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Asymptotic expansion of the one-loop approximation of the Chern–Simons integral in an abstract Wiener space setting

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Abstract

In an abstract Wiener space setting, we construct a rigorous mathematical model of the one-loop approximation of the perturbative Chern–Simons integral, and derive its explicit asymptotic expansion for stochastic Wilson lines.

Keywords: Chern–Simons integral; One-loop approximation; Asymptotic expansion; Abstract Wiener space; Stochastic holonomy; Stochastic Wilson line

Contents

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1. Introduction

Since the pioneering work of Witten [21] in 1989, a multitude of people studied on the relationship between the *Chern–Simons integral*, a formal path integration over an infinite-dimensional space of connections, and *quantum invariants*, new topological invariants of three-manifolds and knots (see, for instance, Atiyah [3] and Ohtsuki [20] for overviews of recent developments in this area). Amongst others, a rigorous mathematical model of the perturbative Chern–Simons integral was constructed by Albeverio and his colleagues; first in the Abelian case as a Fresnel integral [1], and then for the non-Abelian case within the framework of white noise distribution [2].

Recently, an explicit representation of stochastic oscillatory integrals with quadratic phase functions and the formula of changing variables, based on a method of computation of probability via "deformation of the contour integration," have been established on *abstract Wiener spaces* by Malliavin and Taniguchi [17]. Motivated by these antecedent results, the first-named author studied the Chern–Simons integral, in [18,19], from the standpoint of infinite-dimensional stochastic analysis.

The main objective of this paper is, based on the work of Bar-Natan and Witten [5] and the mathematical formulation of the Feynman integral due to Itô [15], to construct, in an abstract Wiener space setting, a rigorous mathematical model of the one-loop approximation of the perturbative Chern–Simons integral of Wilson lines, and derive its explicit asymptotic expansion.

To state our result succinctly, let M be a compact oriented smooth three-manifold, and consider a (trivial) principal G-bundle P over M with a simply connected, connected compact simple gauge group G with Lie algebra $\mathfrak{g}$. We denote by $\Omega^r(M,\mathfrak{g})$ the space of $\mathfrak{g}$-valued smooth r-forms on M equipped with the canonical inner product $(\,,\,)$, and identify a connection on P with a $\mathfrak{g}$-valued 1-form $A\in\Omega^1(M,\mathfrak{g})$. Let

$$Q_{A_0}=(*d_{A_0}+d_{A_0}*)J$$

be a twisted Dirac operator acting on $\Omega^r(M,\mathfrak{g})$, where $*$ is the Hodge $*$-operator defined by a Riemannian metric chosen on M, d_{A_0} is the covariant exterior differentiation defined by a flat connection A_0 on P, and J is an operator defined to be $J\varphi=-\varphi$ if φ is a 0-form or a 3-form, and $J\varphi=\varphi$ if φ is a 1-form or a 2-form. For a sufficiently large integer p, we define the Hilbert subspace $H_p(\Omega_+)$ of $L^2(\Omega_+)=L^2(\Omega^1(M,\mathfrak{g})\oplus\Omega^3(M,\mathfrak{g}))$ with new inner product $(\,,\,)_p$ defined by

$$\big((A,\phi),(B,\varphi)\big)_p=\big(A,\big(I+Q_{A_0}^2\big)^pB\big)+\big(\phi,\big(I+Q_{A_0}^2\big)^p\varphi\big),$$

where I is the identity operator on $L^2(\Omega_+)$.

Now, let $H=H_p(\Omega_+)$ and (B,H,μ) be an abstract Wiener space (see Section 3 for the precise definition). Let λ_i and e_i, $i=1,2,\ldots,$ denote the eigenvalues and eigenvectors of the self-adjoint elliptic operator Q_{A_0}, and $h_i = (1+\lambda_i^2)^{-p/2} e_i$ be the corresponding CONS of H, respectively. Choosing a sufficiently large p satisfying the condition

$$\sum_{i=1}^{\infty} \left(1+\lambda_i^2\right)^{-p} |\lambda_i| < \infty,$$

we define the normalized one-loop approximation of the Lorentz gauge-fixed Chern–Simons integral of the ϵ-regularized Wilson line $F_{A_0}^{\epsilon}(x)$, defined in Section 4, to be

$$I_{CS}\left(F_{A_0}^{\epsilon}\right) = \limsup_{n\to\infty} \frac{1}{Z_n} \int_B F_{A_0}^{\epsilon}\left(\sqrt{n}x\right) e^{\sqrt{-1}kCS(\sqrt{n}x)}\, \mu(dx), \tag{1.1}$$

where

$$Z_n = \int_B e^{\sqrt{-1}kCS(\sqrt{n}x)} \mu(dx), \quad CS(x) = \sum_{i=1}^{\infty} \left(1+\lambda_i^2\right)^{-p} \lambda_i \langle x, h_i\rangle^2,$$

and $\langle , \rangle$ denotes the natural pairing of B and its dual space B^*.

Then we obtain the following expansion theorem.

Theorem. *For any fixed $\epsilon > 0$ and positive integer N,*

$$I_{CS}\left(F_{A_0}^{\epsilon}\right) = \int_B F_{A_0}^{\epsilon}(R_k x) \mu(dx) = \sum_{m<N} k^{-m/2} \cdot J_{CS}^{\epsilon,m} + O\left(k^{-N/2}\right), \tag{1.2}$$

where

$$J_{CS}^{\epsilon,m} = k^{m/2} \cdot \int_B F_{A_0}^{\epsilon,m}(R_k x)\, \mu(dx), \quad R_k = \left\{-2\sqrt{-1}k\left(I + Q_{A_0}^2\right)^{-p} Q_{A_0}\right\}^{-1/2},$$

and $F_{A_0}^{\epsilon,m}(x)$ is defined by (5.3).

The organization of this paper is as follows. In Section 2, we recall relevant basic materials and definitions regarding the one-loop approximation of the perturbative Chern–Simons integral. Then, in Section 3, we define the notion of a stochastic holonomy, and in Section 4, that of a stochastic Wilson line, which is realized as an HC^{∞}-map on an abstract Wiener space. Section 5 is devoted to a rigorous mathematical model of the normalized one-loop approximation of the Lorentz gauge-fixed Chern–Simons integral, which leads to (1.1) defined in an abstract Wiener space setting. Working out this, we then prove our main result, the expansion formula (1.2). In Section 6, as an example, we derive linking numbers of loops from our expansion formula for the ϵ-regularized Wilson line.

Throughout the paper, $\sqrt{z}$ is understood to denote the branch for which $-\pi/2 < \arg\sqrt{z} < \pi/2$.

2. One-loop approximation

Let M be a compact oriented smooth three-manifold, G a simply connected, connected compact simple Lie group, and $P \to M$ a principal G-bundle over M. Since G is simply connected, P is a trivial bundle by topological reason, so that, with a choice of a trivialization of P, we may identify the space of smooth r-forms taking values in the associated adjoint bundle $P \times_{\mathrm{Ad}} \mathfrak{g}$ with $\Omega^r(M, \mathfrak{g})$, the space of $\mathfrak{g}$-valued smooth r-forms on M.

Let $\mathcal{A}$ denote the space of connections on P and $\mathcal{G}$ the group of gauge transformations on P. Note that, by fixing a reference connection on P as the origin, we may identify $\mathcal{A}$ with the (infinite-dimensional) vector space $\Omega^1(M, \mathfrak{g})$, and $\mathcal{G}$ with the space $C^\infty(M, G)$ of smooth maps from M to G, respectively. Then the *Chern–Simons integral* of an integrand $F(A)$ is given by

$$\int_{\mathcal{A}/\mathcal{G}} F(A) e^{L(A)} \mathcal{D}(A), \tag{2.1}$$

where the Lagrangian L is defined by

$$L(A) = -\frac{\sqrt{-1}k}{4\pi} \int_M \mathrm{Tr}\left\{ A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right\}. \tag{2.2}$$

Here $\mathcal{D}(A)$ is the *Feynman measure* integrating over all gauge orbits, that is, over the space $\mathcal{A}/\mathcal{G}$ of equivalence classes of connections modulo gauge transformations, Tr denotes the trace in the adjoint representation of the Lie algebra $\mathfrak{g}$, that is, a multiple of the Killing form of $\mathfrak{g}$, normalized so that the pairing $(X, Y) = -\mathrm{Tr}\, XY$ on $\mathfrak{g}$ is the basic inner product, and the parameter k is a positive integer called the *level of charges*.

Among various integrands, the most typical example of gauge invariant observables is the *Wilson line* defined by

$$F(A) = \prod_{j=1}^{s} \mathrm{Tr}_{R_j} \mathcal{P} \exp \int_{\gamma_j} A, \tag{2.3}$$

where $\mathcal{P}$ denotes the product integral (see [11], or equivalently [7]), γ_j, $j = 1, 2, \ldots, s$, are closed oriented loops, and the trace Tr is taken with respect to some irreducible representation R_j of G assigned to each γ_j. It should be noted that the term $\mathcal{P} \exp \int_{\gamma_j} A$ in (2.3) gives rise to the holonomy of A around γ_j, which is defined to be a solution of the parallel transport equation with respect to A along γ_j. From the standpoint of infinite-dimensional stochastic analysis, we need to regularize the Wilson line (2.3), in a manner similar to that in Albeverio and Schäfer [1], to obtain its ϵ-regularization $F^{\epsilon}_{A_0}(A)$ (see Section 3).

We now recall the perturbative formulation of the Chern–Simons integral [4,5] and adopt the method of superfields in the following manner. Let A_0 be a critical point of the Lagrangian L such that

$$dA_0 + A_0 \wedge A_0 = 0,$$

that is, A_0 is a flat connection. For simplicity, we assume as in [4,5] that A_0 is isolated up to gauge transformations and that the group of gauge transformations fixing A_0 is discrete, or *equivalently* the cohomology $H^*(M, d_{A_0})$ of d_{A_0} vanishes, that is,

$$H^1(M, d_{A_0}) = \{0\}, \qquad H^0(M, d_{A_0}) = \{0\}, \tag{2.4}$$

where d_{A_0} is the covariant exterior differentiation acting on $\Omega^r(M, \mathfrak{g})$, defined by

$$d_{A_0} = d + [A_0, \cdot].$$

Here the bracket $[A, B]$ of $A = \sum A^\alpha \otimes E_\alpha \in \Omega^{r_1}(M, \mathfrak{g})$ and $B = \sum B^\beta \otimes E_\beta \in \Omega^{r_2}(M, \mathfrak{g})$ is defined to be

$$[A, B] = \sum_{\alpha,\beta} A^\alpha \wedge B^\beta \otimes [E_\alpha, E_\beta] \in \Omega^{r_1+r_2}(M, \mathfrak{g}),$$

where $\{E_\alpha\}$ is a basis of the Lie algebra $\mathfrak{g}$.

Then, for the standard gauge fixing, following [4,5], we introduce a Bosonic 3-form ϕ, a Fermionic 0-form c, a Fermionic 3-form $\hat{c}$, which are $\mathfrak{g}$-valued smooth forms on M, and the BRS operator δ. The BRS operator δ is defined by the laws

$$\delta A = -D_A c, \qquad \delta c = \frac{1}{2}[c, c], \qquad \delta \hat{c} = \sqrt{-1}\phi, \qquad \delta\phi = 0,$$

where $D_A = d_{A_0} + [A, \cdot]$. In order to define the Lorentz gauge condition, we now choose a Riemannian metric g on M and denote by $* : \Omega^r(M, \mathfrak{g}) \to \Omega^{3-r}(M, \mathfrak{g})$ the Hodge $*$-operator defined by g, which satisfies $*^2 =$ identity. Then the *Lorentz gauge condition* is given by

$$(d_{A_0})^* A = 0, \tag{2.5}$$

where $(d_{A_0})^* = (-1)^r * d_{A_0} *$ denotes the adjoint operator of d_{A_0}. We set

$$V(A) = \frac{k}{2\pi} \int_M \mathrm{Tr}(\hat{c} * d_{A_0} * A),$$

and define the gauge-fixed Lagrangian of (2.2) by

$$L(A_0 + A) - \delta V(A),$$

where $\delta V(A)$ is given by

$$\delta V(A) = \frac{k}{2\pi} \int_M \mathrm{Tr}\big(\sqrt{-1}\phi * d_{A_0} * A - \hat{c} * d_{A_0} * D_A c\big).$$

Noting that around the critical point A_0 of L, $L(A_0 + A)$ is expanded as

$$L(A_0 + A) = L(A_0) - \frac{\sqrt{-1}k}{4\pi} \int_M \mathrm{Tr}\left\{A \wedge d_{A_0} A + \frac{2}{3} A \wedge A \wedge A\right\},$$

this leads to the Lorentz gauge-fixed Chern–Simons integral written as

$$\int_{\mathcal{A}}\int_{\Phi}\int_{\widehat{\mathcal{C}}}\int_{\mathcal{C}} \mathcal{D}(A)\,\mathcal{D}(\phi)\,\mathcal{D}(\hat{c})\,\mathcal{D}(c)\,F(A_0+A) \times \exp\Big[L(A_0) - \frac{\sqrt{-1}k}{4\pi}\int_M \mathrm{Tr}\Big\{A\wedge d_{A_0}A + \frac{2}{3}A\wedge A\wedge A + 2\phi * d_{A_0} * A + 2\sqrt{-1}\hat{c} * d_{A_0} * D_A c\Big\}\Big]. \tag{2.6}$$

Geometrically, one can derive (2.6) in the following way. First recall that the tangent space $T_{A_0}\mathcal{A} \cong \Omega^1(M,\mathfrak{g})$ of the space of connections $\mathcal{A}$ at A_0 is decomposed as

$$T_{A_0}\mathcal{A} = \mathrm{Im}\, d_{A_0} \oplus \mathrm{Ker}(d_{A_0})^*,$$

since for each $c \in \Omega^0(M,\mathfrak{g})$ we have $(d/dt)|_{t=0}(\exp tc)^*A = d_A c$. Thus the Lorentz gauge condition (2.5) corresponds to the choice of the orthogonal complement of the tangent space to the gauge orbit through A_0. Under the assumption (2.4) we may think that the Lorentz gauge condition $(d_{A_0})^*A = 0$ has a unique solution on each gauge orbit of $\mathcal{G}$. Then, denoting by $\det \mathcal{J}(A)$ the Jacobian of the transformation $\mathcal{G} \ni g \mapsto (d_{A_0})^*(g^*(A_0+A)) \in \Omega^0(M,\mathfrak{g})$ at the identity element of $\mathcal{G}$, we obtain the following basic identity for the Chern–Simons integral (2.1):

$$\int_{\mathcal{A}/\mathcal{G}} F(A)e^{L(A)}\,\mathcal{D}(A) = \int_{\mathcal{A}} \mathcal{D}(A)\,F(A)e^{L(A)}\delta\big((d_{A_0})^*A\big)\det\mathcal{J}(A), \tag{2.7}$$

where δ denotes the Dirac delta function. Here it should be noted that the term $\delta((d_{A_0})^*A)$ can be read into the Lagrangian in the form

$$\int_{\Phi} \mathcal{D}(\phi)\exp\Big[-\sqrt{-1}\int_M \mathrm{Tr}\big\{(d_{A_0})^*A\cdot\phi\big\}\Big],$$

and the term $\det\mathcal{J}(A)$ in the form

$$\int_{\widehat{\mathcal{C}}}\int_{\mathcal{C}} \mathcal{D}(\hat{c})\,\mathcal{D}(c)\exp\Big[\int_M \mathrm{Tr}\big\{\hat{c}\cdot(d_{A_0})^*D_A c\big\}\Big],$$

where $\hat{c}$ and c should be understood as Grassmann (anti-commuting) variables (cf. [22]). Encoding these contributions into (2.7), and taking account of the fact that, when deriving the identity (2.7), the Lorentz gauge condition (2.5) may be replaced by

$$\kappa(d_{A_0})^*A = 0$$

for any non-zero constant $\kappa \in \boldsymbol{C}$, we obtain (2.6), by choosing $\kappa = -k/2\pi$.

Now, noticing that likewise one may simply substitute $\delta(\kappa(d_{A_0})^* A)$ for $\delta((d_{A_0})^* A)$ in (2.7), we set

$$A' = \sqrt{1/2\pi}\, A, \quad \phi' = \sqrt{1/2\pi}\, \phi \quad \text{and} \quad c' = \sqrt{k/2\pi}\, c, \quad \hat{c}' = *\sqrt{k/2\pi}\, \hat{c}$$

in (2.6), and collect the terms that are at most second order in A', ϕ', c' and $\hat{c}'$. In the result, we obtain the following Lorentz gauge-fixed path integral form of the *one-loop approximation* of the Chern–Simons integral, written in variables $c', \hat{c}'$ and (A', ϕ'):

$$\int_{\mathcal{A}'} \int_{\Phi'} \int_{\widehat{\mathcal{C}}'} \int_{\mathcal{C}'} \mathcal{D}(A')\, \mathcal{D}(\phi')\, \mathcal{D}(\hat{c}')\, \mathcal{D}(c')\, F(A_0 + A') \times \exp\bigl[L(A_0) + \sqrt{-1}k\bigl((A', \phi'), Q_{A_0}(A', \phi')\bigr)_+ + (\hat{c}', \Delta_0 c')\bigr] \tag{2.8}$$

(see [5,18] for details). Here we denote by $(\ ,\ )_+$ the inner product of the Hilbert space $L^2(\Omega_+) = L^2(\Omega^1(M, \mathfrak{g}) \oplus \Omega^3(M, \mathfrak{g}))$ given by

$$\bigl((A, \phi), (B, \varphi)\bigr)_+ = (A, B) + (\phi, \varphi),$$

where the inner product and the norm on $\Omega^r(M, \mathfrak{g})$ are defined by

$$(\omega, \eta) = -\int_M \operatorname{Tr} \omega \wedge *\eta, \qquad |\cdot| = \sqrt{(\cdot, \cdot)}. \tag{2.9}$$

Furthermore, Q_{A_0} is a *twisted Dirac operator* defined by

$$Q_{A_0} = (*d_{A_0} + d_{A_0}*)J, \tag{2.10}$$

where $J\varphi = -\varphi$ if φ is a 0-form or a 3-form, and $J\varphi = \varphi$ if φ is a 1-form or a 2-form. It should be noted that Q_{A_0} is a self-adjoint elliptic operator, and $\Delta_0 = (d_{A_0})^* d_{A_0}$ is the Laplacian acting on $\Omega^0(M, \mathfrak{g})$.

Finally, balancing out the contributions coming of the term $L(A_0)$ as well as the Fermi integral

$$\int_{\widehat{\mathcal{C}}'} \int_{\mathcal{C}'} \mathcal{D}(\hat{c}')\, \mathcal{D}(c')\, e^{(\hat{c}', \Delta_0 c')},$$

we arrive at, from (2.8), the *normalized one-loop approximation* of the Lorentz gauge-fixed Chern–Simons integral:

$$\frac{1}{Z} \int_{\mathcal{A}} \int_{\Phi} F(A_0 + A) \exp\bigl[\sqrt{-1}k\bigl((A, \phi), Q_{A_0}(A, \phi)\bigr)_+\bigr]\, \mathcal{D}(A)\, \mathcal{D}(\phi), \tag{2.11}$$

where

$$Z = \int_{\mathcal{A}} \int_{\Phi} \exp\bigl[\sqrt{-1}k\bigl((A, \phi), Q_{A_0}(A, \phi)\bigr)_+\bigr]\, \mathcal{D}(A)\, \mathcal{D}(\phi).$$

Our primary objective is to give a rigorous mathematical meaning to this normalized one-loop approximation of the perturbative Chern–Simons integral (2.11).

3. Stochastic holonomy

To handle the integral (2.11) in an abstract Wiener space setting, we need to extend the holonomy of a smooth connection A around a closed oriented loop γ,

$$\mathcal{P}\exp\int_{\gamma} A,$$

to a rough connection A. To this end we regularize the Wilson line in a manner similar to that in [1], which is suitable for our abstract Wiener space setting.

As in the previous section, let M be a compact oriented smooth three-manifold, G a simply connected, connected compact simple Lie group with Lie algebra $\mathfrak{g}$, and $P \to M$ a principal G-bundle over M. Let $\mathcal{A}$ be the space of connections on P, which is identified with $\Omega^1(M, \mathfrak{g})$, the space of $\mathfrak{g}$-valued smooth 1-forms on M, and denote by $\{E_\alpha\}$, $1 \leqslant \alpha \leqslant d$, a given basis of $\mathfrak{g}$. Let $\gamma : [0,1] \ni \tau \mapsto \gamma(\tau) \in M$ be a closed smooth curve in M, and set $\gamma[s,t] = \{\gamma(\tau) \mid s \leqslant \tau \leqslant t\}$. We regard $\gamma[s,t]$ as a linear functional

$$\bigl(\gamma[s,t]\bigr)[A] = \int_{\gamma[s,t]} A = \int_s^t A\bigl(\dot\gamma(\tau)\bigr)\,d\tau, \quad A \in \mathcal{A},$$

defined on the vector space $\mathcal{A}$. Then $\gamma[s,t]$ is continuous in the sense of distribution and hence defines a ($\mathfrak{g}$-valued) de Rham current of degree two.

To recall the regularization of currents, we first consider the case where γ is a closed smooth curve in $\boldsymbol{R}^3$ and A is a $\mathfrak{g}$-valued smooth 1-form with compact support defined on $\boldsymbol{R}^3$. Let ϕ be a non-negative smooth function on $\boldsymbol{R}^3$ such that the support of ϕ is contained in the unit ball $\boldsymbol{B}^3$ with center $0 \in \boldsymbol{R}^3$ and

$$\int_{\boldsymbol{R}^3} \phi(x)\,dx = 1.$$

Then define $\phi_\epsilon(x) = \epsilon^{-3}\phi(x/\epsilon)$ for each $\epsilon > 0$. If we write

$$A = \sum_\alpha A^\alpha \otimes E_\alpha = \sum_{i,\alpha} {A_i}^\alpha\,dx^i \otimes E_\alpha, \qquad \dot\gamma(\tau) = \sum_i \dot\gamma^i(\tau)\left(\frac{\partial}{\partial x^i}\right)_{\gamma(\tau)}$$

for given A and γ, then we have

$$\lim_{\epsilon\to 0}\sup_{s\leqslant\tau\leqslant t}\left|\int_{\boldsymbol{R}^3} {A_i}^\alpha(x)\phi_\epsilon\bigl(x-\gamma(\tau)\bigr)\,dx - {A_i}^\alpha\bigl(\gamma(\tau)\bigr)\right| = 0, \tag{3.1}$$

and

$$\left|\sum_{i=1}^{3}\int_{s}^{t}\left(\int_{\boldsymbol{R}^3} A_i{}^{\alpha}(x)\phi_\epsilon\big(x-\gamma(\tau)\big)\,dx\right)\dot{\gamma}^i(\tau)\,d\tau\right| \leqslant c_1(\epsilon)\left\|A^\alpha\right\|_{L^2(\boldsymbol{R}^3)}|t-s|. \tag{3.2}$$

Here and in what follows, we denote by $c_k(\star)$ a constant depending on the quantity $\star$ and simply write c_k whenever no confusion may occur.

Now, according to de Rham [10], the regulator of the current $\gamma[s,t]$ is defined by

$$\begin{aligned}\big(\mathcal{R}_\epsilon\gamma[s,t]\big)[A] &= \big(\gamma[s,t]\big)[\mathcal{R}_\epsilon^* A]\\ &= \sum_{i=1}^{3}\int_{s}^{t}\left(\int_{\boldsymbol{R}^3} A_i{}^{\alpha}\big(\gamma(\tau)+y\big)\phi_\epsilon(y)\,dy\right)\dot{\gamma}^i(\tau)\,d\tau\otimes E_\alpha\\ &= \sum_{i=1}^{3}\int_{s}^{t}\left(\int_{\boldsymbol{R}^3} A_i{}^{\alpha}(x)\phi_\epsilon\big(x-\gamma(\tau)\big)\,dx\right)\dot{\gamma}^i(\tau)\,d\tau\otimes E_\alpha,\end{aligned}$$

to which is associated an operator defined by

$$\begin{aligned}\big(\mathcal{A}_\epsilon\gamma[s,t]\big)[B] &= \big(\gamma[s,t]\big)[\mathcal{A}_\epsilon^* B]\\ &= \sum_{i,j=1}^{3}\int_{s}^{t}\left\{\int_{\boldsymbol{R}^3}\left(\int_{0}^{1} y^i B_{ij}{}^{\alpha}\big(\gamma(\tau)+ty\big)\,dt\right)\phi_\epsilon(y)\,dy\right\}\dot{\gamma}^j(\tau)\,d\tau\otimes E_\alpha,\end{aligned}$$

where $B=\sum B_{ij}{}^{\alpha}\,dx^i\wedge dx^j\otimes E_\alpha$ is a $\mathfrak{g}$-valued smooth 2-form with compact support on $\boldsymbol{R}^3$. Then we have the following relation between the operators $\mathcal{R}_\epsilon$ and $\mathcal{A}_\epsilon$, which is known as the homotopy formula (see [10, §15] for details).

Proposition 1. *For each $\epsilon>0$, $\mathcal{R}_\epsilon\gamma[s,t]$ and $\mathcal{A}_\epsilon\gamma[s,t]$ are currents whose supports are contained in the ϵ-tubular neighborhood of $\gamma[s,t]$, and satisfy*

$$\mathcal{R}_\epsilon\gamma[s,t]-\gamma[s,t]=\partial\mathcal{A}_\epsilon\gamma[s,t]+\mathcal{A}_\epsilon\partial\gamma[s,t],$$

where ∂ is the boundary operator of currents.

As in [10], the above construction of regularization generalizes to our case in the following manner. First take a diffeomorphism h of $\boldsymbol{R}^3$ onto the unit ball $\boldsymbol{B}^3$ with center 0 which coincides with the identity on the ball of radius $1/3$ with center 0. Denote by s_y the translation $s_y(x)=x+y$ and let $\boldsymbol{s}_y$ be the map of $\boldsymbol{R}^3$ onto itself which coincides with $h\circ s_y\circ h^{-1}$ on $\boldsymbol{B}^3$ and with the identity at all other points, that is,

$$\boldsymbol{s}_y(x)=\begin{cases} h\circ s_y\circ h^{-1}(x) & \text{if } x\in\boldsymbol{B}^3,\\ x & \text{if } x\notin\boldsymbol{B}^3.\end{cases}$$

Note that with a suitable choice of h we may make $\boldsymbol{s}_y$ to be a diffeomorphism. Then define $\boldsymbol{\mathcal{R}}_\epsilon\gamma[s,t]$ and $\boldsymbol{\mathcal{A}}_\epsilon\gamma[s,t]$ by the same equations above, but now replacing $\gamma(\tau)+y$ and $\gamma(\tau)+ty$ with $\boldsymbol{s}_y(\gamma(\tau))$ and $\boldsymbol{s}_{ty}(\gamma(\tau))$, respectively.

Now, let $\{U_i\}$ be a finite open covering of M such that each U_i is diffeomorphic to the unit ball $\boldsymbol{B}^3$ via a diffeomorphism h_i, which can be extended to some neighborhoods of the closures of U_i and of $\boldsymbol{B}^3$. Using these diffeomorphisms, we transport the transformed operators $\boldsymbol{\mathcal{R}}_\epsilon$ and $\boldsymbol{\mathcal{A}}_\epsilon$ defined on $\boldsymbol{R}^3$ to M. Indeed, let f be a cutoff function which has its support in the neighborhood of the closure of U_i and is equal to 1 on U_i. Set $T=\gamma[s,t]$ for simplicity. Then $T'=fT$ is a current which has its support contained in the neighborhood of the closure of U_i, and h_iT' is a current which has its support contained in the neighborhood of the closure of $\boldsymbol{B}^3$. Note that the support of $T''=T-T'$ does not meet the closure of U_i. We define

$$\mathcal{R}_\epsilon^i T = h_i^{-1}\circ\boldsymbol{\mathcal{R}}_\epsilon\circ h_iT' + T'', \qquad \mathcal{A}_\epsilon^i T = h_i^{-1}\circ\boldsymbol{\mathcal{A}}_\epsilon\circ h_iT'$$

and set inductively

$$\mathcal{R}_\epsilon^{(k)}T=\mathcal{R}_\epsilon^1\circ\mathcal{R}_\epsilon^2\circ\cdots\circ\mathcal{R}_\epsilon^kT, \qquad \mathcal{A}_\epsilon^{(k)}T=\mathcal{R}_\epsilon^1\circ\mathcal{R}_\epsilon^2\circ\cdots\circ\mathcal{R}_\epsilon^{k-1}\circ\mathcal{A}_\epsilon^kT.$$

Then $\mathcal{R}_\epsilon T$ and $\mathcal{A}_\epsilon T$ are obtained to be

$$\mathcal{R}_\epsilon T=\mathcal{R}_\epsilon^{(N)}T, \qquad \mathcal{A}_\epsilon T=\sum_{k=1}^N\mathcal{A}_\epsilon^{(k)}T,$$

where N is the number of open sets in $\{U_i\}$.

The construction of these operators $\mathcal{R}_\epsilon$ and $\mathcal{A}_\epsilon$ is easily generalized to any current T defined on a compact smooth manifold of arbitrary dimension. We remark that the following properties hold for regularization of currents.

Proposition 2. *(See [10].) Let M be a compact smooth manifold. Then for each $\epsilon>0$ there exist linear operators $\mathcal{R}_\epsilon$ and $\mathcal{A}_\epsilon$ acting on the space of de Rham currents with the following properties*:

(1) *If T is a current, then $\mathcal{R}_\epsilon T$ and $\mathcal{A}_\epsilon T$ are also currents and satisfy*

$$\mathcal{R}_\epsilon T - T = \partial\mathcal{A}_\epsilon T + \mathcal{A}_\epsilon\partial T.$$

(2) *The supports of $\mathcal{R}_\epsilon T$ and $\mathcal{A}_\epsilon T$ are contained in an arbitrary given neighborhood of the support of T provided that ϵ is sufficiently small.*
(3) *$\mathcal{R}_\epsilon T$ is a smooth form.*
(4) *For all smooth forms φ we have*

$$\mathcal{R}_\epsilon T[\varphi]\to T[\varphi] \quad \textit{and} \quad \mathcal{A}_\epsilon T[\varphi]\to 0$$

as $\epsilon\to 0$.

Given a closed smooth curve $\gamma : [0,1] \to M$ in M, for each $t \in [0,1]$ and sufficiently small $\epsilon > 0$ we consider a smooth current associated to $\gamma[0,t]$ defined by

$$C_\gamma^\epsilon(t) = *\mathcal{R}_\epsilon \gamma[0,t],$$

where $*$ is the Hodge $*$-operator defined by a Riemannian metric chosen on M, and write $C_\gamma^\epsilon(t) = \sum C_\gamma^\epsilon(t)^\alpha \otimes E_\alpha$. Let U_γ be a tubular neighborhood of $\gamma[0,1]$ in M and $j : U_\gamma \to M$ denote the inclusion. Then

$$j^*\bigl(*C_\gamma^\epsilon(t)\bigr) = j^*\bigl(\mathcal{R}_\epsilon \gamma[0,t]\bigr)$$

is a $\mathfrak{g}$-valued smooth 2-form on U_γ and has a compact support in U_γ from Proposition 2. In particular, for $t = 1$ we see that

$$dj^*\bigl(*C_\gamma^\epsilon(1)\bigr) = dj^*\bigl(\mathcal{R}_\epsilon \gamma[0,1]\bigr) = j^* d\bigl(\mathcal{R}_\epsilon \gamma[0,1]\bigr) = -j^*\mathcal{R}_\epsilon \partial\bigl(\gamma[0,1]\bigr) = 0,$$

since $\mathcal{R}_\epsilon$ and ∂ commute and $\partial(\gamma[0,1]) = \emptyset$.

As a result, each $j^*(*C_\gamma^\epsilon(1)^\alpha)$ determines a cohomology class $[j^*(*C_\gamma^\epsilon(1)^\alpha)] \in H_c^2(U_\gamma)$ in the second de Rham cohomology of U_γ with compact support. Indeed, by virtue of Proposition 2(1), it is not hard to see that

$$\int_{U_\gamma} \omega \wedge j^*\bigl(*C_\gamma^\epsilon(1)^\alpha\bigr) = \int_\gamma i^*\omega$$

holds for any $[\omega] \in H_c^1(U_\gamma)$, where $i : \gamma[0,1] \to U_\gamma$ denotes the inclusion. Namely, we have

Proposition 3. *(See [1].)* $[j^*(*C_\gamma^\epsilon(1)^\alpha)] \in H_c^2(U_\gamma)$ *is the compact Poincaré dual of* γ *in* U_γ *for each* $\alpha = 1, 2, 3$.

Recalling the construction of regulators of currents and noting (3.1) and (3.2), it is not hard to see that we have

$$\lim_{\epsilon \to 0} \sup_{0 \leqslant t \leqslant 1} \left| \sum_{i=1}^{3} \int_0^t \biggl(\int_M A_i{}^\alpha(x)\phi_\epsilon\bigl(x - \gamma(\tau)\bigr)\,dx - A_i{}^\alpha\bigl(\gamma(\tau)\bigr)\biggr) \dot{\gamma}^i(\tau)\,d\tau \right| = 0,$$
$$\left| \int_{\gamma[0,t]} A^\alpha - \int_{\gamma[0,s]} A^\alpha \right| \leqslant c_2(A)|t-s|, \tag{3.3}$$

and

$$\bigl|C_\gamma^\epsilon(t) - C_\gamma^\epsilon(s)\bigr| \leqslant c_1(\epsilon)|t-s|, \tag{3.4}$$

where $|\cdot|$ on the left side of (3.4) is the norm defined in (2.9).

Now, in order to extend the holonomy to a rough connection A, for a non-negative integer p, let $H_p(\Omega_+)$ denote the Hilbert subspace of $L^2(\Omega_+) = L^2(\Omega^1(M,\mathfrak{g}) \oplus \Omega^3(M,\mathfrak{g}))$ with new inner product $(\,,)_p$ defined by

$$\begin{aligned}\big((A,\phi),(B,\varphi)\big)_p &= \big((A,\phi),\big(I+Q_{A_0}^2\big)^p(B,\varphi)\big)_+\\ &= \big(A,\big(I+Q_{A_0}^2\big)^p B\big)+\big(\phi,\big(I+Q_{A_0}^2\big)^p\varphi\big).\end{aligned}\tag{3.5}$$

Here I is the identity operator on $L^2(\Omega_+)$, and the p-norm on $H_p(\Omega_+)$ is defined as usual by $\|\cdot\|_p=\sqrt{(\cdot,\cdot)_p}$. Henceforth we denote $H_p(\Omega_+)$ briefly by H_p whenever no confusion may occur.

Then the holonomy for a smooth connection A is extended to the *stochastic holonomy* of $(A,\phi)\in H_p$ in the following manner. Since

$$\big(A,C_\gamma^\epsilon(t)\big)=\big((A,\phi),\big(I+Q_{A_0}^2\big)^{-p}\big(C_\gamma^\epsilon(t),0\big)\big)_p,$$

by setting

$$\tilde{C}_\gamma^\epsilon(t)=\big(I+Q_{A_0}^2\big)^{-p}\big(C_\gamma^\epsilon(t),0\big),\tag{3.6}$$

we obtain from (3.4) that

$$\big\|\tilde{C}_\gamma^\epsilon(t)-\tilde{C}_\gamma^\epsilon(s)\big\|_p\leqslant c_1(\epsilon)|t-s|.\tag{3.7}$$

Given $(A,\phi)\in H_p$, we now write

$$A_\gamma^\epsilon(t)=\sum_{\alpha=1}^d\big((A,\phi),\tilde{C}_\gamma^\epsilon(t)^\alpha\otimes E_\alpha\big)_p E_\alpha,\tag{3.8}$$

where $\tilde{C}_\gamma^\epsilon(t)=\sum\tilde{C}_\gamma^\epsilon(t)^\alpha\otimes E_\alpha$, and define

$$\bar{A}(t)=\int_{\gamma[0,t]}A.$$

With these understood, recall that for the holonomy for a smooth connection A around A_0, it follows from (3.3) that, in terms of the product integral or Chen's iterated integral (see Theorem 4.3 of [11, p. 31] and also [7]), it is given by

$$\begin{aligned}&\mathcal{P}\exp\int_\gamma A_0+A\\ &\quad=I+\sum_{r=1}^\infty\int_0^1\int_0^{t_1}\cdots\int_0^{t_{r-1}}d(\bar{A}_0+\bar{A})(t_1)\,d(\bar{A}_0+\bar{A})(t_2)\cdots d(\bar{A}_0+\bar{A})(t_r),\end{aligned}\tag{3.9}$$

where $0\leqslant t_{r-1}\leqslant\cdots\leqslant t_1\leqslant t_0=1$. Then, noting (3.7), for each $(A,\phi)\in H_p$ we define the ϵ-regularization of the holonomy by

$$W_\gamma^\epsilon(A)=I+\sum_{r=1}^\infty W_\gamma^{\epsilon,r}(A),\tag{3.10}$$

where

$$W_\gamma^{\epsilon,r}(A) = \int_0^1\int_0^{t_1}\cdots\int_0^{t_{r-1}} d\big(\bar{A}_0 + A_\gamma^\epsilon\big)(t_1)\, d\big(\bar{A}_0 + A_\gamma^\epsilon\big)(t_2)\cdots d\big(\bar{A}_0 + A_\gamma^\epsilon\big)(t_r),$$

and the ϵ-regularized Wilson line by

$$F_{A_0}^\epsilon(A) = \prod_{j=1}^{s} \mathrm{Tr}_{R_j}\, W_{\gamma_j}^\epsilon(A), \tag{3.11}$$

where the trace Tr is taken in the representation R_j of G assigned to each loop γ_j.

4. Stochastic Wilson line

We now proceed to extend the ϵ-regularized Wilson line $F_{A_0}^\epsilon(A)$ in (3.11) even to an abstract Wiener space setting. To this end, let M and G be as in Section 3, and denote by $H_p(\Omega_+)$ the Hilbert subspace of $L^2(\Omega_+) = L^2(\Omega^1(M,\mathfrak{g}) \oplus \Omega^3(M,\mathfrak{g}))$ with inner product $(\,,\,)_p$ defined by (3.5). Then set $H = H_p(\Omega_+)$ and let (B, H, μ) be an *abstract Wiener space* such that μ is a Gaussian measure satisfying

$$\int_B e^{\sqrt{-1}\langle x,\xi\rangle}\,\mu(dx) = e^{-\|\xi\|_p^2/2}$$

for each $\xi \in B^*$. Here B is a real separable Banach space in which the separable Hilbert space H is continuously and densely imbedded, $\langle,\rangle$ denotes the natural pairing of B and its dual space B^*, and B^* is considered as $B^* \subset H$ under the usual identification of H with H^* (cf. [17]).

We first note that the twisted Dirac operator Q_{A_0} of (2.10) has pure point spectrum, since Q_{A_0} is a self-adjoint elliptic operator (cf. [13]). Thus let

$$\lambda_i,\quad e_i = \big(e_i^A, e_i^\phi\big),\quad i = 1, 2, \ldots,$$

be the eigenvalues and eigenvectors of Q_{A_0}. Recall that by our assumption (2.4) the eigenvectors $\{e_i\}$ form a CONS (complete orthonormal system) of $L^2(\Omega_+)$. If we define

$$h_j = \big(1 + \lambda_j^2\big)^{-p/2} e_j,\quad j = 1, 2, \ldots,$$

then the set $\{h_j\}$ gives rise to a CONS of H_p, so that the increasing rate of the eigenvalues of Q_{A_0} guarantees the nuclearity of the system of semi-norms $\|\cdot\|_q$, $q = 1, 2, \ldots$ (see, for instance, Lemma 1.6.3(c) in [13]). Hence there exists some integer p_0 independent of p such that B is realized as H_{-p-p_0} (cf. [12]), where H_{-q} is the dual space of H_q. If we choose a sufficiently large p such that $p > p_0$ and

$$\sum_{i=1}^{\infty}\big(1 + \lambda_i^2\big)^{-p}|\lambda_i| < \infty,$$

if necessary, then we see from (3.6) that

$$\tilde{C}_\gamma^\epsilon(t) \in H_{p+p_0} = B^*.$$

In what follows we take this suitable space as B throughout the paper.

According to (3.8), for each $\epsilon > 0$ and $x \in B$, we define

$$x_\gamma^\epsilon(t) = \sum_{\alpha=1}^{d} \langle x, \tilde{C}_\gamma^\epsilon(t)^\alpha \otimes E_\alpha \rangle E_\alpha,$$

where $\{E_\alpha\}$, $1 \leqslant \alpha \leqslant d$, is a basis of the Lie algebra $\mathfrak{g}$, and briefly denote

$$x_\gamma^{\epsilon,\alpha}(t) = \langle x, \tilde{C}_\gamma^\epsilon(t)^\alpha \otimes E_\alpha \rangle,$$

which is a Gaussian random variable such that

$$E\left[x_\gamma^{\epsilon,\alpha}(t)^2\right] = \left\| \tilde{C}_\gamma^\epsilon(t)^\alpha \otimes E_\alpha \right\|_p^2. \tag{4.1}$$

Since it follows from (3.7) that

$$\left| x_\gamma^{\epsilon,\alpha}(t) - x_\gamma^{\epsilon,\alpha}(s) \right| \leqslant c_1(\epsilon) \|x\|_B |t-s|, \tag{4.2}$$

the Lebesgue–Stieltjes integral

$$\int_0^t dx_\gamma^\epsilon(\tau) = \sum_{\alpha=1}^{d} \int_0^t dx_\gamma^{\epsilon,\alpha}(\tau) \cdot E_\alpha$$

is well defined. Hence, according to (3.10), for each $\epsilon > 0$ we define the *ϵ-regularized stochastic holonomy* for $x \in B$ by

$$W_\gamma^\epsilon(x) = I + \sum_{r=1}^{\infty} W_\gamma^{\epsilon,r}(x), \tag{4.3}$$

where

$$W_\gamma^{\epsilon,r}(x) = \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{r-1}} d\left(\bar{A}_0 + x_\gamma^\epsilon\right)(t_1)\, d\left(\bar{A}_0 + x_\gamma^\epsilon\right)(t_2) \cdots d\left(\bar{A}_0 + x_\gamma^\epsilon\right)(t_r).$$

Then the *ϵ-regularized Wilson line* for $x \in B$ (cf. [1]) is given by

$$F_{A_0}^\epsilon(x) = \prod_{j=1}^{s} \mathrm{Tr}_{R_j} W_{\gamma_j}^\epsilon(x). \tag{4.4}$$

Now, we will see the well-definedness, the smoothness in H-Fréchet differentiation and the integrability of the ϵ-regularized Wilson line $F_{A_0}^\epsilon(x)$ as an analytic function in the sense of

Malliavin and Taniguchi [17]. Indeed, in the representation R_j of G assigned to each loop γ_j, if we define for a given basis $\{E_\alpha\}$ of $\mathfrak{g}$ and an $n \times n$ matrix $A = (a_{ij})$,

$$c_E = \max_{1 \leqslant \alpha \leqslant d} \|E_\alpha\|, \qquad \|A\| = \sum_{i,j=1}^{n} |a_{ij}|,$$

then we have the following

Lemma 1. *For $\epsilon > 0$ and $x \in B$, define the ϵ-regularizations $W_\gamma^\epsilon(x)$ and $F_{A_0}^\epsilon(x)$ by* (4.3) *and* (4.4)*, respectively. Then the following hold.*

(1) *$W_\gamma^\epsilon(x)$ is well defined and C^∞ in H-Fréchet differentiation.*
(2) *For any positive integer q we have*

$$E\big[\big\|W_\gamma^\epsilon(x)\big\|^{2q}\big] < \infty.$$

(3) *For any positive integer q and positive number s we have*

$$\sum_{k=0}^{\infty} \frac{s^k}{k!} E\bigg[\bigg(\sum_{i_1,i_2,\ldots,i_k} \big\|D^k W_\gamma^\epsilon(x)(h_{i_1}, h_{i_2}, \ldots, h_{i_k})\big\|^2\bigg)^q\bigg]^{1/2q} < \infty$$

and

$$\sum_{k=0}^{\infty} \frac{s^k}{k!} E\bigg[\bigg(\sum_{i_1,i_2,\ldots,i_k} \big|D^k F_{A_0}^\epsilon(x)(h_{i_1}, h_{i_2}, \ldots, h_{i_k})\big|^2\bigg)^q\bigg]^{1/2q} < \infty,$$

where $\{h_j\}$ is a CONS *of H.*

Proof. First we prove (1). It follows from (3.3) and (4.2) that for any $t \geqslant 0$ we have

$$\bigg\|\int_0^t d\bar{A}_0\bigg\| \leqslant \sigma c_2(A_0)t, \qquad \bigg\|\int_0^t dx_\gamma^\epsilon(\tau)\bigg\| \leqslant \sigma c_1(\epsilon)\|x\|_B t,$$

where $\sigma = d \cdot c_E$. Then it is not hard to see that for $x \in B$

$$\begin{aligned} \big\|W_\gamma^\epsilon(x)\big\| &\leqslant \sum_{r=0}^{\infty} \bigg\| \int_0^1\int_0^{t_1} \cdots \int_0^{t_{r-1}} d\big(\bar{A}_0 + x_\gamma^\epsilon\big)(t_1)\, d\big(\bar{A}_0 + x_\gamma^\epsilon\big)(t_2) \cdots d\big(\bar{A}_0 + x_\gamma^\epsilon\big)(t_r) \bigg\| \\ &\leqslant \sum_{r=0}^{\infty} \big(\sigma(c_2(A_0) + c_1(\epsilon)\|x\|_B)\big)^r \int_0^1\int_0^{t_1} \cdots \int_0^{t_{r-1}} dt_1\, dt_2 \cdots dt_r \\ &\leqslant \sum_{r=0}^{\infty} \big(\sigma\big(c_2(A_0) + c_1(\epsilon)\|x\|_B\big)\big)^r / r! = e^{\sigma(c_2(A_0)+c_1(\epsilon)\|x\|_B)}, \end{aligned} \tag{4.5}$$

which implies the well-definedness of $W_\gamma^\epsilon(x)$.

To see the smoothness of $W_\gamma^\epsilon(x)$ in H-Fréchet differentiation, we first note that for $h \in H$

$$
\begin{aligned}
DW_\gamma^\epsilon(x)(h) &= \lim_{s\to 0}\bigl\{W_\gamma^\epsilon(x+sh) - W_\gamma^\epsilon(x)\bigr\}/s \\
&= \lim_{s\to 0}\frac{1}{s}\sum_{r=1}^{\infty}\Biggl\{\int_0^1\int_0^{t_1}\cdots\int_0^{t_{r-1}} d\bigl(\bar{A}_0 + (x+sh)_\gamma^\epsilon\bigr)(t_1)\cdots d\bigl(\bar{A}_0 + (x+sh)_\gamma^\epsilon\bigr)(t_r) \\
&\quad - \int_0^1\int_0^{t_1}\cdots\int_0^{t_{r-1}} d\bigl(\bar{A}_0 + x_\gamma^\epsilon\bigr)(t_1)\cdots d\bigl(\bar{A}_0 + x_\gamma^\epsilon\bigr)(t_r)\Biggr\}.
\end{aligned}
$$

Then, in a manner similar to the previous estimate, we have for $|s| \leqslant 1$

$$
\begin{aligned}
&\Biggl\|\frac{1}{s}\sum_{r=1}^{\infty}\Biggl\{\int_0^1\int_0^{t_1}\cdots\int_0^{t_{r-1}} d\bigl(\bar{A}_0 + (x+sh)_\gamma^\epsilon\bigr)(t_1)\cdots d\bigl(\bar{A}_0 + (x+sh)_\gamma^\epsilon\bigr)(t_r) \\
&\quad - \int_0^1\int_0^{t_1}\cdots\int_0^{t_{r-1}} d\bigl(\bar{A}_0 + x_\gamma^\epsilon\bigr)(t_1)\cdots d\bigl(\bar{A}_0 + x_\gamma^\epsilon\bigr)(t_r)\Biggr\}\Biggr\| \\
&\leqslant \Biggl\|\sum_{r=1}^{\infty}\sum_{m=1}^{r}\int_0^1\int_0^{t_1}\cdots\int_0^{t_{r-1}} d\bigl(\bar{A}_0 + x_\gamma^\epsilon\bigr)(t_1)\cdots d\bigl(\bar{A}_0 + x_\gamma^\epsilon\bigr)(t_{m-1}) \\
&\quad \cdot dh_\gamma^\epsilon(t_m)\, d\bigl(\bar{A}_0 + (x+sh)_\gamma^\epsilon\bigr)(t_{m+1})\cdots d\bigl(\bar{A}_0 + (x+sh)_\gamma^\epsilon\bigr)(t_r)\Biggr\| \\
&\leqslant \sum_{r=1}^{\infty}\sum_{m=1}^{r}\sigma^r\bigl(c_2(A_0) + c_1(\epsilon)\|x\|_B\bigr)^{m-1} \\
&\quad \times c_1(\epsilon)\|h\|_B\bigl(c_2(A_0) + c_1(\epsilon)\bigl\{\|x\|_B + \|h\|_B\bigr\}\bigr)^{r-m}/r! \\
&\leqslant \sum_{r=1}^{\infty}\sigma^r\bigl(c_2(A_0) + c_1(\epsilon)\bigl\{\|x\|_B + \|h\|_B\bigr\}\bigr)^{r-1}c_1(\epsilon)\|h\|_B/(r-1)! \\
&= \sigma c_1(\epsilon)\|h\|_B e^{\sigma(c_2(A_0)+c_1(\epsilon)(\|x\|_B+\|h\|_B))} < \infty.
\end{aligned}
$$

This, together with Lebesgue's convergence theorem, implies that $W_\gamma^\epsilon(x)$ is H-Fréchet differentiable. Repeating this argument, we then obtain that $W_\gamma^\epsilon(x)$ is C^∞ in H-Fréchet differentiation.

For the proof of (2) we recall the following lemma due to Fernique (see [16]).

Lemma 2. *There exists $\delta > 0$ such that*

$$
\int_B e^{\delta\|x\|_B^2}\,\mu(dx) < \infty.
$$

Then it follows from (4.5) that

$$E\big[\big\|W_\gamma(x)\big\|^{2q}\big] \leqslant E\big[e^{2q\sigma(c_2(A_0)+c_1(\epsilon)\|x\|_B)}\big],$$

which together with Lemma 2 shows (2) of Lemma 1.

Before proceeding to the proof of (3), we remark the following

Lemma 3. *Let q be a positive integer and $X_{i,j}$, $i, j = 1, 2, \dots,$ be real numbers. Then*

$$\sum_i \Big|\sum_j X_{i,j}\Big|^{2q} \leqslant \bigg(\sum_j \Big(\sum_i |X_{i,j}|^{2q}\Big)^{1/2q}\bigg)^{2q}.$$

Proof of Lemma 3. Note that

$$\Big(\sum_j |X_{i,j}|\Big)^{2q} = \sum_{j_1, j_2, \dots, j_{2q}} |X_{i,j_1}||X_{i,j_2}|\cdots|X_{i,j_{2q}}|,$$

and by using Hölder's inequality recursively we have

$$\begin{aligned}
&\sum_i |X_{i,j_1}||X_{i,j_2}|\cdots|X_{i,j_{2q}}| \\
&\quad\leqslant \Big(\sum_i |X_{i,j_1}|^{2q}\Big)^{1/2q} \Big(\sum_i \big(|X_{i,j_2}|\cdots|X_{i,j_{2q}}|\big)^{2q/(2q-1)}\Big)^{(2q-1)/2q} \\
&\quad\leqslant \Big(\sum_i |X_{i,j_1}|^{2q}\Big)^{1/2q} \Big(\sum_i |X_{i,j_2}|^{2q}\Big)^{1/2q} \\
&\qquad\times \Big(\sum_i \big(|X_{i,j_3}|\cdots|X_{i,j_{2q}}|\big)^{2q/(2q-2)}\Big)^{(2q-2)/2q}
\end{aligned}$$

and so on. Hence we obtain

$$\begin{aligned}
&\sum_i \Big|\sum_j X_{i,j}\Big|^{2q} \leqslant \sum_i \Big(\sum_{j_1, j_2, \dots, j_{2q}} |X_{i,j_1}||X_{i,j_2}|\cdots|X_{i,j_{2q}}|\Big) \\
&\quad= \sum_{j_1, j_2, \dots, j_{2q}} \Big(\sum_i |X_{i,j_1}||X_{i,j_2}|\cdots|X_{i,j_{2q}}|\Big) \\
&\quad\leqslant \sum_{j_1, j_2, \dots, j_{2q}} \Big(\sum_i |X_{i,j_1}|^{2q}\Big)^{1/2q} \Big(\sum_i |X_{i,j_2}|^{2q}\Big)^{1/2q} \cdots \Big(\sum_i |X_{i,j_{2q}}|^{2q}\Big)^{1/2q} \\
&\quad= \bigg(\sum_j \Big(\sum_i |X_{i,j}|^{2q}\Big)^{1/2q}\bigg)^{2q},
\end{aligned}$$

which completes the proof of Lemma 3.

Now we proceed to proving (3) of Lemma 1. Noting that

$$\sum_{i_1,i_2,\ldots,i_k} \left\| D^k W_\gamma^\epsilon(x)(h_{i_1}, h_{i_2}, \ldots, h_{i_k}) \right\|^2 \leqslant \sum_{i_1,i_2,\ldots,i_k} \left(\sum_{r=k}^{\infty} \left\| D^k W_\gamma^{\epsilon,r}(x)(h_{i_1}, h_{i_2}, \ldots, h_{i_k}) \right\| \right)^2,$$

and by making use of Lemma 3 recursively, it is immediate to see that the right side of the above inequality is dominated by

$$\left(\sum_{r=k}^{\infty} \left(\sum_{i_1,i_2,\ldots,i_k} \left\| D^k W_\gamma^{\epsilon,r}(x)(h_{i_1}, h_{i_2}, \ldots, h_{i_k}) \right\|^2 \right)^{1/2} \right)^2.$$

Let us denote for simplicity

$$\sum_{\substack{1\leqslant l_1<l_2<\cdots<l_k\leqslant r,\\ \{j(l_1),j(l_2),\ldots,j(l_k)\}=\{1,2,\ldots,k\}}} \quad \text{by} \quad \sum_{l_1,l_2,\ldots,l_k}.$$

Then, employing Lemma 3 again, we see that

$$\begin{aligned}
&\sum_{i_1,i_2,\ldots,i_k} \left\| D^k W_\gamma^{\epsilon,r}(x)(h_{i_1}, h_{i_2}, \ldots, h_{i_k}) \right\|^2 \\
&= \sum_{i_1,i_2,\ldots,i_k} \Bigg\| \sum_{l_1,l_2,\ldots,l_k} \int_0^1 d\big(\bar{A}_0 + x_\gamma^\epsilon\big)(t_1) \cdots \int_0^{t_{l_1-1}} dh^\epsilon_{i_{j(l_1)}}(t_{l_1}) \cdots \\
&\quad \cdot \int_0^{t_{l_k-1}} dh^\epsilon_{i_{j(l_k)}}(t_{l_k}) \cdots \int_0^{t_{r-1}} d\big(\bar{A}_0 + x_\gamma^\epsilon\big)(t_r) \Bigg\|^2 \\
&\leqslant \sum_{i_1,i_2,\ldots,i_k} \Bigg(c_E^r \sum_{l_1,l_2,\ldots,l_k} \sum_{\alpha_1,\alpha_2,\ldots,\alpha_r=1}^{d} \Bigg| \int_0^1 d(\bar{A}_0^{\alpha_1} + x_\gamma^{\epsilon,\alpha_1})(t_1) \cdots \\
&\quad \cdot \int_0^{t_{l_1-1}} d\big\langle h_{i_{j(l_1)}}, \tilde{C}_\gamma^{\epsilon,\alpha_{l_1}}(t_{l_1})\big\rangle \cdots \int_0^{t_{l_k-1}} d\big\langle h_{i_{j(l_k)}}, \tilde{C}_\gamma^{\epsilon,\alpha_{l_k}}(t_{l_k})\big\rangle \\
&\quad \cdot \cdots \int_0^{t_{r-1}} d\big(\bar{A}_0^{\alpha_r} + x_\gamma^{\epsilon,\alpha_r}\big)(t_r) \Bigg| \Bigg)^2 \\
&\leqslant \Bigg(c_E^r \sum_{l_1,l_2,\ldots,l_k} \sum_{\alpha_1,\alpha_2,\ldots,\alpha_r=1}^{d} \Bigg(\sum_{i_1,i_2,\ldots,i_k} \Bigg| \int_0^1 d\big(\bar{A}_0^{\alpha_1} + x_\gamma^{\epsilon,\alpha_1}\big)(t_1) \cdots
\end{aligned}$$

$$\cdot \int_0^{t_{l_1-1}} d\big\langle h_{i_{j(l_1)}}, \tilde{C}_\gamma^{\epsilon,\alpha_{l_1}}(t_{l_1})\big\rangle \cdots \int_0^{t_{l_k-1}} d\big\langle h_{i_{j(l_k)}}, \tilde{C}_\gamma^{\epsilon,\alpha_{l_k}}(t_{l_k})\big\rangle$$
$$\cdot \cdots \int_0^{t_{r-1}} d\big(\bar{A}_0^{\alpha_r} + x_\gamma^{\epsilon,\alpha_r}\big)(t_r)\bigg|^2\bigg)^{1/2}\bigg)^2,$$

where we write $\tilde{C}_\gamma^{\epsilon,\alpha}(t) = \tilde{C}_\gamma^\epsilon(t)^\alpha \otimes E_\alpha$ for simplicity.

Noticing that, for example,

$$\begin{aligned}
&\sum_{i_j}\bigg|\int_0^s d\big\langle h_{i_j}, \tilde{C}_\gamma^{\epsilon,\alpha}(v)\big\rangle \int_0^v d\big(\bar{A}_0^\beta + x_\gamma^{\epsilon,\beta}\big)(w)\bigg|^2 \\
&\quad= \sum_{i_j}\bigg|\lim_{m\to\infty}\sum_{t=0}^m \big\langle h_{i_j}, \tilde{C}_\gamma^{\epsilon,\alpha}(\tau_{t+1}) - \tilde{C}_\gamma^{\epsilon,\alpha}(\tau_t)\big\rangle \int_0^{\tau_t} d\big(\bar{A}_0^\beta + x_\gamma^{\epsilon,\beta}\big)(w)\bigg|^2 \\
&\quad\leqslant \bigg(\lim_{m\to\infty}\sum_{t=0}^m\bigg(\sum_{i_j}\big|\big\langle h_{i_j}, \tilde{C}_\gamma^{\epsilon,\alpha}(\tau_{t+1}) - \tilde{C}_\gamma^{\epsilon,\alpha}(\tau_t)\big\rangle\big|^2\bigg|\int_0^{\tau_t} d\big(\bar{A}_0^\beta + x_\gamma^{\epsilon,\beta}\big)(w)\bigg|^2\bigg)^{1/2}\bigg)^2 \\
&\quad\leqslant \bigg(\lim_{m\to\infty}\sum_{t=0}^m \big\|\tilde{C}_\gamma^{\epsilon,\alpha}(\tau_{t+1}) - \tilde{C}_\gamma^{\epsilon,\alpha}(\tau_t)\big\|_p\bigg|\int_0^{\tau_t} d\big(\bar{A}_0^\beta + x_\gamma^{\epsilon,\beta}\big)(w)\bigg|\bigg)^2 \\
&\quad\leqslant \bigg(c_1(\epsilon)\big(c_2(A_0) + c_1(\epsilon)\|x\|_B\big)\int_0^s\int_0^v dv\,dw\bigg)^2,
\end{aligned}$$

we obtain as in the proof of (4.5) that

$$\begin{aligned}
&\sum_{i_1,i_2,\ldots,i_k}\big\|D^k W_\gamma^{\epsilon,r}(x)(h_{i_1}, h_{i_2}, \ldots, h_{i_k})\big\|^2 \\
&\quad\leqslant \bigg(\sigma^r\frac{r!}{(r-k)!}\big(c_2(A_0) + c_1(\epsilon)\|x\|_B\big)^{r-k}c_1(\epsilon)^k \\
&\qquad\times \int_0^1\cdots\int_0^{t_{l_1-1}}\cdots\int_0^{t_{l_k-1}}\cdots\int_0^{t_{r-1}} dt_1\cdots dt_{l_1-1}\cdots dt_{l_k-1}\cdots dt_r\bigg)^2 \\
&\quad\leqslant \bigg(\sigma^r\frac{r!}{(r-k)!r!}\big(c_2(A_0) + c_1(\epsilon)\|x\|_B\big)^{r-k}c_1(\epsilon)^k\bigg)^2.
\end{aligned}$$

Hence, noting that

$$\sum_{r=k}^{\infty}\sigma^{r}\frac{1}{(r-k)!}\big(c_2(A_0)+c_1(\epsilon)\|x\|_B\big)^{r-k}c_1(\epsilon)^k$$
$$=\sum_{r=0}^{\infty}\sigma^{r+k}\frac{1}{r!}\big(c_2(A_0)+c_1(\epsilon)\|x\|_B\big)^{r}c_1(\epsilon)^k$$
$$=\big(\sigma c_1(\epsilon)\big)^k e^{\sigma(c_2(A_0)+c_1(\epsilon)\|x\|_B)},$$

we see with Lemma 2 that

$$\sum_{k=0}^{\infty}\frac{s^k}{k!}E\bigg[\bigg(\sum_{i_1,i_2,\ldots,i_k}\big\|D^kW_\gamma^\epsilon(x)(h_{i_1},h_{i_2},\ldots,h_{i_k})\big\|^2\bigg)^q\bigg]^{1/2q}$$
$$\leqslant\sum_{k=0}^{\infty}\frac{s^k}{k!}\big(\sigma c_1(\epsilon)\big)^k E\big[e^{2q\sigma(c_2(A_0)+c_1(\epsilon)\|x\|_B)}\big]^{1/2q}<\infty,$$

which verifies the first part of (3).

By a similar argument we can also obtain the second half of (3), so is omitted the detail. □

5. Definition and expansion theorem

The aim of this section is to give a rigorous mathematical meaning, in an abstract Wiener space setting, to the normalized one-loop approximation of the Lorentz gauge-fixed Chern–Simons integral (2.11). We keep the notation in Section 4.

First, recall that for each $x=(A,\phi)\in L^2(\Omega_+)=L^2(\Omega^1\oplus\Omega^3)$ we have

$$(x,Q_{A_0}x)_+=\sum_{i=1}^{\infty}\lambda_i(x,e_i)_+^2=\sum_{j=1}^{\infty}\big(1+\lambda_j^2\big)^{-p}\lambda_j(x,h_j)_p^2.$$

Then, adopting an idea due to Itô [15], we implement convergent factors

$$\exp\bigg[-\frac{(x,x)}{2n}\bigg]\quad\text{with } n>0$$

into each finite-dimensional approximation of $L^2(\Omega_+)$. This leads us to the following m-dimensional approximation of (2.11) written as

$$\lim_{n\to\infty}\frac{1}{Z_{m,n}}\int_{\boldsymbol{R}^m}F_{A_0}^\epsilon(x_m)\exp\bigg[\sqrt{-1}k(x,Qx)_{m,+}-\frac{(x,x)_m}{2n}\bigg]\frac{\mu_m(dx)}{(\sqrt{2\pi})^m},$$

where μ_m is the m-dimensional Lebesgue measure,

$$x_m=\sum_{j=1}^{m}x_jh_j,\qquad (x,Qx)_{m,+}=\sum_{j=1}^{m}\big(1+\lambda_j^2\big)^{-p}\lambda_jx_j^2,\qquad (x,x)_m=\sum_{j=1}^{m}x_j^2$$

and

$$Z_{m,n} = \int_{\boldsymbol{R}^m} \exp\left[\sqrt{-1}k(x, Qx)_{m,+} - \frac{(x,x)_m}{2n}\right] \frac{\mu_m(dx)}{(\sqrt{2\pi})^m}.$$

Note that, by setting $x = \sqrt{n}y$, this can be rewritten in the form

$$\lim_{n\to\infty} \frac{1}{Z_{m,n}} \int_{\boldsymbol{R}^m} F^{\epsilon}_{A_0}\left(\sqrt{n}y_m\right) \exp\left[\sqrt{-1}k\left(\sqrt{n}y, Q\sqrt{n}y\right)_{m,+}\right] \times \frac{1}{(\sqrt{2\pi})^m} \exp\left[-\frac{(y,y)_m}{2}\right] \mu_m(dy),$$

where

$$Z_{m,n} = \int_{\boldsymbol{R}^m} \exp\left[\sqrt{-1}k\left(\sqrt{n}y, Q\sqrt{n}y\right)_{m,+}\right] \frac{1}{(\sqrt{2\pi})^m} \exp\left[-\frac{(y,y)_m}{2}\right] \mu_m(dy).$$

We then look for the limit

$$\lim_{n\to\infty} \lim_{m\to\infty} \frac{1}{Z_{m,n}} \int_{\boldsymbol{R}^m} F^{\epsilon}_{A_0}\left(\sqrt{n}y_m\right) \exp\left[\sqrt{-1}k\left(\sqrt{n}y, Q\sqrt{n}y\right)_{m,+}\right] \times \frac{1}{(\sqrt{2\pi})^m} \exp\left[-\frac{(y,y)_m}{2}\right] \mu_m(dy). \tag{5.1}$$

However, the canonical Gaussian measure cannot be defined on the Hilbert space $L^2(\Omega_+)$, so that we shall achieve a realization of (5.1) in an abstract Wiener space setting as follows.

Thus, let $H = H_p$ and (B, H, μ) the abstract Wiener space described in Section 4. Then, within this framework, we now define the *normalized one-loop approximation of the perturbative Chern–Simons integral* of the ϵ-regularized Wilson line to be

$$I_{CS}\left(F^{\epsilon}_{A_0}\right) = \limsup_{n\to\infty} \frac{1}{Z_n} \int_B F^{\epsilon}_{A_0}\left(\sqrt{n}x\right) e^{\sqrt{-1}kCS(\sqrt{n}x)} \mu(dx), \tag{5.2}$$

where

$$Z_n = \int_B e^{\sqrt{-1}kCS(\sqrt{n}x)} \mu(dx),$$

$$CS(x) = \left\langle x, \left(I + Q_{A_0}^2\right)^{-p} Q_{A_0} x\right\rangle = \sum_{j=1}^{\infty} \left(1 + \lambda_j^2\right)^{-p} \lambda_j \langle x, h_j\rangle^2,$$

and

$$\limsup_{n\to\infty}\left(x_n + \sqrt{-1}y_n\right) = \limsup_{n\to\infty} x_n + \sqrt{-1} \limsup_{n\to\infty} y_n$$

for real numbers x_n and y_n.

Given $\epsilon > 0$, we also set

$$Z_\gamma^{\epsilon,0}(0) = I,$$

$$Z_\gamma^{\epsilon,r}(i) = \sum_{1\leqslant l_1<l_2<\cdots<l_i\leqslant r} \int_0^1 d\bar{A}_0(t_1)\cdots \int_0^{t_{l_1-1}} dx_\gamma^\epsilon(t_{l_1})\cdots \int_0^{t_{l_i-1}} dx_\gamma^\epsilon(t_{l_i})\cdots \int_0^{t_{r-1}} d\bar{A}_0(t_r)$$

and

$$Z_\gamma^\epsilon(i) = \sum_{r=i}^{\infty} Z_\gamma^{\epsilon,r}(i).$$

It should be noted that

$$\begin{aligned} W_\gamma^{\epsilon,r}(x) &= \int_0^1\int_0^{t_1}\cdots\int_0^{t_{r-1}} d\big(\bar{A}_0 + x_\gamma^\epsilon\big)(t_1)\, d\big(\bar{A}_0 + x_\gamma^\epsilon\big)(t_2)\cdots d\big(\bar{A}_0 + x_\gamma^\epsilon\big)(t_r) \\ &= \sum_{i=0}^{r} Z_\gamma^{\epsilon,r}(i), \end{aligned}$$

which combined with (4.3) yields

$$W_\gamma^\epsilon(x) = I + \sum_{r=1}^{\infty} W_\gamma^{\epsilon,r}(x) = \sum_{i=0}^{\infty} Z_\gamma^\epsilon(i).$$

Thus we define

$$F_{A_0}^{\epsilon,m}(x) = \sum_{i_1+i_2+\cdots+i_s=m} \prod_{j=1}^{s} \mathrm{Tr}_{R_j}\, Z_{\gamma_j}^\epsilon(i_j) \tag{5.3}$$

and set

$$R_{n,k} = \big\{I - 2\sqrt{-1}nk\big(I + Q_{A_0}^2\big)^{-p} Q_{A_0}\big\}^{-1/2}\sqrt{n}I. \tag{5.4}$$

Then, by applying the formula due to Malliavin and Taniguchi [17, Theorem 7.8], we obtain the following expansion theorem.

Theorem 1. *For any fixed $\epsilon > 0$ and positive integer N,*

$$\begin{aligned} I_{CS}\big(F_{A_0}^\epsilon\big) &= \limsup_{n\to\infty} \int_B F_{A_0}^\epsilon(R_{n,k}x)\,\mu(dx) = \int_B F_{A_0}^\epsilon(R_k x)\,\mu(dx) \\ &= \sum_{m<N} k^{-m/2}\cdot J_{CS}^{\epsilon,m} + O\big(k^{-N/2}\big), \end{aligned}$$

where

$$R_k = \left\{-2\sqrt{-1}k\left(I + Q_{A_0}^2\right)^{-p} Q_{A_0}\right\}^{-1/2}, \tag{5.5}$$

and

$$J_{CS}^{\epsilon,m} = k^{m/2} \cdot \int_B F_{A_0}^{\epsilon,m}(R_k x)\,\mu(dx).$$

Proof. *Step* 1. By making use of the so-called Fresnel integral formula

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left[-\frac{zx^2}{2}\right] dx = \frac{1}{\sqrt{z}}, \quad z \in \boldsymbol{C},$$

separately, we obtain

$$Z_n = \left[\det\left\{I - 2\sqrt{-1}nk\left(I + Q_{A_0}^2\right)^{-p} Q_{A_0}\right\}\right]^{-1/2}.$$

Also, it follows from (3.7) that

$$\left\|\sqrt{n}\left(\tilde{C}_\gamma^\epsilon(t) - \tilde{C}_\gamma^\epsilon(s)\right)\right\|_p \leqslant c_3(\epsilon)|t-s|.$$

Hence, by mimicking the proof of (3) of Lemma 1, we see that for any sufficiently small fixed $\epsilon > 0$, the same inequalities in the course of the proof hold with $W_\gamma^\epsilon(x)$ being replaced by $W_\gamma^\epsilon(\sqrt{n}x)$. This, together with (1) of Lemma 1, then yields that

$$\sum_{k=0}^{\infty} \frac{s^k}{k!} E\left[\left(\sum_{i_1,i_2,\ldots,i_k} \left|D^k F_{A_0}^\epsilon\left(\sqrt{n}x\right)(h_{i_1}, h_{i_2}, \ldots, h_{i_k})\right|^2\right)^q\right]^{1/2q} < \infty$$

for any positive number s, implying the analyticity of $F_{A_0}^\epsilon(\sqrt{n}x)$.

Therefore, we can apply the formula of Malliavin and Taniguchi [17, Theorem 7.8] to the right side of (5.2) to obtain, for any sufficiently small fixed $\epsilon > 0$, that

$$I_{CS}\left(F_{A_0}^\epsilon\right) = \limsup_{n\to\infty} \int_B F_{A_0}^\epsilon(R_{n,k}x)\,\mu(dx). \tag{5.6}$$

Step 2. In order to determine the limit in (5.6), we first note that for any positive integer q we have

$$E\left[\left\|W_\gamma^\epsilon(R_{n,k}x)\right\|^{2q}\right] < \infty. \tag{5.7}$$

To see this and for later use as well, we now carry out a more precise estimate than that of proving (2) of Lemma 1 in the following way.

For the twisted Dirac operator Q_{A_0}, we define $a^j_{n,k}, b^j_{n,k} \in \boldsymbol{R}$ by

$$a^j_{n,k} + \sqrt{-1} b^j_{n,k} = \frac{\sqrt{n}}{\sqrt{1 - 2\sqrt{-1} nk (1+\lambda_j^2)^{-p} \lambda_j}},$$

where λ_j are eigenvalues of Q_{A_0} as above. Then we set

$$R^1_{n,k} \tilde{C}^\epsilon_\gamma(t)^\alpha \otimes E_\alpha = \sum_{j=1}^\infty a^j_{n,k} \big(\tilde{C}^\epsilon_\gamma(t)^\alpha \otimes E_\alpha, h_j\big)_p h_j,$$

$$R^2_{n,k} \tilde{C}^\epsilon_\gamma(t)^\alpha \otimes E_\alpha = \sum_{j=1}^\infty b^j_{n,k} \big(\tilde{C}^\epsilon_\gamma(t)^\alpha \otimes E_\alpha, h_j\big)_p h_j.$$

Note that, for each $x \in B$ and $t \in [0,1]$, the operator $R_{n,k}$ defined by (5.4) gives rise to an element

$$R_{n,k} x^\epsilon_\gamma(t) = \sum_{\alpha=1}^d \big\langle x, R_{n,k} \tilde{C}^\epsilon_\gamma(t)^\alpha \otimes E_\alpha \big\rangle E_\alpha \tag{5.8}$$

in the complexification of $\mathfrak{g}$, where $R_{n,k} \tilde{C}^\epsilon_\gamma(t)^\alpha \otimes E_\alpha$ is defined by

$$R_{n,k} \tilde{C}^\epsilon_\gamma(t)^\alpha \otimes E_\alpha = R^1_{n,k} \tilde{C}^\epsilon_\gamma(t)^\alpha \otimes E_\alpha + \sqrt{-1} R^2_{n,k} \tilde{C}^\epsilon_\gamma(t)^\alpha \otimes E_\alpha.$$

For convenience we denote the accompanying Gaussian random variables by

$$R^1_{n,k} x^{\epsilon,\alpha}_\gamma(t) = \big\langle x, R^1_{n,k} \tilde{C}^\epsilon_\gamma(t)^\alpha \otimes E_\alpha \big\rangle, \qquad R^2_{n,k} x^{\epsilon,\alpha}_\gamma(t) = \big\langle x, R^2_{n,k} \tilde{C}^\epsilon_\gamma(t)^\alpha \otimes E_\alpha \big\rangle \tag{5.9}$$

and set

$$R_{n,k} x^{\epsilon,\alpha}_\gamma(t) = R^1_{n,k} x^{\epsilon,\alpha}_\gamma(t) + \sqrt{-1} R^2_{n,k} x^{\epsilon,\alpha}_\gamma(t).$$

Now, noting that

$$\int_0^1 \int_0^{t_1} \cdots \int_0^{t_{r-1}} d\big(\bar{A}_0 + R_{n,k} x^\epsilon_\gamma\big)(t_1)\, d\big(\bar{A}_0 + R_{n,k} x^\epsilon_\gamma\big)(t_2) \cdots d\big(\bar{A}_0 + R_{n,k} x^\epsilon_\gamma\big)(t_r)$$

$$= \sum_{m=0}^r \sum_{1 \leqslant l_1 < l_2 < \cdots < l_m \leqslant r} \int_0^1 d\bar{A}_0(t_1) \cdots \int_0^{t_{l_1 - 1}} dR_{n,k} x^\epsilon_\gamma(t_{l_1}) \cdots \int_0^{t_{l_m - 1}} dR_{n,k} x^\epsilon_\gamma(t_{l_m})$$

$$\cdot \cdots \int_0^{t_{r-1}} d\bar{A}_0(t_r),$$

we obtain, by the same reasoning as in Lemma 3, that for any positive integer q and $x \in B$

$$
\begin{aligned}
&E\big[\big\|W_\gamma^\epsilon(R_{n,k}x)\big\|^{2q}\big]\\
&\leqslant E\Bigg[\Bigg(\sum_{r=0}^{\infty}\sum_{m=0}^{r}\sum_{1\leqslant l_1<l_2<\cdots<l_m\leqslant r}\sum_{\alpha_1,\alpha_2,\ldots,\alpha_r=1}^{d} c_E^r\Bigg|\int_0^1 d\bar{A}_0^{\alpha_1}(t_1)\cdots\\
&\quad\cdot\int_0^{t_{l_1-1}} dR_{n,k}x_\gamma^{\epsilon,\alpha_{l_1}}(t_{l_1})\cdots\int_0^{t_{l_m-1}} dR_{n,k}x_\gamma^{\epsilon,\alpha_{l_m}}(t_{l_m})\cdots\int_0^{t_{r-1}} d\bar{A}_0^{\alpha_r}(t_r)\Bigg|\Bigg)^{2q}\Bigg]\\
&\leqslant E\Bigg[\Bigg(\sum_{r=0}^{\infty}\sum_{m=0}^{r}\sum_{\substack{1\leqslant l_1<l_2<\cdots<l_m\leqslant r,\\ \nu_1,\nu_2,\ldots,\nu_m\in\{1,2\}}}\sum_{\alpha_1,\alpha_2,\ldots,\alpha_r=1}^{d} c_E^r\Bigg|\int_0^1 d\bar{A}_0^{\alpha_1}(t_1)\cdots\\
&\quad\cdot\int_0^{t_{l_1-1}} dR_{n,k}^{\nu_1}x_\gamma^{\epsilon,\alpha_{l_1}}(t_{l_1})\cdots\int_0^{t_{l_m-1}} dR_{n,k}^{\nu_m}x_\gamma^{\epsilon,\alpha_{l_m}}(t_{l_m})\cdots\int_0^{t_{r-1}} d\bar{A}_0^{\alpha_r}(t_r)\Bigg|\Bigg)^{2q}\Bigg]\\
&\leqslant\Bigg(\sum_{r=0}^{\infty}\sum_{m=0}^{r}\sum_{\substack{1\leqslant l_1<l_2<\cdots<l_m\leqslant r,\\ \nu_1,\nu_2,\ldots,\nu_m\in\{1,2\}}}\sum_{\alpha_1,\alpha_2,\ldots,\alpha_r=1}^{d} c_E^r E\Bigg[\Bigg|\int_0^1 d\bar{A}_0^{\alpha_1}(t_1)\cdots\\
&\quad\cdot\int_0^{t_{l_1-1}} dR_{n,k}^{\nu_1}x_\gamma^{\epsilon,\alpha_{l_1}}(t_{l_1})\cdots\int_0^{t_{l_m-1}} dR_{n,k}^{\nu_m}x_\gamma^{\epsilon,\alpha_{l_m}}(t_{l_m})\cdots\int_0^{t_{r-1}} d\bar{A}_0^{\alpha_r}(t_r)\Bigg|^{2q}\Bigg]^{1/2q}\Bigg)^{2q}.
\end{aligned}
\tag{5.10}
$$

To estimate the right side of (5.10), let s_i, $i=0,1,\ldots,r$, be non-negative integers and set

$$
t_i^{s_i}=\begin{cases}0 & \text{if } s_i=0,\\ t_i^{s_i-1}+t_{i-1}^{s_{i-1}}/2^{n_i} & \text{if } s_i\geqslant 1,\end{cases}
$$

with $t_0^{s_0}=1$. Also, write for brevity

$$
\begin{aligned}
A_0^{\alpha_i}[s_i]&=\bar{A}_0^{\alpha_i}\big(t_i^{s_i+1}\big)-\bar{A}_0^{\alpha_i}\big(t_i^{s_i}\big),\\
R_{n,k}^{\nu}x_\gamma^{\epsilon,\alpha_i}[s_i]&=R_{n,k}^{\nu}x_\gamma^{\epsilon,\alpha_i}\big(t_i^{s_i+1}\big)-R_{n,k}^{\nu}x_\gamma^{\epsilon,\alpha_i}\big(t_i^{s_i}\big).
\end{aligned}
$$

Then it follows from an estimate similar to that of (2) of Lemma 1 together with Lebesgue's convergence theorem that

$$
E\Bigg[\Bigg|\int_0^1 d\bar{A}_0^{\alpha_1}(t_1)\cdots\int_0^{t_{l_1-1}} dR_{n,k}^{\nu_1}x_\gamma^{\epsilon,\alpha_{l_1}}(t_{l_1})\cdots
$$

$$\cdot \int_0^{t_{l_m-1}} dR_{n,k}^{\nu_m} x_\gamma^{\epsilon,\alpha_{l_m}}(t_{l_m}) \cdots \int_0^{t_{r-1}} d\bar{A}_0^{\alpha_r}(t_r) \Bigg|^{2q} \Bigg]^{1/2q}$$

$$= \lim_{n_1,\dots,n_r \to \infty} E\Bigg[\Bigg| \sum_{s_1=0}^{2^{n_1}-1} A_0^{\alpha_1}[s_1] \cdots \sum_{s_{l_1}=0}^{2^{n_{l_1}}-1} R_{n,k}^{\nu_1} x_\gamma^{\epsilon,\alpha_{l_1}}[s_{l_1}] \cdots$$

$$\cdot \sum_{s_{l_m}=0}^{2^{n_{l_m}}-1} R_{n,k}^{\nu_m} x_\gamma^{\epsilon,\alpha_{l_m}}[s_{l_m}] \cdots \sum_{s_r=0}^{2^{n_r}-1} A_0^{\alpha_r}[s_r] \Bigg|^{2q} \Bigg]^{1/2q}$$

$$\leqslant c_2(A_0)^{r-m} \lim_{n_1,\dots,n_r \to \infty} E\Bigg[\Bigg(\sum_{s_1=0}^{2^{n_1}-1} \cdots \sum_{s_r=0}^{2^{n_r}-1} \big|t_1^{s_1+1} - t_1^{s_1}\big| \cdots \big|R_{n,k}^{\nu_1} x_\gamma^{\epsilon,\alpha_{l_1}}[s_{l_1}]\big|$$

$$\cdot \cdots \big|R_{n,k}^{\nu_m} x_\gamma^{\epsilon,\alpha_{l_m}}[s_{l_m}]\big| \cdots \big|t_r^{s_r+1} - t_r^{s_r}\big| \Bigg)^{2q} \Bigg]^{1/2q},$$

which is, by the same reasoning as in Lemma 3, dominated by

$$c_2(A_0)^{r-m} \lim_{n_1,\dots,n_r \to \infty} \sum_{s_1=0}^{2^{n_1}-1} \cdots \sum_{s_r=0}^{2^{n_r}-1}$$
$$E\Big[\Big(\big|t_1^{s_1+1} - t_1^{s_1}\big| \cdots \big|R_{n,k}^{\nu_1} x_\gamma^{\epsilon,\alpha_{l_1}}[s_{l_1}]\big| \cdots \big|R_{n,k}^{\nu_m} x_\gamma^{\epsilon,\alpha_{l_m}}[s_{l_m}]\big| \cdots \big|t_r^{s_r+1} - t_r^{s_r}\big|\Big)^{2q}\Big]^{1/2q}. \tag{5.11}$$

Furthermore, for the Gaussian random variables (5.9), we see from (3.7) and (4.1) that for $\nu = 1, 2$

$$\begin{aligned}
E\Big[\big|R_{n,k}^{\nu} x_\gamma^{\epsilon,\alpha}(t) - R_{n,k}^{\nu} x_\gamma^{\epsilon,\alpha}(s)\big|^2\Big] &= \big\| R_{n,k}^{\nu} \tilde{C}_\gamma^\epsilon(t)^\alpha \otimes E_\alpha - R_{n,k}^{\nu} \tilde{C}_\gamma^\epsilon(s)^\alpha \otimes E_\alpha \big\|_p^2 \\
&= \sum_{j=1}^{\infty} \Big(\big(a_{n,k}^j \text{ or } b_{n,k}^j\big) \big(\tilde{C}_\gamma^\epsilon(t)^\alpha \otimes E_\alpha - \tilde{C}_\gamma^\epsilon(s)^\alpha \otimes E_\alpha, h_j\big)_p \Big)^2 \\
&\leqslant \sum_{j=1}^{\infty} \frac{1}{2k|\lambda_j|} \big(C_\gamma^\epsilon(t)^\alpha \otimes E_\alpha - C_\gamma^\epsilon(s)^\alpha \otimes E_\alpha, e_j \big)^2 \\
&\leqslant \frac{1}{2k\rho} \big\| C_\gamma^\epsilon(t)^\alpha \otimes E_\alpha - C_\gamma^\epsilon(s)^\alpha \otimes E_\alpha \big\|_0^2 \\
&\leqslant \frac{1}{2k\rho} c_1(\epsilon)^2 |t-s|^2,
\end{aligned} \tag{5.12}$$

where we set

$$\rho = \min_j |\lambda_j| > 0.$$

Now we recall the following well-known lemma (see [8]).

Lemma 4. *Let X_i, $i = 1, 2, \dots, 2l$, be a mean-zero Gaussian system. Then*

$$E[X_1 X_2 \cdots X_{2l}] = \frac{1}{2^l l!} \sum_{\sigma \in \mathfrak{S}_{2l}} E[X_{\sigma(1)} X_{\sigma(2)}] E[X_{\sigma(3)} X_{\sigma(4)}] \cdots E[X_{\sigma(2l-1)} X_{\sigma(2l)}],$$

where $\mathfrak{S}_{2l}$ denotes the group of permutations of $\{1, 2, \dots, 2l\}$.

Then it follows from (5.12) together with Lemma 4 that

$$E\big[\big(\big|R_{n,k}^{\nu_1} x_\gamma^{\epsilon,\alpha_{l_1}}[s_{l_1}]\big| \cdots \big|R_{n,k}^{\nu_m} x_\gamma^{\epsilon,\alpha_{l_m}}[s_{l_m}]\big|\big)^{2q}\big] \leqslant \frac{(2qm)!(c_1(\epsilon)/\sqrt{2k\rho})^{2qm}}{2^{qm}(qm)!} \big|t_{l_1}^{s_{l_1}+1} - t_{l_1}^{s_{l_1}}\big|^{2q} \cdots \big|t_{l_m}^{s_{l_m}+1} - t_{l_m}^{s_{l_m}}\big|^{2q},$$

from which we see that (5.11) is then dominated by

$$\begin{aligned} & c_2(A_0)^{r-m} \lim_{n_1,\dots,n_r \to \infty} \sum_{s_1=0}^{2^{n_1}-1} \cdots \sum_{s_r=0}^{2^{n_r}-1} \left\{ \frac{(2qm)!(c_1(\epsilon)/\sqrt{2k\rho})^{2qm}}{2^{qm}(qm)!} \right\}^{1/2q} \\ & \quad \cdot \big|t_1^{s_1+1} - t_1^{s_1}\big| \cdots \big|t_r^{s_r+1} - t_r^{s_r}\big| \\ & \leqslant c_2(A_0)^{r-m} \left(\frac{c_1(\epsilon)}{\sqrt{2k\rho}} \right)^m \left\{ \frac{(2qm)!}{2^{qm}(qm)!} \right\}^{1/2q} \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{r-1}} dt_1\, dt_2 \cdots dt_r \\ & \leqslant c_4(A_0)^r \left(\frac{\sqrt{2q}}{\sqrt{2k\rho}} \right)^m \frac{\sqrt{m!}}{r!}, \end{aligned} \tag{5.13}$$

since $(qm)! \leqslant (m! q^m)^q$, where $c_4(A_0) = \max\{c_2(A_0), c_1(\epsilon)\}$.

Consequently, summing up these estimates and denoting $\sigma = d \cdot c_E$, we obtain

$$\begin{aligned} E\big[\big\| W_\gamma^\epsilon(R_{n,k} x)\big\|^{2q}\big] &\leqslant \left(\sum_{r=0}^\infty \big(\sigma c_4(A_0)\big)^r \sum_{m=0}^r {}_r C_m \left(2\sqrt{\frac{q}{k\rho}} \right)^m \frac{1}{\sqrt{r!}} \right)^{2q} \\ &= \left(\sum_{r=0}^\infty \left\{ \sigma c_4(A_0) \left(1 + 2\sqrt{\frac{q}{k\rho}} \right) \right\}^r \frac{1}{\sqrt{r!}} \right)^{2q} < \infty \end{aligned} \tag{5.14}$$

with the bound being independent of n.

Step 3. Since B^* is dense in H, for each $h \in H$, there is a sequence $\{\xi_n\}_{n=1}^\infty$ of elements in B^* such that $\lim_{n\to\infty} \|h - \xi_n\|_p = 0$. As is well-known, $\langle \cdot, \xi_n \rangle$ then converges to $\langle \cdot, h \rangle$ in

$L^2(B, \boldsymbol{R}; \mu)$ as $n \to \infty$. Hence, taking a subsequence if necessary, we may assume that $\langle x, \xi_n \rangle$ converges to $\langle x, h \rangle$ for μ-almost every $x \in B$. Then we define for $x \in B$ and $h \in H$

$$\langle x, h \rangle = \begin{cases} \lim_{n\to\infty} \langle x, \xi_n \rangle & \text{if it exists,} \\ 0 & \text{otherwise,} \end{cases} \tag{5.15}$$

as usual.

It should be noted that, given $\xi \in B^*$, the operator R_k defined by (5.5) takes ξ into H; not into B^* in general. This leads us to define, by virtue of (5.15), elements in the complexification of $\mathfrak{g}$, associated with $x \in B$ and $\tilde{C}^\epsilon_\gamma(t) \in B^*$, by

$$R_k x^\epsilon_\gamma(t) = \sum_{\alpha=1}^{d} \bigl\langle x, R_k \tilde{C}^\epsilon_\gamma(t)^\alpha \otimes E_\alpha \bigr\rangle E_\alpha,$$

$$R_k \tilde{C}^\epsilon_\gamma(t)^\alpha \otimes E_\alpha = R^1_k \tilde{C}^\epsilon_\gamma(t)^\alpha \otimes E_\alpha + \sqrt{-1} R^2_k \tilde{C}^\epsilon_\gamma(t)^\alpha \otimes E_\alpha,$$

and the accompanying Gaussian random variables

$$R^1_k x^{\epsilon,\alpha}_\gamma(t) = \bigl\langle x, R^1_k \tilde{C}^\epsilon_\gamma(t)^\alpha \otimes E_\alpha \bigr\rangle, \qquad R^2_k x^{\epsilon,\alpha}_\gamma(t) = \bigl\langle x, R^2_k \tilde{C}^\epsilon_\gamma(t)^\alpha \otimes E_\alpha \bigr\rangle$$

in a manner similar to that in defining $R_{n,k} x^\epsilon_\gamma(t)$ and $R^1_{n,k} x^{\epsilon,\alpha}_\gamma(t)$, $R^2_{n,k} x^{\epsilon,\alpha}_\gamma(t)$ in (5.8) and (5.9), respectively. Then it is immediate from (5.12) that we have

$$E\bigl[\bigl|R_k x^{\epsilon,\alpha}_\gamma(t) - R_k x^{\epsilon,\alpha}_\gamma(s)\bigr|^2\bigr] \leqslant c_5(\epsilon)^2 |t-s|^2. \tag{5.16}$$

Hence, by virtue of the Kolmogorov–Delporte criterion [9], $R_k x^{\epsilon,\alpha}_\gamma(t)$ has a continuous modification in t. Henceforth we denote such continuous modification by the same symbol $R_k x^{\epsilon,\alpha}_\gamma(t)$.

Now, for any positive integer n, set

$$T_n = \sum_{j=1}^{2^n} \left| R_k x^{\epsilon,\alpha}_\gamma\left(\frac{j}{2^n}\right) - R_k x^{\epsilon,\alpha}_\gamma\left(\frac{j-1}{2^n}\right) \right|.$$

Then, since $T_n \leqslant T_{n+1}$, it is easy to see from (5.16) that

$$\begin{aligned} E\Bigl[\lim_{n\to\infty} T_n\Bigr] &= \lim_{n\to\infty} E\left[\sum_{j=1}^{2^n} \left| R_k x^{\epsilon,\alpha}_\gamma\left(\frac{j}{2^n}\right) - R_k x^{\epsilon,\alpha}_\gamma\left(\frac{j-1}{2^n}\right) \right|\right] \\ &\leqslant \lim_{n\to\infty} \sum_{j=1}^{2^n} E\left[\left| R_k x^{\epsilon,\alpha}_\gamma\left(\frac{j}{2^n}\right) - R_k x^{\epsilon,\alpha}_\gamma\left(\frac{j-1}{2^n}\right) \right|^2\right]^{1/2} \\ &\leqslant \lim_{n\to\infty} \sum_{j=1}^{2^n} c_5(\epsilon) \left| \frac{j}{2^n} - \frac{j-1}{2^n} \right| \\ &\leqslant c_5(\epsilon), \end{aligned}$$

which implies that

$$\lim_{n\to\infty} T_n < \infty \quad \mu\text{-almost everywhere.}$$

Since $R_k x_\gamma^{\epsilon,\alpha}(t)$ is continuous in t almost surely, this implies that $R_k x_\gamma^{\epsilon,\alpha}(t)$ is of bounded variation for all $x \in B' \subset B$ with $\mu(B') = 1$. Therefore the Lebesgue–Stieltjes integral

$$\int_0^1\int_0^{t_1}\cdots\int_0^{t_{r-1}} d\big(\bar{A}_0 + R_k x_\gamma^\epsilon\big)(t_1)\, d\big(\bar{A}_0 + R_k x_\gamma^\epsilon\big)(t_2)\cdots d\big(\bar{A}_0 + R_k x_\gamma^\epsilon\big)(t_r) \tag{5.17}$$

is well defined for all $x \in B' \subset B$ with $\mu(B') = 1$. According to (4.3) and (4.4), we then define the stochastic holonomy given by $R_k x$ to be

$$W_\gamma^{\epsilon,r}(R_k x) = \begin{cases} (5.17) & \text{for } x \in B', \\ 0 & \text{for } x \in B \setminus B', \end{cases}$$

$$W_\gamma^\epsilon(R_k x) = I + \sum_{r=1}^\infty W_\gamma^{\epsilon,r}(R_k x),$$

and the associated Wilson line by

$$F_{A_0}^\epsilon(R_k x) = \prod_{j=1}^s \mathrm{Tr}_{R_j}\, W_{\gamma_j}^\epsilon(R_k x).$$

The well-definedness of $W_\gamma^\epsilon(R_k x)$ can be seen as follows. First we note that

$$\begin{aligned}
&E\Bigg[\bigg|\int_0^1\int_0^{t_1}\cdots\int_0^{t_{r-1}} d\big(\bar{A}_0^{\alpha_1} + R_k x_\gamma^{\epsilon,\alpha_1}\big)(t_1)\cdots d\big(\bar{A}_0^{\alpha_r} + R_k x_\gamma^{\epsilon,\alpha_r}\big)(t_r)\bigg|^{2q}\Bigg] \\
&\quad\leqslant E\Bigg[\lim_{n_1,\dots,n_r\to\infty}\bigg|\sum_{s_1=0}^{2^{n_1}-1}\big|A_0^{\alpha_1}[s_1] + R_k x_\gamma^{\epsilon,\alpha_1}[s_1]\big|\cdots\sum_{s_r=0}^{2^{n_r}-1}\big|A_0^{\alpha_r}[s_r] + R_k x_\gamma^{\epsilon,\alpha_r}[s_r]\big|\bigg|^{2q}\Bigg] \\
&\quad\leqslant \lim_{n_1,\dots,n_r\to\infty}\Bigg(\sum_{s_1=0}^{2^{n_1}-1}\cdots\sum_{s_r=0}^{2^{n_r}-1} E\Big[\Big(\big|A_0^{\alpha_1}[s_1] + R_k x_\gamma^{\epsilon,\alpha_1}[s_1]\big| \\
&\qquad\cdot\cdots\big|A_0^{\alpha_r}[s_r] + R_k x_\gamma^{\epsilon,\alpha_r}[s_r]\big|\Big)^{2q}\Big]^{1/2q}\Bigg)^{2q}.
\end{aligned} \tag{5.18}$$

On the other hand, it is easy to see from (5.16) together with Lemma 4 that

$$E\Big[\big|A_0^{\alpha_i}[s_i] + R_k x_\gamma^{\epsilon,\alpha_i}[s_i]\big|^{2m}\Big] \leqslant c_6(A_0, m, \epsilon)\big|t_i^{s_i+1} - t_i^{s_i}\big|^{2m}$$

for any positive integer m, so that (5.18) is dominated by

$$c_7(\epsilon)\left(\int_0^1\int_0^{t_1}\cdots\int_0^{t_{r-1}} dt_1\,dt_2\cdots dt_r\right)^{2q}.$$

This, together with Lebesgue's convergence theorem, then yields that

$$E\left[\left|\int_0^1\int_0^{t_1}\cdots\int_0^{t_{r-1}} d\big(\bar{A}_0^{\alpha_1}+R_k x_\gamma^{\epsilon,\alpha_1}\big)(t_1)\cdots d\big(\bar{A}_0^{\alpha_r}+R_k x_\gamma^{\epsilon,\alpha_r}\big)(t_r)\right|^{2q}\right]$$
$$=\lim_{n_1,\dots,n_r\to\infty} E\left[\left|\sum_{s_1=0}^{2^{n_1}-1}\big(A_0^{\alpha_1}[s_1]+R_k x_\gamma^{\epsilon,\alpha_1}[s_1]\big)\cdots\sum_{s_r=0}^{2^{n_r}-1}\big(A_0^{\alpha_r}[s_r]+R_k x_\gamma^{\epsilon,\alpha_r}[s_r]\big)\right|^{2q}\right], \tag{5.19}$$

which assures that the above estimates obtained for $W_\gamma^\epsilon(R_{n,k}x)$ in (5.10) through (5.14) also hold for $W_\gamma^\epsilon(R_k x)$ without essential change. In consequence, we obtain

$$E\big[\big\|W_\gamma^\epsilon(R_k x)\big\|^{2q}\big]<\infty, \tag{5.20}$$

showing that $W_\gamma^\epsilon(R_k x)$ is well defined for each $x\in B$.

Step 4. Furthermore, since $R_{n,k}^\nu\tilde{C}_\gamma^\epsilon(t)^\alpha\otimes E_\alpha$ converges to $R_k^\nu\tilde{C}_\gamma^\epsilon(t)^\alpha\otimes E_\alpha$ in H as $n\to\infty$ for $\nu=1,2$, it also follows from Lebesgue's convergence theorem that

$$\lim_{n\to\infty} E\big[\big\|W_\gamma^\epsilon(R_{n,k}x)-W_\gamma^\epsilon(R_k x)\big\|^{2q}\big]=0. \tag{5.21}$$

Indeed, as in the estimation in (5.10) it holds that

$$E\big[\big\|W_\gamma^\epsilon(R_{n,k}x)-W_\gamma^\epsilon(R_k x)\big\|^{2q}\big]$$
$$\leqslant\left(\sum_{r=0}^{\infty}\sum_{m=0}^{r}\sum_{\substack{1\leqslant l_1<l_2<\cdots<l_m\leqslant r,\\ \nu_1,\nu_2,\dots,\nu_m\in\{1,2\}}}\sum_{\alpha_1,\alpha_2,\dots,\alpha_r=1}^{d} c_E^r E\big[\big|D^{r,m}\big[R_{n,k}^\nu x, R_k^\nu x\big]\big|^{2q}\big]^{1/2q}\right)^{2q},$$

where for brevity we write

$$D^{r,m}\big[R_{n,k}^\nu x, R_k^\nu x\big]$$
$$=\int_0^1 d\bar{A}_0^{\alpha_1}(t_1)\cdots\int_0^{t_{l_1-1}} dR_{n,k}^{\nu_1}x_\gamma^{\epsilon,\alpha_{l_1}}(t_{l_1})\cdots\int_0^{t_{l_m-1}} dR_{n,k}^{\nu_m}x_\gamma^{\epsilon,\alpha_{l_m}}(t_{l_m})\cdots\int_0^{t_{r-1}} d\bar{A}_0^{\alpha_r}(t_r)$$
$$-\int_0^1 d\bar{A}_0^{\alpha_1}(t_1)\cdots\int_0^{t_{l_1-1}} dR_k^{\nu_1}x_\gamma^{\epsilon,\alpha_{l_1}}(t_{l_1})\cdots\int_0^{t_{l_m-1}} dR_k^{\nu_m}x_\gamma^{\epsilon,\alpha_{l_m}}(t_{l_m})\cdots\int_0^{t_{r-1}} d\bar{A}_0^{\alpha_r}(t_r).$$

Also, setting

$$B_j=\int_0^1 d\bar{A}_0^{\alpha_1}(t_1)\cdots\int_0^{t_{l_1-1}} dR_k^{\nu_1}x_\gamma^{\epsilon,\alpha_{l_1}}(t_{l_1})\cdots\int_0^{t_{l_j-1}} d\big\{R_{n,k}^{\nu_j}x_\gamma^{\epsilon,\alpha_{l_j}}(t_{l_j})-R_k^{\nu_j}x_\gamma^{\epsilon,\alpha_{l_j}}(t_{l_j})\big\}$$
$$\cdot\cdots\int_0^{t_{l_m-1}} dR_{n,k}^{\nu_m}x_\gamma^{\epsilon,\alpha_{l_m}}(t_{l_m})\cdots\int_0^{t_{r-1}} d\bar{A}_0^{\alpha_r}(t_r),$$

we obtain, by the same reasoning as in Lemma 3, that

$$E\big[\big|D^{r,m}\big[R_{n,k}^{\nu}x,R_k^{\nu}x\big]\big|^{2q}\big]^{1/2q}\leqslant\sum_{j=1}^m E\big[|B_j|^{2q}\big]^{1/2q}. \tag{5.22}$$

On the other hand, by an argument similar to that in obtaining (5.11), we see that each term of the right side of (5.22) is dominated by

$$c_2(A_0)^{r-m}\lim_{n_1,\dots,n_r\to\infty}\sum_{s_1=0}^{2^{n_1}-1}\cdots\sum_{s_r=0}^{2^{n_r}-1}E\big[\big(\big|t_1^{s_1+1}-t_1^{s_1}\big|\cdots\big|R_k^{\nu_1}x_\gamma^{\epsilon,\alpha_{l_1}}[s_{l_1}]\big|\cdots$$
$$\cdot\big|R_{n,k}^{\nu_j}x_\gamma^{\epsilon,\alpha_{l_j}}[s_{l_j}]-R_k^{\nu_j}x_\gamma^{\epsilon,\alpha_{l_j}}[s_{l_j}]\big|\cdots\big|R_{n,k}^{\nu_m}x_\gamma^{\epsilon,\alpha_{l_m}}[s_{l_m}]\big|\cdots\big|t_r^{s_r+1}-t_r^{s_r}\big|\big)^{2q}\big]^{1/2q},$$

where it also holds as in (5.12) that

$$\begin{aligned}&E\big[\big|R_{n,k}^{\nu_j}x_\gamma^{\epsilon,\alpha_{l_j}}[s_{l_j}]-R_k^{\nu_j}x_\gamma^{\epsilon,\alpha_{l_j}}[s_{l_j}]\big|^2\big]\\&\quad=\big\|\big(R_{n,k}^{\nu_j}-R_k^{\nu_j}\big)\tilde{C}_\gamma^\epsilon\big(t_{l_j}^{s_{l_j}+1}\big)^\alpha\otimes E_\alpha-\big(R_{n,k}^{\nu_j}-R_k^{\nu_j}\big)\tilde{C}_\gamma^\epsilon\big(t_{l_j}^{s_{l_j}}\big)^\alpha\otimes E_\alpha\big\|_p^2\\&\quad\leqslant\frac{2}{k\rho}c_1(\epsilon)^2\big|t_{l_j}^{s_{l_j}+1}-t_{l_j}^{s_{l_j}}\big|^2.\end{aligned}\tag{5.23}$$

Hence, by the same reasoning as in (5.13), we obtain that

$$\begin{aligned}&E\big[|B_j|^{2q}\big]^{1/2q}\\&\quad\leqslant c_2(A_0)^{r-m}\lim_{n_1,\dots,n_r\to\infty}\sum_{s_1=0}^{2^{n_1}-1}\cdots\sum_{s_r=0}^{2^{n_r}-1}\bigg\{\frac{(2qm)!(\sqrt{2}c_1(\epsilon)/\sqrt{k\rho})^{2qm}}{2^{qm}(qm)!}\bigg\}^{1/2q}\\&\qquad\cdot\big|t_1^{s_1+1}-t_1^{s_1}\big|\cdots\big|t_r^{s_r+1}-t_r^{s_r}\big|\\&\quad\leqslant c_4(A_0)^r\bigg(2\sqrt{\frac{q}{k\rho}}\bigg)^m\frac{\sqrt{m!}}{r!}.\end{aligned}\tag{5.24}$$

Since each $R_{n,k}^{\nu}\tilde{C}_\gamma^\epsilon(t)^\alpha\otimes E_\alpha$ converges to $R_k^{\nu}\tilde{C}_\gamma^\epsilon(t)^\alpha\otimes E_\alpha$ in H as $n\to\infty$, it follows from the first identities in (5.12) and (5.23) combined with Lemma 4 that

$$\lim_{n\to\infty} E\Big[\Big(\big|t_1^{s_1+1}-t_1^{s_1}\big|\cdots\big|R_k^{\nu_1}x_\gamma^{\epsilon,\alpha_{l_1}}[s_{l_1}]\big|\cdots \cdot \big|R_{n,k}^{\nu_j}x_\gamma^{\epsilon,\alpha_{l_j}}[s_{l_j}]-R_k^{\nu_j}x_\gamma^{\epsilon,\alpha_{l_j}}[s_{l_j}]\big|\cdots\big|R_{n,k}^{\nu_m}x_\gamma^{\epsilon,\alpha_{l_m}}[s_{l_m}]\big|\cdots\big|t_r^{s_r+1}-t_r^{s_r}\big|\Big)^{2q}\Big]=0.$$

This, together with the estimates (5.23) and (5.24) with bound independent of n, then yields by Lebesgue's convergence theorem that

$$\lim_{n\to\infty}\Bigg(c_2(A_0)^{r-m}\lim_{n_1,\dots,n_r\to\infty}\sum_{s_1=0}^{2^{n_1}-1}\cdots\sum_{s_r=0}^{2^{n_r}-1}E\Big[\Big(\big|t_1^{s_1+1}-t_1^{s_1}\big|\cdots\big|R_k^{\nu_1}x_\gamma^{\epsilon,\alpha_{l_1}}[s_{l_1}]\big|\cdots \cdot \big|R_{n,k}^{\nu_j}x_\gamma^{\epsilon,\alpha_{l_j}}[s_{l_j}]-R_k^{\nu_j}x_\gamma^{\epsilon,\alpha_{l_j}}[s_{l_j}]\big|\cdots\big|R_{n,k}^{\nu_m}x_\gamma^{\epsilon,\alpha_{l_m}}[s_{l_m}]\big|\cdots\big|t_r^{s_r+1}-t_r^{s_r}\big|\Big)^{2q}\Big]^{1/2q}\Bigg)=0,$$

so that

$$\lim_{n\to\infty}E\big[\big|D^{r,m}\big[R_{n,k}^{\nu}x,\,R_k^{\nu}x\big]\big|^{2q}\big]^{1/2q}=0.$$

Also, noting that it holds

$$(u+v)^m\leqslant 2^m\big(u^m+v^m\big)$$

for $u, v\geqslant 0$, we have

$$\begin{aligned}
&E\big[\big|D^{r,m}\big[R_{n,k}^{\nu}x,\,R_k^{\nu}x\big]\big|^{2q}\big]^{1/2q}\\
&\quad\leqslant 2\Bigg(E\Bigg[\Bigg|\int_0^1 d\bar A_0^{\alpha_1}(t_1)\cdots\int_0^{t_{l_1-1}} dR_{n,k}^{\nu_1}x_\gamma^{\epsilon,\alpha_{l_1}}(t_{l_1})\\
&\qquad\cdot\cdots\int_0^{t_{l_m-1}} dR_{n,k}^{\nu_m}x_\gamma^{\epsilon,\alpha_{l_m}}(t_{l_m})\cdots\int_0^{t_{r-1}} d\bar A_0^{\alpha_r}(t_r)\Bigg|^{2q}\Bigg]^{1/2q}\\
&\qquad+E\Bigg[\Bigg|\int_0^1 d\bar A_0^{\alpha_1}(t_1)\cdots\int_0^{t_{l_1-1}} dR_{k}^{\nu_1}x_\gamma^{\epsilon,\alpha_{l_1}}(t_{l_1})\\
&\qquad\cdot\cdots\int_0^{t_{l_m-1}} dR_{k}^{\nu_m}x_\gamma^{\epsilon,\alpha_{l_m}}(t_{l_m})\cdots\int_0^{t_{r-1}} d\bar A_0^{\alpha_r}(t_r)\Bigg|^{2q}\Bigg]^{1/2q}\Bigg).
\end{aligned}\tag{5.25}$$

Recalling that the estimates in (5.10) through (5.14) are valid for both $R_{n,k}x$ and R_kx, and the bounds in the estimates (5.12) and (5.14) are independent of n, it follows from (5.25) and Lebesgue's convergence theorem that

$$\lim_{n\to\infty}\Bigg(\sum_{r=0}^{\infty}\sum_{m=0}^{r}\sum_{\substack{1\leqslant l_1<l_2<\cdots<l_m\leqslant r,\\ \nu_1,\nu_2,\dots,\nu_m\in\{1,2\}}}\ \sum_{\alpha_1,\alpha_2,\dots,\alpha_r=1}^{d}c_E^r E\big[\big|D^{r,m}\big[R_{n,k}^{\nu}x,\,R_k^{\nu}x\big]\big|^{2q}\big]^{1/2q}\Bigg)^{2q}=0.$$

Hence we obtain (5.21).

As a result, we see that $\mathrm{Tr}_{R_j} W^{\epsilon}_{\gamma}(R_{n,k}x)$ converges to $\mathrm{Tr}_{R_j} W^{\epsilon}_{\gamma}(R_k x)$ in $L^2(B, \boldsymbol{R}; \mu)$ as $n \to \infty$. This combined with (5.7) and (5.20) then verifies that

$$\limsup_{n\to\infty} \int_B F^{\epsilon}_{A_0}(R_{n,k}x)\,\mu(dx) = \int_B F^{\epsilon}_{A_0}(R_k x)\,\mu(dx).$$

Step 5. Finally, taking into account of (5.3), we note that the following integrability can be proved in a manner similar to that in obtaining the estimates described above. Namely, we have

Lemma 5. *For any positive integer N,*

$$E\left[\sum_{m=N}^{\infty} F^{\epsilon,m}_{A_0}(R_{n,k}x)\right] = O\big(k^{-N/2}\big),$$

where $O(k^{-N/2})$ means

$$\lim_{k\to\infty} k^{N/2}\big|O\big(k^{-N/2}\big)\big| < \infty.$$

Then Lemma 5 and the fact that

$$\int_B F_{A_0}(R_k x)\,\mu(dx) = \sum_{m<N} \int_B F^{\epsilon,m}_{A_0}(R_k x)\,\mu(dx) + \int_B \sum_{m=N}^{\infty} F^{\epsilon,m}_{A_0}(R_k x)\,\mu(dx)$$

complete the rest of the proof of Theorem 1. □

6. Example

As an application of Theorem 1, we now calculate the Wilson line integral of two closed oriented loops γ_1 and γ_2 in three-sphere S^3.

To this end, let $G = SU(2)$ and consider its canonical representation R. We denote by $\{E_\alpha\}$, $1 \leqslant \alpha \leqslant 3$, an *orthonormal basis* of the Lie algebra $\mathfrak{g} = \mathfrak{su}(2)$ with respect to the inner product $(X, Y) = -\operatorname{Tr} XY$ for $X, Y \in \mathfrak{g}$. For simplicity, we also assume for the ϵ-regularized Wilson line (4.4) that $A_0 = 0$, and write

$$F^{\epsilon}_0(x) = \prod_{j=1}^{2} \mathrm{Tr}_R\, W^{\epsilon}_{\gamma_j}(x).$$

Step 1. Recalling (4.3), we begin with the evaluation of

$$E\left[\prod_{j=1}^{2} \mathrm{Tr}_R\, W^{\epsilon,2}_{\gamma_j}(R_k x)\right]. \tag{6.1}$$

Writing briefly

$$\big\langle R_k x, \tilde{C}^{\epsilon}_{\gamma}(t)^{\alpha} \otimes E_\alpha\big\rangle \quad \text{by} \quad \big(R_k x^{\alpha}_{\gamma}\big)(t),$$

we see that (6.1) is equal to

$$
\begin{aligned}
&E\big[\operatorname{Tr}_R W^{\epsilon,2}_{\gamma_1}(R_k x)\otimes W^{\epsilon,2}_{\gamma_2}(R_k x)\big]\\
&\quad=\sum_{\alpha_1,\alpha_2,\beta_1,\beta_2=1}^{3}\operatorname{Tr} E_{\alpha_1}E_{\alpha_2}\otimes E_{\beta_1}E_{\beta_2}\\
&\qquad\cdot E\left[\int_0^1\int_0^{t_1} d\big(R_k x^{\alpha_1}_{\gamma_1}\big)(t_1)\,d\big(R_k x^{\alpha_2}_{\gamma_1}\big)(t_2)\int_0^1\int_0^{\tau_1} d\big(R_k x^{\beta_1}_{\gamma_2}\big)(\tau_1)\,d\big(R_k x^{\beta_2}_{\gamma_2}\big)(\tau_2)\right]. \qquad (6.2)
\end{aligned}
$$

Then, by changing the order of taking sum and expectation, in a similar manner as in the proof of (5.19), we obtain

$$
\begin{aligned}
&E\left[\int_0^1\int_0^{t_1} d\big(R_k x^{\alpha_1}_{\gamma_1}\big)(t_1)\,d\big(R_k x^{\alpha_2}_{\gamma_1}\big)(t_2)\int_0^1\int_0^{\tau_1} d\big(R_k x^{\beta_1}_{\gamma_2}\big)(\tau_1)\,d\big(R_k x^{\beta_2}_{\gamma_2}\big)(\tau_2)\right]\\
&\quad=\lim_{\substack{n_1,n_2\to\infty\\ m_1,m_2\to\infty}}\sum_{s_1=0}^{2^{n_1}-1}\sum_{s_2(s_1)=0}^{2^{n_2}-1}\sum_{s_1=0}^{2^{m_1}-1}\sum_{s_2(s_1)=0}^{2^{m_2}-1} E\big[\big(\big(R_k x^{\alpha_1}_{\gamma_1}\big)\big(t_1^{s_1+1}\big)\\
&\qquad-\big(R_k x^{\alpha_1}_{\gamma_1}\big)\big(t_1^{s_1}\big)\big)\big(\big(R_k x^{\alpha_2}_{\gamma_1}\big)\big(t_2^{s_2(s_1)+1}\big)-\big(R_k x^{\alpha_2}_{\gamma_1}\big)\big(t_2^{s_2(s_1)}\big)\big)\\
&\qquad\cdot\big(\big(R_k x^{\beta_1}_{\gamma_2}\big)\big(\tau_1^{s_1+1}\big)-\big(R_k x^{\beta_1}_{\gamma_2}\big)\big(\tau_1^{s_1}\big)\big)\big(\big(R_k x^{\beta_2}_{\gamma_2}\big)\big(\tau_2^{s_2(s_1)+1}\big)-\big(R_k x^{\beta_2}_{\gamma_2}\big)\big(\tau_2^{s_2(s_1)}\big)\big)\big]. \qquad (6.3)
\end{aligned}
$$

Here we set for $i=1,2$,

$$
t_i^{s_i(s_{i-1})}=\begin{cases}0 & \text{if } s_i(s_{i-1})=0,\\ t_i^{s_i(s_{i-1})-1}+t_{i-1}^{s_{i-1}(s_{i-2})}/2^{n_i} & \text{if } s_i(s_{i-1})\geqslant 1,\end{cases}
$$

and

$$
\tau_i^{s_i(s_{i-1})}=\begin{cases}0 & \text{if } s_i(s_{i-1})=0,\\ \tau_i^{s_i(s_{i-1})-1}+\tau_{i-1}^{s_{i-1}(s_{i-2})}/2^{m_i} & \text{if } s_i(s_{i-1})\geqslant 1,\end{cases}
$$

where $s_i(s_{i-1})$ are non-negative integers and we use the convention such that $s_1(s_0)=s_1$, $s_0(s_{-1})=1$ and $t_0^1=\tau_0^1=1$.

Writing for brevity

$$
\boldsymbol{j}_i=\begin{cases}(R_k x^{\alpha_i}_{\gamma_1})(t_i^{s_i(s_{i-1})+1})-(R_k x^{\alpha_i}_{\gamma_1})(t_i^{s_i(s_{i-1})}) & \text{if } i\leqslant 2,\\ (R_k x^{\beta_{i-2}}_{\gamma_2})(\tau_{i-2}^{s_{i-2}(s_{i-3})+1})-(R_k x^{\beta_{i-2}}_{\gamma_2})(\tau_{i-2}^{s_{i-2}(s_{i-3})}) & \text{if } i>2,\end{cases}
$$

we see from Lemma 4 that the right side of (6.3) is equal to

$$\lim_{\substack{n_1,n_2\to\infty\\ m_1,m_2\to\infty}} \sum_{s_1=0}^{2^{n_1}-1} \sum_{s_2(s_1)=0}^{2^{n_2}-1} \sum_{s_1=0}^{2^{m_1}-1} \sum_{s_2(s_1)=0}^{2^{m_2}-1} \frac{1}{2!2^2} \sum_{\sigma\in\mathfrak{S}_4} E[\boldsymbol{j}_{\sigma(1)}\boldsymbol{j}_{\sigma(2)}]E[\boldsymbol{j}_{\sigma(3)}\boldsymbol{j}_{\sigma(4)}]$$

$$= \lim_{\substack{n_1,n_2\to\infty\\ m_1,m_2\to\infty}} \sum_{s_1=0}^{2^{n_1}-1} \sum_{s_2(s_1)=0}^{2^{n_2}-1} \sum_{s_1=0}^{2^{m_1}-1} \sum_{s_2(s_1)=0}^{2^{m_2}-1} \sum_{\sigma\in\mathfrak{S}_2} E[\boldsymbol{j}_1\boldsymbol{j}_{\sigma(1)+2}]E[\boldsymbol{j}_2\boldsymbol{j}_{\sigma(2)+2}] + T_{\mathrm{self}}$$

$$\begin{aligned}
&= \lim_{\substack{n_1,n_2\to\infty\\ m_1,m_2\to\infty}} \sum_{s_1=0}^{2^{n_1}-1} \sum_{s_2(s_1)=0}^{2^{n_2}-1} \sum_{s_1=0}^{2^{m_1}-1} \sum_{s_2(s_1)=0}^{2^{m_2}-1} \sum_{\sigma\in\mathfrak{S}_2} E\big[\big(\big(R_k x_{\gamma_1}^{\alpha_1}\big)\big(t_1^{s_1+1}\big) - \big(R_k x_{\gamma_1}^{\alpha_1}\big)\big(t_1^{s_1}\big)\big) \\
&\quad \cdot \big(\big(R_k x_{\gamma_2}^{\beta_{\sigma(1)}}\big)\big(\tau_{\sigma(1)}^{s_{\sigma(1)}(s_{\sigma(1)-1})+1}\big) - \big(R_k x_{\gamma_2}^{\beta_{\sigma(1)}}\big)\big(\tau_{\sigma(1)}^{s_{\sigma(1)}(s_{\sigma(1)-1})}\big)\big)\big] \\
&\quad \times E\big[\big(\big(R_k x_{\gamma_1}^{\alpha_2}\big)\big(t_2^{s_2(s_1)+1}\big) - \big(R_k x_{\gamma_1}^{\alpha_2}\big)\big(t_2^{s_2(s_1)}\big)\big) \\
&\quad \cdot \big(\big(R_k x_{\gamma_2}^{\beta_{\sigma(2)}}\big)\big(\tau_{\sigma(2)}^{s_{\sigma(2)}(s_{\sigma(2)-1})+1}\big) - \big(R_k x_{\gamma_2}^{\beta_{\sigma(2)}}\big)\big(\tau_{\sigma(2)}^{s_{\sigma(2)}(s_{\sigma(2)-1})}\big)\big)\big] + T_{\mathrm{self}},
\end{aligned}$$

where T_{self} stands for the collection of self-linking terms containing

$$E\big[\big(\big(R_k x_{\gamma_1}^{\alpha_1}\big)\big(t_1^{l+1}\big) - \big(R_k x_{\gamma_1}^{\alpha_1}\big)\big(t_1^{l}\big)\big)\big(\big(R_k x_{\gamma_1}^{\alpha_2}\big)\big(t_2^{l+1}\big) - \big(R_k x_{\gamma_1}^{\alpha_2}\big)\big(t_2^{l}\big)\big)\big]$$

or

$$E\big[\big(\big(R_k x_{\gamma_2}^{\beta_1}\big)\big(\tau_1^{l+1}\big) - \big(R_k x_{\gamma_2}^{\beta_1}\big)\big(\tau_1^{l}\big)\big)\big(\big(R_k x_{\gamma_2}^{\beta_2}\big)\big(\tau_2^{l+1}\big) - \big(R_k x_{\gamma_2}^{\beta_2}\big)\big(\tau_2^{l}\big)\big)\big].$$

Since $R_k x_{\gamma_i}^{\alpha}(t)$ and $R_k x_{\gamma_j}^{\beta}(t)$ are independent if $\alpha \neq \beta$, we then have

$$\begin{aligned}
&E\big[\big(\big(R_k x_{\gamma_1}^{\alpha_1}\big)\big(t_1^{s_1+1}\big) - \big(R_k x_{\gamma_1}^{\alpha_1}\big)\big(t_1^{s_1}\big)\big) \\
&\quad \cdot \big(\big(R_k x_{\gamma_2}^{\beta_{\sigma(1)}}\big)\big(\tau_{\sigma(1)}^{s_{\sigma(1)}(s_{\sigma(1)-1})+1}\big) - \big(R_k x_{\gamma_2}^{\beta_{\sigma(1)}}\big)\big(\tau_{\sigma(1)}^{s_{\sigma(1)}(s_{\sigma(1)-1})}\big)\big)\big] \\
&\quad \times E\big[\big(\big(R_k x_{\gamma_1}^{\alpha_2}\big)\big(t_2^{s_2(s_1)+1}\big) - \big(R_k x_{\gamma_1}^{\alpha_2}\big)\big(t_2^{s_2(s_1)}\big)\big) \\
&\quad \cdot \big(\big(R_k x_{\gamma_2}^{\beta_{\sigma(2)}}\big)\big(\tau_{\sigma(2)}^{s_{\sigma(2)}(s_{\sigma(2)-1})+1}\big) - \big(R_k x_{\gamma_2}^{\beta_{\sigma(2)}}\big)\big(\tau_{\sigma(2)}^{s_{\sigma(2)}(s_{\sigma(2)-1})}\big)\big)\big] \\
&= \delta_{\alpha_1\beta_{\sigma(1)}} E\big[\big(\big(R_k x_{\gamma_1}^{\alpha_1}\big)\big(t_1^{s_1+1}\big) - \big(R_k x_{\gamma_1}^{\alpha_1}\big)\big(t_1^{s_1}\big)\big) \\
&\quad \cdot \big(\big(R_k x_{\gamma_2}^{\beta_{\sigma(1)}}\big)\big(\tau_{\sigma(1)}^{s_{\sigma(1)}(s_{\sigma(1)-1})+1}\big) - \big(R_k x_{\gamma_2}^{\beta_{\sigma(1)}}\big)\big(\tau_{\sigma(1)}^{s_{\sigma(1)}(s_{\sigma(1)-1})}\big)\big)\big] \\
&\quad \times \delta_{\alpha_2\beta_{\sigma(2)}} E\big[\big(\big(R_k x_{\gamma_1}^{\alpha_2}\big)\big(t_2^{s_2(s_1)+1}\big) - \big(R_k x_{\gamma_1}^{\alpha_2}\big)\big(t_2^{s_2(s_1)}\big)\big) \\
&\quad \cdot \big(\big(R_k x_{\gamma_2}^{\beta_{\sigma(2)}}\big)\big(\tau_{\sigma(2)}^{s_{\sigma(2)}(s_{\sigma(2)-1})+1}\big) - \big(R_k x_{\gamma_2}^{\beta_{\sigma(2)}}\big)\big(\tau_{\sigma(2)}^{s_{\sigma(2)}(s_{\sigma(2)-1})}\big)\big)\big] \\
&= E\big[\big(\big(R_k x_{\gamma_1}^{\alpha_1}\big)\big(t_1^{s_1+1}\big) - \big(R_k x_{\gamma_1}^{\alpha_1}\big)\big(t_1^{s_1}\big)\big) \\
&\quad \cdot \big(\big(R_k x_{\gamma_2}^{\alpha_1}\big)\big(\tau_{\sigma(1)}^{s_{\sigma(1)}(s_{\sigma(1)-1})+1}\big) - \big(R_k x_{\gamma_2}^{\alpha_1}\big)\big(\tau_{\sigma(1)}^{s_{\sigma(1)}(s_{\sigma(1)-1})}\big)\big)\big] \\
&\quad \times E\big[\big(\big(R_k x_{\gamma_1}^{\alpha_2}\big)\big(t_2^{s_2(s_1)+1}\big) - \big(R_k x_{\gamma_1}^{\alpha_2}\big)\big(t_2^{s_2(s_1)}\big)\big) \\
&\quad \cdot \big(\big(R_k x_{\gamma_2}^{\alpha_2}\big)\big(\tau_{\sigma(2)}^{s_{\sigma(2)}(s_{\sigma(2)-1})+1}\big) - \big(R_k x_{\gamma_2}^{\alpha_2}\big)\big(\tau_{\sigma(2)}^{s_{\sigma(2)}(s_{\sigma(2)-1})}\big)\big)\big].
\end{aligned}$$

Furthermore, since $R_k x_{\gamma_i}^{\alpha}(t)$ and $R_k x_{\gamma_i}^{\beta}(t)$ are identically distributed if $\alpha \neq \beta$, we obtain

$$\begin{aligned}(6.3) &= \int_0^1 \int_0^{t_1} \int_0^1 \int_0^{\tau_1} \sum_{\sigma \in \mathfrak{S}_2} dE\big[\big(R_k x_{\gamma_1}^{\alpha_1}\big)(t_1)\big(R_k x_{\gamma_2}^{\alpha_1}\big)(\tau_{\sigma(1)})\big] \\ &\quad \cdot dE\big[\big(R_k x_{\gamma_1}^{\alpha_2}\big)(t_2)\big(R_k x_{\gamma_2}^{\alpha_2}\big)(\tau_{\sigma(2)})\big] + T_{\text{self}} \\ &= \int_0^1 \int_0^{t_1} \int_0^1 \int_0^{\tau_1} \sum_{\sigma \in \mathfrak{S}_2} dE\big[\big(R_k x_{\gamma_1}^{\alpha}\big)(t_1)\big(R_k x_{\gamma_2}^{\alpha}\big)(\tau_{\sigma(1)})\big] \\ &\quad \cdot dE\big[\big(R_k x_{\gamma_1}^{\alpha}\big)(t_2)\big(R_k x_{\gamma_2}^{\alpha}\big)(\tau_{\sigma(2)})\big] + T_{\text{self}}. \end{aligned} \tag{6.4}$$

Consequently, (6.2)–(6.4) yield for each $\alpha = 1, 2, 3$ that

$$\begin{aligned} &E\Bigg[\prod_{j=1}^{2} \operatorname{Tr}_R W_{\gamma_j}^{\epsilon,2}(R_k x)\Bigg] \\ &\quad = \operatorname{Tr} \sum_{\alpha_1,\alpha_2=1}^{3} E_{\alpha_1} E_{\alpha_2} \otimes E_{\alpha_1} E_{\alpha_2} \\ &\qquad \times \int_0^1 \int_0^{t_1} \int_0^1 \int_0^{\tau_1} \sum_{\sigma \in \mathfrak{S}_2} dE\big[\big(R_k x_{\gamma_1}^{\alpha}\big)(t_1)\big(R_k x_{\gamma_2}^{\alpha}\big)(\tau_{\sigma(1)})\big]\, dE\big[\big(R_k x_{\gamma_1}^{\alpha}\big)(t_2)\big(R_k x_{\gamma_2}^{\alpha}\big)(\tau_{\sigma(2)})\big] \\ &\qquad + T_{\text{self}}. \end{aligned} \tag{6.5}$$

Now, noting that

$$\begin{aligned} &\int_0^1 \int_0^{\tau_1} \sum_{\sigma \in \mathfrak{S}_2} dE\big[\big(R_k x_{\gamma_1}^{\alpha}\big)(t_1)\big(R_k x_{\gamma_2}^{\alpha}\big)(\tau_{\sigma(1)})\big]\, dE\big[\big(R_k x_{\gamma_1}^{\alpha}\big)(t_2)\big(R_k x_{\gamma_2}^{\alpha}\big)(\tau_{\sigma(2)})\big] \\ &\quad = \int_0^1 \int_0^1 dE\big[\big(R_k x_{\gamma_1}^{\alpha}\big)(t_1)\big(R_k x_{\gamma_2}^{\alpha}\big)(\tau_1)\big]\, dE\big[\big(R_k x_{\gamma_1}^{\alpha}\big)(t_2)\big(R_k x_{\gamma_2}^{\alpha}\big)(\tau_2)\big] \end{aligned}$$

and

$$\begin{aligned} &\int_0^1 \int_0^{t_1} \int_0^1 \int_0^1 dE\big[\big(R_k x_{\gamma_1}^{\alpha}\big)(t_1)\big(R_k x_{\gamma_2}^{\alpha}\big)(\tau_1)\big]\, dE\big[\big(R_k x_{\gamma_1}^{\alpha}\big)(t_2)\big(R_k x_{\gamma_2}^{\alpha}\big)(\tau_2)\big] \\ &\quad = \int_0^1 \int_0^{t_1} \int_0^1 \int_0^1 dE\big[\big(R_k x_{\gamma_1}^{\alpha}\big)(t_1)\big(R_k x_{\gamma_2}^{\alpha}\big)(\tau_2)\big]\, dE\big[\big(R_k x_{\gamma_1}^{\alpha}\big)(t_2)\big(R_k x_{\gamma_2}^{\alpha}\big)(\tau_1)\big], \end{aligned}$$

we see from (6.5) that

$$
\begin{aligned}
&E\left[\prod_{j=1}^{2} \operatorname{Tr}_R W_{\gamma_j}^{\epsilon,2}(R_k x)\right] \\
&= \operatorname{Tr}\left(\sum_{\alpha_1=1}^{3} E_{\alpha_1} \otimes E_{\alpha_1}\right)^2 \\
&\quad \times \frac{1}{2!} \int_0^1 \int_0^1 \int_0^1 \int_0^1 dE\big[\big(R_k x_{\gamma_1}^{\alpha}\big)(t_1)\big(R_k x_{\gamma_2}^{\alpha}\big)(\tau_1)\big]\, dE\big[\big(R_k x_{\gamma_1}^{\alpha}\big)(t_2)\big(R_k x_{\gamma_2}^{\alpha}\big)(\tau_2)\big] + T_{\text{self}} \\
&= \operatorname{Tr}\left(\sum_{\alpha_1=1}^{3} E_{\alpha_1} \otimes E_{\alpha_1}\right)^2 \frac{1}{2!} E\big[\big(R_k x_{\gamma_1}^{\alpha}\big)(1)\big(R_k x_{\gamma_2}^{\alpha}\big)(1)\big]^2 + T_{\text{self}}.
\end{aligned}
$$

On the other hand, it follows from (3.5), (4.1) and (5.5) that

$$
\begin{aligned}
&E\big[\big(R_k x_{\gamma_1}^{\alpha}\big)(1)\big(R_k x_{\gamma_2}^{\alpha}\big)(1)\big] \\
&\quad = E\big[\big\langle x, R_k \tilde{C}_{\gamma_1}^{\epsilon}(1)^{\alpha} \otimes E_{\alpha}\big\rangle\big\langle x, R_k \tilde{C}_{\gamma_2}^{\epsilon}(1)^{\alpha} \otimes E_{\alpha}\big\rangle\big] \\
&\quad = \big(R_k \tilde{C}_{\gamma_1}^{\epsilon}(1)^{\alpha} \otimes E_{\alpha}, R_k \tilde{C}_{\gamma_2}^{\epsilon}(1)^{\alpha} \otimes E_{\alpha}\big)_p \\
&\quad = \big(R_k\big(\tilde{C}_{\gamma_1}^{\epsilon}(1)^{\alpha} \otimes E_{\alpha}, 0\big), \big(1 + Q_0^2\big)^p R_k\big(\tilde{C}_{\gamma_2}^{\epsilon}(1)^{\alpha} \otimes E_{\alpha}, 0\big)\big)_+ \\
&\quad = -\frac{1}{2\sqrt{-1}k}\big(\big(C_{\gamma_1}^{\epsilon}(1)^{\alpha} \otimes E_{\alpha}, 0\big), Q_0^{-1}\big(C_{\gamma_2}^{\epsilon}(1)^{\alpha} \otimes E_{\alpha}, 0\big)\big)_+ \\
&\quad = -\frac{1}{2\sqrt{-1}k}\big(C_{\gamma_1}^{\epsilon}(1)^{\alpha} \otimes E_{\alpha}, \omega_2^{\alpha} \otimes E_{\alpha}\big),
\end{aligned}
$$

where

$$\omega_2 = \text{1-form part of } Q_0^{-1}\big(C_{\gamma_2}^{\epsilon}(1), 0\big).$$

Recall that, as seen in Proposition 3, $*C_{\gamma_2}^{\epsilon}(1)^{\alpha}$ is a representative of the compact Poincaré dual of γ_2 extended by zero to all of S^3, and the second de Rham cohomology $H_{DR}^2(S^3) = \{0\}$, so that we have $d\omega_2^{\alpha} = *C_{\gamma_2}^{\epsilon}(1)^{\alpha}$, since $*C_{\gamma_2}^{\epsilon}(1)^{\alpha}$ is closed and exact. Hence, for each $\alpha = 1, 2, 3$,

$$\big(C_{\gamma_1}^{\epsilon}(1)^{\alpha} \otimes E_{\alpha}, \omega_2^{\alpha} \otimes E_{\alpha}\big) = \int_{S^3} C_{\gamma 1}^{\varepsilon}(1)^{\alpha} \wedge *\omega_2^{\alpha}$$

yields the linking number $L(\gamma_1, \gamma_2)$ of loops γ_1 and γ_2, provided that $\epsilon > 0$ is sufficiently small so that the ϵ-tubular neighborhoods of γ_1 and γ_2 are pairwise disjoint (see [6] for details). Also, by investigating deformed Wilson loops, it has been proved by Hahn [14] that $T_{\text{self}} = 0$ for non-self-intersected links.

Step 2. We proceed to evaluate mth order coefficients of the expansion, that is,

$$E\big[\operatorname{Tr}_R W_{\gamma_1}^{\epsilon,m_1}(R_k x) \operatorname{Tr}_R W_{\gamma_2}^{\epsilon,m_2}(R_k x)\big], \tag{6.6}$$

where $m = m_1 + m_2$. Note that if m is odd, then (6.6) is equal to zero. Even if m is even, when $m_1 \neq m_2$, the term (6.6) belongs to T_{self}, where T_{self} denotes the collection of self-linking terms containing the limits of

$$E\big[\cdots\big(\big(R_k x_{\gamma_1}^{\alpha_1}\big)\big(t_1^{l+1}\big) - \big(R_k x_{\gamma_1}^{\alpha_1}\big)\big(t_1^{l}\big)\big)\big(\big(R_k x_{\gamma_1}^{\alpha_2}\big)\big(t_2^{l'+1}\big) - \big(R_k x_{\gamma_1}^{\alpha_2}\big)\big(t_2^{l'}\big)\big)\big]$$

or

$$E\big[\cdots\big(\big(R_k x_{\gamma_2}^{\beta_1}\big)\big(\tau_1^{l+1}\big) - \big(R_k x_{\gamma_2}^{\beta_1}\big)\big(\tau_1^{l}\big)\big)\big(\big(R_k x_{\gamma_2}^{\beta_2}\big)\big(\tau_2^{l'+1}\big) - \big(R_k x_{\gamma_2}^{\beta_2}\big)\big(\tau_2^{l'}\big)\big)\big]$$

as $|t_j^{l+1} - t_j^{l}|, |\tau_{j'}^{l'+1} - \tau_{j'}^{l'}| \to 0$. Hence it suffices to evaluate the case with $m_1 = m_2$.

Consequently, (6.6) is equal to

$$\begin{aligned}
&E\big[\operatorname{Tr}_R W_{\gamma_1}^{\epsilon,m_1}(R_k x) \otimes W_{\gamma_2}^{\epsilon,m_2}(R_k x)\big] \\
&\quad = \sum_{\alpha_1,\alpha_2,\ldots,\alpha_{m_1}=1}^{3} \sum_{\beta_1,\beta_2,\ldots,\beta_{m_1}=1}^{3} \operatorname{Tr} E_{\alpha_1} E_{\alpha_2} \cdots E_{\alpha_{m_1}} \otimes E_{\beta_1} E_{\beta_2} \cdots E_{\beta_{m_1}} \\
&\qquad \times E\Bigg[\int_0^1 \int_0^{t_1} \cdots \int_0^{t_{m_1-1}} \int_0^1 \int_0^{\tau_1} \cdots \int_0^{\tau_{m_1-1}} d\big(R_k x_{\gamma_1}^{\alpha_1}\big)(t_1)\, d\big(R_k x_{\gamma_1}^{\alpha_2}\big)(t_2) \cdots \\
&\qquad \cdot d\big(R_k x_{\gamma_1}^{\alpha_{m_1}}\big)(t_{m_1})\, d\big(R_k x_{\gamma_2}^{\beta_1}\big)(\tau_1)\, d\big(R_k x_{\gamma_2}^{\beta_2}\big)(\tau_2) \cdots d\big(R_k x_{\gamma_2}^{\beta_{m_1}}\big)(\tau_{m_1})\Bigg] \\
&\qquad + T_{\text{self}}.
\end{aligned} \tag{6.7}$$

Then writing for brevity

$$\boldsymbol{j}_i = \begin{cases} (R_k x_{\gamma_1}^{\alpha_i})(t_i) & \text{if } i \leqslant m_1, \\ (R_k x_{\gamma_2}^{\beta_{i-m_1}})(\tau_{i-m_1}) & \text{if } i > m_1, \end{cases}$$

we obtain, in a manner similar to the derivation of (6.3), that

$$\begin{aligned}
&E\Bigg[\int_0^1 \int_0^{t_1} \cdots \int_0^{t_{m_1-1}} \int_0^1 \int_0^{\tau_1} \cdots \int_0^{\tau_{m_1-1}} d\big(R_k x_{\gamma_1}^{\alpha_1}\big)(t_1)\, d\big(R_k x_{\gamma_1}^{\alpha_2}\big)(t_2) \cdots \\
&\qquad \cdot d\big(R_k x_{\gamma_1}^{\alpha_{m_1}}\big)(t_{m_1})\, d\big(R_k x_{\gamma_2}^{\beta_1}\big)(\tau_1)\, d\big(R_k x_{\gamma_2}^{\beta_2}\big)(\tau_2) \cdots d\big(R_k x_{\gamma_2}^{\beta_{m_1}}\big)(\tau_{m_1})\Bigg] \\
&\quad = \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{m_1-1}} \int_0^1 \int_0^{\tau_1} \cdots \int_0^{\tau_{m_1-1}} \frac{1}{m_1! 2^{m_1}} \sum_{\sigma \in \mathfrak{S}_{2m_1}} dE[\boldsymbol{j}_{\sigma(1)} \boldsymbol{j}_{\sigma(2)}] \\
&\qquad \cdot dE[\boldsymbol{j}_{\sigma(3)} \boldsymbol{j}_{\sigma(4)}] \cdots dE[\boldsymbol{j}_{\sigma(2m_1-1)} \boldsymbol{j}_{\sigma(2m_1)}].
\end{aligned} \tag{6.8}$$

Since in the right side of (6.8) those terms having $\sigma(i-1)$ and $\sigma(i)$ both in $\{1,2,\dots,m_1\}$ or $\{m_1+1, m_1+2,\dots,2m_1\}$ belong to T_{self}, it follows that

$$\begin{aligned}
(6.8) &= \int_0^1\int_0^{t_1}\cdots\int_0^{t_{m_1-1}}\int_0^1\int_0^{\tau_1}\cdots\int_0^{\tau_{m_1-1}}\sum_{\sigma\in\mathfrak{S}_{m_1}} dE[\boldsymbol{j}_1\boldsymbol{j}_{m_1+\sigma(1)}]\,dE[\boldsymbol{j}_2\boldsymbol{j}_{m_1+\sigma(2)}] \\
&\quad \cdots dE[\boldsymbol{j}_{m_1}\boldsymbol{j}_{m_1+\sigma(m_1)}] + T_{\text{self}} \\
&= \int_0^1\int_0^{t_1}\cdots\int_0^{t_{m_1-1}}\int_0^1\int_0^{\tau_1}\cdots\int_0^{\tau_{m_1-1}}\sum_{\sigma\in\mathfrak{S}_{m_1}} dE\big[\big(R_k x_{\gamma_1}^{\alpha_1}\big)(t_1)\big(R_k x_{\gamma_2}^{\beta_{\sigma(1)}}\big)(\tau_{\sigma(1)})\big] \\
&\quad \cdot dE\big[\big(R_k x_{\gamma_1}^{\alpha_2}\big)(t_2)\big(R_k x_{\gamma_2}^{\beta_{\sigma(2)}}\big)(\tau_{\sigma(2)})\big]\cdots dE\big[\big(R_k x_{\gamma_1}^{\alpha_{m_1}}\big)(t_{m_1})\big(R_k x_{\gamma_2}^{\beta_{\sigma(m_1)}}\big)(\tau_{\sigma(m_1)})\big] \\
&\quad + T_{\text{self}}.
\end{aligned}$$

Again, since $(R_k x_{\gamma_1}^{\alpha})(t_1)$ and $(R_k x_{\gamma_1}^{\beta})(t_1)$ are independent and identically distributed if $\alpha\neq\beta$, we have

$$\begin{aligned}
E\big[\big(R_k x_{\gamma_1}^{\alpha_j}\big)(t_j)\big(R_k x_{\gamma_2}^{\beta_{\sigma(j)}}\big)(\tau_{\sigma(j)})\big] &= \delta_{\alpha_j\beta_{\sigma(j)}} E\big[\big(R_k x_{\gamma_1}^{\alpha_j}\big)(t_j)\big(R_k x_{\gamma_2}^{\alpha_j}\big)(\tau_{\sigma(j)})\big] \\
&= \delta_{\alpha_j\beta_{\sigma(j)}} E\big[\big(R_k x_{\gamma_1}^{\alpha}\big)(t_j)\big(R_k x_{\gamma_2}^{\alpha}\big)(\tau_{\sigma(j)})\big]
\end{aligned}$$

from which we see that the right side of (6.8) is equal to

$$\begin{aligned}
&\int_0^1\int_0^{t_1}\cdots\int_0^{t_{m_1-1}}\int_0^1\int_0^{\tau_1}\cdots\int_0^{\tau_{m_1-1}}\sum_{\sigma\in\mathfrak{S}_{m_1}}\prod_{j=1}^{m_1}\delta_{\alpha_j\beta_{\sigma(j)}}\, dE\big[\big(R_k x_{\gamma_1}^{\alpha}\big)(t_1)\big(R_k x_{\gamma_2}^{\alpha}\big)(\tau_{\sigma(1)})\big] \\
&\quad \cdot dE\big[\big(R_k x_{\gamma_1}^{\alpha}\big)(t_2)\big(R_k x_{\gamma_2}^{\alpha}\big)(\tau_{\sigma(2)})\big]\cdots dE\big[\big(R_k x_{\gamma_1}^{\alpha}\big)(t_{m_1})\big(R_k x_{\gamma_2}^{\alpha}\big)(\tau_{\sigma(m_1)})\big] \\
&\quad + T_{\text{self}}. \qquad (6.9)
\end{aligned}$$

It then follows from (6.7)–(6.9) that

$$\begin{aligned}
&E\big[\operatorname{Tr}_R W_{\gamma_1}^{\epsilon,m_1}(R_k x)\operatorname{Tr}_R W_{\gamma_2}^{\epsilon,m_2}(R_k x)\big] \\
&\quad = \sum_{\alpha_1,\alpha_2,\dots,\alpha_{m_1}=1}^{3} \operatorname{Tr} E_{\alpha_1}E_{\alpha_2}\cdots E_{\alpha_{m_1}}\otimes E_{\alpha_1}E_{\alpha_2}\cdots E_{\alpha_{m_1}} \\
&\quad \times \int_0^1\int_0^{t_1}\cdots\int_0^{t_{m_1-1}}\int_0^1\int_0^{\tau_1}\cdots\int_0^{\tau_{m_1-1}}\sum_{\sigma\in\mathfrak{S}_{m_1}} dE\big[\big(R_k x_{\gamma_1}^{\alpha}\big)(t_1)\big(R_k x_{\gamma_2}^{\alpha}\big)(\tau_{\sigma(1)})\big] \\
&\quad \cdot dE\big[\big(R_k x_{\gamma_1}^{\alpha}\big)(t_2)\big(R_k x_{\gamma_2}^{\alpha}\big)(\tau_{\sigma(2)})\big]\cdots dE\big[\big(R_k x_{\gamma_1}^{\alpha}\big)(t_{m_1})\big(R_k x_{\gamma_2}^{\alpha}\big)(\tau_{\sigma(m_1)})\big] \\
&\quad + T_{\text{self}}. \qquad (6.10)
\end{aligned}$$

Now, noting that

$$\int_0^1\int_0^{\tau_1}\cdots\int_0^{\tau_{m_1-1}}\sum_{\sigma\in\mathfrak{S}_{m_1}} dE\big[\big(R_kx^\alpha_{\gamma_1}\big)(t_1)\big(R_kx^\alpha_{\gamma_2}\big)(\tau_{\sigma(1)})\big]$$
$$\cdot dE\big[\big(R_kx^\alpha_{\gamma_1}\big)(t_2)\big(R_kx^\alpha_{\gamma_2}\big)(\tau_{\sigma(2)})\big]\cdots dE\big[\big(R_kx^\alpha_{\gamma_1}\big)(t_{m_1})\big(R_kx^\alpha_{\gamma_2}\big)(\tau_{\sigma(m_1)})\big]$$
$$=\int_0^1\int_0^1\cdots\int_0^1 dE\big[\big(R_kx^\alpha_{\gamma_1}\big)(t_1)\big(R_kx^\alpha_{\gamma_2}\big)(\tau_1)\big]\,dE\big[\big(R_kx^\alpha_{\gamma_1}\big)(t_2)\big(R_kx^\alpha_{\gamma_2}\big)(\tau_2)\big]$$
$$\cdots dE\big[\big(R_kx^\alpha_{\gamma_1}\big)(t_{m_1})\big(R_kx^\alpha_{\gamma_2}\big)(\tau_{m_1})\big],$$

and for any $\sigma\in\mathfrak{S}_{m_1}$

$$\int_0^1\int_0^{t_1}\cdots\int_0^{t_{m_1-1}}\int_0^1\int_0^1\cdots\int_0^1 dE\big[\big(R_kx^\alpha_{\gamma_1}\big)(t_1)\big(R_kx^\alpha_{\gamma_2}\big)(\tau_1)\big]$$
$$\cdot dE\big[\big(R_kx^\alpha_{\gamma_1}\big)(t_2)\big(R_kx^\alpha_{\gamma_2}\big)(\tau_2)\big]\cdots dE\big[\big(R_kx^\alpha_{\gamma_1}\big)(t_{m_1})\big(R_kx^\alpha_{\gamma_2}\big)(\tau_{m_1})\big]$$
$$=\int_0^1\int_0^{t_1}\cdots\int_0^{t_{m_1-1}}\int_0^1\int_0^1\cdots\int_0^1 dE\big[\big(R_kx^\alpha_{\gamma_1}\big)(t_{\sigma(1)})\big(R_kx^\alpha_{\gamma_2}\big)(\tau_1)\big]$$
$$\cdot dE\big[\big(R_kx^\alpha_{\gamma_1}\big)(t_{\sigma(2)})\big(R_kx^\alpha_{\gamma_2}\big)(\tau_2)\big]\cdots dE\big[\big(R_kx^\alpha_{\gamma_1}\big)(t_{\sigma(m_1)})\big(R_kx^\alpha_{\gamma_2}\big)(\tau_{m_1})\big],$$

we find from (6.10) that for each $\alpha=1,2,3$,

$$E\big[\mathrm{Tr}_R\,W^{\epsilon,m_1}_{\gamma_1}(R_kx)\,\mathrm{Tr}_R\,W^{\epsilon,m_2}_{\gamma_2}(R_kx)\big]$$
$$=\mathrm{Tr}\Bigg(\sum_{\alpha_1=1}^3 E_{\alpha_1}\otimes E_{\alpha_1}\Bigg)^{m_1}\frac{1}{m_1!}$$
$$\times\int_0^1\int_0^{t_1}\cdots\int_0^{t_{m_1-1}}\int_0^1\int_0^1\cdots\int_0^1\sum_{\sigma\in\mathfrak{S}_{m_1}} dE\big[\big(R_kx^\alpha_{\gamma_1}\big)(t_{\sigma(1)})\big(R_kx^\alpha_{\gamma_2}\big)(\tau_1)\big]$$
$$\cdot dE\big[\big(R_kx^\alpha_{\gamma_1}\big)(t_{\sigma(2)})\big(R_kx^\alpha_{\gamma_2}\big)(\tau_2)\big]\cdots dE\big[\big(R_kx^\alpha_{\gamma_1}\big)(t_{\sigma(m_1)})\big(R_kx^\alpha_{\gamma_2}\big)(\tau_{m_1})\big]$$
$$+\,T_{\mathrm{self}}$$
$$=\mathrm{Tr}\Bigg(\sum_{\alpha_1=1}^3 E_{\alpha_1}\otimes E_{\alpha_1}\Bigg)^{m_1}\frac{1}{m_1!}$$
$$\times\int_0^1\int_0^1\cdots\int_0^1\int_0^1\int_0^1\cdots\int_0^1 dE\big[\big(R_kx^\alpha_{\gamma_1}\big)(t_1)\big(R_kx^\alpha_{\gamma_2}\big)(\tau_1)\big]$$

$$\cdot\, dE\big[\big(R_k x^{\alpha}_{\gamma_1}\big)(t_2)\big(R_k x^{\alpha}_{\gamma_2}\big)(\tau_2)\big]\cdots dE\big[\big(R_k x^{\alpha}_{\gamma_1}\big)(t_{m_1})\big(R_k x^{\alpha}_{\gamma_2}\big)(\tau_{m_1})\big] + T_{\mathrm{self}}$$
$$= \mathrm{Tr}\left(\sum_{\alpha_1=1}^{3} E_{\alpha_1}\otimes E_{\alpha_1}\right)^{m_1}\frac{1}{m_1!}E\big[\big(R_k x^{\alpha}_{\gamma_1}\big)(1)\big(R_k x^{\alpha}_{\gamma_2}\big)(1)\big]^{m_1} + T_{\mathrm{self}}.$$

Summing up the above argument together with Lebesgue's convergence theorem guaranteed by an estimate similar to that in the proof of (2) of Lemma 1, we finally obtain

$$I_{CS}\big(F_0^{\epsilon}\big) = E\big[F_0^{\epsilon}(R_k x)\big] = E\left[\prod_{j=1}^{2}\mathrm{Tr}_R\, W^{\epsilon}_{\gamma_j}(R_k x)\right]$$
$$= (\mathrm{Tr}\, I)^2 + \sum_{n=1}^{\infty}\mathrm{Tr}\left(\sum_{\alpha_1=1}^{3} E_{\alpha_1}\otimes E_{\alpha_1}\right)^{n}\frac{1}{n!}E\big[\big(R_k x^{\alpha}_{\gamma_1}\big)(1)\big(R_k x^{\alpha}_{\gamma_2}\big)(1)\big]^{n} + T_{\mathrm{self}}.$$

Step 3. Now, noting that an orthonormal basis of $\mathfrak{su}(2)$ is given by

$$E_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}\sqrt{-1} & 0\\ 0 & -\sqrt{-1}\end{bmatrix},\quad E_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix},\quad E_3 = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & \sqrt{-1}\\ \sqrt{-1} & 0\end{bmatrix},$$

so that

$$E_1\otimes E_1 = \frac{1}{2}\begin{bmatrix}-1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{bmatrix},\quad E_2\otimes E_2 = \frac{1}{2}\begin{bmatrix}0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\\ 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\end{bmatrix},$$
$$E_3\otimes E_3 = \frac{1}{2}\begin{bmatrix}0 & 0 & 0 & -1\\ 0 & 0 & -1 & 0\\ 0 & -1 & 0 & 0\\ -1 & 0 & 0 & 0\end{bmatrix},$$

we have

$$\sum_{\alpha_1=1}^{3} E_{\alpha_1}\otimes E_{\alpha_1} = \frac{1}{2}\begin{bmatrix}-1 & 0 & 0 & 0\\ 0 & 1 & -2 & 0\\ 0 & -2 & 1 & 0\\ 0 & 0 & 0 & -1\end{bmatrix}.$$

Since the eigenvalues of $2\sum E_{\alpha_1}\otimes E_{\alpha_1}$ are $-1, -1, -1, 3$, we obtain

$$\mathrm{Tr}\left(\sum_{\alpha_1=1}^{3} E_{\alpha_1}\otimes E_{\alpha_1}\right)^{n} = \frac{(-1)^n + (-1)^n + (-1)^n + 3^n}{2^n}.$$

Consequently, we have

$$\begin{aligned}
I_{CS}\big(F_0^{\epsilon}\big) &= E\big[F_0^{\epsilon}(R_k x)\big] \\
&= (\operatorname{Tr} I)^2 + \sum_{n=1}^{\infty} \operatorname{Tr}\left(\sum_{\alpha_1=1}^{3} E_{\alpha_1} \otimes E_{\alpha_1}\right)^n \frac{1}{n!} E\big[\big(R_k x_{\gamma_1}^{\alpha}\big)(1)\big(R_k x_{\gamma_2}^{\alpha}\big)(1)\big]^n + T_{\mathrm{self}} \\
&= 4 + \sum_{n=1}^{\infty} \frac{(-1)^n + (-1)^n + (-1)^n + 3^n}{2^n} \frac{1}{n!}\left(-\frac{1}{2\sqrt{-1}k} L(\gamma_1, \gamma_2)\right)^n + T_{\mathrm{self}} \\
&= 4 + \sum_{n=1}^{\infty} \frac{\sqrt{-1}^n\{(-1)^n + (-1)^n + (-1)^n + 3^n\}}{(4k)^n} \frac{1}{n!} L(\gamma_1, \gamma_2)^n + T_{\mathrm{self}} \\
&= 3e^{-\sqrt{-1}L(\gamma_1,\gamma_2)/4k} + e^{3\sqrt{-1}L(\gamma_1,\gamma_2)/4k} + T_{\mathrm{self}},
\end{aligned}$$

where

$$L(\gamma_1, \gamma_2) = \text{the linking number of loops } \gamma_1 \text{ and } \gamma_2.$$

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