A remark on simplicity of vertex algebras and Lie conformal algebras

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Abstract

I give a short proof of the following algebraic statement: in a simple vertex algebra, the underlying Lie conformal algebra is either abelian, or it is an irreducible central extension of a simple Lie conformal algebra. This provides many examples of non-finite simple Lie conformal algebras, and should prove useful for classification purposes.
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1. Introduction

The notion of *vertex algebra* was introduced by Borcherds in [Bo] to axiomatize algebraic properties of the Operator Product Expansion (= OPE) of quantum fields in a (chiral) Conformal Field Theory in dimension two. Vertex algebras were defined as vector spaces endowed with infinitely many bilinear operations satisfying involved axioms that are now known as Borcherds identities.

The construction of non-trivial examples of vertex algebras is a complicated matter, because interesting known examples are very large objects—typically graded vector spaces of superpolynomial growth, called *Vertex Operator Algebras*—and finite-dimensional instances degenerate

into differential commutative algebra structures [Bo]. Examples of physical interest are usually described by giving generating quantum fields, after prescribing the singular part of their OPE. This idea can be made precise by axiomatizing the singular part of the OPE into a (*Lie*) *conformal algebra* structure, introduced by Kac in [K]. Lie conformal algebras [DK], and their generalizations [DsK], only determine commutation properties of quantum fields, and the whole vertex algebra can then be recovered by taking a suitable quotient of a certain universal envelope [K,L, P,R] of the Lie conformal algebra.

The Lie conformal algebra theory has proved simpler than the vertex algebra one. On the one hand, it is easy to construct small non-trivial examples; on the other hand, Lie conformal algebras possess a close resemblance to Lie algebras—they are indeed Lie algebras in a suitable pseudo-tensor category, see [BKV,BDK]—and can be treated by means of similar techniques.

It is clear that every vertex algebra is also a Lie conformal algebra: the resulting forgetful functor is adjoint to the above-mentioned universal enveloping vertex algebra functor. Both vertex algebras and Lie conformal algebras have corresponding notions of ideal and simplicity; however, it is easier for a subspace to be an ideal with respect to the Lie conformal algebra structure than with respect to the vertex one. The main result of the present paper is a short and elementary proof of the quite surprising fact that the Lie conformal algebra structure underlying a simple vertex algebra is as simple as it can be: its only ideals are central, and the whole Lie conformal algebra is a central extension of a simple structure. Indeed the one-dimensional vector space spanned by the vacuum element is always a central Lie conformal ideal.

The main tool employed in the paper is identity (2) whose constant (in z) part generalizes a formula devised by Wick [W] to compute the singular OPE of normally ordered products of fields in a free theory, which was independently mentioned in [BK] and [preD], and whose algebraic consequences range beyond the present result.

2. Vertex algebras

Let V be a complex vector space. A *field* on V is a formal distribution $\phi \in (\text{End } V)[[z, z^{-1}]]$ with the property that $\phi(v) \in V((z))$ for every $v \in V$. In other words, if

$$\phi(z) = \sum_{i \in \mathbb{Z}} \phi_i z^{-i-1},$$

then $\phi_n(v) = 0$ for sufficiently large n.

Definition 2.1. (See [K].) A vertex (super)algebra is a complex vector superspace $V = V_0 \oplus V_1$ endowed with a linear parity preserving *state-field correspondence* $Y : V \to (\text{End } V)[[z, z^{-1}]]$, a *vacuum element* $\mathbf{1} \in V_0$ and an even operator $T \in \text{End } V$ satisfying the following properties:

- *Field axiom*: $Y(v, z)$ is a field for all $v \in V$.
- *Locality*: For every choice of $a \in V_{p(a)}$ and $b \in V_{p(b)}$ one has

$$(z - w)^N \big(Y(a, z)Y(b, w) - (-1)^{p(a)p(b)} Y(b, w)Y(a, z) \big) = 0$$

for sufficiently large N.

- *Vacuum axiom*: The vacuum element **1** is such that

$$Y(\mathbf{1}, z) = \mathrm{id}_V, \qquad Y(a, z)\mathbf{1} \equiv a \mod zV[\![z]\!],$$

for all $a \in V$.

- *Translation invariance*: T satisfies

$$\big[T, Y(a, z)\big] = Y(Ta, z) = \frac{d}{dz} Y(a, z),$$

for all $a \in V$.

Notice that the vector space V carries a natural $\mathbb{C}[T]$-module structure. Fields $Y(a, z)$ are called *vertex operators*, or *quantum fields*. Vertex algebra axioms have strong algebraic consequences, among which we recall the following:

- *Skew-commutativity*: $Y(a, z)b = (-1)^{p(a)p(b)} e^{zT} Y(b, -z)a.$

Coefficients of quantum fields

$$Y(a, z) = \sum_{j \in \mathbb{Z}} a_{(j)} z^{-j-1}$$

in a vertex algebra span a Lie (super)algebra under the commutator Lie bracket, and more explicitly satisfy—see [K, Theorem 2.3(iv)],

$$[a_{(m)}, b_{(n)}] = \sum_{j \in \mathbb{N}} \binom{m}{j} (a_{(j)}b)_{(m+n-j)}, \tag{1}$$

for all $a, b \in V, m, n \in \mathbb{Z}$.

If A and B are subspaces of V, then we may define $A \cdot B$ as the $\mathbb{C}$-linear span of all products $a_{(j)}b$, where $a \in A$, $b \in B$, $j \in \mathbb{Z}$. It follows that if A and B are $\mathbb{C}[T]$-submodules of V, then $A \cdot B$ is also a $\mathbb{C}[T]$-submodule of V, as by translation invariance T is a derivation of all j-products. Notice that in this case $A \cdot B = B \cdot A$ by skew-commutativity, and that $A \subset A \cdot V$ by the vacuum axiom.

An *ideal* of V is a $\mathbb{C}[T]$-submodule $I \subset V$ such that $V \cdot I \subset I$. We will say that a vertex algebra is *simple* if its only ideals are trivial.

Remark 2.1. In order for a subspace $A \subset V$ to be an ideal, it suffices to check that $A \cdot V \subset A$: then A is indeed a $\mathbb{C}[T]$-submodule of V, as $a_{(-2)}\mathbf{1} = Ta$; moreover, skew-commutativity gives $V \cdot A = A \cdot V \subset A$.

3. Conformal algebras

Definition 3.1. (See [DK].) A *Lie conformal (super)algebra* is a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}[\partial]$-module $R = R_0 \oplus R_1$ endowed with a parity preserving $\mathbb{C}$-bilinear product $(a, b) \mapsto [a_\lambda b] \in R[\lambda]$ satisfying the following axioms:

(C1) $[a_\lambda b] \in R[\lambda]$,
(C2) $[\partial a_\lambda b] = -\lambda[a_\lambda b]$, $[a_\lambda \partial b] = (\partial + \lambda)[a_\lambda b]$,
(C3) $[a_\lambda b] = -(-1)^{p(a)p(b)}[b_{-\partial-\lambda}a]$,
(C4) $[a_\lambda[b_\mu c]] - (-1)^{p(a)p(b)}[b_\mu[a_\lambda c]] = [[a_\lambda b]_{\lambda+\mu}c]$,

for every choices of homogeneous elements $a, b, c \in V$, $p(r) \in \mathbb{Z}/2\mathbb{Z}$ denoting the parity of the homogeneous element r.

Any vertex (super)algebra V can be given a $\mathbb{C}[\partial]$-module structure by setting $\partial = T$. Then defining

$$[a_\lambda b] = \sum_{n \in \mathbb{N}} \frac{\lambda^n}{n!} a_{(n)} b$$

endows V with a Lie conformal (super)algebra structure. Indeed, (C1) follows from the field axiom, (C2) from translation invariance, (C3) from skew-commutativity, and (C4) from (1). For the sake of simplicity, in all that follows the super- prefix will not be explicitly mentioned, but tacitly understood.

If A and B are subspaces of a Lie conformal algebra R, then we may define $[A, B]$ as the $\mathbb{C}$-linear span of all λ-coefficients in the products $[a_\lambda b]$, where $a \in A$, $b \in B$. It follows from axiom (C2) that if A and B are $\mathbb{C}[\partial]$-submodules of R, then $[A, B]$ is also a $\mathbb{C}[\partial]$-submodule of R. Notice that in this case $[A, B] = [B, A]$ by axiom (C3). A Lie conformal algebra R is *solvable* if, after defining

$$R^0 = R, \qquad R^{n+1} = [R^n, R^n], \quad n \geqslant 0,$$

we find that $R^N = 0$ for sufficiently large N. R is solvable iff it contains a solvable ideal S such that R/S is again solvable. Solvability of a non-zero Lie conformal algebra R trivially fails if R equals its derived subalgebra $R' = [R, R]$. An *ideal* of a Lie conformal algebra R is a $\mathbb{C}[\partial]$-submodule $I \subset R$ such that $[R, I] \subset I$. If I, J are ideals of R, then $[I, J]$ is an ideal as well. An ideal I is said to be *central* if $[R, I] = 0$, i.e., if it is contained in the *center* $Z = \{r \in R \mid [r_\lambda s] = 0$ for all $s \in R\}$ of R. A Lie conformal algebra R is *simple* if its only ideals are trivial, and R is not *abelian*, i.e., $[R, R] \neq 0$.

Notice that, when V is a vertex algebra, we should distinguish between ideals of the vertex algebra structure and ideals of the underlying Lie conformal algebra. Indeed, ideals of the vertex algebra are also ideals of the Lie conformal algebra, but the converse is generally false, as it can be seen by observing that $\mathbb{C}\mathbf{1}$ is always a central ideal of the Lie conformal algebra structure, but it is never an ideal of the vertex algebra.

In order to avoid confusion, we will denote by V^{Lie} the Lie conformal algebra structure underlying a vertex algebra V.

4. A Poisson-like generalization of the Wick formula

The following formula—which is similar to (3.3.7) and (3.3.12) in [K]—relating the vertex and Lie conformal algebra structures is the key tool in the present paper.

Proposition 4.1. *If a, b, c are homogeneous elements of the vertex algebra V, then:*

$$\left[a_\lambda Y(b, z) c\right] = e^{\lambda z} Y\left([a_\lambda b], z\right) c + (-1)^{p(a)p(b)} Y(b, z)[a_\lambda c]. \tag{2}$$

Proof. Multiply both sides of (1) by $\lambda^m z^{-n-1}/m!$, then add up over all $m \in \mathbb{N}$, $n \in \mathbb{Z}$. Applying both sides to $c \in V$ proves the statement. $\square$

Lemma 4.1. *Let $U \subset V$ be vector spaces, and $p(\lambda), q(\lambda)$ be elements of $V((z))[\lambda]$. If all coefficients of*

$$e^{\lambda z} p(\lambda) + q(\lambda)$$

lie in $U((z))$, then the same is true for the coefficients of $p(\lambda)$.

Proof. If m and n are the degrees of p and q as polynomials in λ, we write

$$p(\lambda) = \sum_{i=0}^{m} p_i(z)\lambda^i, \qquad q(\lambda) = \sum_{j=0}^{n} q_i(z)\lambda^j.$$

The expression $e^{\lambda z} p(\lambda) + q(\lambda)$ is a power series in λ, and the coefficient multiplying λ^N is independent of $q(\lambda)$ if $N > n$. If also $N \geqslant m$, it equals

$$\sum_{i=0}^{m} \frac{z^{N-i}}{(N-i)!} p_i(z) = z^N \sum_{i=0}^{m} \frac{1}{(N-i)!} \cdot \frac{p_i(z)}{z^i}.$$

If all λ-coefficients of $e^{\lambda z} p(\lambda) + q(\lambda)$ lie in $U((z))$, then

$$\sum_{i=0}^{m} \frac{1}{(N-i)!} \cdot \frac{p_i(z)}{z^i} \in U((z))$$

for all sufficiently large N. However, the $(m+1) \times (m+1)$ matrix whose (i, j)-entry is $1/(N+j-i)!$ is non-singular,[2] hence

$$\frac{p_i(z)}{z^i} \in U((z)),$$

thus showing that $p(\lambda) \in U((z))[\lambda]$. $\square$

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[2] Its determinant can be computed by induction, and is easily showed to be equal to $m!(m-1)!(m-2)!\ldots 3!2!1!/((N+m)!(N+m-1)!\ldots N!)$.

5. A simplicity argument

Theorem 5.1. *Let V be a vertex algebra, and $I \subset V$ a subspace. Then $[I, V]$ is an ideal of V.*

Proof. Choose $a \in I$, $b, c \in V$. The linear span of all coefficients of $[a_\lambda Y(b, z)c]$, when $a \in I$, $b, c \in V$ equals $[I, V \cdot V] = [I, V]$. By Lemma 4.1 applied to (2), all coefficients of $Y([a_\lambda b], z)c$, $a \in I$, $b, c \in V$ lie in $[I, V]$, thus $[I, V] \cdot V \subset [I, V]$. Then Remark 2.1 ensures that $[I, V]$ is an ideal of V. $\square$

Corollary 5.1. *Let V be a simple vertex algebra. Then either V^{Lie} is abelian, or it is an irreducible central extension of a simple Lie conformal algebra.*

Proof. Let I be a proper ideal of V^{Lie}. Then $[I, V] \subset I$ is a proper ideal of V, forcing $[I, V] = 0$ by simplicity. Thus all proper ideals of V^{Lie} lie in the center Z of V^{Lie}, hence either $V^{Lie} = Z$, or V^{Lie}/Z has no non-trivial ideal; in the former case V^{Lie} is abelian.

In the latter, $[V, V]$ is a non-zero ideal of V, hence $V = [V, V]$. Then V^{Lie} is not solvable, as it equals its derived subalgebra, so V^{Lie}/Z cannot be abelian, and is therefore simple. As V^{Lie} equals its derived algebra, it is an irreducible central extension. $\square$

Remark 5.1. Corollary 5.1 can be used, along with the classification [DK] of finite simple (purely even) Lie conformal algebras and a known characterization of irreducible central extensions of the Virasoro Lie conformal algebra, to show that all simple (purely even) vertex algebra structures on a finitely generated $\mathbb{C}[\partial]$-module are *abelian*, i.e. have a trivial underlying Lie conformal algebra structure. From this it follows that if V is a finite vertex algebra, then V^{Lie} is solvable.

A more detailed investigation of such finite vertex algebras can show that V^{Lie} is indeed nilpotent, as soon as V contains no element a such that $Y(a, z)a = 0$.[3] Such claims are proved in a separate paper [D].

Remark 5.2. Let V be a vertex algebra, and assume that whenever a subspace $U \subset V$ is invariant under the action of coefficients of all quantum fields, then U is a $\mathbb{C}[\partial]$-module, and therefore an ideal. This happens, for instance, if $\partial = T$ is a coefficient of some quantum field, e.g., in a (conformal) Vertex Operator Algebra, where $T = L_{-1}$.

By a Schur Lemma argument, one may then show that if the vertex algebra V is simple and countable-dimensional—as it is always the case when V is a $\mathbb{Z}$-graded vector space with finite-dimensional homogeneous components—then the only central elements in the underlying Lie conformal algebra are scalar multiples of the vacuum element. Then V^{Lie} is an irreducible central extension of a simple Lie conformal algebra by the one-dimensional ideal $\mathbb{C}\mathbf{1}$.

Remark 5.3. If R is an irreducible central extension of a simple Lie conformal algebra by a one-dimensional center $\mathbb{C}\mathbf{1}$, and a grading is given on R which is compatible with its Lie conformal algebra structure, then there exists at most one simple vertex algebra structure compatible with the same grading, in which $\mathbf{1}$ is the vacuum element. Indeed, the universal enveloping vertex algebra of R has a unique maximal graded ideal, which must intersect R trivially because of the vacuum axiom, and of the simplicity of $R/\mathbb{C}\mathbf{1}$.

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[3] This is analogous to demanding that a commutative algebra possess no nilpotent elements.

This provides a strategy for finding simple vertex algebras, by first looking for simple Lie conformal algebra structures and their possible central extensions R, and then checking whether the unique simple quotient of the universal enveloping vertex algebra is R or a larger space. This strategy might become effective for families of Lie conformal algebras for which a classification of simple objects is likely to be achieved, e.g., under a polynomial growth or a finite Gelfand–Kirillov dimension assumption [X,Z1,Z2]. Recall that no Vertex Operator Algebra of physical interest is of this kind, as the presence of a Virasoro field forces superpolynomial growth.

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