Stability of numerical methods for delay differential equations

Lucio TORELLI
Dipartimento di Scienze Matematiche, Università degli Studi, 34127 Trieste, Italy

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Abstract: Consider the following delay differential equation (DDE)
\[ y'(t) = f(t, y(t), y(t - r(t))), \quad t \geq t_0, \]  
with the initial condition
\[ y(t) = \Phi(t) \quad \text{for} \quad t \leq t_0, \]  
where \( f \) and \( \Phi \) are such that (0.1), (0.2) has a unique solution \( y(t) \). The author gives sufficient conditions for the asymptotic stability of the equation (0.1) for which he introduces new definitions of numerical stability. The approach is reminiscent of that from the nonlinear, stiff ordinary differential equation (ODE) field. He investigates stability properties of the class of one-point collocation rules. In particular, the backward Euler method turns out to be stable with respect to all the given definitions.

Keywords: Numerical analysis, delay equation, Runge–Kutta methods, stability.

1. Introduction

Let us consider the following linear delay differential equation
\[ y'(t) = ay(t) + by(t - \tau), \quad t > 0, \]
\[ y(t) = \Phi(t), \quad -\tau \leq t \leq 0, \]  
where \( y : [-\tau, +\infty) \to \mathbb{C} \), \( a \) and \( b \) are complex numbers, \( \tau \) is a positive constant delay and \( \Phi \) is a specified initial function.

It is known that (see [1] and [2]), if \( \Phi \) is continuous and if
\[ |b| < -\text{Re}(a), \]  
then the initial-value problem (1.1) has a unique solution, which approaches zero as \( t \) tends to infinity, for every \( \tau \).

In this case equation (1.1) is said to be asymptotically stable.

Concerning numerical methods for the initial-value problem (1.1), let us recall Barwell’s [3] definitions of P-stability and GP-stability:
Definition 1.1. A numerical method for DDE's is called \textit{P-stable} if, for all coefficients $a, b$ satisfying (1.2), the numerical solution $y_k$ of (1.1) at the mesh points $t_k = kh$, $k \geq 0$ satisfies

$$y_k \to 0 \quad \text{as} \quad k \to \infty,$$

for every stepsize $h$ such that

$$h = \tau/m, \quad \text{where} \quad m \text{ is a positive integer.} \quad (1.3)$$

Definition 1.2. A numerical method for DDE's is called \textit{GP-stable} if, under condition (1.2), $y_k \to 0$ as $k \to \infty$ for every stepsize $h > 0$.

Recently, results on P and GP-stability have been given by Al-Mutib [1], Jackiewicz [7] and Zennaro [9]. The main difficulty in studying either the P-stability or the GP-stability properties of a numerical method for DDE's is that one has to deal with difference equations of arbitrary high order (depending on the ratio $\tau/h = m$). In [9], Zennaro shows that the study of vector difference equations of arbitrary order $m + 1$ is equivalent to the solution of a constrained minimum problem in the complex plane. In this way one is able to find out general P-stability properties of numerical methods for DDE’s.

In particular, as one can see in [8] and [9] every A-stable projection method (hence every A-stable collocation method for stiff DDE's) is such that the corresponding method for DDE's is P-stable. In this way one can prove that the one-step collocation method at Gaussian points for DDE's is P-stable (but not GP-stable).

So far, only the linear test equation (1.1) was used to study stability properties of numerical methods for DDE's. It is the purpose of this paper to investigate stability properties of some simple Runge-Kutta methods for DDE's with respect to the nonlinear test equation (2.1) defined in the next section. For such equation we shall introduce RN (and GRN) stability. Equation (2.1) includes as a special case the linear test equation (3.2) (defined in Section 3.) for which we shall introduce PN (and GPN) stability. Our stability analysis is reminiscent of the numerical stability analysis of Runge-Kutta methods for stiff, nonlinear DDE's [5].

2. Stability of nonlinear delay differential equations

Consider the following nonlinear equations

$$y'(t) = f(t, y(t), y(t - \tau(t))), \quad t \geq t_0,$$

$$y(t) = \Phi(t), \quad t \leq t_0,$$ \hspace{1cm} (2.1)

and

$$z'(t) = f(t, z(t), z(t - \tau(t))), \quad t \geq t_0,$$

$$z(t) = \Gamma(t), \quad t \leq t_0,$$ \hspace{1cm} (2.2)

where

$$f: [t_0, + \infty) \times C^n \times C^n \to C^n, \quad y, z: R \to C^n, \quad \tau(t) \geq \tau_0 > 0.$$
The solution of (2.1) is stable (with respect to the initial function) if there exists a norm on $C^n$ such that, for every $t \geq t_0$,

$$\| y(t) - z(t) \| \leq \max_{t \leq t_0} \| \Phi(t) - \Gamma(t) \|$$

where $y(t)$ and $z(t)$ are solutions of (2.1) and (2.2) respectively, and $\Phi$, $\Gamma$ are distinct and continuous functions.

In this section we shall formulate sufficient conditions for the stability of (2.1) in the sense described above.

**Theorem 2.1.** Assume that the delay $\tau(t)$ is continuous and that there exists $(\cdot | \cdot)$, an inner product on $C^n$, such that

$$\gamma(t) \leq -\sigma(t) \quad \text{for every } t \geq t_0,$$

where

$$\sigma(t) := \sup_{y, z \in C^n} \text{Re} \left( \left( f(t, y), (z - \tau(t)) - f(t, z) \right) \right) / \| y - z \|^2.$$

$$\gamma(t) := \sup_{y, z \in C^n} \| f(t, y, z) - f(t, y, z) \| / \| z - z \|$$

and $\| x \|^2 = (x | x)$ for every $x \in C^n$.

Then

$$\| y(t) - z(t) \| \leq \max_{t \leq t_0} \| \Phi(t) - \Gamma(t) \|$$

for every $t \geq t_0$.

Before proving the theorem, observe that, if

$$f(t, y(t), y(t - \tau(t))) = ay(t) + by(t - \tau)$$

as in (1.1), then $\sigma(t) = \text{Re}(a)$ and $\gamma(t) = |b|$. In this case, if $|b| \leq -\text{Re}(a)$, by Theorem 2.1 we get that, for every $t \geq t_0$,

$$|y(t)| \leq \max_{t \leq t_0} |\Phi(t)|.$$

We shall need the following theorems.

**Theorem 2.2.** Consider the linear equation

$$y'(t) = a(t)y(t) + f(t), \quad t \geq t_0,$$

$$y(t_0) = y_0.$$  \hspace{1cm} (2.6)

with $y$, $a$, $f$: $[t_0, +\infty) \to C$ and $\text{Re}(a(t)) < 0$ for every $t \geq t_0$.

Then the solution $y(t)$ of the initial-value problem (2.6) is such that

$$|y(t)| \leq \max \left\{ |y_0|, \max_{t_0 \leq x \leq t} \left| f(x) / (-\text{Re}(a(x))) \right| \right\}.$$
Proof. Define $A(t) := \int_{t_0}^{t} a(x) \, dx$; we note that $\text{Re}(A(t)) < 0$ for every $t \geq t_0$.

The solution of (2.6) is

$$y(t) = e^{A(t)} y_0 + e^{A(t)} \int_{t_0}^{t} e^{-A(x)} f(x) \, dx.$$ 

We have that

$$\left| \int_{t_0}^{t} e^{-\text{Re}(A(x))} f(x) \, dx \right|$$

$$= \left| \int_{t_0}^{t} \left[ -\text{Re}(a(x)) \right] e^{-\text{Re}(A(x))} f(x) / (-\text{Re}(a(x))) \right| \, dx$$

$$\leq \max_{t_0 < x < t} \left\{ \left| f(x) / (-\text{Re}(a(x))) \right| \right\} \left| \int_{t_0}^{t} -\text{Re}(a(x)) \, e^{-\text{Re}(A(x))} \, dx \right|$$

and

$$\int_{t_0}^{t} -\text{Re}(a(x)) \, e^{-\text{Re}(A(x))} \, dx = e^{-\text{Re}(A(t))} - 1.$$ 

Therefore

$$\left| \int_{t_0}^{t} e^{-\text{Re}(A(x))} f(x) \, dx \right| \leq \max_{t_0 < x < t} \left\{ \left| f(x) / (-\text{Re}(a(x))) \right| \right\} \left| e^{-\text{Re}(A(t))} - 1 \right|.$$ 

Hence

$$|y(t)| \leq e^{\text{Re}(A(t))} |y_0| + (1 - e^{\text{Re}(A(t))}) \max_{t_0 < x < t} \left| f(x) / (-\text{Re}(a(x))) \right|$$

and so, for every $t \geq t_0$.

$$|y(t)| \leq \max\left\{ |y_0|; \max_{t_0 < x < t} \left| f(x) / (-\text{Re}(a(x))) \right| \right\}. \quad \Box$$

Theorem 2.3. Let us consider the following equations

$$y'(t) = f(t, y(t), u(t)), \quad t \geq t_0, \quad (2.7)$$

$$y(t_0) = y_0,$$

and

$$z'(t) = f(t, z(t), v(t)), \quad t \geq t_0,$$

$$z(t_0) = z_0,$$

with $f : [t_0, +\infty) \times C^n \times C^n \to C^n$, $y$, $z$, $u$, $v : [t_0, +\infty) \to C^n$ and $y_0 \neq z_0$.

Assume that there exists $(\cdot, \cdot)$, an inner product on $C^n$ such that $\sigma(t) < 0$ for every $t \geq t_0(0(t), \gamma(t)$ defined in Theorem 2.1, $\|x\|^2 = (x \cdot x)$ for every $x \in C^n$). Then, for every $t \geq t_0$,

$$\|y(t) - z(t)\| \leq \max\{\|y_0 - z_0\|; \max_{t_0 < x < t} \{\gamma(x) \| u(x) - v(x) \|(\sigma(x))\}\}.$$
Proof. We have
\[ \frac{1}{2} \frac{d}{dt} \| y(t) - z(t) \|^2 = \text{Re}(y'(t) - z'(t) \mid y(t) - z(t)) \]
\[ = \text{Re}(f(t, y(t), u(t)) - f(t, z(t), v(t)) \mid y(t) - z(t)) \]
\[ = \text{Re}(f(t, y(t), u(t)) - f(t, y(t), v(t)) \mid y(t) - z(t)) \]
\[ + \text{Re}(f(t, y(t), v(t)) - f(t, z(t), v(t)) \mid y(t) - z(t)) \]
It follows from the definitions of \( \sigma(t) \) and \( \gamma(t) \) and from the Schwartz inequality that
\[ \frac{1}{2} \frac{d}{dt} \| y(t) - z(t) \|^2 \leq \| f(t, y(t), u(t)) - f(t, y(t), v(t)) \| \| y(t) - z(t) \| + \sigma(t) \| y(t) - z(t) \|^2 \]
\[ \leq \gamma(t) \| u(t) - v(t) \| \| y(t) - z(t) \| + \sigma(t) \| y(t) - z(t) \|^2 \]
We define
\[ Y(t) := \| y(t) - z(t) \| \]
(note that \( Y(t) > 0 \) for every \( t > t_0 \) because we assume that the function \( f \) is such that (2.7) has a unique solution \( y(t) \) for every initial condition \( y(t_0) = y_0 \)).
Then
\[ \frac{d}{dt} \| y(t) - z(t) \|^2 = 2 \| y(t) - z(t) \| \frac{d}{dt} \| y(t) - z(t) \| = 2Y(t)Y'(t) \]
so we have that
\[ Y(t)Y'(t) \leq \sigma(t)Y^2(t) + \gamma(t) \| u(t) - v(t) \| \]
and hence
\[ Y'(t) \leq \sigma(t)Y(t) + \gamma(t) \| u(t) - v(t) \|. \]
We define
\[ f(t) := \gamma(t) \| u(t) - v(t) \|, \]
therefore
\[ Y'(t) \leq \sigma(t)Y(t) + \gamma(t) \]
and, by Theorem 2.2, for every \( t > t_0 \)
\[ Y(t) \leq \max \left\{ Y_0, \max_{t_0 \leq x \leq t} f(x)/(-\sigma(x)) \right\}, \]
i.e.
\[ \| y(t) - z(t) \| \leq \max \left\{ \| y_0 - z_0 \|, \max_{t_0 \leq x \leq t} \gamma(x) \| u(x) - v(x) \|/(-\sigma(x)) \right\}. \]

Proof of Theorem 2.1. By Theorem 2.3 we know that, for every \( t > t_0 \) the solutions \( y(t) \) and \( z(t) \) of (2.1), (2.2) respectively are such that
\[ \| y(t) - z(t) \| \leq \max \left\{ \| \Phi(t_0) - \Gamma(t_0) \|, \max_{t_0 \leq x \leq t} \gamma(x) \| y(x - \tau(x)) - z(x - \tau(x)) \|/(-\sigma(t)) \right\} \]
We assumed that \( y(t) \leq -u(t) \) and \( \tau(t) \geq \tau_0 > 0 \) for every \( t \geq t_0 \), therefore

\[
\| y(t) - z(t) \| \leq \max_{t_0 \leq x \leq t} \left( \| \Phi(t_0) - \Gamma(t_0) \|; \max_{t_0 \leq x \leq t} \| y(x - \tau(x)) - z(x - \tau(x)) \| \right),
\]

i.e., for every \( t \geq t_0 \)

\[
\| y(t) - z(t) \| \leq \max_{t \leq t_0} \| \Phi(t) - \Gamma(t) \|. \quad \Box
\]

(One can prove Theorem 2.1 in an indirect way using the Razumikhin-type theorems in Hale [6, Chapter 5.4]).

3. RN and GRN-stability, PN and GPN-stability

Let us make a more extensive stability analysis of numerical methods for the initial-value problem (2.1), introducing some new definitions of stability based on more general classes of test equations. From now on, we consider the delay \( \tau(t) = \tau \) as a constant in the test equations (2.1), (2.2) and we refer to the norm of Theorem 2.1.

**Definition 3.1.** A numerical method for DDE’s is called **RN-stable** if, under the condition (2.3)

\[
|y_k - z_k| \leq \max_{t \leq t_0} \| \Phi(t) - \Gamma(t) \|
\]

for every \( k \geq 0 \) and for every stepsize \( h \) such that \( h = \tau/m \), where the delay \( \tau \) is constant and where \( m \) is a positive integer.

**Definition 3.2.** A numerical method for DDE’s is called **GRN-stable** if, under condition (2.3), (3.1) is satisfied for every \( k \geq 0 \) and for every stepsize \( h > 0 \).

As a special case of (2.1) let us consider the following linear test DDE with variable coefficients

\[
\begin{align*}
y'(t) &= a(t)y(t) + b(t)y(t - \tau), & t \geq 0, \\
y(t) &= \Phi(t), & t \leq 0,
\end{align*}
\]

where \( y: \mathbb{R} \to \mathbb{C}, a, b: [0, +\infty) \to \mathbb{C} \) and \( \tau > 0 \). In this case \( \sigma(t) = \Re(a(t)), \gamma(t) = |b(t)| (\sigma(t) \) and \( \gamma(t) \) defined by (2.4), (2.5)) and hence, by Theorem 2.1, the solution \( y(t) \) of (3.2) is bounded by \( \Phi(t) \), provided that, for every \( t \geq 0 \),

\[
|b(t)| \leq -\Re(a(t)).
\]

**Definition 3.3.** A numerical method for DDE’s is said to be **PN-stable** if, under the condition (3.3), the numerical solution \( y_k \) of (3.2) is such that

\[
|y_k| \leq \max_{t \leq 0} |\Phi(t)|
\]

for every \( k \geq 0 \) and for every stepsize \( h \) such that \( h = \tau/m \), where \( m \) is a positive integer.
Definition 3.4. A numerical method for DDE’s is called \( GPN\)-stable if, under condition (3.3), the numerical solution of (3.2) satisfies (3.4) for every \( k \geq 0 \) and for every stepsize \( h > 0 \).

The first result we provide is a negative result concerning collocation methods at the roots of the Legendre polynomials (Gaussian collocation). These methods, which are \( A \)-stable, are \( P \)-stable when applied to DDE’s (see Zennaro [9]). Nevertheless not all \( AN \)-stable Gaussian collocation methods are \( PN \)-stable. We shall prove this for the one-point Gaussian collocation method.

In general the one-point collocation rule for the nonlinear equation (2.1) reads

\[
y_{k+1} = y_k + h f(t, \theta y_k + \theta y_{k+1}, (1 - \theta) y_k - \theta y_{k-m} + \theta y_{k-m+1})
\]

where the stepsize \( h \) is such that \( h = \tau / m \) (\( m \) a positive integer) and \( \theta \in [0, 1] \).

We apply the mid-point collocation method (one-point collocation with \( \theta = \frac{1}{2} \)), which is \( AN \)-stable, to the equation

\[
y' = -c(t) y + c(t) y(t-1), \quad t \geq 0,
y(0) = \Phi(0),
\]

where \( c(t) \) is a real function. In this case, by Theorem 2.1, the solution \( y(t) \) of (3.5) is bounded by \( \Phi(t) \), provided that \( c(t) \geq 0 \) for every \( t \geq 0 \). Hence, if the coefficient \( c(t) \) is nonnegative, then

\[|y(t)| \leq \max_{t \leq 0} |\Phi(t)| =: S \quad \text{for every } t \geq 0.\]

With the stepsize \( h = \frac{1}{2} \), we get the difference equation

\[
y_{k+1} = y_k + \frac{1}{4} c_k (-y_k - y_{k+1} + y_{k-2} + y_{k-1})
\]

where \( c_k = c(\frac{1}{2} k + \frac{1}{2}) \), \( k \geq 0 \).

Hence

\[
y_{k+1} = \frac{1 - \frac{1}{4} c_k}{1 + \frac{1}{4} c_k} y_k + \frac{\frac{1}{4} c_k}{1 + \frac{1}{4} c_k} (y_{k-2} + y_{k-1}).
\]

We put

\[
p_k := \frac{1}{4} c_k/(1 + \frac{1}{4} c_k),
\]

hence \( 0 \leq p_k < 1 \) for every \( k \geq 0 \) and

\[1 - 2p_k = (1 - \frac{1}{4} c_k)/(1 + \frac{1}{4} c_k).
\]

With this notation we can write (3.6) in a different way:

\[
y_{k+1} = (1 - 2p_k) y_k + p_k (y_{k-1} + y_{k-2}).
\]

It is possible to choose a sequence \( \{p_k\} \) such that the one-point Gaussian collocation method is not stable. We put \( p_0 = 0 \) \( (c(\frac{1}{4}) = 0) \); \( 0 \leq \rho := p_1 = p_2 < 1 \) \( (c(\frac{3}{4}) = c(\frac{5}{4}) =: c) \). Hence

\[
\rho - \frac{1}{4} c/(1 + \frac{1}{4} c) \quad \text{and} \quad c - 4\rho/(1 - \rho).
\]

Then we choose the sequence \( \{p_k\} \) such that it is \( T \)-periodic, with \( T = 3 \), i.e. \( p_0 = p_3 = \cdots = 0 \); \( p_1 = p_4 = \cdots = \rho \); \( p_2 = p_5 = \cdots = \rho \). We define \( Y_k = (y_k, y_{k-1}, y_{k-2})^T \). Hence equation (3.7) is equivalent to

\[
Y_{k+1} = A_k Y_k
\]
where
\[
A_k = \begin{pmatrix}
1 - 2\rho_k & \rho_k & \rho_k \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}.
\]
The periodicity of the sequence \(\{\rho_k\}\) yields
\[
Y_{k+3} = BY_k \quad \text{where} \quad B = A_{k+2}A_{k+1}A_k, \quad k = 0, 3, 6, 9, \ldots .
\]
We have
\[
B = \begin{pmatrix}
1 - \rho + 2\rho^2 & \rho - 2\rho^2 & 0 \\
1 - \rho & \rho & 0 \\
1 & 0 & 0
\end{pmatrix}
\]
and therefore
\[
\det(B - \lambda I) = -\lambda \left[ \lambda^2 - (1 + 2\rho^2)\lambda + 2\rho^2 \right] = 0
\]
iff \(\lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = 2\rho^2\).

If \(\lambda_3 = 2\rho^2 = 1\), then the eigenvalue \(\lambda = 1\) has algebraic multiplicity equal to 2 and geometric multiplicity equal to 1 and the sequence \(Y_{k+3}\) is bounded if and only if \(\lambda_3 = 2\rho^2 \leq 1\), i.e. \((0 \leq \rho \leq \frac{1}{\sqrt{2}})\).

We can conclude that, if the function \(c(t)\) gives rise to a sequence \(\{\rho_k\}\) such that \(\frac{1}{\sqrt{2}} \leq \rho < 1\), then the one-point Gaussian collocation method with stepsize \(h = \frac{1}{2}\) yields a numerical solution \(y_k\) which blows up as \(k \to +\infty\). Thus the one-point Gaussian collocation method is not PN-stable.

**Definition 3.5.** Given a numerical method for DDE’s, we define “region of PN-stability” of the method the set \(\mathcal{S}\) of the complex valued numbers \((ha(t), hb(t))\) for which stability (in the sense of Definition 3.3) holds.

If the region \(\mathcal{S}\) includes all the pairs \((ha(t), hb(t))\) with \(|b(t)| \leq -\Re(a(t))\), then the method is PN-stable.

Now, let us consider the collocation method at the point \(t_k + \theta h\), \(0 \leq \theta \leq 1\), for the equation (3.2) with \(a(t), b(t)\) real coefficients such that \(|b(t)| \leq -a(t)\) for every \(t \geq 0\). We prove that,
for each \( \theta \in [\frac{1}{2}, 1] \), the region of PN-stability \( S_\theta \) includes the pairs \((ha(t), hb(t))\) such that for every \( t \), \(|b(t)| \leq -a(t)\) and
\[
h|b(t)|(1 - \theta) \leq 1 \tag{3.9}
\]
In particular for \( \theta = 1 \) (backward Euler method) (3.9) is true for all \( b(t) \) and then the method is PN-stable.

If we apply the collocation method at the point \( t_k + \theta h \) to (3.2) \((a(t), b(t))\) real coefficients such that \(|b(t)| \leq -a(t)\) for every \( t \geq 0 \), we get the difference equation
\[
y_{k+1} = y_k + a_k h (\theta y_{k+1} + (1 - \theta) y_k) + b_k h (\theta y_{k-m+1} + (1 - \theta) y_{k-m})
\]
where, for every \( k \geq 0 \), \( a_k = a(t_k + \theta h) \), \( b_k = b(t_k + \theta h) \).

The last equation yields
\[
y_{k+1} = \frac{1 + a_k h (1 - \theta)}{1 - a_k h \theta} y_k + \frac{b_k h}{1 - a_k h \theta} (\theta y_{k-m+1} + (1 - \theta) y_{k-m}).
\]
Putting \( \alpha_k = -h a_k \) and \( \beta_k = h b_k \), we obtain
\[
y_{k+1} = \frac{1 - \alpha_k (1 - \theta)}{1 + \alpha_k \theta} y_k + \frac{\beta_k (1 - \theta)}{1 + \alpha_k \theta} \left( \theta y_{k-m+1} + (1 - \theta) y_{k-m} \right).
\]

Hence the method is stable if
\[
\left| \frac{1 - \alpha_k (1 - \theta)}{1 + \alpha_k \theta} \right| + \left| \frac{\beta_k (1 - \theta)}{1 + \alpha_k \theta} \right| \leq 1.
\]

Two cases are possible:

(i) \( 0 \leq \alpha_k (1 - \theta) \leq 1 \). We have
\[
\left| \frac{1 - \alpha_k (1 - \theta)}{1 + \alpha_k \theta} \right| + \left| \frac{\beta_k (1 - \theta)}{1 + \alpha_k \theta} \right| = \frac{1 - \alpha_k (1 - \theta)}{1 + \alpha_k \theta} + \frac{\beta_k (1 - \theta)}{1 + \alpha_k \theta} \leq 1
\]
because \( |\beta_k| \leq \alpha_k \).

(ii) \( \alpha_k (1 - \theta) > 1 \). In this case
\[
\left| \frac{1 - \alpha_k (1 - \theta)}{1 + \alpha_k \theta} \right| + \left| \frac{\beta_k (1 - \theta)}{1 + \alpha_k \theta} \right| = \frac{\alpha_k (1 - \theta) - 1 + |\beta_k| (1 - \theta) + |\beta_k| (1 - \theta)}{1 + \alpha_k \theta} = \frac{\alpha_k - \alpha_k \theta - 1 + |\beta_k|}{1 + \alpha_k \theta}
\]
\[
= -1 + \frac{\alpha_k + |\beta_k|}{1 + \alpha_k \theta} \leq 1.
\]

In fact we know that \( |\beta_k|(1 - \theta) \leq 1 \), so \( |\beta_k|(2 - 2\theta) \leq 2 \), \( |\beta_k|(1 + 1 - 2\theta) \leq 2 \), \( |\beta_k| \leq
2 + \beta_k |(-1 + 2\theta); now, |\beta_k| \leq \alpha_k and, if \theta \geq \frac{1}{2}, -1 + 2\theta \geq 0, hence |\beta_k| \leq 2 + \alpha_k(1 + 2\theta), 
\alpha_k(1 - 2\theta) \leq 2 - |\beta_k|, \alpha_k + |\beta_k| \leq 2 + 2\alpha_k\theta. In this way we have that (\alpha_k + |\beta_k|)/(1 + \alpha_k\theta) \leq 2

if \theta \geq \frac{1}{2}.

Now let us consider the one-step collocation method without the restriction (1.3). In this case \( m = \tau/h \) is not an integer (now \( m + \tau/h =: m' + \epsilon \) with \( m' \) integer, \( \epsilon \in [0, 1] \), and we get the difference equation

\[
y_{k+1} = \frac{1 - \alpha_k(1 - \theta)}{1 + \alpha_k\theta} y_k + \frac{\beta_k}{1 + \alpha_k\theta} \left[ \mu y_{k-m+1} + (1 - \mu) y_{k-m} \right]
\]

with \( \mu \in [0, 1] \) and

\[
m := \begin{cases} 
m' & \text{if } \theta - \epsilon \in [0, 1], \\
m' - 1 & \text{if } \theta - \epsilon \in [-1, 0].
\end{cases}
\]

Also in this situation (stepsize \( h \) without restriction (1.3) we can as before prove that the method is stable for \( \theta \geq \frac{1}{2} \) if, for every \( t \), \( |b(t)| \leq -a(t), \ h |b(t)|(1 - \theta) \leq 1 \). Note that this result implies GPN-stability of the backward Euler method.

From the literature [5] we know that the backward Euler method is algebraically stable (hence BN-stable, B-stable, AN-stable, A-stable) when it is applied to an ODE and that it is P and GP-stable when it is applied to a DDE. In the last section we proved that it is also PN and GPN-stable.

We conclude the paper by proving that the implicit Euler method is GRN-stable and hence RN-stable. The implicit Euler rule, applied to (2.1), (2.2), yields

\[
y_{k+1} = y_k - h f(t_{k+1}, y_{k+1}, \mu y_{k-m} + (1 - \mu) y_{k+1-m}),
\]

\[
z_{k+1} = z_k + h f(t_{k+1}, z_{k+1}, \mu z_{k-m} + (1 - \mu) z_{k+1-m}),
\]

with \( \mu \in [0, 1] \).

Put

\[
\delta_k := y_k - z_k, \ Y_\mu := \mu y_{k-m} + (1 - \mu) y_{k+1-m}, \ Z_\mu := \mu z_{k-m} - (1 - \mu) z_{k+1-m}.
\]

Then

\[
\| \delta_{k+1} \|^2 = \| \delta_k + h \left[ f(t_{k+1}, y_{k+1}, Y_\mu) - f(t_{k+1}, z_{k+1}, Z_\mu) \right] \|^2
\]

\[
= \| \delta_k \|^2 + 2h \text{Re}(\delta_k \mid \Delta f) + h^2 (\Delta f \mid \Delta f).
\]

where \( \Delta f := f(t_{k+1}, y_{k+1}, Y_\mu) - f(t_{k+1}, z_{k+1}, Z_\mu) \).

Substituting \( \delta_k = \delta_{k+1} - h \Delta f \) in the inner product \( \text{Re}(\delta_k \mid \Delta f) \), we get

\[
\| \delta_{k+1} \|^2 = \| \delta_k \|^2 + 2h \text{Re}(\delta_{k+1} \mid \Delta f) - h^2 (\Delta f \mid \Delta f)
\]

\[
< \| \delta_k \|^2 + 2h \text{Re}(\delta_{k+1} \mid \Delta f).
\]
Adding and subtracting the term \( f(t_{k+1}, z_{k+1}, Y_{\mu}) \), routine calculations yield
\[
\| \delta_{k+1} \|^2 \leq \| \delta_k \|^2 + 2h(\delta_{k+1} | f(t_{k+1}, y_{k+1}, Y_{\mu}) - f(t_{k+1}, z_{k+1}, Y_{\mu}) |)
\]
\[
+ 2h(\delta_{k+1} | f(t_{k+1}, z_{k+1}, Y_{\mu}) - f(t_{k+1}, z_{k+1}, Z_{\mu}) |)
\]
\[
\leq \| \delta_k \|^2 + 2h\sigma(t_{k+1}) \| \delta_{k+1} \|^2
\]
\[
+ 2h \| \delta_{k+1} \| \| f(t_{k+1}, z_{k+1}, Y_{\mu}) - f(t_{k+1}, Z_{k+1}, Z_{\mu}) \|
\]
\[
\leq \| \delta_k \|^2 + 2h\gamma(t_{k+1}) \| \delta_{k+1} \|^2 + 2h(\gamma(t_{k+1}) \| Y_{\mu} - Z_{\mu} \|
\]
\[
\leq \| \delta_k \|^2 + 2h\gamma(t_{k+1}) \| \delta_{k+1} \|^2 + 2h(\gamma(t_{k+1}) \| \delta_{k+1} \| \| Y_{\mu} - Z_{\mu} \|
\]
\[
= \| \delta_k \|^2 + 2h \| \delta_{k+1} \| \gamma(t_{k+1})(\| Y_{\mu} - Z_{\mu} \| - \| \delta_{k+1} \|)
\]
We have used the Schwartz inequality, the Definitions 2.4 and 2.5 and the inequality \( \sigma(t) \leq -\gamma(t) \) for every \( t \geq t_0 \) from (2.3).

We define \( Q_{k+1} := 2h\gamma(t_{k+1}) \) and we get
\[
\| \delta_{k+1} \|^2 \leq \| \delta_k \|^2 + Q_{k+1} \| \delta_{k+1} \| (\| Y_{\mu} - Z_{\mu} \| - \| \delta_{k+1} \|).
\]  \( 3.10 \)

Note that
\[
\| \delta_k \| \leq S \left( S := \max_{t \leq t_0} \| \Phi(t) - \Gamma(t) \| \right) \text{ for } k \leq 0.
\]

Assume \( \| \delta_k \| \leq S \) for every \( k \leq j \) \((j \geq 0)\), and consider two cases:

(i) \( \| \delta_{j+1} \| \leq S \) if \( \| \delta_{j+1} \| \leq \| \delta_j \| \),

(ii) if \( \| \delta_{j+1} \| > \| \delta_j \| \), the inequality (3.10) yields
\[
0 < \| \delta_{j+1} \|^2 - \| \delta_j \|^2 \leq Q_{j+1} \| \delta_{j+1} \| (\| Y_{\mu} - Z_{\mu} \| - \| \delta_{j+1} \|)
\]
and \( Q_{j+1} \) is nonnegative, so
\[
\| Y_{\mu} - Z_{\mu} \| - \| \delta_{j+1} \| \geq 0,
\]
and then
\[
\| \delta_{j+1} \| \leq \| \mu y_{j-m} + (1 - \mu) y_{j+1-m} - \mu z_{j-m} - (1 - \mu) z_{j+1-m} \|
\]
\[
\leq \mu \| \delta_{j-m} \| + (1 - \mu) \| \delta_{j+1-m} \| \leq S.
\]

We can conclude that \( \| \delta_k \| \leq S \) for every \( k \), i.e.
\[
\| y_k - z_k \| \leq S \text{ for every } k.
\]

It follows that implicit Euler method is GRN-stable and, by Definition 3.1, it is RN-stable.

References


