} \Xi_{\sigma, i}
\]
for any \( 0 \leq i \leq m \). It is not hard to see that all of these maps are idempotents in \( T \). Note that \( d_{m, 0}^+ = d_m^+ \) for the power \( X^{(m)} \), and \( d_{m, m}^+ = d_m^+ \) for \( Y^{(m)} \), where \( d_m^\pm \) are the idempotents defined in Section 2.1. Similarly, \( e_{m, 0}^+ = d_m^\pm \) for \( X^{(m)} \), and \( e_{m, m}^+ = d_m^\pm \) for \( Z^{(m)} \). Therefore we say that \( d_m^\pm \) are pure idempotents and that \( d_{m, i}^\pm \) and \( e_{m, i}^\pm \) are mixed ones.
Summing the vertical maps in the last commutative diagram, we obtain the mixed idempotents of the distinguished triangle

\[
\begin{array}{ccccccc}
D_m^i & \xrightarrow{w_{m,i}} & D_m^{i+1} & \xrightarrow{D_m^{i+1}} & E_m^{i+1} & \xrightarrow{\Sigma D_m^i} & \Sigma D_m^i \\
\downarrow d_{m,i}^- & & \downarrow d_{m,i+1}^- & & \downarrow e_{m,i+1}^+ & & \downarrow \Sigma d_{m,i}^- \\
D_m^i & \xrightarrow{w_{m,i}} & D_m^{i+1} & \xrightarrow{D_m^{i+1}} & E_m^{i+1} & \xrightarrow{\Sigma D_m^i} & \Sigma D_m^i \\
\end{array}
\]

Similarly, summing with the signs \(\text{sgn}(\sigma)\) near the maps \(\Gamma_{\sigma,i}, \Gamma_{\sigma,i+1}\) and \(\Xi_{\sigma,i}\), we obtain the mixed idempotent of the distinguished triangle

\[
\begin{array}{ccccccc}
D_m^i & \xrightarrow{w_{m,i}} & D_m^{i+1} & \xrightarrow{D_m^{i+1}} & E_m^{i+1} & \xrightarrow{\Sigma D_m^i} & \Sigma D_m^i \\
\downarrow d_{m,i}^+ & & \downarrow d_{m,i+1}^+ & & \downarrow e_{m,i+1}^- & & \downarrow \Sigma d_{m,i}^+ \\
D_m^i & \xrightarrow{w_{m,i}} & D_m^{i+1} & \xrightarrow{D_m^{i+1}} & E_m^{i+1} & \xrightarrow{\Sigma D_m^i} & \Sigma D_m^i \\
\end{array}
\]

We will denote both of them by the common symbol \((d_{m,i}^\pm, d_{m,i+1}^\pm, e_{m,i+1}^\pm)\).

**Lemma 11.** Let \(X\) be a pseudoabelian triangulated category and let

\[
\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\
\downarrow a & & \downarrow b & & \downarrow c & & \downarrow \Sigma a \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\
\end{array}
\]

be an endomorphism of a distinguished triangle in \(X\). Assume that \(a, b\) and \(c\) are idempotents, and let \(f', g'\) and \(h'\) be the morphisms induced on their images by the morphisms \(f, g\) and \(h\), respectively. Then the triangle

\[
\text{im}(a) \xrightarrow{f'} \text{im}(b) \xrightarrow{g'} \text{im}(c) \xrightarrow{h'} \Sigma \text{im}(a)
\]

is distinguished in \(X\).

**Proof.** The chain of morphisms

\[
\text{im}(a) \xrightarrow{f'} \text{im}(b) \xrightarrow{g'} \text{im}(c) \xrightarrow{h'} \Sigma \text{im}(a)
\]

is a candidate triangle in \(X\), [13, Definition 1.1.1]. By symmetry we also have the candidate triangle

\[
\text{im}(1_X - a) \xrightarrow{f''} \text{im}(1_X - b) \xrightarrow{g''} \text{im}(1_X - c) \xrightarrow{h''} \Sigma \text{im}(1_X - a).
\]
At the same time, the distinguished triangle $XYZ$ is a direct sum of these two candidate triangles. Therefore, both triangles are distinguished, [13, Proposition 1.2.3]. □

**Proposition 12.** For each $m$ and $i$, there exists a distinguished triangle

$$I_{m,i}^\pm \xrightarrow{w_{m,i}^\pm} I_{m,i+1}^\pm \rightarrow J_{m,i+1}^\pm \rightarrow \Sigma I_{m,i}^\pm$$

in the category $\mathcal{T}$, where $I_{m,i}^\pm = \text{im}(d_{m,i}^\pm)$, $J_{m,i}^\pm = \text{im}(e_{m,i}^\pm)$, and $w_{m,i}^\pm$ is induced on images by the morphism $w_{m,i}$.

**Proof.** Apply Lemma 11 to $(d_{m,i}^\pm, d_{m,i+1}^\pm, e_{m,i+1}^\pm)$. □

In order to compute $J_{m,i}^\pm$ in terms of the objects $X$ and $Z$ we need the following lemma.

**Lemma 13.** Let $X$ be a $\mathbb{Q}$-linear pseudoabelian category, and let

$$A \xrightarrow{u} B \xrightarrow{d} A$$

be a diagram in $X$, commutative up to a scalar $\alpha$, i.e., $\alpha a = dbu$ for some $\alpha \in \mathbb{Q}$. Assume, furthermore, that $b^2 = b$ and $ud = \alpha 1_B$. Then $a^2 = a$, $ua = bu$, $ad = db$ and the induced morphism $\iota' : \text{im}(a) \to \text{im}(b)$ is an isomorphism.

**Proof.** Indeed, since $\alpha a = dbu$ and $ud = \alpha$, it follows that $\alpha ua = \alpha bu$, whence $ua = bu$. Similarly, multiplying $\alpha a = dbu$ on $d$ from the right, we have that $\alpha ad = dbud$. Since $ud = \alpha$, we get $\alpha ad = dbu$, whence $ad = db$.

Furthermore, $a^2 = \alpha^{-2} (dbu)(dbu) = \alpha^{-2} db(ud)bu = \alpha^{-2} db \alpha bu = \alpha^{-1} dbu$. Since $b^2 = b$ by assumption, we get $a^2 = \alpha^{-1} dbu = a$.

Now let us consider the commutative diagram

$$
\begin{array}{ccccccc}
B & \xrightarrow{d} & A & \xrightarrow{u} & B & \xrightarrow{d} & A \\
\downarrow{\pi_B} & & \downarrow{\pi_A} & & \downarrow{\pi_B} & & \downarrow{\pi_A} \\
I_B & \xrightarrow{l_B} & I_A & \xrightarrow{\iota_A} & I_B & \xrightarrow{l_B} & I_A \\
\downarrow{\iota_B} & & \downarrow{\iota_A} & & \downarrow{\iota_B} & & \downarrow{\iota_A} \\
B & \xrightarrow{d} & A & \xrightarrow{u} & B & \xrightarrow{d} & A
\end{array}
$$

where the columns are splittings of the idempotents $a$ and $b$, $I_A = \text{im}(a)$, $I_B = \text{im}(b)$, etc. An easy diagram chase shows that $\iota_B l_A l_B \pi_B = bud = \alpha l_B \pi_B$, whence $\iota_B (\alpha^{-1} l_A l_B) \pi_B = \ldots$. \n
$t_B \pi_B$. Since $t_B$ is a right inverse for $\pi_B$, it follows that $\alpha^{-1} I_u I_d = 1_B$. Finally, $I_d I_u = (\pi_A d I_B)(\pi_B u I_A) = \pi_A d (t_B \pi_B) u I_A = \pi_A (\alpha a) I_A = \alpha \pi_A a I_A = \alpha \pi_A I_A \pi_A I_A = \alpha$, whence $\alpha^{-1} I_d I_u = 1_A$. □

**Proposition 14.** The images $J_{m,i}^\pm$ of the idempotents $e_{m,i}^\pm$ can be computed by the following formulas:

$$J_{m,i}^+ \cong Z^{(i)} \otimes X^{(m-i)}, \quad J_{m,i}^- \cong Z^{(i)} \otimes X^{(m-i)}.$$ 

**Proof.** Embed $\Sigma_i \times \Sigma_{m-i}$ into $\Sigma_m$ in the standard way, i.e., $\Sigma_i \times \Sigma_{m-i}$ is the subgroup in $\Sigma_m$ consisting of permutations $\sigma \times \tau$, where $\sigma$ acts on the set $\{1, \ldots, i\}$ and $\tau$ acts on $\{i+1, \ldots, m\}$.

Recall that $T = Ho(C)$ is a triangulated category (see Section 2.2). In particular, it is an additive category. Therefore, finite direct products and finite direct sums agree in $T$. Since $E^i_m = \bigoplus_{v \in U^i_m} E_v$, it follows that for each $v \in U^i_m$ there exists the canonical embedding

$$i_v : E_v \longrightarrow E^i_m$$

and the canonical projection

$$\pi_v : E^i_m \longrightarrow E_v$$

corresponding to the vertex $v$. Fix a permutation $\varsigma_v \in \Sigma_m$, such that

$$\varsigma_v(v) = (1, \ldots, 1, 0, \ldots, 0),$$

where the units are placed on the first $i$ places, and the zeros are on the remaining $m-i$ places. The isomorphisms

$$\Gamma_{\varsigma_v} : E_v \cong Z^{(i)} \otimes X^{(m-i)},$$

permuting factors by $\varsigma_v$ according to the commutativity and associativity constraints in $T$, induce the universal map

$$u^+ : E^i_m \longrightarrow Z^{(i)} \otimes X^{(m-i)},$$

such that $u^+ \circ i_v = \Gamma_{\varsigma_v}$. Note that for any permutation $\sigma$ the morphism $\Gamma_{\sigma}$ is an isomorphism, and

$$\Gamma_{\sigma}^{-1} = \Gamma_{\sigma^{-1}}.$$ 

The inverse isomorphisms

$$\Gamma_{\varsigma_v}^{-1} : Z^{(i)} \otimes X^{(m-i)} \cong E_v$$
induce the universal map
\[ d^- : Z^{(i)} \otimes X^{(m-i)} \longrightarrow E^i_m, \]
such that \( \pi_v \circ d^- = \Gamma_{\varepsilon \varepsilon}^{-1} \). Since \( \Gamma_{\varepsilon \varepsilon} \) is a morphism in the additive category \( T \), one can consider also the isomorphisms
\[ \tilde{\Gamma}_{\varepsilon \varepsilon} = \text{sgn}(\varepsilon_{\varepsilon}) \cdot \Gamma_{\varepsilon \varepsilon}. \]
These isomorphisms \( \tilde{\Gamma}_{\varepsilon \varepsilon} \) induce the universal map
\[ u^+ : E^i_m \longrightarrow Z^{(i)} \otimes X^{(m-i)}, \]
where \( u^+ \circ \iota_v = \tilde{\Gamma}_{\varepsilon \varepsilon} \), and the inverse isomorphisms
\[ \tilde{\Gamma}_{\varepsilon \varepsilon}^{-1} = \text{sgn}(\varepsilon_{\varepsilon}) \cdot \Gamma_{\varepsilon \varepsilon}^{-1} \]
yield the universal map
\[ d^+ : Z^{(i)} \otimes X^{(m-i)} \longrightarrow E^i_m \]
with \( \pi_v \circ d^+ = \tilde{\Gamma}_{\varepsilon \varepsilon}^{-1} \).

For any \( \sigma \in \Sigma_m \) and any \( v \in U^i_m \) the permutation
\[ \varepsilon_{\sigma(v)} \varepsilon_{\varepsilon}^{-1} \]
is in \( \Sigma_i \times \Sigma_{m-1} \). Let \( \sigma_v \in \Sigma_i \) and \( \sigma'_v \in \Sigma_{m-1} \) be the two uniquely defined permutations, such that
\[ \sigma = \varepsilon_{\sigma(v)}^{-1} \cdot (\sigma_v \times \sigma'_v) \cdot \varepsilon_v. \]
Applying \( \Gamma \) to this decomposition of \( \sigma \) we get the commutative diagram
\[
\begin{array}{ccc}
E_v & \xrightarrow{\Gamma_{\varepsilon \varepsilon} \varepsilon} & Z^{(i)} \otimes X^{(m-i)} \\
\downarrow{\Gamma_{\sigma}} & & \downarrow{\Gamma_{\varepsilon \varepsilon} \otimes \Gamma_{\varepsilon \varepsilon} \varepsilon} \\
E_{\sigma(v)} & \xleftarrow{\Gamma_{\varepsilon \varepsilon}^{-1} \varepsilon_{\sigma(v)}} & Z^{(i)} \otimes X^{(m-i)}
\end{array}
\]
Analogously, one can get the commutative diagram

\[
\begin{array}{ccc}
E_{\uparrow \sigma} & \xrightarrow{\tilde{\Gamma}_{\downarrow \sigma}} & Z(i) \otimes X^{(m-i)} \\
\downarrow \Gamma_{\sigma} & & \downarrow \Gamma_{\sigma_{\uparrow \sigma}} \otimes \Gamma_{\sigma'_{\downarrow \sigma}} \\
E_{\sigma(\downarrow \sigma)} & \xleftarrow{\tilde{\Gamma}_{\downarrow \sigma}^{-1}} & Z(i) \otimes X^{(m-i)}
\end{array}
\]

where \( \tilde{\Gamma}_{\sigma} = \text{sgn}(\sigma) \cdot \Gamma_{\sigma} \), \( \tilde{\Gamma}_{\sigma_{\uparrow \sigma}} = \text{sgn}(\sigma_{\uparrow \sigma}) \cdot \Gamma_{\sigma_{\uparrow \sigma}} \) and \( \tilde{\Gamma}_{\sigma'_{\downarrow \sigma}} = \text{sgn}(\sigma'_{\downarrow \sigma}) \cdot \Gamma_{\sigma'_{\downarrow \sigma}} \).

Now let

\[
\hat{d}_{i}^{\pm} = i!d_{i}^{\pm}, \quad \hat{d}_{m-i}^{\pm} = (m-i)!d_{m-i}^{\pm}, \quad \hat{e}_{m,i}^{\pm} = m!e_{m,i}^{\pm}.
\]

Since \( \{\varsigma_{L}^{-1}\}_{t \in U_{m}^{i}} \) is a set of representatives of the left cosets of the group \( \Sigma_{m} \) modulo \( \Sigma_{i} \times \Sigma_{m-i} \), for any vertex \( \uparrow \sigma \in U_{m}^{i} \) one has

\[
\Sigma_{m} = \Sigma_{m}\varsigma_{\uparrow \sigma} = \bigcup_{t \in U_{m}^{i}} (\varsigma_{L}^{-1} \cdot (\Sigma_{i} \times \Sigma_{m-i}) \cdot \varsigma_{\uparrow \sigma}).
\]

Therefore, after summing over all the permutations in \( \Sigma_{m} \), the last two commutative diagrams give rise to the commutative diagrams

\[
\begin{array}{ccc}
E_{\uparrow \sigma} & \xrightarrow{\Gamma_{\sigma_{\downarrow \sigma}}} & Z(i) \otimes X^{(m-i)} \\
\downarrow \pi_{L}\hat{d}_{m,i}^{-\uparrow} & & \downarrow \hat{d}_{m,i}^{-\uparrow} \otimes \hat{d}_{m-i}^{-\uparrow} \\
E_{L} & \xleftarrow{\Gamma_{\downarrow \sigma_{\downarrow \sigma}}} & Z(i) \otimes X^{(m-i)}
\end{array}
\]

and

\[
\begin{array}{ccc}
E_{\uparrow \sigma} & \xrightarrow{\tilde{\Gamma}_{\downarrow \sigma_{\downarrow \sigma}}} & Z(i) \otimes X^{(m-i)} \\
\downarrow \pi_{L}\hat{d}_{m,i}^{+\uparrow} & & \downarrow \hat{d}_{m,i}^{+\uparrow} \otimes \hat{d}_{m-i}^{+\uparrow} \\
E_{L} & \xleftarrow{\tilde{\Gamma}_{\downarrow \sigma_{\downarrow \sigma}}} & Z(i) \otimes X^{(m-i)}
\end{array}
\]
respectively. By the universality of the morphisms \( u^\pm \) and \( d^\pm \), we have the commutative diagram

\[
\begin{array}{ccc}
E^i_m & \xrightarrow{u^\pm} & Z^{(i)} \otimes X^{(m-i)} \\
\downarrow e^\pm_{m,i} & & \downarrow d^\pm \otimes d^\pm_{m-i} \\
E^i_m & \xleftarrow{d^\pm} & Z^{(i)} \otimes X^{(m-i)}
\end{array}
\]

It follows that

\[
\left( \begin{array}{c} m \\ i \end{array} \right) e^\pm_{m,i} = d \circ \left( d^\pm_i \otimes d^\pm_{m-i} \right) \circ u.
\]

Moreover, both the compositions \( u^- d^- \) and \( u^+ d^+ \) coincide with the multiplication by \( \left( \begin{array}{c} m \\ i \end{array} \right) \) because the number of the left cosets in \( \Sigma_m \) modulo the subgroup \( \Sigma_i \times \Sigma_{m-i} \) is equal to \( \left( \begin{array}{c} m \\ i \end{array} \right) \) (and it coincides with the number of elements in the set \( U^i_m \)). Now it remains just to apply Lemma 13, and observe that

\[
\text{im}(d^\pm_i \otimes d^\pm_{m-i}) = \text{im}(d^\pm_i) \otimes \text{im}(d^\pm_{m-i}). \quad \square
\]

### 3.3. The proof of Theorem 1

Let \( X \) be an arbitrary triangulated category and let \( X \) be an object in \( X \). The following definition is standard: a finite (increasing) filtration \( F^\bullet \) on \( X \) is a sequence of morphisms

\[
0 \xrightarrow{F^{-1}X} A_0 \xrightarrow{F^0X} A_1 \xrightarrow{F^1X} \cdots \xrightarrow{F^{m-1}X} A_m = X
\]

in \( X \). The graded pieces of such a filtration are defined by the formula

\[
\text{Gr}^F_i X = \text{cone}(F^{i-1} X).
\]

**Proposition 15.** Let \( T = \text{Ho}(\mathbb{C}) \), where \( \mathbb{C} \) is a pointed simplicial model monoidal category, and let

\[
X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X
\]

be a distinguished triangle in \( T \) (see Section 2.2). Then, for any natural number \( m \), there exists a finite increasing filtration \( F^\bullet Y^{[m]} \) on \( Y^{[m]} \) and a finite increasing filtration \( F^\bullet Y^{(m)} \) on \( Y^{(m)} \), such that

\[
\text{Gr}^F_i Y^{[m]} \cong Z^{(i)} \otimes X^{(m-i)} \quad \text{and} \quad \text{Gr}^F_i Y^{(m)} \cong Z^{(i)} \otimes X^{(m-i)}
\]

for any \( i \).
Proof. Put $A_i = \mathcal{I}_{m,i}^\pm$ for any $0 \leq i \leq m$, $F^i = \mathcal{W}_{m,i}^\pm$ for any $0 \leq i \leq m - 1$ and apply Propositions 12 and 14. □

Now we can finish the proof of Theorem 1. Assume that $X$ and $Z$ in the triangle from Theorem 1 are evenly finite-dimensional. This means that $X^{(i)} = 0$ and $Z^{(i)} = 0$ for some natural $t$. Then, for $m \geq 2t + 1$, all the graded pieces $Gr^i_p Y^{(m)} \cong Z^{(i)} \otimes X^{(m-i)}$ of the even filtration are equal to zero, whence $Y^{(m)} = 0$. The odd case is similar.

As to Remark 2, it is just the equivalent reformulation of Theorem 1. To see this, we only need to observe that the shift endofunctor in any $\mathbb{Q}$-linear monoidal triangulated category $X$ carries evenly (respectively, oddly) finite-dimensional objects into oddly (respectively, evenly) finite-dimensional objects. This is because of the axioms encoding the compatibility of the monoidal and the triangulated structures in $X$, see [10, A8].

4. Motives of quasiprojective curves

In this section we prove Theorem 3. The word “curve” means a quasi-projective curve over a field. Let $k$ be a field of characteristic zero. We work in Voevodsky’s triangulated category $\mathcal{DM} = \mathcal{DM}(k) \otimes \mathbb{Q}$ of motives over $k$ with coefficients in $\mathbb{Q}$ and denote the shift functor by $M \mapsto M[1]$. The unit motive in $\mathcal{DM}$ is denoted by $\mathbb{Q}$, and the Lefschetz motive is denoted by $\mathbb{Q}(1)[2]$ (see [17]).

Let $X$ be a curve and assume, for simplicity, that $X$ is integral over $k$ (it will be clear from the below arguments how to extend them to the case when $X$ is reducible). The curve $X$ can be considered as a Zariski open subset in an irreducible projective curve $Y$. Let $p : W \to Y$ be a resolution of singularities of $Y$ and let $U = p^{-1}(X)$, so that we have the commutative square

$$
\begin{array}{ccc}
U & \longrightarrow & W \\
\downarrow p & & \downarrow p \\
X & \longrightarrow & Y
\end{array}
$$

Let also $Z = Y - X$ and $V = W - U$ be the complements of $Y$ and $U$ in the projective curves $X$ and $W$, respectively. By Lemma 8 we may assume that $\sqrt{-1} \in k$ and all the data in the above commutative square is rational over $k$.

Lemma 16. Let $X$ be a triangulated category with the shift functor $\Sigma$. Assume that we have a distinguished triangle

$$
\begin{array}{c}
A \oplus B \xrightarrow{(a \ b \ c \ d)} A \oplus C \longrightarrow D \longrightarrow \Sigma(A \oplus B)
\end{array}
$$
in $X$, where $a$ is an automorphism of the object $A$. Then this triangle is isomorphic to the direct sum of the triangles

$$A \xrightarrow{1} A \xrightarrow{} 0 \xrightarrow{} \Sigma A$$

and

$$B \xrightarrow{t} C \xrightarrow{} D \xrightarrow{} \Sigma B,$$

where $t = d - ca^{-1}b$.

**Proof.** This is just a reformulation of Lemma 1.2.4 from [13].

If the curve $X$ is smooth and projective, then Theorem 3 holds by Kimura’s theorem, see [7, Corollary 4.4]. Otherwise there are three cases:

(a) $X$ is projective and not smooth;
(b) $X$ is not projective and smooth;
(c) $X$ is not projective and not smooth.

For simplicity, we will consider them separately.

(a) $X$ is projective and not smooth. Then $X = Y$ and $p : U \rightarrow X$ is a resolution of singularities of $X$. For simplicity, we will assume that $X$ has only one singular point (the other case can be proved by the same methods, but with more cumbersome formulas). Then $p$ contracts points $\{u_1, \ldots, u_n\}$ onto a singular point in $X$. Let

$$\mathbb{Q}^{\oplus n} \rightarrow \mathbb{Q} \oplus M(U) \rightarrow M(X) \rightarrow \mathbb{Q}^{\oplus n}[1] \quad (1)$$

be the blow up distinguished triangle corresponding to the map $p$, [15, Theorem 5.2], where $\mathbb{Q}^{\oplus n}$ is the motive of the finite set $\{u_1, \ldots, u_n\}$. The composition $\mathbb{Q} \rightarrow \mathbb{Q}^{\oplus n} \rightarrow \mathbb{Q} \oplus M(U) \rightarrow \mathbb{Q}$ induced by the point $u_1$ is an isomorphism. By Lemma 16 the triangle (1) is isomorphic to the direct sum of the distinguished triangles

$$\mathbb{Q}^{\oplus n-1} \xrightarrow{t} M(U) \rightarrow M(X) \rightarrow \mathbb{Q}[1]^{\oplus n-1} \quad (2)$$

and $\mathbb{Q} \rightarrow \mathbb{Q} \rightarrow 0 \rightarrow \mathbb{Q}[1]$. Since $U$ is smooth and projective, we have the decomposition

$$M(U) = \mathbb{Q} \oplus M^1(U) \oplus \mathbb{Q}(1)[2]$$

induced by a $k$-rational point on $U$ different from the points $\{u_1, \ldots, u_n\}$. For any $i$ let $\nu_i : \text{Spec}(k) \rightarrow U$ be the map induced by $u_i$, and let $\gamma : U \rightarrow \text{Spec}(k)$ be the structure map for $U$. If $i > 1$, the composition $\mathbb{Q} \rightarrow \mathbb{Q}^{\oplus n-1} \xrightarrow{t} M(U)$ induced by $u_i$ coincides with the difference $M(\nu_i) - M(\nu_1)$ (here we use the general expression for the morphism $t$ given by Lemma 16). The projection $\mathbb{Q} \oplus M^1(U) \oplus \mathbb{Q}(1)[2] \rightarrow \mathbb{Q}$ is, in fact,
the morphism $M(\gamma) : M(U) \to M(\text{Spec}(k))$. Therefore, for any $u_i$, $i > 1$, the composition $Q \to \mathbb{Q}^\oplus n-1 \to M(U) \to \mathbb{Q}$ coincides with the difference $M(\gamma v_i) - M(\gamma v_1)$, which is equal to zero. In addition, any map from $\mathbb{Q}$ to $\mathbb{Q}(1)[2]$ is zero. This shows that the triangle (2) is a direct sum of two distinguished triangles

$$Q^\oplus n-1 \to M^1(U) \to G \to \mathbb{Q}[1]^\oplus n-1$$

and

$$0 \to \mathbb{Q} \oplus \mathbb{Q}(1)[2] \to \mathbb{Q} \oplus \mathbb{Q}(1)[2] \to 0.$$ 

In particular,

$$M(X) = Q \oplus G \oplus \mathbb{Q}(1)[2].$$

The motive $M^1(U)$ is oddly finite-dimensional by [7, Theorem 4.2]. Then $G$ is oddly finite-dimensional by Theorem 1 and Corollary 9, whence finite dimensionality of $M(X)$ follows.

(b) $X$ is not projective, but smooth. In that case $X = U$, $V = W - U \neq \emptyset$, and we have the canonical distinguished triangle

$$M(V) \to M(W) \to M^c(U) \to M(V)[1],$$

(3)

where $M^c(U)$ is the motive of $U$ “with compact support,” see [17, 4.1].

Let $M(W) = Q \oplus M^1(W) \oplus \mathbb{Q}(1)[2]$ be the decomposition determined by some point of $V$. Splitting the isomorphism induced by this point from the triangle (3) (using Lemma 16) we get the distinguished triangle

$$Q^\oplus n-1 \to M^1(W) \oplus \mathbb{Q}(1)[2] \to M^c(U) \to \mathbb{Q}[1]^\oplus n-1.$$ 

This triangle gives rise to the distinguished triangle

$$Q^\oplus n-1 \to M^1(W) \to N \to \mathbb{Q}[1]^\oplus n-1,$$

(4)

where $N$ is such that

$$M^c(U) \cong N \oplus \mathbb{Q}(1)[2].$$

The motive $M^1(W)$ is oddly finite-dimensional because $W$ is smooth and projective. Applying Theorem 1 and Corollary 9 to (4) we see that $N$ is oddly finite-dimensional. Since $U$ is smooth and of pure dimension one,

$$M(U) \cong N^*(1)[2] \oplus \mathbb{Q}$$
by [17, Theorem 4.3.7(3)], where $N^*$ is the motive dual to $N$. Recall that $\text{DM}$ is rigid and that the dualization is a tensor endofunctor on $\text{DM}$, whence $N^*$ is oddly finite-dimensional because $N$ is so. The motive $N^*(1)[2]$ is oddly finite-dimensional as a product of motives with different parities, see Section 2.1. Thus, $M(U)$ is finite-dimensional.

(c) $X$ is not projective and not smooth. Here we argue similarly to the case (a) with the use of the result from (b). Again, for simplicity, assume that $p : U \to X$ contracts $n$ points in $U$ onto one singular point in $X$. Starting from the blow up distinguished triangle associated with the contraction $p$ and splitting one point by Lemma 16 we get the distinguished triangle

$$\mathbb{Q}^{\oplus n-1} \to M(U) \to M(X) \to \mathbb{Q}[1]^{\oplus n-1}. \quad (5)$$

Since $U$ is smooth and not projective, $M(U) \cong N^*(1)[2] \oplus \mathbb{Q}$ by (b), where $N^*(1)[2]$ is oddly finite-dimensional. If $M(U) = \tilde{M}(U) \oplus \mathbb{Q}$ is the splitting induced by a $k$-rational point of $U$, then $\tilde{M}(U) \cong N^*(1)[2]$ by [1, Proposition 9.1.10], whence $\tilde{M}(U)$ is oddly finite-dimensional. Similarly to the case (a), if the last splitting is induced by a point different from the points contracted by $p$, the composition of $t$ with the projection $\tilde{M}(U) \oplus \mathbb{Q} \to \mathbb{Q}$ is zero. Therefore the triangle (5) gives rise to the distinguished triangle

$$\mathbb{Q}^{\oplus n-1} \to \tilde{M}(U) \to D \to \mathbb{Q}[1]^{\oplus n-1}, \quad (6)$$

where $D \oplus \mathbb{Q} \cong M(X)$. Applying Theorem 1 together with Corollary 9 to (6) we have that $D$ is oddly finite-dimensional. Then $M(X)$ is finite-dimensional as the direct sum of $D$ and $\mathbb{Q}$.

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