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Note

On the paper "Symmetry analysis of wave equation on sphere" by H. Azad and M.T. Mustafa

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ABSTRACT

Using the scalar curvature of the product manifold $\mathbb{S}^2 \times \mathbb{R}$ and the complete group classification of nonlinear Poisson equation on (pseudo) Riemannian manifolds, we extend the previous results on symmetry analysis of homogeneous wave equation obtained by H. Azad and M.T. Mustafa [H. Azad, M.T. Mustafa, Symmetry analysis of wave equation on sphere, J. Math. Anal. Appl. 333 (2007) 1180–1188] to nonlinear Klein–Gordon equations on the two-dimensional sphere.

1. Introduction

In a previous work, Azad and Mustafa [1] considered the Lie point symmetries of the homogeneous wave equation induced by the 2-sphere $\mathbb{S}^2$ metric

$$u_{tt} = u_{xx} + (\cot x)\, u_x + \frac{1}{\sin^2 x} u_{yy}. \tag{1}$$

Eq. (1) is a particular case of the general equation

$$\Delta_g u + f(u) = 0, \tag{2}$$

where

$$\Delta_g u = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(\sqrt{g} g^{ij} \frac{\partial u}{\partial x^j}\right) = g^{ij}\nabla_i \nabla_j u = \nabla^j \nabla_j u = \nabla_i \nabla^i u,$$

where Δ_g is the Laplace–Beltrami operator on an arbitrary (pseudo) Riemannian manifold (M^n, g) and ∇_i is the covariant derivative corresponding to the Levi-Civita connection and the Einstein summation convention over repeated indices is understood.

Eq. (2) covers Poisson and Klein–Gordon semilinear equations, depending on if (M^n, g) is a Riemannian or a pseudo-Riemannian manifold, respectively. Eq. (1) can be obtained from (2) taking on $\mathbb{S}^2 \times \mathbb{R}$ the metric

$$ds^2 = dt^2 - dx^2 - \sin^2 x\, dy^2 \tag{3}$$

and $f(u) = 0$.

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We shall denote the product manifold $\mathbb{S}^2 \times \mathbb{R}$ endowed with the metric (3) as $(\mathbb{S}^2 \times \mathbb{R}, g)$.

Group classification of equations with coefficients depending on metric tensor on specific Riemannian manifolds are well known. See [1,4,6–8,11].

The Lie point symmetries of Eq. (2) on flat manifolds, with some functions $f(u)$ are performed in [11]. In [4,3,5,10] the Lie point symmetries, the Noether symmetries and the conservation laws of the Kohn–Laplace equations were studied. In [6] the symmetry analysis of Eq. (2) was carried out on an arbitrary (pseudo) Riemannian manifold. The Lie symmetries of the Poisson equation with Euclidean metric are well known, see [20]. The group classification of the Poisson equation on the hyperbolic plane with metric of Klein's model of Lobachevskian geometry and in $\mathbb{S}^2$ was carried out in [7,8], respectively.

In this article we are interested in the Lie point symmetries, the Noether symmetries and the conservation laws of equation

$$u_{tt} = u_{xx} + (\cot x)\, u_x + \frac{1}{\sin^2 x} u_{yy} + f(u), \tag{4}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a smooth function. Existence, uniqueness and behavior of solutions of initial value problems of (4) are established in [19].

Denoting by $Isom(\mathbb{S}^2 \times \mathbb{R})$ the Lie algebra of the Killing vector fields of $(\mathbb{S}^2 \times \mathbb{R}, g)$, our main result can be formulated as follows:

Theorem 1. *Except to the linear cases, the symmetry Lie algebra of Eq.* (4) *with an arbitrary function $f(u)$ is generated by* $Isom(\mathbb{S}^2 \times \mathbb{R}, g)$, *that is,*

$$S_0 = \frac{\partial}{\partial t}, \qquad S_1 = \frac{\partial}{\partial y}, \qquad S_2 = \sin y \frac{\partial}{\partial x} + \cot x \cos y \frac{\partial}{\partial y}, \qquad S_3 = \cos y \frac{\partial}{\partial x} - \cot x \sin y \frac{\partial}{\partial y}. \tag{5}$$

If $f(u) = c\,u$, $c = const$, in (2), *in addition to* $Isom(\mathbb{S}^2 \times \mathbb{R})$, *we have the following generators*:

$$S_4 = u \frac{\partial}{\partial u} \tag{6}$$

and

$$S_\infty = b(x, y, t) \frac{\partial}{\partial u}, \tag{7}$$

where

$$b_{tt} = b_{xx} + (\cot x)\, b_x + \frac{1}{\sin^2 x} b_{yy} + c\,b. \tag{8}$$

The case $f(u) = k = const \neq 0$ is reduced to the case $f(u) = 0$ under the change $u \to u + kt^2/2$.

As a consequence, we have the following classification of the Noether symmetries.

Theorem 2. *For any function $f(u)$ in* (4), *the isometry algebra* $Isom(\mathbb{S}^2 \times \mathbb{R})$ *generates a variational symmetry Lie algebra. If $f(u) = c\,u$, the symmetry* (7) *is a Noether symmetry, where $b = b(x, y, t)$ satisfies* (8).

From Theorem 2 and the Noether's theorem, we have:

Corollary 1. *Let $F = F(u)$ be a function such that $F'(u) = f(u)$. The conservation laws of the Noether symmetries of Eq.* (4), *for any function $f(u)$, are*:

1. *For the symmetry S_0, the conservation law is $Div(A) = 0$, where $A = (A^0, A^1, A^2)$ and*

$$A^0 = -\frac{\sin x}{2} u_t^2 - \frac{\sin x}{2} u_x^2 - \frac{1}{2 \sin x} u_y^2 - \sin x\, F(u), \qquad A^1 = \sin x\, u_t u_x, \qquad A^2 = \frac{1}{\sin x} u_t u_y. \tag{9}$$

2. *For the symmetry S_1, the conservation law is $Div(B) = 0$, where $B = (B^0, B^1, B^2)$ and*

$$B^0 = -\sin x\, u_t u_y, \qquad B^1 = \sin x\, u_x u_y, \qquad B^2 = \frac{\sin x}{2} u_t^2 - \frac{\sin x}{2} u_x^2 + \frac{1}{2 \sin x} u_y^2 - \sin x\, F(u). \tag{10}$$

3. *For the symmetry S_2, the conservation law is Div$(C)=0$, where $C=(C^0,C^1,C^2)$ and*

$$
\begin{aligned}
C^0 &= -\sin x\sin y\,u_t u_x - \cos x\cos y\,u_t u_y,\\
C^1 &= \frac{\sin x\sin y}{2}u_t^2 + \frac{\sin x\sin y}{2}u_x^2 - \frac{\sin y}{2\sin x}u_y^2 + \cos x\cos y\,u_x u_y - \sin x\sin y\,F(u),\\
C_2 &= \frac{\cos x\cos y}{2}u_t^2 - \frac{\cos x\cos y}{2}u_x^2 + \frac{\cos x\cos y}{2\sin^2 x}u_y^2 + \frac{\sin y}{\sin x}u_x u_y - \cos x\cos y\,F(u).
\end{aligned}
\tag{11}
$$

4. *For the symmetry S_3, the conservation law is Div$(D)=0$, where $D=(D^0,D^1,D^2)$ and*

$$
\begin{aligned}
D^0 &= -\sin x\cos y\,u_t u_x + \cos x\sin y\,u_t u_y,\\
D^1 &= \frac{\sin x\cos y}{2}u_t^2 + \frac{\sin x\cos y}{2}u_x^2 - \frac{\cos y}{2\sin x}u_y^2 - \cos x\sin y\,u_x u_y - \sin x\cos y\,F(u),\\
D^2 &= -\frac{\cos x\sin y}{2}u_t^2 + \frac{\cos x\sin y}{2}u_x^2 - \frac{\cos x\sin y}{2\sin^2 x}u_y^2 + \frac{\cos y}{2\sin x}u_x u_y + \cos x\sin x\,F(u).
\end{aligned}
\tag{12}
$$

5. *If $F(u)=c\,u^2/2$, then the conservation law for the symmetry* (7)*, with b satisfying* (8)*, is Div$(\alpha)=0$, where $\alpha=(\alpha^0,\alpha^1,\alpha^2)$ and*

$$
\alpha^0 = \sin x(bu_t - b_t u),\qquad \alpha^1 = \sin x(b_x u - bu_x),\qquad \alpha^2 = \frac{1}{\sin x}(bu_t - b_t u).
\tag{13}
$$

We shall not present preliminaries concerning Lie point symmetries of differential equations supposing that the reader is familiar with the basic notions and methods of contemporary group analysis. See [2,11,15]. For a geometric viewpoint of Lie point symmetries, see [13,14].

This paper is organized as follows. In Section 2 we recall some geometric results regarding to $(\mathbb{S}^2\times\mathbb{R},g)$. These results will be used in Section 3 to prove Theorem 1. The Noether's symmetries and the conservation laws are obtained in Section 4. In Section 5 we identify the classical Lie algebras that the symmetry Lie algebras are isomorphic to.

2. The product manifold $\mathbb{S}^2\times\mathbb{R}$

Let $x^0=t$, $x^1=x$ and $x^2=y$ be local coordinates of $(\mathbb{S}^2\times\mathbb{R},g)$. We observe that the Riemann and the Ricci tensors used in this paper coincide with those in Yano's book [16] and in Dubrovin, Fomenko and Novikov's book [9].

Lemma 1. *The scalar curvature R of the product manifold $(\mathbb{S}^2\times\mathbb{R},g)$ is constant.*

Proof. The Riemann tensor of $(\mathbb{S}^2\times\mathbb{R},g)$ is

$$R^i_{jks} = -\left(\delta^{2i}\delta_{1j}\delta_{2k}\delta_{1s} - \delta^{2i}\delta_{1j}\delta_{1k}\delta_{2s}\right) + \sin^2 x\left(\delta^{2i}\delta_{2j}\delta_{2k}\delta_{1s} - \delta^{1i}\delta_{2j}\delta_{1k}\delta_{2s}\right).$$

Then $R^i_s = -\delta^{2i}\delta_{2s} - \delta^{2i}\delta_{1s}$ and $R=-1$. □

Lemma 2. *The sectional curvature of $(\mathbb{S}^2\times\mathbb{R},g)$ is non-constant.*

Proof. Let $K(p,X,Y)$ be the sectional curvature of $(\mathbb{S}^2\times\mathbb{R},g)$ at a point $p=(t,x,y)$. (See [12] for the definitions.) Let $X=(X^0,X^1,X^2)$ and $Y=(Y^0,Y^1,Y^2)$. Then, we obtain

$$K(p,X,Y) = \frac{-X^2\sin^2 x + X^1}{2X^1}. \qquad \square$$

Lemma 3. *The isometry group of $(\mathbb{S}^2\times\mathbb{R},g)$ is generated by the vector fields S_0, S_1, S_2 and S_3.*

Proof. It is clear that the vector fields (5) satisfy the equation

$$L_X g_{ij} = \xi^s\frac{\partial g_{ij}}{\partial x^s} + g_{kj}\frac{\partial \xi^k}{\partial x^i} + g_{ik}\frac{\partial \xi^k}{\partial x^j} = 0.$$

From Lemma 2, the sectional curvature of $(\mathbb{S}^2\times\mathbb{R},g)$ is non-constant. Then, from Yano [16, p. 57, Theorem 6.2], *dim*(*Isom*$(\mathbb{S}^2\times\mathbb{R})$) <6. From Fubini's theorem (see Yano [16]), *dim*(*Isom*$(\mathbb{S}^2\times\mathbb{R})$) cannot be 5. Thus, *dim*(*Isom*$(\mathbb{S}^2\times\mathbb{R})$) $\leqslant 4$. Since (5) are isometries, we conclude that the isometry algebra *Isom*$(\mathbb{S}^2\times\mathbb{R})$ is generated by S_0, S_1, S_2 and S_3. □

3. The group classification

In this section we perform the group classification of Eq. (4). To begin with we need of the following lemma:

Lemma 4. *Let (M^n, g) be a manifold with non-null constant scalar curvature. Then the Lie point symmetry group of the Poisson equation* (2) *with an arbitrary $f(u)$ coincides with the isometry group of M^n.*

In the particular cases $f(u) = c\,u$, where $c = const$, in addition to the isometry group, we have the generators

$$U = u\frac{\partial}{\partial u} \quad \text{and} \quad X_\infty = b(x)\frac{\partial}{\partial u},$$

where b satisfies (2)*.*

Proof. See [6]. □

We observe that Lemma 4 is a particular case of the main result obtained in [6]. In this work the authors carried out the group classification of Eq. (2) on an arbitrary (pseudo) Riemannian manifold.

Proof of Theorem 1. From Lemma 1, the scalar curvature of $(\mathbb{S}^2 \times \mathbb{R}, g)$ is $R = -1$. Then, Theorem 1 follows from Lemmas 3 and 4. □

4. The Noether's symmetries and the conservation laws

In this section we prove Theorem 2.

It is easy to check that if $X \in Isom(\mathbb{S}^2 \times \mathbb{R})$, then X is a variational symmetry of Eq. (2), for any function $f(u)$. That is,

$$X^{(1)}\mathcal{L} + \mathcal{L}D_i\xi^i = 0,$$

where

$$\mathcal{L} = \frac{\sin x}{2}u_t^2 - \frac{\sin x}{2}u_x^2 - \frac{1}{2\sin x}u_y^2 + \sin x\, F(u) \tag{14}$$

is the function of Lagrange of Eq. (2). For more details, see [6].

Let us consider the symmetry (7). It is easy to verify that

$$X^{(1)}\mathcal{L} + \mathcal{L}D_i\xi^i = Div\left(\sin x\, b_t u, -\sin x\, b_x u, -\frac{1}{\sin x}b_y u\right),$$

where $F(u) = c\,u^2/2$ in (14). These observations prove Theorem 2.

The following lemma establishes the conservation laws (9)–(13):

Lemma 5. *The conservation laws of the Noether symmetries of Eq.* (2)*, where (M^n, g) is a manifold with constant, non-null scalar curvature, are $D_i A^i = 0$, where*

$$A^k = \sqrt{g}\left(\frac{1}{2}g^{ij}\xi^k - g^{kj}\xi^i\right)u_i u_j - \sqrt{g}\xi^k F(u), \tag{15}$$

for any function $f(u)$. If $f(u) = c\,u$, then the conservation law corresponding to the symmetry (7) *is*

$$A^k = \sqrt{g}g^{jk}(bu_j - b_j u). \tag{16}$$

Proof. It is a consequence of [6] when the scalar curvature of (M^n, g) is constant. □

Proof of Corollary 1. Substituting the symmetries and the metric coefficients into (15)–(16), we obtain (9)–(13). □

5. Symmetry Lie algebras

Let $\mathfrak{S}_1$, $\mathfrak{S}_2$ be the finite dimensional symmetry Lie algebras for an arbitrary $f(u)$ and $f(u) = c\,u$, $c = const$, respectively. Following the notations of [17,18], the symmetry Lie algebras are:

1. If $f(u)$ is an arbitrary function, then $[S_1, S_2] = S_3$, $[S_1, S_3] = -S_2$, $[S_2, S_3] = S_1$. Thus, $\mathfrak{S}_1 = Isom(\mathbb{S}^2 \times \mathbb{R}) \approx A_{3,9} \oplus A_1$, where $A_{3,9} = \mathfrak{so}(3)$.
2. If $f(u) = c\,u$, then $\mathfrak{S}_2 \approx A_{3,9} \oplus 2A_1$.

We have the following one-dimensional subalgebras of $\mathfrak{S}_1$: $\mathfrak{L}_1 = \langle S_0 + a\,S_1\rangle$ and $\mathfrak{L}_2 = \langle S_1\rangle$.

If $f(u) = c\,u$, we have the following classification of subalgebras of $\mathfrak{S}_2$:

1. Dimension 1: $\mathfrak{L}_1 = \langle a\,S_0 + S_1 + b\,S_4\rangle \approx A_1$ and $\mathfrak{L}_2 = \langle a\,S_0 + b\,S_4\rangle \approx A_1$.
2. Dimension 2: $\mathfrak{L}_3 = \langle a\,S_0 + b\,S_4, S_1\rangle \approx 2A_1$ and $\mathfrak{L}_4 = \langle S_0, S_4\rangle \approx 2A_1$.
3. Dimension 3: $\mathfrak{L}_5 = \langle S_1, S_2, S_3\rangle \approx A_{3,9}$ and $\mathfrak{L}_6 = \langle S_0, S_1, S_4\rangle \approx 3A_1$.
4. Dimension 4: $\mathfrak{L}_7 = \langle a\,S_0 + b\,S_4, S_1, S_2, S_3\rangle \approx A_{3,9} \oplus A_1$.

We observe that the subalgebras (1–4) above were obtained by Azad and Mustafa when $f(u) = 0$ in (4).

The invariant solutions of (4) can be obtained following the same procedure employed by Azad and Mustafa in [1] with addition of the corresponding term $f(u)$ in (2). Thus we shall omit the details.

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