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T-colorings of graphs: recent results and open problems

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Abstract

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Suppose G is a graph and T is a set of nonnegative integers. A T-coloring of G is an assignment of a positive integer $f(x)$ to each vertex x of G so that if x and y are joined by an edge of G, then $|f(x)-f(y)|$ is not in T. T-colorings were introduced by Hale in connection with the channel assignment problem in communications. Here, the vertices of G are transmitters, an edge represents interference, $f(x)$ is a television or radio channel assigned to x, and T is a set of disallowed separations for channels assigned to interfering transmitters. One seeks to find a T-coloring which minimizes either the number of different channels $f(x)$ used or the distance between the smallest and largest channel. This paper surveys the results and mentions open problems concerned with T-colorings and their variations and generalizations.

1. Introduction

T-colorings of graphs arose in connection with the *channel assignment problem* in communications. In this problem, there are n transmitters in a region, $x_1, x_2, \ldots, x_n$. We wish to assign to each transmitter x a frequency $f(x)$ over which it can operate. In the simplest version of this problem, but one with many practical applications, $f(x)$ is assumed to be a positive integer. We shall make this assumption throughout. Some transmitters can interfere. The *interference graph* $G=(V, E)$ is defined as follows: $V=\{x_1, x_2, \ldots, x_n\}$, and $\{x_i, x_j\}$ is in E if and only if x_i and x_j interfere. (Interference can be due to geographic proximity, meteorological factors, etc.) There is a set T of nonnegative integers which represents *disallowed separations* between channels which are assigned to transmitters which interfere. It is assumed that 0 belongs to T. The requirement can be summarized by the following equation:

$$\{x, y\} \in E \Rightarrow |f(x)-f(y)| \notin T. \tag{1}$$

A function f from V into the set of positive integers satisfying (1) is called a *T-coloring*.

To illustrate the definition, we note that if $T = \{0\}$, then a T-coloring is just an ordinary graph coloring. Another important case arises when $T = \{0, 1\}$. In this case, if two channels interfere, they get not only different but also non-adjacent channels. Rather unusual sets T can arise in practical channel assignment problems. For instance, in UHF television, sets such as $T = \{0, 7, 14, 15\}$ and $\{0, 1, 2, 3, 4, 5, 7, 8, 14, 15\}$ arise. (See Middlekamp [32] or Pugh et al. [34].) This formulation of the channel assignment problem is due to Hale [21]. Earlier graph-theoretical approaches to this problem were due to Metzger [31] and Zoellner and Beall [50].

The spectrum of radio and television frequencies is becoming increasingly crowded. This implies that it will be more and more necessary to make channel assignments efficiently, so as to conserve spectrum. An entire issue of the Proceedings of the IEEE (Vol. 68, No. 12, December 1980) was devoted to this problem. From a T-coloring point of view, there are two important criteria for efficiency. These involve first the *order* of a T-coloring f, i.e., the number of different colors $f(x)$, and second the *span* of f, i.e., the maximum of $|f(x) - f(y)|$. We can measure the efficiency of a T-coloring f by comparing it to the *T-chromatic number* $\chi_T(G)$, the minimum order of a T-coloring of G, or to the *T-span* $\text{sp}_T(G)$, the minimum span of a T-coloring of G.

To illustrate these ideas, consider the case where $G = K_3$, the complete graph on 3 vertices, and $T = \{0, 1, 4, 5\}$. Suppose we try to color G by being greedy, i.e., by picking the lowest possible channel each time. It is easy to see that in this case, we get a coloring using the channels 1, 3, and 9. This certainly is a minimum order T-coloring. However, by using channels 1, 4, and 7, we must also get a T-coloring, and this coloring has a smaller span: 7-1 as opposed to 9-1. This is in fact the minimum span. As pointed out by Hale [21], it is possible to give examples where no optimal order T-coloring gives an optimal span and vice versa. For instance, if G is the 5-cycle and $T = \{0, 1, 4, 5\}$, then $\chi_T(G) = 3$ and $\text{sp}_T(G) = 4$, and optimal-order and optimal-span T-colorings are shown in Fig. 1.

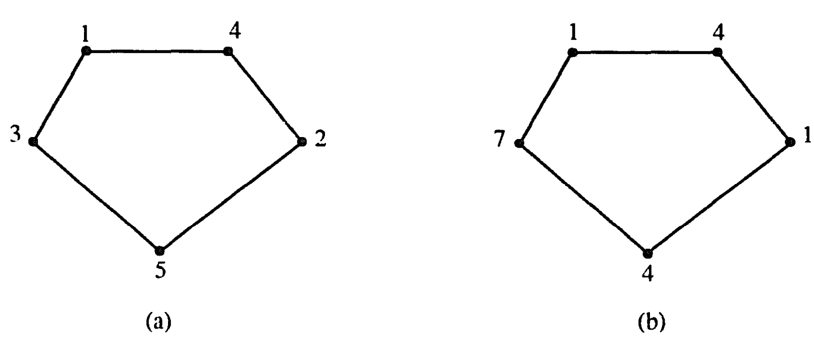

Fig. 1. A minimum span assignment (a) and a minimum order assignment (b) for the 5-cycle if $T = \{0, 1, 4, 5\}$.

In general, the problems of computing χ_T and sp_T, and of finding optimal-order and -span T-colorings, are NP-complete. The easiest way to see this is to observe that if $T=\{0\}$, then $\chi_T(G)=\chi(G)$, the ordinary chromatic number, and $\mathrm{sp}_T(G)=\chi(G)-1$. (Just color using the channels $1,2,\ldots,\chi$.) While the general problems are hard, considerable progress has been made on special versions of these problems. In particular, progress has been made under special assumptions about the graph G or about the nature of the set T, or both.

In this paper, we survey the results and mention some open questions on the T-coloring problem. We start with some general results in the next section. Then we talk about results under special assumptions about the set T in Section 3 and about the graph G in Section 4. In subsequent sections, we talk briefly about other aspects of the T-coloring problem, namely edge span, optimal T-colorings under restrictions on order or span, channel assignments when there are several levels of interference, set T-colorings, list T-colorings, and no-hole T-colorings.

The notion of T-coloring has given rise to a large literature in recent years. It has given rise to parts of five Ph.D. Theses, by Wang [48], Raychaudhuri [36], Tesman [45], Bonias [7a] and Liu [29a]. Tesman [45] gives a comprehensive survey of the literature of T-colorings. Other references on the subject include Baybars [1], Berry and Cronin [2–3], Cozzens and Roberts [12–12a], Cozzens and Wang [13], Füredi, Griggs, and Kleitman [16], Hale [21–24], Lanfear [28–29], Metzger [31], Rabinowitz and Proulx [35], Raychaudhuri [37], Roberts [40–41], Sakai and Wang [42], Smith [43], Zoellner [49], and Zoellner and Beall [50].

2. A few general results

Following the graph-theoretical formulation of the T-coloring problem by Hale [21], some basic results were obtained by Cozzens and Roberts [12]. They can be summarized in the following theorems.

Theorem 1 (Cozzens and Roberts [12]). *For all graphs G and sets T, $\chi_T(G)=\chi(G)$.*

Thus, χ_T is not a new number. The emphasis in the literature has been on sp_T.

Theorem 2 (Cozzens and Roberts [12]). *For all graphs G and sets T,*

$$\chi(G)-1\leq \mathrm{sp}_T(G)\leq (r+1)(\chi(G)-1),$$

where $r=\max T$.

The upper bound has recently been improved by Tesman [45–46] to

$$\mathrm{sp}_T(G)\leq t(\chi(G)-1),$$

where $t=|T|$.

Theorem 3 (Cozzens and Roberts [12]). *For all graphs G and sets T,*

$$\mathrm{sp}_T(K_{\omega(G)}) \leq \mathrm{sp}_T(G) \leq \mathrm{sp}_T(K_{\chi(G)}),$$

where $\omega(G)$ is the size of the largest clique of G.

It follows from this result that if $\chi(G) = \omega(G)$, i.e., if G is weakly γ-perfect, then $\mathrm{sp}_T(G)$ can be computed by computing the T-span of an appropriate complete graph. We shall say more about T-spans of complete graphs in Section 4.

Cozzens and Roberts [12] also observed that the following *greedy algorithm* sometimes gives us $\mathrm{sp}_T(G)$. Order V as $x_1, x_2, \ldots, x_n$. Pick $f(x_1) = 1$. Having assigned $f(x_1), f(x_2), \ldots, f(x_k)$, let $f(x_{k+1})$ be the smallest channel so that $f(x_1), f(x_2), \ldots, f(x_{k+1})$ do not violate the T-coloring requirements. One of the general questions of interest in the theory of T-colorings is to determine for what graphs and what sets T the greedy algorithm actually gives an f of optimal span. In later sections, we shall discuss some results on this question.

3. Results under special assumptions about T

In their 1982 paper, Cozzens and Roberts obtained results for T-colorings when T is an *r-initial set*, a set of the form

$$T = \{0, 1, \ldots, r\} \cup S,$$

where S contains no multiple of $r + 1$. In particular, they showed the following.

Theorem 4 (Cozzens and Roberts [12]). *If T is r-initial, then for all graphs G,* $\mathrm{sp}_T(G) = \mathrm{sp}_T(K_{\chi(G)}) = (r + 1)(\chi(G) - 1)$.

Raychaudhuri [36–37] showed that the first equality in Theorem 4 also holds for *k multiple of s sets*, sets of the form

$$T = \{0, s, 2s, \ldots, ks\} \cup S,$$

where $s \geq 1$, $k \geq 1$, and S is a subset of $\{s + 1, s + 2, \ldots, ks - 1\}$. (The idea of considering such sets was due to D. de Werra.)

Theorem 5 (Raychaudhuri [36–37]). *If T is a k multiple of s set, then for all graphs G,*

$$\mathrm{sp}_T(G) = \mathrm{sp}_T(K_{\chi(G)}) = \begin{cases} st + skt - sk - 1 & \text{if } \chi(G) = st, \\ st + skt + m - 1 & \text{if } \chi(G) = st + m, \quad 1 \leq m \leq s - 1. \end{cases}$$

Cozzens and Wang [13] and Wang [48] have studied T-sets of the form

$$T = \{0, 1, \ldots, r\} \cup \{p(r + 1)\} \cup S,$$

where all elements of S are larger than $p(r+1)$. They calculate $\mathrm{sp}_T(K_q)$ for some sets of this form. (Recall that in many cases, the problem of computing $\mathrm{sp}_T(G)$ reduces to the problem of computing $\mathrm{sp}_T(K_q)$ for $q = \chi(G)$.) Details on these results will be included in the next section. Tesman [45] has studied T-sets arising in part from UHF television problem, for example $T = \{0, r, 2r, 2r+1\}$, $T = \{0, r, r+1, \ldots, kr+l\}$, and $T = \{0, r, r+1, 2r+1\}$. Other sets are studied in [7a], [12a], [29a].

4. Results under special assumptions about G

We have already pointed out that the following result is an immediate corollary of Theorem 3.

Theorem 6 (Cozzens and Roberts [12]). *If G is weakly γ-perfect, then for all sets T, $\mathrm{sp}_T(G) = \mathrm{sp}_T(K_{\chi(G)})$.*

Of course, Theorem 6 is not particularly useful without first being able to compute $\chi(G)$. If G is a perfect graph, this can be done in $O(n^2)$ steps (cf. Grötschel et al. [20]). Most of the results about T-colorings for special graphs have been obtained for special kinds of perfect graphs. We discuss results for indifference graphs, chordal graphs, and perfectly orderable graphs. Then we summarize what is known for complete graphs and describe other important classes of graphs for which nothing is known.

Indifference graphs are graphs for which there exists a real-valued function u on the set of vertices so that

$$\{x, y\} \in E \iff |u(x) - u(y)| \leq d,$$

where d is a fixed positive number. Such graphs arise if transmitters are lined up along a corridor and two transmitters interfere if and only if they are at most d miles apart. For references on indifference graphs and their applications and for definitions of terms not defined here, see Roberts [38].

Theorem 7 (Cozzens and Roberts [12]). *Suppose G is an indifference graph and T is any set. Then:*

(1) *For a special vertex ordering called compatible (which always exists), the greedy algorithm gives a T-coloring of χ_T colors.*

(2) *Moreover, if T is r-initial, the algorithm also gives a T-coloring of span equal to sp_T.*

(3) *The algorithm has complexity $O(n^2 t)$ given the ordering, where $n = |V|$ and $t = |T|$. If T is r-initial, the algorithm has complexity $O(n^2)$.*

(4) *The compatible ordering can be found in $O(n^2)$ steps and hence if T is r-initial, optimal-order and optimal-span T-colorings can be found in $O(n^2)$ steps.*

Note that the greedy algorithm described in Cozzens and Roberts is slightly different from that defined in Section 2 of this paper. However, it is easy to see that the same result holds for that greedy algorithm. That can readily be proved directly, or as a corollary of the following theorem. Note also that the result in (2) of Theorem 7 is false if T is not r-initial. For instance, if $G = K_3$ and $T = \{0, 1, 4, 5\}$, then any vertex ordering gives a non-optimal coloring using the colors 1, 3, and 9.

A graph is called *chordal* if it does not have a cycle of length $4, 5, \ldots$ as a generated subgraph. See Golumbic [18] for information about chordal graphs and definitions of terms not defined here.

Theorem 8 (Raychaudhuri [36–37]). *Suppose G is a chordal graph and T is any set. Then*:

(1) *For a special vertex ordering, namely the reverse of a perfect elimination ordering (which always exists), the greedy algorithm gives a T-coloring of* χ_T *colors.*

(2) *Moreover, if T is an r-initial or k multiple of s set, the algorithm also gives a T-coloring of span equal to* sp_T.

(3) *The algorithm has complexity* $O(n^2t)$ *given the ordering. If T is an r-initial or k multiple of s set, the algorithm has complexity* $O(n^2)$.

(4) *The reverse of a perfect elimination ordering can be found in* $O(n^2)$ *steps and hence, if T is an r-initial or k multiple of s set, optimal-order and optimal-span T-colorings can be found in* $O(n^2)$ *steps.*

These results generalize those of Theorem 7 since every indifference graph is chordal and every compatible order is the reverse of a perfect elimination ordering.

A graph G is called *perfectly orderable* if the vertices can be ordered so that the orientation of the edges of G induced by this ordering has no generated subgraph of the form shown in Fig. 2. Such a vertex ordering is called *admissible*. Perfectly orderable graphs were introduced by Chvátal [9].

Theorem 9 (Raychaudhuri [36–37]). *Suppose G is a perfectly orderable graph and T is any set. Then*:

(1) *For an admissible ordering of G, the greedy algorithm gives a T-coloring of* χ_T *colors.*

(2) *Moreover, if T is an r-initial or k multiple of s set, the algorithm also gives a T-coloring of span equal to* sp_T.

(3) *The algorithm has complexity* $O(n^2t)$ *given the admissible ordering. If T is an r-initial or k multiple of s set, then the algorithm has complexity* $O(n^2)$.

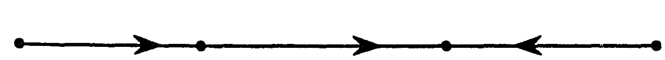

Fig. 2. A forbidden generated subgraph for the orientation of a perfectly orderable graph.

These results generalize those of Theorem 8 since every chordal graph is perfectly orderable and the reverse of a perfect elimination ordering is always admissible. However, admissible orderings may not be easy to find—good algorithms for finding them are unknown and recognizing perfectly orderable graphs has been proven NP-complete in [31a].

As we have already observed, the T-span of complete graphs is fundamental because for many graphs, in particular for the weakly γ-perfect ones, $\mathrm{sp}_T(G) = \mathrm{sp}_T(K_{\chi(G)})$. Hence, it is not surprising that a great deal of emphasis has been placed on computing the T-span of complete graphs. Theorems 4 and 5 give formulas for $\mathrm{sp}_T(K_q)$ in case T is an r-initial or k multiple of s set. Other sets T of the form

$$T = \{0, 1, \ldots, r\} \cup \{p(r+1)\} \cup S \tag{2}$$

are studied by Cozzens and Wang [13] and Wang [48]. Some of the main results are summarized in the following theorem.

Theorem 10 (Wang [48]). *Suppose T is of the form* (2). *Then*:

(1) *If $r \geqslant 2$, $p > 1$, and $S = \{p(r+1)+1\}$, then*

$$\mathrm{sp}_T(K_q) = (q-1)(r+1) + 2\left\lfloor \frac{(q-1)}{p} \right\rfloor.$$

(2) *If $r \geqslant 2$, $p > 1$, and $S = \{p(r+1)+1, \ldots, p(r+1)+r-1\}$, then*

$$\mathrm{sp}_T(K_q) = (q-1)(r+1) + r\left\lfloor \frac{(q-1)}{p} \right\rfloor.$$

(3) *If $r \geqslant 2$, $p > 1$, and $S = \{p(r+1)+1, \ldots, p(r+1)+r\}$, then the result in* (2) *is false. However, if $r \geqslant 1$ and $p > 1$, then for such S,*

$$\mathrm{sp}_T(K_{p+1}) = p(r+1) + (r+1) = (p+1)(r+1).$$

Moreover, if $p = r+1$, then for all $q > p$, $\mathrm{sp}_T(K_q) = (q-1)(p+1)$.

Other results about $\mathrm{sp}_T(K_q)$ are obtained in [45], [7a], [12a], [29a].

In spite of Theorems 4, 5, 10, and related results in the literature, it remains open to compute $\mathrm{sp}_T(K_q)$ even for some relatively simple sets T. For instance $\{0,1,4,5\}$, $\{0,1,4,6\}$, or $\{0,1,4,7\}$ have only recently been handled, in [7a], the former not completely.

Also open is a characterization of sets T and integers q such that the greedy algorithm gives a T-coloring of span equal to $\mathrm{sp}_T(K_q)$. Results on this problem will have wide significance, for we have the following theorem.

Theorem 11. *Suppose G is weakly γ-perfect. Then there is an ordering of G for which the greedy algorithm gives a T-coloring of span equal to $\mathrm{sp}_T(G)$ if and only if (there is an ordering of $K_{\chi(G)}$ for which) the greedy algorithm gives a T-coloring of span equal to $\mathrm{sp}_T(K_{\chi(G)})$.*

Proof. We begin by observing that if f is a greedy T-coloring of a graph G and f uses p colors, then the colors are determined only by T, and not by G. Let us call them

$$1 = a_1 < a_2 < \cdots < a_p. \tag{3}$$

Moreover, a greedy coloring of any graph in $m < p$ colors uses the colors $a_1, a_2, \ldots, a_m$. Suppose now that G is weakly γ-perfect. Suppose there is an ordering o of G so that the greedy algorithm with o gives a T-coloring f of G with span equal to $\mathrm{sp}_T(G)$. Let f use the colors in (3). Since $\chi = \omega$, G has $K_{\chi(G)}$ as a generated subgraph. Hence $p \geq \chi(G) = m$. But the colors $a_1, a_2, \ldots, a_m$ give a greedy coloring of $K_m = K_{\chi(G)}$, and this has span

$$a_m - 1 \leq a_p - 1 = \mathrm{sp}_T(G) = \mathrm{sp}_T(K_{\chi(G)}).$$

It follows that a greedy coloring of $K_{\chi(G)}$ in any order gives a T-coloring of span equal to $\mathrm{sp}_T(K_{\chi(G)})$.

Conversely, suppose o is an ordering of $K_{\chi(G)}$ and f is a greedy coloring obtained using o and f has span equal to $\mathrm{sp}_T(K_{\chi(G)})$. Suppose f uses the colors in (3). Now $p = \chi(G) = m$. We can find an ordinary coloring of G in colors $1, 2, \ldots, m$. The ordering o of G is defined as follows. First list all vertices of color 1, then all vertices of color 2, and so on. The order of vertices of the same color is arbitrary. Now a greedy coloring of G in order o is obtained by coloring all color 1 vertices using color a_1, all color 2 vertices using color $a_2, \ldots$, all color m vertices using color a_m. This coloring has span

$$a_m - 1 = \mathrm{sp}_T(K_{\chi(G)}) = \mathrm{sp}_T(G).$$

Hence, the coloring has span equal to $\mathrm{sp}_T(G)$. □

Note that Theorem 11 can fail if G is not weakly γ-perfect. For instance, if G is the 5-cycle and $T = \{0, 1, 4, 6\}$, then (the only) greedy coloring of G and an optimal-span coloring of G are shown in Fig. 3, and the latter has a smaller span. On the other hand, a greedy coloring of $K_{\chi(G)} = K_3$ uses the colors 1, 3, and 6, and is optimal.

A little while ago, very little was known about what sets T and integers q have the property that the greedy algorithm gives an optimal-span T-coloring

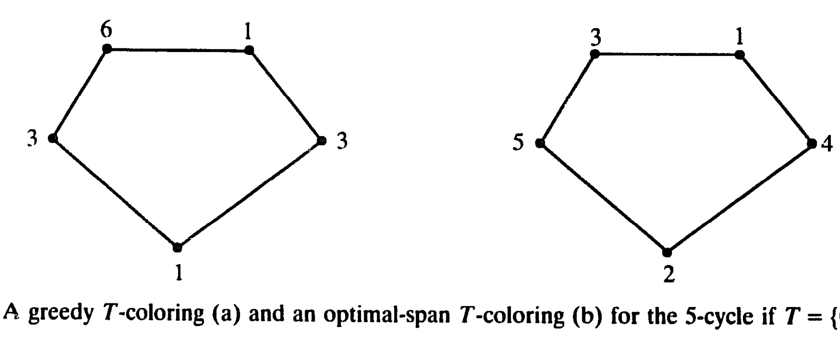

Fig. 3. A greedy T-coloring (a) and an optimal-span T-coloring (b) for the 5-cycle if $T = \{0, 1, 4, 6\}$.

of K_q. However, see [7a], [12a], [29a] for recent major progress. To give simple result, if $q = 3$, it is easy to show the following: the greedy algorithm gives a T-coloring of K_3 of span equal to $\text{sp}_T(K_3)$ if:

(1) 1 and 2 are not in T; or

(2) $T = \{0, 1, \ldots, r\} \cup S$, with $r + 1 \notin S$ and $2r + 2 \notin S$; or

(3) $T = \{0, 2, 4, \ldots, 2k\} \cup S$, with $1 \notin S$, $2k + 1 \notin S$, and $2k + 2 \notin S$.

(In case (2), the coloring uses colors 1, $r + 2$, $2r + 3$. In case (3), it uses colors 1, $2k + 2$, $2k + 3$.)

Before leaving complete graphs, we note that some work has been done on asymptotic behavior. We have the following theorem. (See also [29a].)

Theorem 12 (Rabinowitz and Proulx [35]). *Each T-set has a rate* $\text{rt}(T)$ *and* $\text{sp}_T(K_q)$ *is asymptotic to* $q/\text{rt}(T)$.

Perhaps the most important class of graphs for which to study T-colorings is the class of *2-unit sphere graphs,* the class of graphs for which there exists a function u which assigns to each vertex x a point in Euclidean 2-space so that

$$\{x, y\} \in E \iff d(u(x), u(y)) \leq d,$$

where d is a fixed positive number. Such graphs arise if transmitters lie in the plane and two transmitters interfere if and only if they are at most d miles apart. Such graphs have been studied by Hale [21], Havel [25], Havel, Kuntz, and Crippen [26], Havel et al. [27], and Maehara [30], but very little is known about them. The 2-unit sphere graphs are analogous to the indifference graphs, which arise in the same way if the mapping is into Euclidean 1-space. Orlin (unpublished) shows that computation of even the ordinary chromatic number is NP-complete for 2-unit sphere graphs. Hence, either special assumptions must be made about the 2-unit sphere graphs being studied, or only approximate or heuristic algorithms should be emphasized.

5. Edge span

In a graph G, the *edge span* $\text{esp}_T(G)$ is defined to be the maximum of $|f(x) - f(y)|$ for $\{x, y\}$ an edge of G and f a T-coloring. The concept of bandwidth of a graph, which has a large literature (see Chinn et al. [8] and Chvatalova [10] for surveys), is the same as esp_T where $T = \{0\}$ and T-colorings are assumed to be one-to-one. Basic results about edge span are summarized in the following theorems. See [29a] for recent results on edge span.

Theorem 13 (Cozzens and Roberts [12]). *For all graphs G and sets T,*

(1) $\chi(G) - 1 \leq \text{esp}_T(G) \leq \text{sp}_T(G)$.

(2) $\text{sp}_T(K_{\omega(G)}) \leq \text{esp}_T(G) \leq \text{sp}_T(K_{\chi(G)})$.

This theorem is analogous to Theorems 2 and 3.

Theorem 14 (Cozzens and Roberts [12]). *If G is weakly γ-perfect, then for all sets T,* $\mathrm{esp}_T(G) = \mathrm{sp}_T(G) = \mathrm{sp}_T(K_{\chi(G)})$. *Moreover, if T is also r-initial, then*

$$\mathrm{esp}_T(G) = (r+1)(\chi(G)-1). \tag{4}$$

This theorem is analogous to Theorems 4 and 6. Note however that in contrast to Theorem 4, to derive (4) one needs both assumptions weakly γ-perfect and r-initial. For instance, if $T = \{0, 1\}$ and G is the 5-cycle, then esp_T is 3 while $(r+1)(\chi-1)$ is 4.

Theorem 15 (Cozzens and Roberts [12], Raychaudhuri [36–37]). *The greedy algorithm computes a T-coloring of G of edge span equal to* $\mathrm{esp}_T(G)$ *in* $O(n^2)$ *steps if*:

(1) *G is an indifference graph and T is an r-initial or k multiple of s set*; *or*

(2) *G is a chordal graph and T is an r-initial or k multiple of s set*; *or*

(3) *G is a perfectly orderable graph with a given admissible ordering, and T is an r-initial or k multiple of s set.*

This theorem is analogous to the results for span in Theorems 7, 8, and 9.

6. Restricted T-colorings

Sometimes restrictions are placed on T-colorings. Graceful numberings, which have a long literature (see for instance Bloom and Golomb [4–5]) can be thought of as restricted T-colorings where $T = \{0\}$ and all the differences $|f(x) - f(y)|$ along edges x, y are distinct. In practical applications, we sometimes want to restrict $f(x)$ to belong to a set C of allowable channels. This idea is mentioned in Hale [21] and is discussed in Section 7 below. More common restrictions would be to restrict the order or the span of a T-coloring. Under all of these restrictions, we would then be interested in finding T-colorings of minimum order or span. For instance, we would like to be able to compute the minimum span of a T-coloring of order χ_T (Hale [21]) or the minimum order of a T-coloring of span sp_T. Very little work has been done on these problems.

7. List T-colorings

A special case of restricted T-colorings arises when we specify the channels acceptable for assignment to a particular transmitter. We then have for each vertex x a set or *list* $C(x)$ of possible channels or colors to be assigned to x. A *list T-coloring* of G is a T-coloring f in which each $f(x)$ belongs to the set $C(x)$. This idea is mentioned by Hale [21], was introduced in an early draft of this paper, and is studied extensively by Tesman [45]. If $T = \{0\}$, we have a *list coloring*, an idea introduced by Erdős, Rubin, and Taylor [15]. These authors introduce the *choice number* of a graph G, the smallest k so that G can be list-colored for *any* assignment of lists $C(x)$ with each $C(x)$ having exactly k elements. (Bollobás and Harris [7] consider a similar concept for edge colorings.) Tesman introduces the *T-choice number* of G, $T\text{-ch}(G)$, the smallest k so that G is list T-colorable for every assignment of lists $C(x)$ in which each list has k elements. For instance, it is easy to show that if $G = K_3$, $T = \{0, 1\}$, and we choose each list to be $\{1, 2, 3, 4\}$, then G is not list T-colorable. However, we can show that G is list T-colorable whenever each list has seven elements, and so $4 < T\text{-ch}(K_3) \leq 7$.

A sample result of Tesman [45] is the following.

Theorem 16 (Tesman [45]). *If G is chordal, T is any set, and $t = |T|$, then $T\text{-ch}(G) \leq (2t - 1)(\chi(G - 1) + 1$.*

Tesman [45] also studies $T\text{-ch}(G)$ for G a tree or a cycle. For instance, if C_k is the cycle of k vertices, he shows the following.

Theorem 17 (Tesman [45]). *If $T = \{0, 1, \ldots, r\}$, $T\text{-ch}(C_{2n+1}) = 2r + 3$, $n \geq 1$.*

However, the exact value of $T\text{-ch}(C_{2n})$ is not yet known even if $T = \{0, 1, \ldots, r\}$.

8. Set T-colorings

Sometimes in making channel assignments, we might want a transmitter to be able to operate over more than one channel. This suggests considering assignments of a set $S(x)$ of colors or channels to each vertex x of a graph G. A natural condition is that

$$\{x, y\} \in E \Rightarrow |a - b| \notin T \quad \text{for all } a \in S(x), \quad b \in S(y).$$

An assignment $S(x)$ satisfying this condition is called a *set T-coloring*. If $T = \{0\}$, this is the set-coloring which was defined by Roberts [39] and has been studied extensively. (Recent results on set-colorings are surveyed by Roberts [40].) Set colorings are studied in the literature under a variety of special assumptions about the types of sets $S(x)$. The most relevant to T-coloring is the special case that each $S(x)$ is a set of n positive integers. In that case, the assignment $S(x)$ is called an *n-tuple coloring*. Such colorings were introduced by Gilbert [17] in connection with a practical problem of assigning frequencies to mobile radio transmitters. Gilbert introduced the parameter $\chi_n(G)$ for the smallest $|\cup S(x)|$ over all n-tuple colorings $S(x)$ of G. Some results about n-tuple colorings can be found in the papers by Stahl [44], Roberts [39], and Opsut and Roberts [33].

Tesman [45, 47] introduces the study of set T-colorings by studying *n-tuple T-colorings,* set T-colorings where each set $S(x)$ is a set of n positive integers. Analogously to the original concepts of Hale [21], Tesman defines the *order* of such an n-tuple T-coloring to be the number of distinct integers used in all of the sets $S(x)$ and the *span* to be the difference between the largest and smallest integers used in any of the sets $S(x)$. Then he denotes by $\chi_T^n(G)$ the minimum order of an n-tuple T-coloring of G and $\mathrm{sp}_T^n(G)$ the minimum span of such an n-tuple T-coloring. Tesman [45, 47] obtains generalizations of many of the earlier results of Cozzens and Roberts [12] and Raychaudhuri [36–37] about ordinary T-colorings. For instance, we have the following results.

Theorem 18 (Tesman [45, 47]). *For all graphs G, all sets T, and all positive integers n, $\chi_T^n(G) = \chi_n(G)$ and*

$$\mathrm{sp}_T^n(K_{\omega(G)}) \leqslant \mathrm{sp}_T^n(G) \leqslant \mathrm{sp}_T^n(K_{\chi(G)}).$$

Theorem 19 (Tesman [45, 47]). *The greedy algorithm on a compatible ordering and the reverse of a perfect elimination ordering computes χ_T^n and sp_T^n for indifference graphs and chordal graphs, respectively, if $T = \{0, 1, \ldots, r\}$.*

Interesting bounds on the span are given by the following theorem.

Theorem 20 (Tesman [45], Füredi, Griggs, and Kleitman [16]). *Suppose $T = (0, 1, \ldots, r\}$. Then if G has an edge,*

$$(r+1)(\chi(G)-1) + 2(n-1) \leqslant \mathrm{sp}_T^n(G) \leqslant (n+r)\chi(G) - (r+1).$$

Füredi, Griggs, and Kleitman [16] have conjectured that all values in the interval in the above theorem are attained by suitable graphs G. While this conjecture remains unsettled, Füredi, Griggs, and Kleitman [16] have proved it for the case $n = 2$.

The special case $T = \{0, 1\}$, $n = 2$, has led to the following interesting result.

Theorem 21 (Füredi, Griggs, and Kleitman [16]). *If G is any graph with $\chi(G) = 3$ and $T = \{0, 1\}$, then $\mathrm{sp}_T^2(G) = 6$ if and only if G is homomorphic to C_5; otherwise $\mathrm{sp}_T^2(G) = 7$.*

Füredi, Griggs, and Kleitman [16] conjecture that if $\chi(G) = 3$ and $T = \{0, 1\}$, it is NP-complete to determine whether $\mathrm{sp}_T^2(G)$ is 6 or 7.

It is natural to combine the ideas of both this and the previous section, to obtain the notion of a set list T-coloring, or an n-tuple list T-coloring. This combination, which is defined in the obvious way, has not yet been studied.

9. Levels of interference

In making channel assignments, we sometimes consider several different levels of interference. For instance, transmitters at most 10 miles apart might interfere at one level, while transmitters at most 50 miles apart might interfere at a second level. To take into account these different levels of interference, we consider k different graphs, $G_0, G_1, \ldots, G_{k-1}$, each on the same vertex set V, the set of transmitters, with an edge between transmitters x_u and x_v appearing in graph G_i if and only if x_u and x_v interfere at level i. In UHF television, k is 5. (See Middlekamp [32], Pugh et al. [34], or Hale [21].) For each level i, we have a disallowed set of separations $T(i)$ for transmitters interfering at level i. Typically,

$$G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots \supseteq G_{k-1} \tag{5}$$

and

$$T(0) \subseteq T(1) \subseteq T(2) \subseteq \cdots \subseteq T(k-1). \tag{6}$$

We seek a function f which assigns to each transmitter a channel, a positive integer, so that f is simultaneously a $T(i)$-coloring of G_i for all i, i.e., so that for $i = 0, 1, \ldots, k-1$,

$$\{x, y\} \in E(G_i) \Rightarrow |f(x) - f(y)| \notin T(i). \tag{7}$$

For instance, if $k = 2$ and $T(0) = \{0\}$ and $T(1) = \{0, 1\}$, then if x and y interfere at level 0, they must get different channels, but if they interfere at level 1, they must get not only different but also non-adjacent channels. If $G = (V, G_0, G_1, \ldots, G_{k-1})$ satisfies (5), we call it a *nested graph*. If $T = (T(0), T(1), \ldots, T(k-1))$ satisfies (6), we call a function f satisfying (7) a *T-coloring*. The *order* and *span* of a T-coloring are defined as before, as are χ_T and sp_T.

A few results about T-colorings when there are several levels of interference are known. We give several of these results here. See [7a], [45] for recent results.

Theorem 22 (Cozzens and Wang [13]). *Let $G = (V, G_0, G_1, \ldots, G_{k-1})$ be a nested graph and let $T = (T(0), T(1), \ldots, T(k-1))$ satisfy* (6). *Then*:

(1) $\chi_T(G) = \chi(G_0)$.

(2) $\mathrm{sp}_T(G) \geq \chi(G_0) - 1$.

(3) $\max \mathrm{sp}_{T(m)}(G_m) \leq \mathrm{sp}_T(G) \leq \mathrm{sp}_{T(k-1)}(G_0)$.

(4) *If each $T(i)$ is r_i-initial, $i = 0, 1, \ldots, k-1$, then*

$$\max_{0 \leq m \leq k-1} [(r_m + 1)(\chi(G_m) - 1)] \leq \mathrm{sp}_T(G) \leq (r_{k-1} + 1)(\chi(G_0) - 1).$$

(5) *If $\chi(G_i) = \chi(G_j)$, for all i, j, and if $T(m)$ is r_m-initial for all m, then* $\mathrm{sp}_T(G) = \mathrm{sp}_{T(k-1)}(G_0) = (r_{k-1} + 1)(\chi(G_0) - 1)$.

(6) *If $\chi(G_i) = \chi(G_j)$, for all i,j, and if $T(m)$ is r_m-initial for all m, and if G_0 is chordal, then the greedy algorithm finds T-colorings of order χ_T and span sp_T in $O(n^2)$ time, where $n = |V|$.*

Theorem 23 (Raychaudhuri [36–37]). *Suppose $k=2$, $T(0)=\{0\}$, $T(1)=\{0, 1\}$, and G_0 is complete. Then $\mathrm{sp}_T(G)$ is the (weighted) length of the shortest hamiltonian path in the complete graph of $n=|V|$ vertices which has weight 1 on edges $\{x, y\}$ not in G_1 and weight 2 on all other edges.*

Raychaudhuri [36–37] also observes that it follows from Theorem 23 that, using results of Goodman and Hedetniemi [19] and of Boesch et al. [6], one can compute $\mathrm{sp}_T(G)$ in $O(n^2)$ time in the situation of Theorem 23 if in addition G_1 is a tree or a forest.

T-colorings are especially interesting for nested graphs where each G_i is a 2-unit sphere graph on the same set of points in Euclidean 2-space, i.e., where the transmitters are thought of as points in the plane and there are positive numbers $d_0>d_1>\cdots>d_{k-1}$ so that

$$\{x, y\} \in E(G_i) \Leftrightarrow d(x, y) \leq d_i.$$

Such nested graphs have not as yet been characterized. However, the corresponding graphs in Euclidean 1-space, i.e., nested families of indifference graphs arising from the same set of points on the line, have been studied by Cozzens and Roberts [11], and some related work has been done by Doignon [14]. Unfortunately, very little positive is known about T-colorings even in this situation, except for some results of Tesman [45] for the special case when there are two levels of interference, the two graphs G_0 and G_1 are indifference graphs with a common compatible ordering, and $T(0)=\{0\}$, $T(1)=\{0, 1\}$. There are also some negative results known. For instance, even if G is a nested family of indifference graphs arising from the same set of points on the line, and each $T(i)$ is an r_i-initial set, it does not follow that a greedy algorithm will find a T-coloring which is both optimal in order and in span. Cozzens and Wang [13] give an example of such G and $T(i)$, $i=0,1$, where no minimum order T-coloring has minimum span. Thus, the situation with several levels of interference will be even more difficult to handle than is the situation with only one level.

10. No-hole T-colorings

The special case of multiple levels of interference when $k=2$, $T(0)=\{0\}$, $T(1)=\{0, 1\}$ has been studied by a variety of authors, including Cozzens and Wang [13], Lanfear [29], Raychaudhuri [36–37], and Tesman [45]. In this setting, Lanfear suggests a heuristic for obtaining an optimal 2-level T-coloring which, at one point, seeks an ordinary T-coloring of a certain graph where $T=\{0, 1\}$ and the set of colors $f(V)=\{f(x): x \in V(G)\}$ is a consecutive set of integers. Lanfear suggests that such ordinary T-colorings should be close to optimal in span. Roberts [41] calls a *T-coloring* a *no-hole* T-coloring if the set $f(V)$ is a consecutive set of integers, and studies no-hole T-colorings when $T = \{0, 1\}$. Let us call such a coloring *near-optimal* if it has a span at most one more than the optimal.

Theorem 24 (Roberts [41]). *Suppose that G is an indifference graph and $T = \{0, 1\}$. Then G has a no-hole T-coloring if $|V(G)| > 2\chi(G) - 1$ and does not have a no-hole T-coloring if $|V(G)| < 2\chi(G) - 1$. If G has a no-hole T-coloring, then it has a near-optimal one.*

Sakai and Wang [42] show that there are graphs which have no-hole T-colorings with $T = \{0, 1\}$, but do not have near-optimal ones; indeed, there are graphs where the optimal span of a no-hole T-coloring can be arbitrarily larger than the optimal span of a T-coloring. Sakai and Wang also study no-hole T-colorings for other sets T.

Note added in proof

Motivated by early drafts of this paper, references [7a], [12a], [29a], which were prepared recently, make substantial progress on some of the problems posed here. Unfortunately, it was impossible in press to make more than a brief mention of these references.

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