Discrete Applied Mathematics 159 (2011) 232–239

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Planar graphs without cycles of length 4, 7, 8, or 9 are 3-choosable☆

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ARTICLE INFO

Article history:
Received 8 September 2009
Received in revised form 28 October 2010
Accepted 4 November 2010
Available online 7 December 2010

Keywords:
Planar graph
Cycle
Choosability
Coloring

ABSTRACT

It is known that planar graphs without cycles of length 4, i, j, or 9 with $4 < i < j < 9$, except that $i = 7$ and $j = 8$, are 3-choosable. This paper proves that planar graphs without cycles of length 4, 7, 8, or 9 are also 3-choosable.

1. Introduction

Graphs considered in this paper are finite, simple and undirected. Let $G = (V, E)$ be a graph with the set of vertices V and the set of edges E. A mapping $\phi : V \longrightarrow \{1, \ldots, k\}$ is called a *k-coloring* of G if $\phi(u) \neq \phi(v)$ whenever $uv \in E$. G is said to be *k-colorable* if it admits a k-coloring. For every $v \in V$, assign a list of available colors to v, say $L(v) \subset \{1, 2, \ldots\}$, then $L = \{L(v)|v \in V\}$ is called a *list-assignment* of G. If there is a mapping $\phi : V \longrightarrow \{1, 2, \ldots\}$ such that $\phi(v) \in L(v)$ for each $v \in V$ and $\phi(u) \neq \phi(v)$ whenever $uv \in E$, then G is said to be *L-colorable*. G is said to be *k-list-colorable*, or *k-choosable*, if it is L-colorable for every list-assignment L with $|L(v)| \geq k$ for all $v \in V$. Clearly, if G is k-choosable, then it is k-colorable. However, the converse is generally not true. For example, given any positive integer k, there are bipartite (2-colorable) graphs which are not k-choosable, see [1].

Call a graph *planar* if it can be embedded into the plane so that its edges only meet at their ends. Any such embedding of a planar graph is called a *plane* graph. For a positive integer k, a *k-cycle* is a cycle of length k. A 3-cycle is also called a *triangle*.

For choosability of planar graphs, in 1979, Erdös et al. [1] conjectured that every planar graph is 5-choosable and there are planar graphs which are not 4-choosable. More than one decade later, Voigt [7] constructed a planar graph which is not 4-choosable; Thomassen [4] proved that every planar graph is 5-choosable. A natural problem on choosability of planar graphs is to determine whether a given planar graph is 3-choosable, or 4-choosable. In 1996, Gutner [2] proved that these two problems are NP-hard. Thus, sufficient conditions for a planar graph to be 3-, or 4-choosable is of interest.

This paper mainly concerns 3-choosability of planar graphs with some forbidden short cycles. Note that odd cycles are not 2-colorable, hence, not 2-choosable. It follows that every (planar) graph with odd cycles is not 2-choosable. What conditions can ensure a (planar) graph with odd cycles to be 3-choosable? Thomassen [5] proved that every planar graph with girth at least 5 (with neither 3- nor 4-cycle) is 3-choosable. What conditions can ensure a (planar) graph with triangles to be 3-choosable? Montassier [3] conjectured that planar graphs without cycles of length 4, 5 or 6 are 3-choosable. Generally,

☆ Supported by the Natural Science Foundation of Zhejiang Province, China, Grant No. Y6090699, partially supported by the Natural Science Foundation of China, Grant No. 10971198, and Zhejiang Innovation Project, Grant No. T200905.

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0166-218X/$ – see front matter
doi:10.1016/j.dam.2010.11.002

what is the set of pairs of integers (i, j), if any, every planar graph without cycles of length 4, i, or j is 3-choosable? Towards this problem, we would like to summarize some known related results as follows.

Theorem A. *A planar graph is* 3-*choosable if it has no*

- [11] 4-, 5-, 6-, *or* 9-*cycles; or*
- [10] 4-, 5-, 7-, *or* 9-*cycles; or*
- [9] 4-, 5-, 8-, *or* 9-*cycles; or*
- [8] 4-, 6-, 7-, *or* 9-*cycles; or*
- [6] 4-, 6-, 8-, *or* 9-*cycles.*

This paper will prove the following result.

Theorem 1. *Every planar graph without cycles of length* 4, 7, 8, *or* 9 *is* 3-*choosable.*

Clearly, Theorem A together with Theorem 1 completes one interesting stage conclusion on 3-choosability of planar graphs without some short cycles as follows.

Theorem B. *Planar graphs without cycles of length* 4, i, j, *or* 9 *with* $4 < i < j < 9$ *are* 3-*choosable.*

The rest of this section is devoted to some terminology and notation used later. Let $G = (V, E, F)$ be a plane graph with the set of faces F. For a face $f \in F$, its *boundary*, denoted by $b(f)$, is the closed walk around f. The steps of $b(f)$, denoted by $d(f)$, is called the *degree* of f. A face f is *simple* if its boundary is a cycle. The set of vertices on the boundary of f is denoted by $V(f)$. We often specify a simple face in a plane graph by the sequences of its vertices in the clockwise order or in the anticlockwise order. A vertex v and a face f are *incident* if $v \in V(f)$. Two faces are *adjacent* if they have at least one edge in common. Two faces are *normally* adjacent if they have exactly one edge in common. Furthermore, two faces are *strict* normally adjacent if they are normally adjacent and have no more vertices in common other than the ends of their unique common edge. Let $S \subset V$, $G - S$ is the graph obtained from G by deleting all vertices in S. As usual, $G[S]$ is the subgraph of G induced by S. Call $v \in V$ a *k-vertex*, or a k^+-*vertex*, or a k^--*vertex* if its degree $d(v)$ is equal to k, or at least k, or at most k, respectively. The notions of a *k-face*, a k^+-*face* and a k^--*face* are similarly defined. The *minimum degree* of vertices of G is denoted by δ. An edge $e = xy$ is often said to be a $(d(x), d(y))$-edge.

A *chord* of a cycle is an edge that connects two non-consecutive vertices of the cycle. Let C be a cycle in G and xy a chord of C. If xy lies in the region inside C, then xy is called an *internal* chord of C. Otherwise, xy is called an *external* chord of C. Call a vertex-induced subgraph H in G a *special* Θ-*like* subgraph, in short, an $S\Theta$, if H satisfies

(1) $\delta(H) = 2$;
(2) having a spanning cycle C;
(3) after deleting all external chords (if any) of C, the resulting graph is just C with exactly one internal chord that is a $(3, 4^-)$-edge in G;
(4) except possibly one vertex that, being an end of the unique internal chord of C, may be a 4-vertex, all the other vertices of H are 3-vertices in G.

2. Proof of Theorem 1

We shall prove Theorem 1 by showing a structural theorem as follows.

Theorem 2. *Let G be a plane graph with $\delta \geq 3$. If G has no cycles of length* 4, 7, 8, *or* 9, *then G has either an $S\Theta$ or a* 10-*face incident with ten* 3-*vertices.*

Assuming Theorem 2, we can easily prove Theorem 1:

Suppose that G is a counterexample to Theorem 1 with minimum number of vertices, then $\delta \geq 3$. Embedding G into the plane, we get a plane graph, still denoted by G. Since G has no i-cycles for all $i = 4, 7, 8, 9$, according to Theorem 2, G has either an $S\Theta$, or a 10-face incident with ten 3-vertices. Let V' be the vertex-set of a possible $S\Theta$ or a possible 10-face incident with ten 3-vertices and L a list-assignment of G with $|L(v)| \geq 3$ for all $v \in V$ such that G is not L-colorable. Setting $G' = G - V'$, by the choice of G, G' admits an L'-coloring ϕ where L' is the restriction of L to G'. For $v \in V'$, let $L'(v) = L(v) \setminus \{\phi(u)\}$, where u is the unique neighbor of v in G', if any. Thus, $|L'(v)| \geq 3$ if v is not adjacent to any vertices in G', $|L'(v)| \geq 2$ otherwise. Note that $G[V']$ is isomorphic to an $S\Theta$, or a 10-cycle, or a 10-cycle with exactly one external chord that evenly divides the 10-cycle. In each case, $G[V']$ is L'-colorable by Lemma 1. Thus, G is L-colorable, a contradiction. □

Lemma 1. (1) *An even cycle is* 2-*choosable.*
(2) *A cycle C with exactly one (external) chord uw is L-colorable, if $|L(u)|, |L(w)| \geq 3$, and $|L(v)| \geq 2$ for $v \in V(C) \setminus \{u, w\}$.*
(3) *Let H be an $S\Theta$ in G with the unique internal chord xy and L an list-assignment of H satisfying*
 (i) $|L(v)| = d_H(v)$ *for* $v \in V(H) \setminus \{y\}$.
 (ii) $|L(y)| \geq d_H(y) - 1$. *Then, H is L-colorable.*

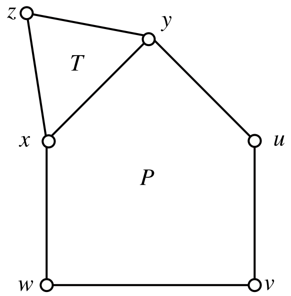

Fig. 1. The strict normal adjacency of a 3-face to a 5-face.

Proof. (1) and (2) are obvious. (3) Let C be the spanning cycle of H and z a 2-vertex of H that is closest to y on C. Without loss of generality, we may assume that x, z and y appear on C in the anticlockwise order. If there is a 3-vertex other than y in H on the segment of C from z to y in the anticlockwise order, then z', the neighbor of z on the segment, is a 3-vertex. We choose a color from $L(z') \setminus L(z)$ to color z', and then we can color all other vertices along C in the anticlockwise order until z. Otherwise, z is adjacent to y on C. If $|L(y)| = 2$, then we choose a color from $L(x) \setminus L(y)$ to color x, and then color all other vertices along C in the anticlockwise order. If $|L(y)| \geq 3$, then we choose a color from $L(y) \setminus L(z)$ to color y, and then color all other vertices along C in the anticlockwise order. (Note that, if u is an end of an external chord of C, then $|L(u)| = 3$. In the coloring procedure above, before coloring u, at most two neighbors of u on C have been colored; hence, u can always be properly colored.) □

3. Proof of Theorem 2

First, if there is a plane graph having

(1) $\delta \geq 3$;
(2) no cycles of length 4, 7, 8, or 9;
(3) no $S\Theta$;
(4) no 10-face incident with ten 3-vertices,

then there is a connected plane graph also having (1), (2), (3) and (4), since any component of G is clearly the graph that we are looking for.

Lemma 2. *If there is a connected plane graph having* (1), (2), (3) *and* (4), *then there is at least one 2-connected plane graph also having* (1), (2), (3) *and* (4).

Proof. Let G be a connected plane graph having (1), (2), (3) and (4). If G itself is 2-connected, then we are done. Otherwise, let B be an end-block of G and u the unique cut-vertex belonging to B. If the degree of u in B is at least 5, then B is the graph that we are looking for, since it satisfies (1), (2), (3) and (4), and is 2-connected. Otherwise, we can construct a graph H as follows. Let $v \in B$ be a neighbor of u lying on the outer face of B. Take 11 copies $B_0, B_1, \ldots, B_{10}$ of B and identify v_i with u_{i+1} for $0 \leq i \leq 10$, where the indices are modulo 11. Clearly, the graph H satisfies (1), (2), (3) and (4), and is 2-connected. □

Lemma 3. *Let G be a* 2*-connected plane graph with* $\delta \geq 3$ *and without cycles of length* 4, 7, 8, *or* 9 . *Then G has no the following configurations:*

C1 *a* 3*-face adjacent to a* 3*-face ;*
C2 *a* 3*-face adjacent to a* 6*-face ;*
C3 *a* 5*-face adjacent to a* 5*-face;*
C4 *a* 3*-face adjacent to two* 5*-faces ;*
C5 *a* 5*-face adjacent to two* 3*-faces.*

Proof. We only show that G has neither C3 nor C4, since the others can be similarly (even more easily) proved. Note that every face in G is simple since G is 2-connected.

First, we claim that if a 3-face and a 5-face are adjacent, then they are strict normally adjacent. To see this, let $T = xyz$ be a 3-face adjacent to a 5-face $P = uvwxy$, see Fig. 1. If $z = u$, then $uvwxz(=u)$ is a 4-cycle in G, a contradiction. By symmetry, $z \neq w$. If $z = v$, then $uyxz(=v)u$ is again a 4-cycle in G, a contradiction.

Next, we claim that two adjacent 5-faces, if any, must be strict normally adjacent, too. To see this, let $P = uvwxy$ and $P' = xyu'v'w'$ be two adjacent 5-faces, see Fig. 2. If $u = u'$, then, as edges, $yu = yu'$ since G is simple. Thus, $d(y) = 2$ since P and P' are faces. This clearly contradicts $\delta \geq 3$. If $u = v'$, then $uyxw'v'(=u)$ is a 4-cycle in G, a contradiction. If $u = w'$, then $uyu'v'w'(=u)$ would be again a 4-cycle. By symmetry, $w \neq w', v', u'$. It is easy to check that $v \neq u', v', w'$. Therefore, two adjacent 5-faces, if any, must be strict normally adjacent in G. However, this would imply that G has an 8-cycle. So G has no C3.

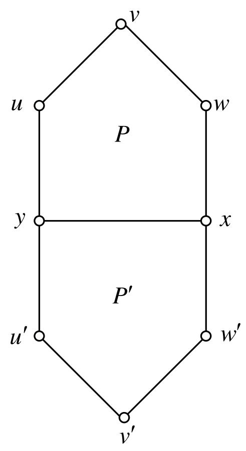

Fig. 2. The non-adjacency of a 5-face to a 5-face in G.

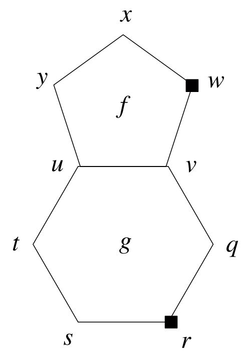

Fig. 3. The adjacency between a 5-face and a 6-face.

To prove C4. Let $T = xyz$ be a 3-face, $P = uvwxz$ and $P' = yzu'v'w'$, two 5-faces adjacent to T. First we have $y \notin \{u, v, w\}$ and $x \notin \{u', v', w'\}$ since a 3-face and a 5-face can only be strict normally adjacent. Next, $u \neq u'$ since a 5-face cannot be adjacent to a 5-face. Lastly, $\{u, v, w\} \cap \{u', v', w'\} = \emptyset$ just because G has no 4-cycles. It follows that G has a 9-cycle $C = xwvuzu'v'w'yx$, a contradiction showing that G has no C4. □

Lemma 4. *If there exists a* 2*-connected plane graph having* (1), (2), (3) *and* (4)*, then there exists a* 2*-connected plane graph having not only* (1), (2), (3) *and* (4) *but also* (5) *no* 5*-face is adjacent to a* 6*-face.*

Proof. Let G be a 2-connected plane graph having (1), (2), (3) and (4). If no 5-face is adjacent to a 6-face in G, then G is the desired graph, we are done. Otherwise, let $f = vuyxw$ be a 5-face that is adjacent to a 6-face $g = qrstuv$ in G. We claim that either $w = r$, or else, $y = s$. First, f and g cannot be strict normally adjacent since otherwise G would have a 9-cycle. Next, as in Lemma 3, it is easy to check that (see Fig. 3, two black boxes are identical) $w \neq q, s, t$, $x \neq q, r, s, t$ and $y \neq q, r, t$. If $w = r$ and $y = s$, then G has a 4-cycle $yuvwy$, a contradiction. The claim follows. Suppose without loss of generality that $w = r$. We call a triangle between a 5-face and a 6-face *special*; furthermore, *simply special* if no other special triangle is in its interior. Since G is finite, G has at least one simply special triangle. As indicated in Fig. 3, let $T = qvw(w = r)$ be a simply special triangle in G and H the subgraph of G induced by T and its interior. If $d_H(v) \geq 5$ and $d_H(w) \geq 5$, then H is the graph that we are looking for. Otherwise, H could have at least one 2-vertex that belongs to $\{v, w = r\}$, or a 10-face f that is incident with ten 3-vertices (v or w or both belongs to $V(f)$), or an $S\Theta$ (with v or w being an end of an internal chord of the spanning cycle of H). Now, we can construct the desired graph G^* as follows: take 11 copies $H_0, H_1, \ldots, H_{10}$ of H and identify v_i with w_{i+1} for $0 \leq i \leq 10$, where the indices are modulo 11. □

From now on, $G = (V, E, F)$ is a 2-connected plane graph satisfying (1), (2), (3), (4) and (5) stated in Lemma 4. Our goal is to derive a contradiction if such a G exists. The desired contradiction is obtained by a discharging procedure. In the procedure, the *initial charge* ch on $V \cup F$ is defined as: $\mathrm{ch}(v) = 2d(v) - 6$ for $v \in V$, $\mathrm{ch}(f) = d(f) - 6$ for $f \in F$. Applying $\sum_{v\in V} d(v) = 2|E| = \sum_{f\in F} d(f)$, Euler's formula $|V| - |E| + |F| = 2$ can be rewritten as

$$\sum_{x\in V\cup F} \mathrm{ch}(x) = -12.$$

We use ch′ to denote the *final charge* when a discharging procedure is over. If we can define suitable discharging rules such that $\mathrm{ch}'(x) \geq 0$ for every $x \in V \cup F$, then we get an obvious contradiction $-12 = \sum_{x\in V\cup F} \mathrm{ch}(x) = \sum_{x\in V\cup F} \mathrm{ch}'(x) \geq 0$, which completes the proof of Theorem 2.

Call a vertex *big* if it is a 4^+-vertex; *triangular* if it is incident with a triangle. A 5-face is *light* if it is incident with at most one big vertex. Let $g = uvwxy$ be a 5-face (strict normally) adjacent to a 3-face $T = uvz$. We say that x is 2-*apart from* T, in short, 2-*apart*. A 10-face f is *weak* if it is incident with one 4-vertex and nine 3-vertices, and adjacent to five faces that are mutually disjoint on the boundary of f; moreover, each of the five faces is either a 3-face or a light 5-face, and the face incident with the unique 4-vertex among the five faces is a 3-face. A 4-vertex is called a *cross* if it is incident with one 3-face T, one 5-face g, and one weak face f, where T and f are adjacent. In the definition of a cross, the face adjacent to T other than f is called the *handle* of the cross. Note that the handle of a cross is neither a 3-face by C1 of Lemma 3 nor a 6-face by C2 of Lemma 3. Namely, a handle is either a 5-face or a 10^+-face. Also note that a 10-face, as a handle, is not weak since otherwise there would be a 5-face adjacent to a 5-face or two 3-faces by the definition of a weak face, contradicting C3 or C5 of Lemma 3.

Let us make our discharging procedure with the following discharging rules:

R1. A big vertex sends 1 to each incident 3- or 5-face.
R2. A 3-face gets 1 or $\frac{1}{2}$ from each adjacent face according to whether their common edge is a (3, 3)- or a $(3, 4^+)$-edge, respectively.
R3. Let g be a light 5-face and f a 10^+-face adjacent to g with the common edge uv. Then
  R3.1 f sends 1 to g if each of u and v is a non-triangular 3-vertex.
  R3.2 f sends $\frac{1}{2}$ to g if one of u and v is not big and the other is big and 2-apart from an adjacent 3-face of g.
R4. A cross gets $\frac{1}{2}$ from its handle.
R5. A 5^+-vertex sends $\frac{1}{2}$ to each incident 10-face and a 4-vertex sends $\frac{1}{2}$ to each of incident weak faces. (Note that if a 4-vertex is a cross, then it is incident with exactly one weak face by the definition of a weak face and C3 or C5 of Lemma 3.)

The rest of the paper is devoted to checking that $\mathrm{ch}'(x) \geq 0$ for every $x \in V \cup F$. This consists of two parts as follows.

The final charge of vertices. In this paragraph, we shall show that $\mathrm{ch}'(v) \geq 0$ for every $v \in V$. If $d(v) = 3$, then $\mathrm{ch}'(v) = \mathrm{ch}(v) = 2d(v) - 6 = 2 \times 3 - 6 = 0$, since no charge is sent from or to v according to our rules. Let $d(v) \geq 4$. Note that both R2 and R3 do not apply here. Namely, only R1, R4 and R5 may apply. If $d(v) \geq 6$, then $\mathrm{ch}'(v) \geq \mathrm{ch}(v) - d(v) \times 1 = (2d(v) - 6) - d(v) = d(v) - 6 \geq 0$ by R1 and R5. If $d(v) = 5$, then $\mathrm{ch}'(v) \geq \mathrm{ch}(v) - 3 \times 1 - 2 \times \frac{1}{2} = 0$ since v is incident with at most three 5^--faces by Lemma 3. Finally, let $d(v) = 4$. By Lemma 3, v is incident with at most two 5^--faces. If v is not incident with any weak face, then $\mathrm{ch}'(v) \geq \mathrm{ch}(v) - 2 \times 1 = 0$ by R1. Assume that v is incident with at least one weak face. Let f be a weak face incident with v. By the definition of a weak face, there is a 3-face g that is incident with v and adjacent to f. Let the remaining two adjacent faces at v be f' and g' where f' is adjacent to f and g' is adjacent to g. By C1 of Lemma 3, g' is not a 3-face, and by the definition of a weak face, f' is not a 3-face, too. According to C3 of Lemma 3, at most one of g' and f' is a 5-face. If neither f' nor g' is a 5-face, then $\mathrm{ch}'(v) \geq \mathrm{ch}(v) - 1 - 2 \times \frac{1}{2} = 0$ since f' is not weak. If exactly one of f' and g' is a 5-face, then v is a cross; hence, v can get $\frac{1}{2}$ from its handle g' by R4. It follows that $\mathrm{ch}'(v) = \mathrm{ch}(v) - 2 \times 1 - \frac{1}{2} + \frac{1}{2} = 0$.

The final charge of faces. Now, let us analyze the final charge of a face $f \in F$. First, G has no 4-, 7-, 8-, or 9-faces, since it has no cycles of length 4, 7, 8, or 9.

Let f be a 3-face. If all vertices of f are big, then $\mathrm{ch}'(v) = \mathrm{ch}(v) + 3 = 0$ by R1. If f is incident with exactly two big vertices, then v gets 1 from each incident big vertex by R1 and $\frac{1}{2}$ from each adjacent 5^+-face that shares a $(3, 4^+)$-edge with f by R2; hence, $\mathrm{ch}'(v) = \mathrm{ch}(v) + 2 \times 1 + 2 \times \frac{1}{2} = 0$. If f is incident with exactly one big vertex, then $\mathrm{ch}'(v) = \mathrm{ch}(v) + 1 \times 1 + 1 \times 1 + 2 \times \frac{1}{2} = 0$ by R1 and R2. If no big vertex is incident with f, then $\mathrm{ch}'(v) = \mathrm{ch}(v) + 3 \times 1 = 0$ by R2.

Let f be a 5-face. By C3 of Lemma 3, f is not adjacent to any 5-face. By C5 of Lemma 3, f is adjacent to at most one 3-face. Namely, f is adjacent to at least four 10^+-faces by hypothesis (5). Note that $\mathrm{ch}(f) = -1$. If f is not a handle of a cross, then f may only send 1 or $\frac{1}{2}$ to a possible adjacent 3-face by R2. If f is a handle of a cross, then f send $\frac{1}{2}$ to the cross by R4 and may also send $\frac{1}{2}$ to the 3-face that is incident with the cross and adjacent to f by R2. To conclude, f send totally at most 1 to its adjacent faces or incident vertices. If we can show that f can get totally at least 2 from its incident vertices or adjacent faces, then we are done. This clearly holds by R1 if f is incident with at least two big vertices. If no big vertex is incident with f, then there are at least two 10^+ faces totally sending 2 to f by R3, since f has at most two adjacent triangular vertices by C5 of Lemma 3. Assume that f is incident with exactly one big vertex. In this case, f gets 1 from its unique incident big vertex by R1, and gets at least 1 totally from its adjacent faces: if f has a (3, 3)-edge with two non-triangular ends, this is clearly true by R3.1, otherwise, f has a big vertex that is 2-apart from an adjacent 3-face of f; hence, R3.2 plays, giving the desired result.

Let f be a 6-face. Since f is adjacent to neither a 3-face nor a 5-face, $\mathrm{ch}'(f) = \mathrm{ch}(f) = 0$ by our rules.

Let us make some preparations before checking $\mathrm{ch}'(f) \geq 0$ for a 10^+-face f. From now on, f is a 10^+-face in G. For an edge e of f, let $g(e)$ be the face that shares e with f, and $c_0(e)$ the amount of the charge sent from f to $g(e)$ by R2 or R3. More precisely,

$$c_0(e) = \begin{cases} 1, & \text{if } e \text{ is a (3,3)-edge and } g(e) \text{ is a 3-face,} \\ \dfrac{1}{2}, & \text{if } e \text{ is an edge with exactly one big end and } g(e) \text{ is a 3-face,} \\ 1, & \text{if } e \text{ is a (3,3)-edge with two non-triangular ends and } g(e) \text{ is a light 5-face,} \\ \dfrac{1}{2}, & \text{if } e \text{ is an edge with exactly one big end being 2-apart and } g(e) \text{ is a light 5-face,} \\ 0, & \text{otherwise.} \end{cases}$$

Let X be the set of edges $e \in E(f)$ such that e is incident with a cross whose handle is f and e is also incident with the 5-face defined in the definition of a cross.

Observation 1. *If $e \in X$, then $c_0(e) = 0$.*

Proof. First, $c_0(e) \neq 1$ since e is not a (3, 3)-edge. Next, $c_0(e) \neq \frac{1}{2}$ since otherwise, as noted above or by R3, the end of e being a cross is 2-apart from a 3-face that is adjacent to $g(e)$ and $g(e)$ is light, but then, according to that the face incident with the cross and adjacent to $g(e)$ other than f is weak, we can easily see that $g(e)$ is either adjacent two 3-faces or adjacent to a light 5-face, contradicting C5 or C3 of Lemma 3. The conclusion follows. □

Observation 2. *If $e \in X$ is incident with a 3-vertex w, then the face adjacent to f and incident with w but not incident with e is neither a 3-face nor a 5-face by C5 and C3 of Lemma 3.* □

Define $c(e) := \frac{1}{2}$ if $e \in X$ and $c(e) := c_0(e)$ otherwise. Observe that, according to R2, R3 and R4, $\sum_{e\in E(f)} c(e)$ is the amount of charge sent by f.

Lemma 5. *Let e_1, e_2 and e_3 be consecutive edges of f. If $c(e_2) = 1$, then $c(e_1) = 0$ and $c(e_3) = 0$.*

Proof. According to R2 and R3, e_2 is a (3, 3)-edge and $g(e_2)$ is either a 3-face or a light 5-face.

If $g(e_2)$ is a 3-face, then neither $g(e_1)$ nor $g(e_3)$ is a 3-face by C1 of Lemma 3. Namely, R2 does not apply when we consider moving charge from f to $g(e_i)$ for $i \in \{1, 3\}$. R3 does not apply, too: first, R3.1 does not apply since e_i has an triangular end; next, R3.2 does not apply since otherwise $g(e_i)$ is a 5-face, and e_i has an end that is big and 2-apart from a 3-face T_i that is adjacent to $g(e_i)$, it follows that $g(e_i)$ is adjacent to two 3-faces, say T_i and $g(e_2)$, a contradiction. Hence, $c_0(e_1) = 0$ and $c_0(e_3) = 0$.

If $g(e_2)$ is a light 5-face, then neither $g(e_1)$ nor $g(e_3)$ is a 5-face by C3 of Lemma 3; hence, R3 does not apply. Note that the two ends of e_2 are non-triangular by R3, namely, neither $g(e_1)$ nor $g(e_3)$ is a 3-face, so R2 does not apply here, too. Hence, $c_0(e_1) = 0$ and $c_0(e_3) = 0$.

By Observation 2, no cross with f as its handle is incident with e_1 or e_3. Hence $c(e_1) = c_0(e_1) = 0$ and $c(e_3) = c_0(e_3) = 0$. The conclusion follows. □

According to Lemma 5, we can equivalently redistribute the charge given by c among the edges of f: in the situation described above, move $\frac{1}{4}$ from e_2 to each of e_1 and e_3. Do this for every edge $e \in E(f)$ with $c(e) = 1$. Let q be the resulting assignment of the charge to the edges of f. Clearly, $\sum_{e\in E(f)} c(e) = \sum_{e\in E(f)} q(e)$, and $q(e) \leq \frac{1}{2}$ for each $e \in E(f)$.

If f is a 12^+-face, then $\mathrm{ch}'(f) \geq \mathrm{ch}(f) - \frac{1}{2}d(f) = \frac{1}{2}(d(f) - 12) \geq 0$.

Observation 3. *The number of edges of f such that $q(e) = \frac{1}{4}$ is even.* □

Lemma 6. *Three edges, each of them satisfies $c(e) = \frac{1}{2}$, cannot be consecutive on f.*

Proof. Let e_1, e_2 and e_3 be three consecutive edges on f such that $c(e_i) = \frac{1}{2}$ for $i = 1, 2, 3$. According to our discharging rules and the definition of the function c, there are three cases producing $c(e_i) = \frac{1}{2}$:

(1) e_i is a $(3, 4^+)$-edge and $g(e_i)$ is a 3-face (by R2);
(2) e_i is a $(3, 4^+)$-edge with the 4^+-end being 2-apart and $g(e_i)$ is a light 5-face (by R3.2);
(3) e_i is a (3, 4)-edge with the end of degree 4 being a cross with handle f (by R4).

We shall derive a contradiction showing the non-existence of such three consecutive edges on f. This will be done by considering three possible cases as follows.

(1) *The face $g(e_2)$ is a 3-face and e_2 is a $(3, 4^+)$-edge.*

Let $e_2 = uv$ with end u being a 3-vertex. Without loss of generality, we may assume that u belongs to $g(e_1)$. By C1 of Lemma 3, $g(e_1)$ is not a 3-face; hence, R2 does not apply for $c(e_1) = \frac{1}{2}$. In other word, $c(e_1) = \frac{1}{2}$ is a result by applying R3.2 or R4. R3.2 cannot apply since the end of e_1 other than u cannot be 2-apart from an adjacent 3-face of $g(e_1)$. So, R4 applies. It follows that $g(e_1)$ is a 5-face and f is the handle of the end of e_1 other than u. By Observation 2, $g(e_2)$ is not a 3-face, a contradiction.

(2) *The face $g(e_2)$ is a light 5-face and e_2 is a $(3, 4^+)$-edge with the 4^+-end being 2-apart from an adjacent 3-face of $g(e_2)$.*

With out loss of generality, we may assume that $e_2 = uv$ with $u \in g(e_1)$ being a 3-vertex. By C3 of Lemma 3, $g(e_1)$ is not a 5-face; hence, both R3 and R4 do not apply for $c(e_1) = \frac{1}{2}$. So, R2 applies. It follows that $g(e_1)$ is a 3-face. On the other hand, that v is 2-apart from an adjacent 3-face of $g(e_2)$ implies that $g(e_2)$ is adjacent to two 3-faces, a contradiction.

(3) *The face $g(e_2)$ is a 5-face and e_2 is a $(3, 4)$-edge with the end of degree 4 being a cross with handle f.*

Let $e_2 = uv$ with u being the cross with handle f and v a 3-vertex (without loss of generality) belonging to $g(e_3)$. By Observation 2, $g(e_3)$ is neither a 3-face nor a 5-face; hence, R2, R3 and R4 do not apply for $c(e_3) = \frac{1}{2}$, a contradiction. □

Now, let f be an 11-face. Note that $\mathrm{ch}'(f) = \mathrm{ch}(f) - \sum_{e\in E(f)} q(e) = 5 - \sum_{e\in E(f)} q(e)$ and $q(e) \in \left\{0, \frac{1}{4}, \frac{1}{2}\right\}$ for every $e \in E(f)$. If there is at least one edge $e \in E(f)$ such that $q(e) = 0$, then $\mathrm{ch}'(f) \geq 0$. If there are at least two edges such that $q(e) = \frac{1}{4}$, then $\mathrm{ch}'(f) \geq 0$. So, $\mathrm{ch}'(f) < 0$ only if f has ten edges such that $q(e) = \frac{1}{2}$ and one edge such that $q(e) \geq \frac{1}{4}$. By Observation 3, we conclude that $q(e) = \frac{1}{2}$ for every $e \in E(f)$. By Lemma 6, equation $q(e) = c(e)$ cannot hold for every $e \in E(f)$. Namely there exists at least one edge $e \in E(f)$ such that $c(e) = 0$. Let $A = \{e \in E(f) | c(e) = 1\}$ and $B = \{e \in E(f) | c(e) = 0\}$. Since $q(e) = \frac{1}{2}$ for every $e \in E(f)$, the edges in A and those in B must be alternatively appeared on f; hence, f is even, a contradiction excluding the possibility of $\mathrm{ch}'(f) < 0$.

Finally, Let f be a 10-face. In order to show $\mathrm{ch}'(f) \geq 0$, we first show a structural lemma as follows.

Lemma 7. *Let G be a 2-connected plane graph with $\delta \geq 3$ and without cycles of length 4, 7, 8, or 9.*

(1) *If $g = xyy_1y_2y_3$ is a 5-face adjacent to a 10-face $f = yxx_1x_2 \ldots x_8$ in G, then g is strict normally adjacent to f.*

(2) *Furthermore, if g is light with y being a 4-vertex and f is incident with nine 3-vertices in G, then g and f form an $S\Theta$.*

Proof. (1) Let $X = \{x_1, x_2, \ldots, x_8\}$, $Y = \{y_1, y_2, y_3\}$. We only need to prove $X \cap Y = \emptyset$. If $y_3 = x_1$, then $d(x) = 2$, a contradiction. If $y_3 = x_2$, then G would have a 9-cycle $y_3xyx_8x_7 \ldots x_3x_2(=y_3)$. Similarly $y_3 \neq x_3, x_4$, since G has no 8-, 7-cycle, respectively. By symmetry, $y_1 \neq x_8, x_7, x_6, x_5$. Since G has no 4-cycle, $y_3 \neq x_8, x_7, y_1 \neq x_1, x_2$ and $y_2 \neq x_1, x_2, x_8, x_7$. Suppose $y_3 = x_5$. If $y_2 = x_6$, then $xx_1x_2 \ldots x_5x_6(=y_2)y_1yx$ is 9-cycle in G (note that $y_1 \neq x_3, x_4$ since y_1 is separated from $\{x_3, x_4\}$ by cycle $y_3xyx_8x_7x_6x_5(=y_3)$), a contradiction. If $y_2 \neq x_6$, then $y_3y_2y_1yx_8x_7x_6x_5(=y_3)$ is a 7-cycle in G, a contradiction proving $y_3 \neq x_5$. Similarly $y_1 \neq x_4$. Now, if $y_3 = x_6$, then $y_3xx_1x_2 \ldots x_6(=y_3)$ would be a 7-cycle in G; hence, $y_3 \neq x_6$. Similarly $y_1 \neq x_3$. To conclude, $y_1, y_3 \notin X$. Now, if $y_2 = x_3$, then G has a 8-cycle $y_2y_1yx_8x_7x_6x_5x_4x_3$, a contradiction. If $y_2 = x_4$, then G would have a 7-cycle. Hence $y_2 \neq x_3, x_4$. By symmetry $y_2 \neq x_5, x_6$. To conclude, $y_2 \notin X$.

(2) To examine the definition of an $S\Theta$, we only need to show that the subgraph of G induced by $V(g) \cup V(f)$, denoted by H, satisfies $\delta(H) = 2$, since (2), (3), (4) in the definition of an $S\Theta$ are obvious. First note that $b(f)$, the boundary of f, has at most one external chord that evenly divide $b(f)$, since G has no 4-, 7-, or 9-cycles, and $b(g)$ has no external chord simply by no 4-cycle in G. Thus, except at most one external chord, every external chord of H has one end in $V(g) \setminus \{x, y\}$ and the other in $V(f) \setminus \{x, y\}$. Note that every vertex of H is incident with at most one external chord. It follows that H has at most four external chord. Hence, there are at least five vertices in $V(f)$ that are not incident with any external chord. Namely f has at least three vertices of degree 2 in H. □

Now we are going to analyze the final charge for a 10-face f. By hypothesis (4), f has at least one big vertex. Suppose that $\mathrm{ch}'(f) < 0$. Let us derive a contradiction. According to $\mathrm{ch}'(f) = \mathrm{ch}(f) - \sum_{e\in E(f)} q(e) = 4 - \sum_{e\in E(f)} q(e)$, $q(e) \in \{0, \frac{1}{4}, \frac{1}{2}\}$ and Observation 3, $\mathrm{ch}'(f) < 0$ only if

(a) $q(e) = \frac{1}{2}$ for every edge of f, or

(b) $q(e) = \frac{1}{2}$ for all but one edge of f and $q(e) = 0$ for the unique exceptional edge, or

(c) $q(e) = \frac{1}{2}$ for all but two edges of f and $q(e) = \frac{1}{4}$ for the two exceptional edges.

If (a) happens, then there is at least one edge $e \in E(f)$ such that $q(e) \neq c(e)$ by Lemma 6. It follows that $c(e) = 0$ has a solution in $E(f)$. Consequently, edges in $A = \{e \in E(f) | c(e) = 1\}$ cover all vertices of f. By R2 and R3, every edge in A is a $(3, 3)$-edge; hence, f has ten 3-vertices, a contradiction.

If (b) happens, then, by examining $c(e)$ and $q(e)$ starting at the unique exceptional edge, we find that $c = q$ on $E(f)$, contradicting Lemma 6.

Finally assume that (c) happens. Let $e_1, \ldots, e_{10}$ be the edges of f in a cyclic order, and $q(e_1) = q(e_k) = \frac{1}{4}$ for some $k > 1$. According to the definitions of the functions q and c on $E(f)$, we may assume that $c(e)$ first takes value in $\{0, 1\}$ alternatively from e_1 to e_k and then takes $\frac{1}{2}$ for every one of the remaining edges. Namely, $c(e_2) = 1, c(e_3) = 0, \ldots, c(e_{k-1}) = 1$, $c(e_k) = 0$; hence, k is odd, and $c(e_{k+1}) = \cdots = c(e_{10}) = \frac{1}{2}$. By Lemma 6, $k = 9$. If u, the common vertex of e_1 and e_{10}, is a 5^+-vertex, then u sends $\frac{1}{2}$ to f by R5, making $\mathrm{ch}'(f) = 0$, a contradiction. Assume that u is a 4-vertex. Note that $c(e_{10}) = \frac{1}{2}$ implies that $g(e_{10})$ is a light 5-face by R3.2, or a 3-face by R2, or $e_{10} \in X$ by R4. If $g(e_{10})$ is a light 5-face, then G has an $S\Theta$ by Lemma 7, contradicting hypothesis (3). If $g(e_{10})$ is a 3-face, then f is weak. By R5, f receives $\frac{1}{2}$ from u, making $\mathrm{ch}'(f) = 0$, a contradiction. If $e_{10} \in X$, then u is a cross and f is the handle of u. By the definitions of a cross and its handle and $c(e_{10}) = \frac{1}{2}$, $g(e_{10})$ is a 5-face; hence, $g(e_1)$ is a 3-face. Note that $c_0(e_1) = 0$ and e_1 is a $(3, 4)$-edge. By R2, $g(e_1)$ is not a 3-face, a contradiction. The proof of Theorem 2 is completed. □

Acknowledgements

The authors are grateful to the referees for their wise suggestions to simplify the original proof of this paper.

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