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Homotopy, the Codimension 2 Correspondence and Sections of Rank 2 Vector Bundles

Raja Sridharan*

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400 005, India

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INTRODUCTION

Let A be a commutative neotherian ring with dim $A = n$. Let P be a projective A-module of rank n having trivial determinant. In [RS], assuming that $n \geq 3$, we proved certain results about the existence of unimodular elements in P. In this paper, we prove the dimension 2 analogues of these results. We note that in this case, due to Serre's codimension 2 correspondence, the question of existence of unimodular elements is equivalent to that of efficient generation of ideals. We therefore prove results on efficient generation of ideals. We now briefly outline the main results of this paper.

The results in this paper were motivated by the following theorem proved by Mohan Kumar [MK].

THEOREM. *Let A be an affine algebra over an algebraically closed field with* dim $A = n$. *Let I and J be two comaximal ideals of A which are local complete intersections of height n. Then*

(a) *If I and J are generated by n elements then so is $I \cap J$.*

(b) *If I and $I \cap J$ are generated by n elements then so is J.*

In [RS], we establish the validity of (a) if A is a noetherian ring with $\dim A \geq 3$. In this paper we consider the 2-dimensional case. We prove the following addition principles (cf. Theorems 2.3 and 2.7).

THEOREM (Addition Principle). *Let A be a noetherian ring with* dim $A = 2$. *Let I_1 and I_2 be two comaximal ideals of height 2 in A, which are*

* E-mail: sraja@tifrvax.bitnet.

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both intersections of finitely many maximal ideals. Suppose that both I_1 and I_2 are generated by 2 elements. Then so is $I_1 \cap I_2$.

THEOREM (Addition Principle). *Let A be an affine algebra over a field F with* $\dim A = 2$. *Let J be an ideal of A which is the intersection of finitely many maximal ideals each of which is two generated. Assume further that the residue field of any of the maximal ideals containing J is isomorphic to either F or $\bar{F}$. Then, any projective A-module of rank 2 and having trivial determinant that maps onto J is free.*

We also prove the following subtraction principles (cf. Theorems 2.6 and 2.10).

THEOREM (Subtraction Principle). *Let A be an affine algebra over a field F such that* $\dim A = 2$. *Let J_1 and J_2 be two comaximal ideals of height 2 in A which are both intersections of finitely many distinct maximal ideals. Suppose that A/J_1 and A/J_2 are products of algebraically closed fields and that J_1 and $J_1 \cap J_2$ are generated by 2 elements. Then so is J_2.*

THEOREM (Subtraction Principle). *Let A be an affine algebra over a field F with* $\dim A = 2$. *Let m_1 and m_2 be two distinct maximal ideals in A, which correspond to F-rational points of* Spec A. *Suppose that both m_1 and $m_1 \cap m_2$ are generated by 2 elements. Then so is m_2.*

The question whether the subtraction principle is true in general is still open.

The idea of the proofs of these theorems is taken from [RS]. We use the codimension 2 correspondence and Quillen's localisation theorem instead of Mandal's theorem as in [RS].

If A is a finitely generated algebra over either an algebraically closed field or over $\mathbb{Z}$, the theorems of this paper were proved by Mohan Kumar [MK] in all dimensions. However, the proofs given here work when A is an affine algebra over a field that is not necessarily algebraically closed. For the higher dimensional analogues of the theorems of this paper, we refer to [RS].

In Section 1 we recall some known results used in the sequel. For instance we discuss the codimension 2 correspondence due to Serre which is crucial for the proof the main results.

In this paper, all rings considered are commutative, noetherian, and have identity elements. All modules considered are assumed to be finitely generated.

A word about the notation: the symbol $\tilde{\to}$ means isomorphism, $M \xrightarrow{u} M$ or $M \xrightarrow{\lambda_u} M$ denotes homothesy by u, and A^* denotes units of A. We say that a maximal ideal m of a ring A is *regular* if the localization A_m of A

at m is a regular local ring. For all other unexplained notation and definitions used in this paper we refer to [Ba].

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1. SOME KNOWN LEMMAS AND RESULTS

In this section, we state some lemmas and results which will be used to prove the main theorems.

LEMMA 1. *Let A be a commutative noetherian ring. Let $I \subset A$ be an ideal which is generated by 2 elements a_1, a_2. Let $P_1, P_2, \ldots, P_r$ be a finite set of prime ideals of* Spec A. *Suppose $I \not\subset P_i$ for every i. Then we can find an element λ in A, such that $a_1 + \lambda a_2 \notin \bigcup_{i=1}^{r} P_i$.*

The lemma follows from a simple prime avoidance argument. We omit the proof.

Let A be a commutative ring. A row $(a_0, a_1, \cdots, a_n) \in A^{n+1}$ is said to be a *unimodular row* (of length $n + 1$) if there exist $b_0, b_1, \ldots, b_n$ in A such that $a_0 b_0 + a_1 b_1 + \cdots + a_n b_n = 1$.

For any commutative ring A, let $U_{n+1}(A)$ denote the set of unimodular rows of length $n + 1$ in A.

We state a theorem due to Swan and Towber and independently due to Suslin.

THEOREM. *Let $(a, b, c) \in U_3(A)$. Then there exists a matrix $C \in SL_3[A]$ having (a^2, b, c) as its first row.*

For a proof of this theorem, we refer to [ST].

We state a theorem which is due to Quillen [Q]. We will use this theorem often in Section 2.

THEOREM (Quillen). *Let A be a noetherian ring. Let P be a finitely generated projective $A[t]$-module. Then P is extended from A if and only if P_m is extended from A_m for every maximal ideal m of A.*

The following results are due to Serre [Se]. We state them without proofs. We state these results only in the generality that we need them.

Let A be a noetherian ring with $\dim A = 2$. Let J be an ideal of height 2 in A which is a product of regular maximal ideals. Then we see easily, using regularity, that J is locally generated by a regular sequence of length

2 and that J/J^2 is a free A/J module of rank 2. With the above notation, we have the following

THEOREM A. (1) *We have an isomorphism* $\varphi : \mathrm{Ext}^1_A(J, A) \xrightarrow{\sim} \mathrm{Hom}_{A/J}(\wedge^2 J/J^2, A/J)$.

(2) *If* i *is an isomorphism from* $\wedge^2 J/J^2$ *to* A/J *then* $\varphi^{-1}(i)$ *is an extension* $0 \to A \to Q \to J \to 0$ *where* Q *is a projective* A*-module of rank* 2.

For a proof of the above theorem, we refer to [Mu, Lemmas 1.1, 1.3].

Suppose we choose any isomorphism $i : \wedge^2 J/J^2 \to A/J$. Then, any map $j \in \mathrm{Hom}_{A/J}(\wedge^2 J/J^2, A/J)$ is given by $j : \wedge^2 J/J^2 \xrightarrow{i} A/J \xrightarrow{\lambda_{\bar{u}}} A/J$ where $\lambda_{\bar{u}}$ denotes homothesy by $\bar{u} \in A/J$.

By (1) of Theorem A, $\varphi^{-1}(j)$ is an extension of J by A and it is obtained by the following usual pushout diagram (see [Mac, p. 66] for details). We choose any lift $u \in A$ of $\bar{u}$ and consider the diagram

$$\begin{array}{llllll} \varphi^{-1}(i) : 0 \to A & \xrightarrow{t} & Q & \xrightarrow{s} J & \to 0 \\ \quad\quad\quad\quad \downarrow u & & \downarrow & \downarrow \mathrm{Id} & \\ \varphi^{-1}(j) : 0 \to A & \to & (Q \oplus A)/(-t(1), u) & \to J & \to 0 \end{array}$$

An easy local checking shows that $(Q \oplus A)/(-t(1), u)$ is a projective A-module if and only if $\bar{u} \in (A/J)^*$.

In particular, we have the following:

THEOREM B. (1) *Let* $j : \wedge^2 J/J^2 \to A/J$ *be a homomorphism.*

Let $\varphi^{-1}(j)$ *be the extension* $0 \to A \to M \to J \to 0$.

Then M *is a projective* A*-module of rank* 2 *if and only if* j *is an isomorphism.*

(2) *If* $j' = \lambda_{\bar{u}} o j$, *then* $\varphi^{-1}(j')$ *is given by the pushout diagram (where* $u \in A$ *is any lift of* $\bar{u}$*)*

$$\begin{array}{llllll} \varphi^{-1}(j) : 0 \to A & \xrightarrow{t} & M & \xrightarrow{s} J & \to 0 \\ \quad\quad\quad\quad \downarrow u & & \downarrow & \downarrow \mathrm{Id} & \\ \varphi^{-1}(j') : 0 \to A & \to & (M \oplus A)/(t(1), -u) & \to J & \to 0 \end{array}$$

This theorem follows easily from Theorem A.

Let, as above, A be a noetherian ring with $\dim A = 2$. Let $I \subset A[t]$ be an ideal such that $\dim A[t]/I = 1$. Suppose that I/I^2 is a free $A[t]/I$ module of rank 2, and that I is locally generated by a regular sequence of length 2. Then with the above notation, we have the following Theorems A′ and B′, whose proofs are similar to those of Theorems A and B. We again refer to [Mu, Lemmas 1.1, 1.3].

THEOREM A′. (1) *There exists an isomorphism*

$$\varphi[t] : \mathrm{Ext}^1_{A[t]}(I, A[t]) \tilde{\rightarrow} \mathrm{Hom}_{A[t]/I}(\wedge^2 I/I^2, A[t]/I).$$

(2) *If* $i[t] : \wedge^2 I/I^2 \rightarrow A[t]/I$ *is an isomorphism, then* $\varphi[t]^{-1}(i[t])$ *is an extension* $0 \rightarrow A[t] \rightarrow Q \rightarrow I \rightarrow 0$, *where* Q *is a projective* $A[t]$*-module of rank* 2.

THEOREM B′. (1) *Let* $j[t] : \wedge^2 I/I^2 \rightarrow A[t]/I$ *be a homomorphism. Let*

$$\varphi[t]^{-1}(j[t]) : 0 \rightarrow A[t] \rightarrow M \rightarrow I \rightarrow 0.$$

Then M *is a projective* $A[t]$*-module of rank* 2 *if and only if* $j[t]$ *is an isomorphism.*

(2) *If* $j'[t] = \lambda_{\overline{u[t]}} \circ j[t]$, *where* $\overline{u[t]} \in A[t]/I$. *Then* $\varphi[t]^{-1}(j'[t])$ *is obtained from* $\varphi[t]^{-1}(j[t])$ *as in Theorem B.*

We also have the following theorem whose proof follows that of [A-K, Theorem 4.5, p. 13].

THEOREM C′ (Functoriality). *Let* $I \subset A[t]$ *be an ideal as in Theorem* A′. *Let* $p : A[t] \rightarrow A$ *be the homomorphism which takes any polynomial to its constant coefficient. Assume that* $p(I) = J$, *where* J *is as in Theorem* A. *Then the following diagram is commutative*:

$$\begin{array}{ccc} \mathrm{Ext}^1(I, A[t]) & \overset{\varphi[t]}{\rightarrow} & \mathrm{Hom}(\wedge^2 I/I^2, A[t]/I) \\ \downarrow {\scriptstyle t=0} & & \downarrow {\scriptstyle t=0} \\ \mathrm{Ext}^1(J, A) & \overset{\varphi}{\rightarrow} & \mathrm{Hom}(\wedge^2 J/J^2, A/J) \end{array}$$

2. THE MAIN THEOREMS

In this section, we prove the main theorems. We follow the notation of Section 1. We first make a simple remark that we will use in the rest of section.

Remark 2.1. Let A be a noetherian ring with $\dim A = 2$. Let I be an ideal of height 2 in A which is a product of distinct maximal ideals m_i of A. Suppose that I is generated by 2 elements. Then all the m_i are regular maximal ideals of A.

Using this remark, it is easy to verify that all the maximal ideals that are considered in this section are regular maximal ideals. Thus, one may apply the results of Section 1.

We now prove a lemma, which is essentially a restatement of Schanuel's lemma.

LEMMA 2.2. *Let B be a commutative ring. Let I be an ideal of B and P a projective B-module of rank* 2. *Let*

$$0 \to B \xrightarrow{t} B^2 \xrightarrow{s} I \to 0$$
$$0 \to B \xrightarrow{t'} P \xrightarrow{s'} I \to 0$$

be two exact sequences. Let $s(1,0) = a$ and $s(0,1) = b$. Then $P \simeq B^3/(u, b, -a)$, where $u \in B$ is a unit modulo $(b, -a)$.

Proof. We choose ϕ: $B^2 \to P$ such that $s'\phi = s$. Then ϕ induces a map $\bar{\phi}$ from $\operatorname{Ker} s = B$ to $\operatorname{Ker} s' = B$, which is multiplication by an element u of B. Schanuel's Lemma [Go, p. 12] yields an exact sequence

$$0 \to B \to B \oplus B^2 \to P \to 0,$$

where the map from $B \oplus B^2$ to P is given by $-t' \oplus \phi$ and the map from B to $B \oplus B^2$ sends 1 to $(u, b, -a)$. Thus P is isomorphic to $B^3/(u, b, -a)$. One easily checks that u is a unit modulo $(b, -a)$.

We prove the next theorem in some detail. In the proofs of the other theorems, we will use without further comment the reductions obtained in the proof this theorem.

THEOREM 2.3 (Addition Principle). *Let A be a noetherian ring with* $\dim A = 2$. *Let I_1 and I_2 be two comaximal ideals of height* 2 *in A which are both intersections of maximal ideals. Suppose that both I_1 and I_2 are generated by* 2 *elements. Then so is $I_1 \cap I_2$.*

Proof. Let $I_1 = \cap_i m_i$ and $I_2 = \cap_j m'_j$. Since I_1 and I_2 are generated by 2 elements, we may apply the preliminary remark to conclude that the m_i and m'_j are regular maximal ideals for every i and j.

We claim that not every element of I_1 is a zero divisor of A. Suppose on the contrary, that $I_1 \subset \bigcup p_i$, $p_i \in \operatorname{Ass} A$. If this were so, I_1 would be contained in p_i for some $p_i \in \operatorname{Ass} A$, hence, m_j would be contained in p_i, for some m_j, which is minimal over I_1. Therefore, m_j would be associated to A and this would contradict the fact that m_j is a regular maximal ideal of A.

Since $I_1 + I_2 = A$, $I_1 \not\subset m'_j$ for any j. Therefore, $I_1 \not\subset \bigcup p_i \cup m'_j$, where $p_i \in \operatorname{Ass} A$. Let $I_1 = (f_1, f_2)$. Using Lemma 1 of Section 1, we may assume, by replacing f_1 by $f_1 + \lambda f_2$, that f_1 is not a zero divisor of A and that $(f_1) + I_2 = A$.

Let $I_1' = (f_1, t-1) \subset A[t]$ and let $I_2' = I_2 A[t]$. Since f_1 is not a zero divisor of A, we see that f_1, $t-1$ is a regular sequence and hence $I_1'/I_1'^2 \simeq (A[t]/I_1')^2$. Since all the m_j' are regular maximal ideals of A, we see that $I_2'/I_2'^2 \simeq (A[t]/I_2')^2$.

Let $I = I_1' \cap I_2'$. We now check that the hypotheses needed to apply Theorem A′ (Section 1) are satisfied.

Since (f_1) and I_2 are comaximal, $I_1' + I_2' = A$. Applying the Chinese Remainder Theorem, we see that $I/I^2 \simeq (A[t]/I)^2$.

We now check that I is locally generated by a regular sequence of length 2. Since I_1' and I_2' are comaximal, any maximal ideal m of $A[t]$ contains either I_1' or I_2', but not both. We check that after localising at a maximal ideal containing I_1', I is generated by a regular sequence of length 2. The case of a maximal ideal containing I_2' is similar. If m contains I_1', we have $I_m = I'_{1_m}$, which is generated by the regular sequence $f_1, t-1$.

Therefore the hypotheses needed to apply Theorem A′ are satisfied. Let $j'[t]\colon \wedge^2 I/I^2 \to A[t]/I$ be any isomorphism. Since I_2 is generated by 2 elements, we have a Koszul resolution

$$E: 0 \to A \to A^2 \to I_2 \to 0.$$

Let $\phi(E) = j\colon \wedge^2 I_2/I_2^2 \to A/I_2$ be the isomorphism associated to E. We show that we can alter $j'[t]$ by a unit $u[t]$ of $A[t]/I$ to an isomorphism $j[t]\colon \wedge^2 I/I^2 \to A[t]/I$, which satisfies the property that $j(0) = j$ (where $j(0)$ is obtained from $j[t]$ by setting $t = 0$).

We note that $j'(0)$ and j differ by a unit u of A/I_2. We consider the homomorphism $A[t]/I \to A/I_2$ which sends t to 0. We show that we can lift u via this map to a unit $u[t]$ of $A[t]/I$. This $u[t]$ will clearly satisfy the required property. By the Chinese Remainder Theorem, $A[t]/I \simeq A[t]/I_1' \times A[t]/I_2'$. One checks easily that $u[t] = (1, u)$ satisfies the required property. By altering $j'[t]$ by the unit $u[t]$ to $j[t]$, we may assume that we have an isomorphism $j[t]\colon \wedge^2 I/I^2 \to A[t]/I$ such that $j[0] = j$.

Let $\phi[t]^{-1}(j[t]) = E[t]: 0 \to A[t] \to P' \to I \to 0$. We claim that P' is extended from A. We grant the claim and proceed to prove the theorem. Since $j(0) = j = \phi(E)$, by Theorem C′,

$$E(0): 0 \to A \to P'/tP' \to I_2 \to 0 = E: 0 \to A \to A^2 \to I_2 \to 0$$

(where $E(0)$ is obtained from $E[t]$ by setting $t = 0$). Comparing extensions we have $P'/tP' \simeq A^2$ and since P' is extended, $P' \simeq A[t]^2$. Now specialising $E[t]$ at $t = 1 + f_2$, we see that $I_1 \cap I_2$ is generated by 2 elements.

We now check that P' is extended from A. In order to check this, it is enough to check by Quillen's theorem (Section 1), that P'_m is extended from A_m for every maximal ideal m of A. Suppose $m \neq m'_j$ for all j, then P'_m maps onto I'_{1_m}, which is generated by the regular sequence f_1, $t - 1$. An easy application of Lemma 2.2 shows that $P'_m \simeq A_m[t]^3/(v(t), f_1, t-1)$, which is free by [RR, Proposition 2.1], since $t - 1$ is a monic polynomial.

Similarly we can check that $P_{m'_j}$ is free for every m'_j.

This concludes the proof of the theorem.

COROLLARY 2.4. *Let A be an affine algebra over the field of real numbers with* $\dim A = 2$. *If m_i, $1 \leq i \leq n$, are maximal ideals of A which are 2-generated, then $\cap_i m_i$ is 2-generated.*

THEOREM 2.5. *Let A be an affine algebra over a field F such that* $\dim A = 2$. *Let J be an ideal of height 2 in A which is the intersection of finitely many maximal ideals $m_1, \cdots, m_l$ of A, where the m_i satisfy the property that A/m_i is algebraically closed for every i. Assume that J is generated by 2 elements. Let P be a projective A-module of rank 2 with trivial determinant. Assume that there exists a surjection s from P to J. Then P is free.*

Proof. Let J be generated by f_1, f_2. We will be through as in (2.2), if we show that (u, f_1, f_2) is completable to a matrix in $SL_3(A)$, for any $u \in A$ such that u is a unit mod(f_1, f_2). Since A/J is a product of algebraically closed fields, we see that u is a square mod(f_1, f_2). Thus, by the Swan–Towber–Suslin Theorem, (u, f_1, f_2) is completable. Hence P is free.

As a consequence of the above theorem, we deduce the following

THEOREM 2.6 (Subtraction Principle). *Let A be an affine algebra over a field F such that* $\dim A = 2$. *Let J_1 and J_2 be two comaximal ideals of height 2 in A, which are both intersections of finitely many distinct maximal ideals. Suppose that A/J_1 and A/J_2 are products of algebraically closed fields. Assume further that J_1 and $J_1 \cap J_2$ are generated by 2 elements. Then so is J_2.*

Proof. We may assume as in the proof of (2.3), that $J_1 = (f_1, f_2)$, with f_1 a non-zero divisor in A and $(f_1) + J_2 = A$. Let $I_1 = (f_1, t - f_2) \subset A[t]$, $I_2 = J_2 A[t]$, and $I = I_1 \cap I_2$. Let $j[t] := \wedge^2(I/I^2) \to A[t]/I$ be any isomorphism and $E[t]: 0 \to A[t] \to P[t] \to I \to 0$ be the corresponding extension. Specialising $E[t]$ at $t = 0$ and using (2.5), we see that P is free. Now, specialising $E[t]$ at $t = 1 + f_2$ we see that J_2 is 2-generated.

THEOREM 2.7. *Let A be an affine algebra over a field F with* $\dim A = 2$. *Let J be an ideal of A which is the intersection of finitely many maximal ideals each of which is two generated. Assume further that the residue field of any of* *the maximal ideals containing J is isomorphic to either F or $\bar{F}$. Then, any projective A-module of rank 2 and having trivial determinant that maps onto J is free.*

Proof.

Step 1. Let $J = (\bigcap_1^r m_i) \cap (\bigcap_1^s n_j)$ be such that $A/m_i \simeq F$ and $A/n_j \simeq \bar{F}$. By (2.3), J is two generated. If $r = 0$, then the theorem follows from (2.5). (If $s = 0$ and $r = 1$, a proof similar to (2.5) can be given.) We assume therefore that $r \geq 1$. Let $J_2 = (\bigcap_2^r)m_i \cap (\bigcap_1^s n_j)$. Let $m_1 = (f_1, f_2)$. We may assume as in the proof of (2.3), that $(f_1) + J_2 = A$ and that f_1 is a non-zero divisor in A. Let $I_1 = (f_1, t - f_2) \subset A[t]$ and let $I_2 = J_2 A[t]$. If $I = I_1 \cap I_2$, we see, as in (2.3), that $I/I^2 \tilde{\to} (A[t]/I)^2$. Let $j[t]$: $\wedge^2 I/I^2 \to A[t]/I$ be any isomorphism and let $\varphi[t]^{-1}(j[t]) = E[t]: 0 \to A[t] \to P[t] \to I \to 0$. If we specialise $E[t]$ at $t = f_2 + 1$, we obtain a surjection from P to J_2. By induction on r, we see that P is free, therefore so is $P[t]$.

Step 2. Let $s: P' \to J$ be any surjection with P' a projective A-module of rank 2 having trivial determinant. We want to show that P' is free. We consider the Koszul resolution

$$E' := 0 \to A \to P' \xrightarrow{s} J \to 0.$$

Let $j' = \varphi(E'): \wedge^2 (J/J^2) \to A/J$ be the isomorphism associated to E'. Let u be the unit in A/J which is the difference between j' and $j[0]$ (where $j[0]$ is obtained from $j[t]$ be setting at $t = 0$ and $j[t]$ is as in Step 1). We have, by the Chinese Remainder Theorem, $A/J \simeq \prod_1^r A/m_i \times \prod_1^s A/n_i$. Let $u = (u_1, \cdots, u_r, u_{r+1}, \cdots, u_{r+s})$. We first show, that there exists an unit $u[t]$ in $A[t]/I$, such that $u[0] = (u_1, \cdots, u_r, \lambda_1, \cdots, \lambda_s)$ for some $\lambda_j \in A/n_j$. We note that $A[t]/I \simeq A[t]/I_1 \times \prod_2^r A/m_i[t] \times \prod_1^s A/n_j[t]$ and that the homomorphism $A[t]/I \to A/J$ sending t to 0 maps $A[t]/I_1$ onto A/m_1, $A/m_i[t]$ onto A/m_i, and $A/n_j[t]$ onto A/n_j. Since $A/m_i \simeq F$ for $1 \leq i \leq r$, clearly a unit $u[t]$ exists with the required property. Let $j'[t] = u[t]j[t]: \wedge^2 (I/I^2) \to A[t]/I$ and $E'[t] = \varphi[t]^{-1}(j'[t])$. If we specialise $E'[t]$ at $t = 0$, we obtain using Step 1, a surjection from A^2 to J, which sends the two coordinate functions to a and b, respectively. Now using (2.2), we see that $P' \simeq A^3/(u', a, b)$ where $J = (a, b)$ and $u' = (1, \cdots, 1, \lambda'_1, \cdots, \lambda'_s)$. Since $A/n_j \simeq \bar{F}$, we see that u' is square modulo J. Arguing as in (2.5), we see that P' is free. This completes the proof of the theorem.

COROLLARY 2.8. *Let A be an affine algebra over the field of real numbers with* $\dim A = 2$. *Let J be is the intersection of finitely many maximal ideals of A each of which is two generated. Then any projective A-module of rank 2 having trivial determinant that maps onto J is free.*

COROLLARY 2.9. *Let A be a regular affine algebra over the field of real numbers with* $\dim A = 2$. *Assume that* $\operatorname{Pic} A = 0$. *Then the following are equivalent*:

(i) *Every projective A-module is free.*

(ii) *All maximal ideals of A are two generated.*

The proof of the above corollary follows easily from (2.7) and the codimension 2 correspondence.

THEOREM 2.10 (Subtraction Principle). *Let A be an affine algebra over a field F with* $\dim A = 2$. *Let m_1 and m_2 be two distinct maximal ideals in A, which correspond to F-rational points of* $\operatorname{Spec} A$. *Suppose that both m_1 and $m_1 \cap m_2$ are generated by 2 elements. Then so is m_2.*

Proof. We may assume as in (2.3), that $m_1 = (f_1, f_2)$ with f_1 a nonzero divisor in A and $(f_1) + m_2 = A$. Let $I_1 = (f_1, t - f_2) \subset A[t]$, $I_2 = m_2 A[t]$, and $I = I_1 \cap I_2$. Let $j[t] := \wedge^2 I/I^2 \to A[t]/I$ be any isomorphism and $E' : 0 \to A \to A^2 \to J \to 0$ be a Koszul resolution, where $J = m_1 \cap m_2$. Let $\varphi[E'] = j' : \wedge^2 J/J^2 \to A/J$. By altering $j[t]$ by a unit $u[t] \in (A[t]/I)^*$, we may assume as in (2.3), that we have an isomorphism $j'[t]: \wedge^2 I/I^2 \xrightarrow{\sim} A[t]/I$, such that, when we specialise at $t = 0$, $j'(0) = j' : \wedge^2 J/J^2 \to A/J$. Let $\varphi[t]^{-1}(j'[t]) = E'[t] : 0 \to A[t] \to P'[t] \to I \to 0$. Then by Theorem C′, $\varphi^{-1}(j') = E'(0) : 0 \to A \to P' \to J \to 0$. But $\varphi^{-1}(j') = E'$. Comparing extensions, we have $P' \xrightarrow{\sim} A^2$ which implies $P'[t] \xrightarrow{\sim} A[t]^2$. Specializing $E'[t]$ at $t = 1 + f_2$, we see that m_2 is generated by 2 elements.

The results of this paper lead one to the following question: Let A be a regular noetherian ring with $\dim A = 2$. Let S be the set of pairs (m, i), where m is a maximal ideal of A and $i : \wedge^2 (m/m^2) \to A/m$ an isomorphism. Let G be the free abelian group generated by S. Let I be an ideal of $A[t]$ which is locally generated by regular sequence of length 2 and such that $\dim A[t]/I = 1$. Suppose that $I|_{t=0} = \cap_1^r m_i$ and $I|_{t=1} = \cap_1^s m'_j$. Then any isomorphism $j[t] : \wedge^2 (I/I^2) \to A[t]/I$, gives rise, when we specialise at $t = 0$ and 1, to elements $g_0, g_1 \in G$. Let H be the subgroup of G generated by $g_0 - g_1$ where g_0 and g_1 are obtained as above. Let P be a projective A-module of rank 2 having trivial determinant. We fix an isomorphism $\theta : \wedge^2 P \to A$, and let $s : P \to J$ a generic section with $J = \bigcap_1^r m_i$. Then, by Serre's codimension 2 correspondence, we obtain an isomorphism $\wedge^2 J/J^2 \to A/J$ associated to θ and s. That is, we obtain an element $s_1 + \cdots + s_k$ of G with $s_i \in S$. Suppose there exist $s'_1, \ldots, s'_l \in S$ such that $\Sigma_1^k s_i - \Sigma_1^l s'_i \in H$. Then, does there exist a surjection s': $P \to \bigcap m'_j$, such that $\Sigma s'_j$ is the element of G associated to θ and s'?

Remark. After this paper was sent for publication we realised that the same proofs work to yield the following theorems.

THEOREM 1. *Let A be a noetherian ring with* dim $A = 2$. *Let I_1 and I_2 be two comaximal ideals of height* 2 *in A, both of which are intersections of finitely many maximal ideals. Suppose that I_1 is generated by* 2 *elements. Let P be a projective A-module with rank $P = 2$ and having trivial determinant. Suppose that there exists a surjective map from P to I_2. Then, there also exists a surjective map from P to $I_1 \cap I_2$.*

THEOREM 2. *Let A be a noetherian ring with* dim $A = 2$. *Let I_1 and I_2 be two comaximal ideals of height* 2 *in A, both of which are intersections of finitely many maximal ideals. Suppose that I_1 is generated by* 2 *elements. Let P be a projective A-module with rank $P = 2$ and having trivial determinant. Suppose that there exists a surjective map from P to $I_1 \cap I_2$. Then, there exists surjective map from P to I_2 in the following cases*:

(a) *A is an affine algebra over a field F and I_1 is a maximal ideal which corresponds to an F-rational point of* Spec A.

(b) *A is an affine algebra over a field F and A/I_1 is a product of algebraically closed fields.*

We note that the main results of this paper and in fact stronger ones can be deduced from Theorems 1 and 2.

The higher dimensional analogues of Theorems 1 and 2 are also valid and will appear in a joint work with Satya Mandal.

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