Dual Hopf orders in group rings of elementary abelian $p$-groups

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Let $R$ be the valuation ring of $K$, a finite extension of $\mathbb{Q}_p$ containing a primitive $p$th root of unity, and let $G$ be an elementary abelian $p$-group of order $p^n$, with dual group $\hat{G}$. We construct a new family of pairs of Hopf orders over $R$ in $KG$ and in $K\hat{G}$ whose construction extends the truncated exponential construction of [8].

For $G$ a finite abelian group of order $p^n$, the classification of Hopf orders over $R$ in $KG$ is known only for $n = 1$ [12] and 2 [1,7,13]. For $n > 2$ only a few families of Hopf orders were known until recently: [10] for $G$ elementary abelian of order $p^n$, [9] for arbitrary $G$ (“Larson orders”), [14] for $G$ cyclic of order $p^3$, [4,8] for $G$ elementary abelian of order $p^n$. For $G$ of order $p^n$, Larson orders are described completely by $n$ valuation parameters (that determine the discriminant of the Hopf order); the examples of [4,7,8] suggest that a general Hopf order of rank $p^n$ should involve, in addition, $n(n - 1)/2$ unit parameters, which can conveniently be laid out as entries of a lower triangular matrix.

There are two ways that these unit parameters can arise. In the formal group construction of [4], the matrix of unit parameters is used to construct an isogeny of polynomial formal groups whose kernel is represented by a Hopf order. The matrix entries then show up as coefficients of group elements in the algebra generators of the Hopf order. In the constructions of [4,14], the unit parameters appear directly in the algebra generators, gen-
eralizing Greither’s construction in [7]. This approach was codified in [15] in the definition of triangular Hopf orders.

This is the fourth in a recent series of papers that construct new families of Hopf orders with the desired number of parameters. [5] used formal groups to construct families of Hopf orders in $K G$ when $G$ is cyclic of order $p^n$; [15] obtained several families of triangular Hopf orders in $K G$, $G$ cyclic of order $p^3$ and strengthened the formal group construction of [5] when $n = 3$, and [6] identified a subfamily of the formal group Hopf orders of [5] in the cyclic $p^n$ case whose duals are triangular. In this paper we obtain new families of triangular Hopf orders in the elementary abelian $p^n$ case.

The construction in [8] used unit parameters in the form of truncated exponentials. Parameters of that form are particularly attractive for constructing dual pairs of Hopf orders, because the matrices of unit parameters associated to dual pairs are essentially inverses of each other. In this paper we construct a family of dual pairs of Hopf orders using truncated exponentials, similar to that in [8], but with less restrictive conditions on the parameters.

Here is an outline of the paper. In Section 1 we apply the strategy of utilizing Larson orders inside arbitrary Hopf orders, used in [8], to construct (Theorem 3) a new class of triangular Hopf orders in $K G$ for arbitrary $n$. In Section 2 the construction is extended (Theorem 5) to construct dual pairs of triangular Hopf orders in $K G, K \hat{G}$. In Section 3 we construct (Theorem 9) dual pairs of triangular Hopf orders using the truncated exponential function introduced in [8]. To do this construction requires subjecting the $n(n − 1)/2$ unit parameters to an additional set of inequalities beyond those required for Theorem 5. Examples can be found using the simplex algorithm that show that the three constructions are in fact increasingly restrictive. In the final section, we describe how our families of examples fit into the existing landscape of Hopf orders in $K G$. We show in particular that there are rank $p^3$ examples obtained from Theorem 3 that do not arise from polynomial formal groups, and rank $p^n$ examples for all $n$ from Theorem 9 that have no rank $p^2$ Larson subquotients and do not arise from [8]. In obtaining these examples we exhibit a way to manage the large number of inequality constraints that arise in Theorem 9.

The results in this paper extend and refine results from [11].

1. Hopf orders

Let $K, R$ be as noted above. Let $\pi$ be a parameter for (generator of the maximal ideal of) $R$, let $e$ be the absolute ramification index of $K$, assume $K$ contains a primitive $p$th root of unity $\zeta$, and let $e'= \text{ord}(\zeta − 1) = e/(p − 1)$. Let $G$ be an elementary abelian $p$-group of order $p^n$, $G = G_1 \times \cdots \times G_n$ with $G_r = \langle \sigma_r \rangle$ cyclic of order $p$. In $K G_r = K \langle \sigma_r \rangle$, let

$$e_j^{(r)} = \frac{1}{p} \sum_{i=1}^{p-1} \zeta^{-ij} \sigma_r^i, \quad j = 1, \ldots, p − 1,$$
be the primitive idempotents. For \( u \) in \( K \) let

\[ a^{(r)}_u = \sum_{k=0}^{p-1} u^k e^{(r)}_k. \]

Then \( a^{(r)}_u \) is a multiplicative homomorphism from \( K^\times \) to \( KG^\times \) satisfying \( a^{(r)}_\zeta = \sigma_r. \)

Let \( i_1, \ldots, i_n \) be numbers satisfying \( 0 \leq i_j \leq e' \) for all \( j \), and let \( i'_j = e' - i_j \) as usual. Let \( U = (u_{i,j}) \) be a lower triangular matrix with diagonal entries \( = \zeta \). Let \( a_{ij} = a^{(j)}_{ui,j} \) in \( KGj \). Then \( a_{jj} = a^{(j)}_{\zeta} = \sigma_j. \) Consider the \( R \)-algebra

\[ H_n = R \left[ \frac{\sigma_1 - 1}{\pi^{i_1}}, \frac{a_{2,1}\sigma_2 - 1}{\pi^{i_2}}, \ldots, \frac{a_{n,1}a_{n,2} \cdots a_{n,n-1}\sigma_n - 1}{\pi^{i_n}} \right]. \]

In this section we find conditions on the entries of \( U \) for \( H_n \) to be a Hopf order over \( R \) in \( KG \).

Our approach uses the following result of Greither [7], for which a convenient reference is [3, (31.8), (31.10)]:

**Proposition 1.** Let \( 0 \leq i, k \leq e' \) and \( k < pi \). Let \( G, G' \) be cyclic of order \( p \) with generators \( \sigma, \sigma' \), respectively. For \( v \) in \( R \), let \( a_v \) be in \( KG \) and \( t = (a_v\sigma' - 1)/\pi^k \). Let \( H(i) = R[(\sigma - 1)/\pi^i] \) in \( KG \). Then \( E_v = H(i)[t] \) is an order in \( K[G \times G'] \), free of rank \( p \) as an \( H(i) \)-module with power basis \( 1, t, \ldots, t^{p-1} \), if and only if \( a_v - 1 \in \pi^{pk} H(i) \), iff \( \text{ord}(v - 1) \geq i' + pk \); \( H \) is then a Hopf order in \( K[G \times G'] \) if

\[ \Delta(a_v) \equiv a_v \otimes a_v \quad (\text{mod } \pi^k (H(i) \otimes H(i))), \]

iff \( \text{ord}(v - 1) \geq i' + k/p \).

We proceed to construct the order \( H_n \) in \( KG \) with \( G = G_1 \times \cdots \times G_n \), inductively.

We begin by fixing the numbers \( i_1, \ldots, i_n \) with \( 0 \leq i_k \leq e' \), and assume \( i_r \leq pi_s \) for all \( r > s \). These will be called the valuation parameters for \( H_n \). The valuation parameters determine the discriminant of \( H_n \).

For each \( r \) with \( 1 \leq r < n \), having constructed \( H_r \), the construction of \( H_{r+1} \) involves finding a Larson order \( H(\mu_r) = R[(\sigma_r - 1)/\pi^{i_r}] \) inside \( H_r \). We call \( \mu_1, \mu_2, \ldots, \mu_n \) a set of Larson parameters for \( H_n \) if \( H(\mu_r) \subseteq H_r \) for \( r = 1, \ldots, n \).

To begin with \( r = 1 \), let

\[ H_1 = R \left[ \frac{\sigma_1 - 1}{\pi^{i_1}} \right] = R \left[ \frac{a_{1,1} - 1}{\pi^{i_1}} \right], \]
a Larson order in $KG_1$, and set $\mu_1 = i_1$, the first Larson parameter. Recalling the matrix $U$, let

$$H_2 = H_1 \left[ \frac{a_{2,1}\sigma_2 - 1}{\pi^{\mu_2}} \right] = H_1 \left[ \frac{a_{2,1}a_{2,2} - 1}{\pi^{\mu_2}} \right].$$

By Proposition 1, this is a Hopf order in $K[G_1 \times G_2]$ iff

$$\text{ord}(u_{2,1}^{\mu_2} - 1) \geq i'_1 + pi_2$$

and

$$\text{ord}(u_{2,1} - 1) \geq i'_1 + \frac{i_2}{p}.$$ 

Given that $H_2$ is a Hopf order, we have

$$H_2 \cap K[\sigma_1] = H_1 = H(i_1) = R \left[ \frac{\sigma_1 - 1}{\pi^{i_1}} \right],$$

and we need to find a Larson parameter $\mu_2$ so that

$$H(\mu_2) = R \left[ \frac{\sigma_2 - 1}{\pi^{\mu_2}} \right] \subseteq H_2 \cap K[\sigma_2].$$

**Proposition 2.** $H(\mu_2) \subseteq H_2 \cap K[\sigma_2]$ if $\mu_2 \leq \min\{i_2, \text{ord}(u_{2,1} - 1) - i'_1\}$.

**Proof.** We have

$$\frac{a_{2,1}\sigma_2 - 1}{\pi^{\mu_2}} = \frac{a_{2,1}(\sigma_2 - 1)}{\pi^{\mu_2}} + \frac{a_{2,1} - 1}{\pi^{\mu_2}}.$$

Assume that $\mu_2 \leq i_2$, so that $(a_{2,1}\sigma_2 - 1)/\pi^{\mu_2}$ is in $H_2$. Then $a_{2,1}$ is a unit of $H_2$, and so $(\sigma_2 - 1)/\pi^{\mu_2}$ will be in $H_2$ if and only if $(a_{2,1} - 1)/\pi^{\mu_2}$ is in $H_1$, iff

$$\text{ord}(u_{2,1} - 1) \geq i'_1 + \mu_2$$

by Proposition 1. Thus if

$$\mu_2 \leq \min\{i_2, \text{ord}(u_{2,1} - 1) - i'_1\},$$

then $H(\mu_2) \subseteq H_2$. \qed

We proceed by induction. Suppose we have found conditions on the entries of the first $r - 1$ rows of the matrix $U$ so that $H_{r-1}$ is an $R$-Hopf order in $K[G_1 \times \cdots \times G_{r-1}]$ in such a way that for $2 \leq j \leq r - 1$, $H_j$ is free of rank $p$ over $H_{j-1}$ on powers of the algebra
generator $t_j = (a_{j,1}a_{j,2} \cdots a_{j,j-1}\sigma_j - 1)/\pi^{i_j}$ of $H_j$ over $H_{j-1}$. Suppose also we have found $r - 1$ Larson parameters $\mu_1 = i_1, \mu_2, \ldots, \mu_{r-1}$ so that for each $j, 1 \leq j \leq r - 1,$

$$H_{r-1} \cap K[G_j] = H_j \cap K[G_j] \supseteq H(\mu_j) = R\left[\frac{\sigma_j - 1}{\pi^{\mu_j}}\right].$$

Consider

$$H_r = H_{r-1}\left[\frac{a_{r,1}a_{r,2} \cdots a_{r,r-1}\sigma_r - 1}{\pi^{i_r}}\right].$$

For $H_r$ to be free of rank $p$ over $H_{r-1}$ with power basis generated by

$$t = \frac{a_{r,1}a_{r,2} \cdots a_{r,r-1}\sigma_r - 1}{\pi^{i_r}}$$

it suffices that

$$\frac{a_{r,1}^p a_{r,2}^p \cdots a_{r,r-1}^p - 1}{\pi^{pi_r}} \in H_{r-1},$$

which follows if

$$\frac{a_{r,j}^p - 1}{\pi^{pi_r}} \in H(\mu_j) \subset H_{r-1} \cap KG_j$$

for all $j < r$, since

$$a_{r,1}^p a_{r,2}^p \cdots a_{r,r-1}^p - 1 = a_{r,1} - 1 + a_{r,1}^p (a_{r,2}^p - 1) + \cdots + a_{r,1}^p a_{r,2}^p \cdots a_{r,r-2}^p (a_{r,r-1}^p - 1).$$

But by Proposition 1,

$$\frac{a_{r,j}^p - 1}{\pi^{pi_r}} = \frac{a_{u_{r,j}}^p - 1}{\pi^{pi_r}} \in H(\mu_j)$$

iff

$$\text{ord}(u_{r,j}^p - 1) - pi_r \geq \mu_j'.$$

Since $\text{ord}(u_{r,j}^p - 1) \geq \min\{p \text{ord}(u_{r,j} - 1), \text{ord}(u_{r,j} - 1) + e\}$ where $e = \text{ord}(p)$, this follows if

$$\text{ord}(u_{r,j} - 1) \geq \mu_j' + pi_r - e$$

and

$$\text{ord}(u_{r,j} - 1) \geq \frac{\mu_j'}{p} + i_r$$

for $j = 1, \ldots, r - 1$. 
To show that $H_r$ is a Hopf order, that is, the comultiplication on $KG$ maps $H_r$ to $H_r \otimes H_r$, we need

$$\Delta(a_{r,1} \cdots a_{r,r-1}) \equiv a_{r,1} \cdots a_{r,r-1} \otimes a_{r,1} \cdots a_{r,r-1} \pmod{\pi^{i_r} H_{r-1} \otimes H_{r-1}}.$$ 

Now

$$\Delta(a_{r,1} \cdots a_{r,r-1}) = \Delta(a_{r,1}) \cdots \Delta(a_{r,r-1})$$

and

$$a_{r,1} \cdots a_{r,r-1} \otimes a_{r,1} \cdots a_{r,r-1} = (a_{r,1} \otimes a_{r,1}) \cdots (a_{r,r-1} \otimes a_{r,r-1}).$$

So it suffices that for each $j$,

$$\Delta(a_{r,j}) \equiv a_{r,j} \otimes a_{r,j} \pmod{\pi^{i_r} H_{r-1} \otimes H_{r-1}}.$$ 

But $a_{r,j} \in H_{r-1} \cap KG_j \supseteq H(\mu_j)$. So the comultiplication formula holds if

$$\Delta(a_{r,j}) \equiv a_{r,j} \otimes a_{r,j} \pmod{\pi^{i_r} H(\mu_j) \otimes H(\mu_j)},$$

which holds if $i_r \leq p \mu_j$ and

$$\text{ord}(a_{r,j} - 1) \geq \mu_j' + \frac{i_r}{p}$$

for all $j$, by Proposition 1.

To complete the inductive construction, we need an $r$th Larson parameter $\mu_r$ so that

$$H_r \cap K[G_r] \supseteq H(\mu_r).$$

We have

$$\frac{a_{r,1} a_{r,2} \cdots a_{r,r-1} \sigma_r - 1}{\pi^{i_r}}$$

$$= \frac{a_{r,1} - 1}{\pi^{i_r}} + \sum_{i=2}^{r-1} \frac{a_{r,1} a_{r,2} \cdots a_{r,i-1}}{\pi^{i_r}} (a_{r,i} - 1) + \frac{a_{r,1} a_{r,2} \cdots a_{r,r-1}}{\pi^{i_r}} (\sigma_r - 1).$$

Now

$$\frac{a_{r,1} a_{r,2} \cdots a_{r,r-1} \sigma_r - 1}{\pi^{i_r}}$$

is in $H_r$ if $\mu_r \leq i_r$. If

$$\frac{a_{r,k} - 1}{\pi^{i_r}} \in H_k$$
for $k = 1, \ldots, r - 1$, then the $a_{r,k}$ will be units of $H_r$, and so

$$\frac{\sigma_r - 1}{\pi^{\mu_r}} \in H_r,$$

hence

$$R \left[ \frac{\sigma_r - 1}{\pi^{\mu_r}} \right] \subseteq H_r \cap K[G_r].$$

Now since

$$R \left[ \frac{a_{r,k} - 1}{\pi^{\mu_r}} \right] = R \left[ \frac{\sigma_k - 1}{\pi^{\nu_k}} \right]$$

with $\text{ord}(u_{r,k} - 1) - \mu_r = e' - \nu_k$ by [15, Proposition 2.1], and

$$R \left[ \frac{\sigma_k - 1}{\pi^{\nu_k}} \right] \subseteq H_k$$

if $\nu_k \leq \mu_k$, it follows that

$$\frac{a_{r,k} - 1}{\pi^{\mu_r}} \in H_k$$

if $\text{ord}(u_{r,k} - 1) - \mu_r \geq e' - \mu_k$ for $k = 1, \ldots, r - 1$.

Thus we require $\mu_r$ to satisfy

$$\mu_r \leq i_r$$

and

$$\text{ord}(u_{r,k} - 1) - \mu_r \geq e' - \mu_k = \mu'_k$$

for $k = 1, \ldots, r - 1$.

To summarize:

**Theorem 3.** Given $i_1, \ldots, i_n$ with $0 \leq i_r \leq e'$ for all $r$ and $i_r \leq p \xi_s$ for $r > s$, suppose $U = (u_{r,s})$ is a lower triangular $n \times n$ matrix with entries in $R$ and diagonal entries $\xi$. Set $H_0 = R$ and for $1 \leq r \leq n$, define $H_r$ by $H_r = H_{r-1}[t_r]$, where

$$t_r = \frac{a_{r,1}a_{r,2} \cdots a_{r,r-1}\sigma_r - 1}{\pi^{i_r}}.$$

Then $H_r$ is free over $H_{r-1}$ with basis $\{1, t_r, \ldots, t_r^{n-1}\}$ and $H(U) = H_n$ is a Hopf order with Larson parameters $\mu_1 = i_1, \mu_2, \ldots, \mu_n$ if for all $r, s$ with $1 \leq s < r \leq n$:

$$\mu_r \leq i_r \leq p \mu_s$$
and

$$\text{ord}(u_{r,s} - 1) \geq \max \left\{ \frac{\mu'_s}{p} + i_r, \mu'_s + ir, \mu'_s + \mu_r, \mu'_s + pi_r - e \right\}.$$ 

Thus given a set $i_1, \ldots, i_n$ of valuation parameters satisfying the conditions of Theorem 3, the matrix $U$ yields a Hopf order if the valuations $\text{ord}(ur,s - 1)$ of the $n(n - 1)/2$ off-diagonal entries of $U$ and the $n - 1$ Larson parameters $\mu_1, \mu_2, \ldots, \mu_{n-1}$ satisfy a collection of $(5n^2 - 5n)/2$ linear inequalities.

2. Duality

Let $G = \langle \sigma_1 \rangle \times \cdots \times \langle \sigma_n \rangle$ and $\hat{G} = \langle \gamma_1 \rangle \times \cdots \times \langle \gamma_n \rangle$ with

$$\langle \sigma_r, \gamma_s \rangle = \begin{cases} 1, & \text{if } s \neq r, \\ \zeta, & \text{if } s = r. \end{cases}$$

Then $KG$ and $K\hat{G}$ are dual group rings.

Given $H(U)$ as in Theorem 3 with valuation parameters $i_1, \ldots, i_n$, let $i'_k = e'_k - i_k$ for $k = 1, \ldots, n$. Here is Theorem 3 for a potential dual:

**Theorem 4.** Given $i'_n, \ldots, i'_1$ with $0 \leq i'_s \leq e'$ for all $s$ and $i'_s \leq pi'_r$ for $s < r$, suppose $W = (w_{s,r})$ is an upper triangular $n \times n$ matrix with entries in $R$ and diagonal entries $\zeta$. Set $J_{n+1} = R$ and for $1 \leq s \leq n$, define $J_s$ by $H_s = H_{s+1}[q_s]$, where

$$q_s = \frac{b_{s,n}b_{s,n-1} \cdots b_{s,s+1}y_s - 1}{\pi'^s_s}.$$ 

Then $J_s$ is free over $J_{s+1}$ with basis $\{1, q_s, \ldots, q_s^{p-1}\}$ and $J(W) = J_1$ is a Hopf order with Larson parameters $\delta'_n = i_n, \delta'_{n-1}, \ldots, \delta'_1$ if for all $s, r$ with $1 \leq s < r \leq n$

$$\delta'_s \leq i'_s \leq p\delta'_r$$

and

$$\text{ord}(w_{s,r} - 1) \geq \max \left\{ \frac{\delta'_r}{p} + i'_s, \delta'_r + i'_s, \delta'_s, \delta'_r + pi'_s - e \right\}.$$ 

Assuming that $H(U), J(W)$ are Hopf orders in $KG$ and $K\hat{G}$, respectively, then since $H(U)$ has valuation parameters $i_1, \ldots, i_n$ and $J(W)$ has valuation parameters $i'_n, \ldots, i'_1$, $J(W)$ will be the dual of $H(U)$ iff $\langle H(U), J(W) \rangle \subset R$, by a routine discriminant argument. So we seek conditions on $U$ and $W$ so that $\langle H(U), J(W) \rangle \subset R$. 
For $x$ and $y$ units of $R$ the quantity

$$G(x, y) = \frac{1}{p} \sum_{0 \leq i, j \leq p-1} x^i \zeta_1^{-ij} y^j,$$

is the Gauss sum of $x$ and $y$ [8]. Note that $G(x, 1) = 1$. Also,

$$G(\zeta_1^k, w) = \frac{1}{p} \sum_{i,j=0}^{p-1} \zeta_1^{ki} \zeta_1^{-ij} w^j = \frac{1}{p} \sum_{j=0}^{p-1} \left( \sum_{i=0}^{p-1} \zeta_1^{(k-j)i} \right) w^j = w^k.$$

The Gauss sum arises in connection with duality because $G(x, y) = \langle ax, ay \rangle$ (where $ax \in K \cdot \mathbb{C}_p$, $ay \in K \hat{\cdot} \mathbb{C}_p$), as is easily verified (cf. [8]).

Since both $H(U)$ and $J(W)$ are assumed to be Hopf orders, for $\langle H(U), J(W) \rangle \subseteq R$ it suffices that the duality map applied to generators maps into $R$, that is, for all $r, s$,

$$\left\{ \frac{a_{r,1} a_{r,2} \cdots a_{r,r-1} \sigma_r - 1}{\pi^{ir}}, \frac{b_{s,n} b_{s,n-1} \cdots b_{s,s+1} \gamma_s - 1}{\pi^{is}} \right\} \subset R,$$

that is,

$$D_{r,s} - 1 := (a_{r,1} a_{r,2} \cdots a_{r,r-1} \sigma_r - 1, b_{s,n} b_{s,n-1} \cdots b_{s,s+1} \gamma_s - 1) - 1 \in \pi^{ir+is} R.$$

One sees easily that $D(r, r) = \zeta$ and $D_{r,s} = 1$ if $r < s$, and so $D_{r,s} - 1$ is in $\pi^{ir+is} R$ if $r \leq s$. For $r > s$,

$$D_{r,s} = (a_{r,s}, \gamma_s)(a_{r,s+1}, b_{s,s+1}) \cdots (a_{r,r-1}, b_{s,r-1})(a_{r,s}, \gamma_s)$$

$$= u_{r,s} G(u_{r,s+1}, w_{s,s+1}) \cdots G(u_{r,r-1}, w_{s,r-1}) w_{s,r}$$

(cf. [8, Lemma 2.1]).

In order to construct $W$ so that $D_{r,s} \equiv 1 \pmod{\pi^{ir+is}}$ for $r > s$, we make the assumptions on the entries of $U$:

$$\text{ord}(u_{r,s} - 1) > 0$$

for all $r > s$, and

$$\text{ord}(u_{r,k} - 1) + \text{ord}(u_{k,s} - 1) > e' + \text{ord}(u_{r,s} - 1)$$

for all $r > k > s$. The first assumption follows from the inequalities of Theorem 3 provided that $i_r > 0$. The second assumption implies that

$$\text{ord}(G(u_{r,k}, u_{k,s}) - 1) = \text{ord}(u_{r,k} - 1) + \text{ord}(u_{k,s} - 1) - e'$$

by [15, Proposition 2.3].
Assuming these inequalities, we define the off-diagonal entries of $W$ inductively.

For all $r$ define $w_{r-1,r}$ by

$$D_{r,r-1} = u_{r,r-1} w_{r-1,r} = 1.$$ 

Then $D_{r,r-1} - 1 \in \pi^{l_r'} R$ and $\text{ord}(u_{r,r-1} - 1) = \text{ord}(w_{r-1,r} - 1)$.

Define $w_{r-2,r}$ by

$$D_{r,r-2} = u_{r,r-2} G(u_{r,r-1}, w_{r-2,r-1}) w_{r-2,r} = 1.$$ 

This definition makes sense because

$$\text{ord}(u_{r,r-1} - 1) + \text{ord}(w_{r-2,r-1} - 1) - e' = \text{ord}(u_{r,r-1} - 1) + \text{ord}(w_{r-2,r-1} - 1) - e'$$

$$> \text{ord}(u_{r,r-2} - 1) > 0$$

and so

$$\text{ord}(G(u_{r,r-1}, w_{r-2,r-1}) - 1) = \text{ord}(u_{r,r-1} - 1) + \text{ord}(w_{r-2,r-1} - 1) - e'$$

$$> \text{ord}(u_{r,r-2} - 1),$$

hence both $G(u_{r,r-1}, w_{r-2,r-1})$ and $u_{r,r-2}$ are units. Also,

$$\text{ord}(u_{r,r-2} - 1) = \text{ord}(w_{r-2,r} - 1)$$

by the isosceles triangle inequality applied to the equation $0 = D_{r,r-2} - 1$, since

$$D_{r,r-2} - 1 = u_{r,r-2} G(u_{r,r-1}, w_{r-2,r-1})(w_{r-2,r} - 1)$$

$$+ u_{r,r-2} (G(u_{r,r-1}, w_{r-2,r-1}) - 1) + (u_{r,r-2} - 1).$$

Assume that $w_{s,r}$ has been defined for all $r, s$ with $r - s = d > 0$ so that $\text{ord}(w_{s,r} - 1) = \text{ord}(u_{r,s} - 1)$. We have

$$u_{r+1,s} G(u_{r+1,s+1}, w_{s,s+1}) \cdots G(u_{r+1,r}, w_{s,r}) w_{s,r+1} = 1.$$ 

Now for all $k$ with $r + 1 > k > s$,

$$\text{ord}(u_{r+1,k} - 1) + \text{ord}(w_{s,k} - 1) - e' = \text{ord}(u_{r+1,k} - 1) + \text{ord}(u_{k,s} - 1) - e'$$

$$> \text{ord}(u_{r+1,s} - 1) > 0,$$

hence for $r + 1 > k > s$, $G(u_{r+1,k}, w_{s,k})$ and $u_{r+1,s}$ are units of $R$. Therefore we may define $w_{s,r+1}$ by

$$D_{r+1,s} = u_{r+1,s} G(u_{r+1,s+1}, w_{s,s+1}) \cdots G(u_{r+1,r}, w_{s,r}) w_{s,r+1} = 1.$$
Since
\[
\text{ord}(G(u_{r+1, k}, w_{s, k}) - 1) = \text{ord}(u_{r+1, k} - 1) + \text{ord}(w_{s, k} - 1) - e' > \text{ord}(u_{r+1, s} - 1)
\]
for all \( k \) with \( r + 1 > k > s \), it follows that
\[
\text{ord}(u_{r+1, s} - 1) = \text{ord}(w_{s, r+1} - 1)
\]
by the isosceles triangle inequality.

In this way we may define the entries of \( W \) so that
\[
D_{r,s} - 1 = 0
\]
for all \( r > s \), and so we obtain a dual pair of Hopf orders, \( H(U) \) and \( J(W) \), provided that both \( H(U) \) and \( J(W) \) are Hopf orders.

We collect the needed inequalities for both \( H(U) \) and \( J(W) \) to be Hopf orders and duals of each other.

**Theorem 5.** Let \( i_1, \ldots, i_n \) be valuation parameters satisfying \( 0 \leq i_r \leq e' \) for all \( r \) and \( i_r \leq p_i \) and \( i'_s \leq p_i \) for all \( r > s \). Let \( U = (u_{r, s}) \) be a lower triangular \( n \times n \) matrix with entries in \( R \) and diagonal entries \( \zeta \). Define the upper triangular matrix \( W = (w_{s, r}) \) by \( w_{s,s} = \zeta \) and for \( r > s \),

\[
u_{r,s} G(u_{r,s+1}, w_{s,s+1}) \cdots G(u_{r,r-1}, w_{s,r-1}) w_{s,r} = 1.
\]

Then \( H(U) \) and \( J(W) \) are a dual pair of Hopf orders with Larson parameters \( \mu_1 = i_1, \mu_2, \ldots, \mu_n, \delta'_n = i'_n, \delta'_{n-1}, \ldots, \delta'_1 \) if the following inequalities hold for all \( 1 \leq s < k < r \leq n \):

\[
\begin{align*}
\mu_r &\leq i_r \leq p \mu_s, \quad \delta'_s \leq i'_s \leq p \delta'_r; \\
\text{ord}(u_{r,s} - 1) &\geq \max \left\{ 1, i_r + \frac{\mu'_s}{p}, \frac{\mu'_r}{p} + \mu'_s, \mu'_s + pi_r - e \right\}, \\
\text{ord}(u_{r,s} - 1) &\geq \max \left\{ \frac{\delta_r}{p} + i'_s, \delta_r + \frac{i'_s}{p}, \delta_r + pi'_s - e \right\}; \\
\text{ord}(u_{r,s} - 1) &\geq \max \left\{ \mu_r + \mu'_s, \delta_r + \delta'_s \right\}; \\
\text{ord}(u_{r,k} - 1) + \text{ord}(u_{k,s} - 1) &> e' + \text{ord}(u_{r,s} - 1).
\end{align*}
\]

We will give examples in Section 4.
3. Truncated exponentials

Now we adapt the construction of dual Hopf orders in $K C_p^n$ of Theorem 5 to generalize the construction of [8]. This construction uses the truncated exponential function,

$$\exp(x) = \sum_{k=0}^{p-1} \frac{x^k}{k!},$$

which behaves well with respect to duality. (We never explicitly use the untruncated series and so will not require special notation.) Here is the first of two results that facilitate the use of the truncated exponential:

**Lemma 6.** For $x, y \in \pi^l R$ with $l \geq 1$, we have

$$\exp(x + y) \equiv \exp(x) \exp(y) \pmod{\pi^{pl} R}.$$

**Proof.** As noted in [8, Remark 1.1], the proof is a matter of showing that the difference of the two sides can be written as a power series of order at least $p$ with coefficients in $R$ in which all terms have valuation $\geq l$. Define $\lambda \in R$ by the equation $\exp(\lambda) = \zeta$. (This can be done explicitly by solving $\exp(x) = \zeta$ for $x$ modulo higher and higher powers of $\pi$.) From the definition of $\exp(-)$, we get that $\ord(\lambda) = \ord(\zeta - 1) = e'$. Then for $x$ in $K$, $\ord(\exp(\lambda x) - 1) = \ord(x) + e'$. Thus $\exp(\lambda x)$ is a unit of $R$ iff $\ord(\lambda x) \geq 1$, iff $\ord(x) \geq -e' + 1$.

Let $Y = (y_{r,s})$ be a lower triangular matrix of elements of $K$ with diagonal entries $y_{r,r} = 1$ and $\ord(y_{r,s}) > -e'$ for all $r > s$. Then the matrix $U = (u_{r,s})$ with $u_{r,s} = \exp(\lambda y_{r,s})$ for $r \geq s$, $u_{r,s} = 0$ for $r < s$, is lower triangular with entries in $R$ and diagonal entries $\zeta$. The valuation conditions on $U$ in order that $H(U)$ be a Hopf order as in Theorem 3 translate immediately to valuation conditions on $Y$, since $\ord(u_{r,s} - 1) = \ord(y_{r,s}) + e'$.

Denote $H(U) = H^e(Y)$.

Similarly, let $Z = (z_{r,s})$ be an upper triangular matrix of elements of $K$ with diagonal entries $z_{r,r} = 1$ and $\ord(z_{s,r}) > -e'$ for all $s < r$. Then the matrix $W = (w_{s,r})$ with $w_{s,r} = \exp(\lambda z_{s,r})$ for $s \leq r$, $z_{s,r} = 0$ for $s > r$, is upper triangular with entries in $R$ and diagonal entries $\zeta$. The valuation conditions on $W$ in order that $J(W)$ be a Hopf order as in Theorem 4 translate immediately to valuation conditions on $Z$, since $\ord(w_{s,r} - 1) = \ord(z_{s,r}) + e'$. Denote $J(W) = J^e(Z)$.

The attractiveness of using matrices $U$ and $W$ with entries that are truncated exponentials of entries in $Y, Z$, respectively, is that the dual of $H^e(Y)$ is $J^e(Z)$ where the transpose of $Z$ is the inverse of $Y$.

Along with Lemma 6 we need for duality the following extension of a result on Gauss sums [8, Theorem 1.4]:
Theorem 7. Let $x, y$ be elements of $K$ with $\min\{\text{ord}(\lambda x), \text{ord}(\lambda y)\} = g$ where $0 < g \leq e'$. Then

$$G(\exp(\lambda x), \exp(\lambda y)) - \exp(\lambda xy) \in \pi^{(2p-1)g-(p-1)e'} R.$$ 

Proof. Let $P(X, Y) = G(\exp(\lambda X), \exp(\lambda Y))$. Theorem 1.4 of [8] asserts that if $X, Y$ are indeterminates, then the polynomial $Q(X, Y) = P(X, Y) - \exp(\lambda XY)$ satisfies

$$Q(X, Y) = \pi^{pe'} F(X, Y),$$

where $F(X, Y)$ is a polynomial with coefficients in $R$. Now

$$Q(X, Y) = \frac{1}{p} \sum_{i,j=0}^{p-1} \left( \sum_{m=0}^{p-1} \frac{\lambda X^m}{m!} \right)^i \zeta^{-ij} \left( \sum_{n=0}^{p-1} \frac{\lambda Y^n}{n!} \right)^j - \sum_{k=0}^{p-1} \frac{\lambda XY^k}{k!}$$

for some coefficients $s_{m,n} \in R$. Since $Q(X, Y)$ has coefficients in $\pi^{pe'} R$, it follows that for all $m + n = d \geq 0$,

$$\text{ord}(s_{m,n}) - \text{ord}(p) + de' \geq pe',$$

hence

$$\text{ord}(s_{m,n}) \geq (2p - 1 - d)e'.$$

Since $s_{m,n}$ is in $R$, we also have

$$\text{ord}(s_{m,n}) \geq 0$$

for all $m, n$.

Suppose $\min\{\text{ord}(\lambda x), \text{ord}(\lambda y)\} = g$ with $0 < g \leq e'$. Then for each $m, n$ with $m + n = d$,

$$\text{ord}\left( \frac{1}{p} s_{m,n} (\lambda X)^m (\lambda Y)^n \right) \geq -(p-1)e' + gd + ((2p-1) - d)e'$$

for $d \leq 2p - 1$, and

$$\text{ord}\left( \frac{1}{p} s_{m,n} (\lambda X)^m (\lambda Y)^n \right) \geq -(p-1)e' + gd$$

for $d \geq 2p - 1$. Thus the term with minimal valuation in $Q(x, y)$ has valuation $\geq -(p-1)e' + (2p-1)g$, completing the proof. \(\Box\)
Now we repeat the construction in Section 2.

Assuming that \( H^e(Y), J^e(Z) \) are Hopf orders in \( KG \) and \( K\hat{G} \), respectively, then by the choice of denominators (valuation parameters), \( J^e(Z) \) will be the dual of \( H^e(Y) \) iff \( \langle H^e(Y), J^e(Z) \rangle \subset R \). Since both are Hopf orders, it suffices that the duality map applied to generators lands in \( R \), that is, for all \( r, s \),

\[
\left( \frac{a_{r,1}a_{r,2}\cdots a_{r,r-1}\sigma_r - 1}{\pi^{i_r}}, \frac{b_{s,n}b_{s,n-1}\cdots b_{s,s+1}\gamma_s - 1}{\pi^{i'_s}} \right) \subset R.
\]

As before, set

\[
D_{r,s} = \langle a_{r,1}a_{r,2}\cdots a_{r,r-1}\sigma_r, b_{s,n}b_{s,n-1}\cdots b_{s,s+1}\gamma_s \rangle.
\]

We require that

\[
D_{r,s} - 1 \in \pi^{i_r+i'_s}R.
\]

One sees easily that \( D_{r,s} = 1 \) if \( r < s \), and \( = \zeta \) if \( r = s \). For \( r > s \),

\[
D_{r,s} = \langle a_{r,s}\gamma_s \rangle a_{r,s+1}b_{s,s+1} \cdots a_{r,r-1}b_{s,r-1}\langle a_{r,s}\gamma_s \rangle
\]

\[= P(y_{r,s}, z_{s,s}) P(y_{r,s+1}, z_{s,s+1}) \cdots P(y_{r,r-1}, z_{s,r-1}) P(y_{r,r}, z_{s,r})
\]

(cf. [8, Lemma 2.1]), where \( P(X, Y) = G(\exp(\lambda X), \exp(\lambda Y)) \) as in Theorem 7. Thus we want

\[
P(y_{r,s}, z_{s,s}) P(y_{r,s+1}, z_{s,s+1}) \cdots P(y_{r,r-1}, z_{s,r-1}) P(y_{r,r}, z_{s,r}) \in \pi^{i_r+i'_s}R
\]

for all \( r > s \).

As in the previous section, in order to construct \( Z \) so that \( D_{r,s} \equiv 1 \mod \pi^{i_r+i'_s}R \) for \( r > s \), we assume that for all \( r > k > s \), we have

\[
\text{ord}(y_{r,k}) + \text{ord}(y_{k,s}) > \text{ord}(y_{r,s}).
\]

For \( u_{r,s} = \exp(\lambda y_{r,s}) \), this assumption is equivalent to the assumption

\[
\text{ord}(u_{r,k} - 1) + \text{ord}(u_{k,s} - 1) > e' + \text{ord}(u_{r,s} - 1)
\]

made in the construction of Theorem 5.

**Proposition 8.** Suppose \( Z' = Y^{-1} \), and assume that for all \( r > k > s \),

\[
\text{ord}(y_{r,s}) < \text{ord}(y_{r,k}) + \text{ord}(y_{k,s}).
\]

Then for all \( r > s \), \( \text{ord}(z_{s,r}) = \text{ord}(y_{r,s}) \).
**Proof.** For \( r > s \) we have

\[
y_{r,s} + y_{r,s+1}z_{s,s+1} + \cdots + y_{r,r-1}z_{s,r-1} + z_{s,r} = 0.
\]

Thus \( \text{ord}(y_{r,r-1}) = \text{ord}(z_{r-1,r}) \) for all \( r \). Proceeding by induction, assume \( \text{ord}(z_{t,r}) = \text{ord}(y_{r,t}) \) for \( r - t < r - s \), then

\[
\text{ord}(y_{r,s}) < \text{ord}(y_{r,k}) + \text{ord}(y_{k,s}) = \text{ord}(y_{r,k}z_{s,k})
\]

for \( k = s + 1, \ldots, r - 1 \), so by the isosceles triangle inequality we have

\[
\text{ord}(y_{r,s}) = \text{ord}(z_{s,r}).
\]

Assume the entries of \( Y \) satisfy the hypotheses of Proposition 8. Then by Theorem 7, for all \( r > k > s \),

\[
P(y_{r,k}, z_{s,k}) \equiv \exp(\lambda y_{r,k}z_{s,k}) \pmod{\pi^{(2p-1)g_{r,k,s}-(p-1)e'}}
\]

where

\[
g_{r,k,s} = \min\{\text{ord}(\lambda y_{r,k}), \text{ord}(\lambda z_{s,k})\} = \min\{\text{ord}(\lambda y_{r,k}), \text{ord}(\lambda y_{k,s})\}
\]

by Proposition 8. So assume

\[
(2p - 1)\text{ord}(\lambda y_{r,k}) - (p - 1)e' \geq i_r + i'_s
\]

and

\[
(2p - 1)\text{ord}(\lambda y_{k,s}) - (p - 1)e' \geq i_r + i'_s
\]

for all \( r > k > s \). Then modulo \( \pi^{i_r+i'_s} R \)

\[
D_{r,s} = P(y_{r,s}, z_{s,s})P(y_{r,s+1}, z_{s,s+1}) \cdots P(y_{r,r-1}, z_{s,r-1})P(y_{r,r}, z_{s,r})
\]

\[
\equiv \exp(\lambda y_{r,s}) \exp(\lambda y_{r,s+1}z_{s,s+1}) \cdots \exp(\lambda y_{r,r-1}z_{s,r-1}) \exp(\lambda z_{s,r}).
\]

Now we apply Lemma 6: assume that for all \( r \geq k \geq s \),

\[
\text{ord}(\lambda y_{r,k}z_{s,k}) \geq \frac{i_r + i'_s}{p}.
\]

Then

\[
\exp(\lambda y_{r,s}) \exp(\lambda y_{r,s+1}z_{s,s+1}) \cdots \exp(\lambda y_{r,r-1}z_{s,r-1}) \exp(\lambda z_{s,r})
\]

\[
\equiv \exp\left(\lambda (y_{r,s} + y_{r,s+1}z_{s,s+1} + \cdots + y_{r,r-1}z_{s,r-1} + z_{s,r})\right)
\]

\[
= \exp(0) = 1 \pmod{\pi^{i_r+i'_s}}
\]
and so for all $r > s$, 

$$D_{r,s} - 1 \in \pi^{i_r+i'_s} R,$$

which implies that

$$\langle H^e(Y), J^e(Z) \rangle \subseteq R.$$

We have shown nearly all of

**Theorem 9.** Suppose $Y = (y_{r,s})$ is an $n \times n$ lower triangular matrix with entries in $\pi^{-e'+1} R$ and diagonal entries 1. Suppose for all $r > k > s$,

$$\text{ord}(y_{r,k}) + \text{ord}(y_{k,s}) > \text{ord}(y_{r,s}).$$

Let $Z^t = Y^{-1}$, let $U = (u_{r,s})$ be lower triangular with $u_{r,s} = \exp(\lambda_{y_{r,s}})$ for $r \geq s$, and let $W = (w_{s,r})$ be upper triangular with $w_{s,r} = \exp(\lambda_{z_{s,r}})$ for $r \geq s$. If $\{u_{r,s}, \mu_{r}, \delta'_{s}\}$ satisfy the inequalities of Theorem 5 together with the inequalities:

\[
(2p - 1)\text{ord}(u_{r,k} - 1) \geq (p - 1)e' + i_r + i'_s,
\]

\[
(2p - 1)\text{ord}(u_{k,s} - 1) \geq (p - 1)e' + i_r + i'_s
\]

for all $r > k > s$, then $J(W)$ and $H(U)$ are a dual pair of Hopf orders.

**Proof.** Most of this result follows from Theorem 5 and the discussion just above the statement of Theorem 9. The only remaining observation to make is that the inequality

$$\text{ord}(\lambda_{y_{r,k}z_{s,k}}) \geq \frac{i_r + i'_s}{p}$$

required to apply Lemma 6 is equivalent to

$$\text{ord}(u_{r,k} - 1) + \text{ord}(u_{k,s} - 1) - e' \geq \frac{i_r + i'_s}{p},$$

and that follows from

$$\text{ord}(u_{r,k} - 1) + \text{ord}(u_{k,s} - 1) - e' > \text{ord}(u_{r,s} - 1)$$

and the inequalities

$$\text{ord}(u_{r,s} - 1) \geq \frac{i_r}{p} + \mu'_{s}$$
and

$$\mu_s \leq i_s,$$

for then

$$\frac{i_r}{p} + \mu'_s \geq \frac{i_r}{p} + i'_s \geq \frac{i_r}{p} + i'_s .$$

**Remark 10.** The extra inequalities

$$(2p - 1)\text{ord}(u_{r,k} - 1) \geq (p - 1)e' + i_r + i'_s,$$

$$(2p - 1)\text{ord}(u_{k,s} - 1) \geq (p - 1)e' + i_r + i'_s$$

of Theorem 9 impose a mild extra restriction on the orders of the elements of $Y$ beyond the inequalities of Theorem 5. They may be restated as

$$\text{ord}(u_{r,k} - 1) \geq \frac{(p - 1)e' + i_r + i'_s}{2p - 1},$$

$$\text{ord}(u_{k,s} - 1) \geq \frac{(p - 1)e' + i_r + i'_s}{2p - 1},$$

which, when added, yield

$$\text{ord}(u_{r,k} - 1) + \text{ord}(u_{k,s} - 1) \geq 2 \frac{(p - 1)e' + i_r + i'_s}{2p - 1}$$

an inequality that follows from the inequality

$$\text{ord}(u_{r,s} - 1) \geq \frac{i_r}{p} + \frac{i'_s}{p}$$

found above, since

$$\text{ord}(u_{r,k} - 1) + \text{ord}(u_{k,s} - 1)$$

$$\geq e' + \text{ord}(u_{r,s} - 1) \geq e' + \frac{i_r}{p} + \frac{i'_s}{p}$$

$$= \frac{2(p - 1)}{2p - 1} e' + \frac{1}{2p - 1} e' + \frac{2p(i_r + i'_s)}{p(2p - 1)} - \frac{(i_r + i'_s)}{p(2p - 1)}$$

$$\geq \frac{2(p - 1)}{2p - 1} e' + \frac{2(i_r + i'_s)}{2p - 1}.$$
Remark 11. Since the inequalities in Theorem 9 are all linear, one can use the simplex algorithm to construct examples. With \( n = 5 \) there are 14 variables and 50 inequality constraints from Theorem 3, 18 variables and 110 constraints from Theorem 5, and 18 variables and 130 constraints from Theorem 9. Picking the sum of some or all of the \( \text{ord}(u_{r,s} - 1) \) as the objective function \( S \), it is a routine computer computation to find the minimum value of \( S \) subject to the inequalities of Theorem 3 as constraints, and not difficult to find examples where the minimum value of \( S \) increases when the additional inequalities of Theorem 5 are imposed; and similarly when going from the inequalities of Theorem 5 to those of Theorem 9. In this way one may verify that the family of examples obtained by Theorem 3 is strictly larger than the family obtained from Theorem 5, which in turn is strictly larger than that obtained from Theorem 9.

4. The landscape of Hopf orders

The four known general classes of Hopf orders in \( KC_n^p \) for \( n > 2 \) are those of Raynaud [10], Larson [9], Childs and Sauerberg [4], and Greither and Childs [8]. In this section we show that the Hopf orders constructed in the previous sections include examples not in these known families.

Larson orders

A Larson order in \( KC_p^n \) is a Hopf order of the form

\[
L = R \left[ \frac{\sigma_1 - 1}{\pi^{i_1}}, \frac{\sigma_2 - 1}{\pi^{i_2}}, \ldots, \frac{\sigma_n - 1}{\pi^{i_n}} \right].
\]

If \( H(U) \) is a Hopf order obtained as in Theorem 3, then \( H(U) \) is a Larson order if \( U \) is diagonal. The dual of a Larson order is also Larson, as can be seen either directly or from Theorem 5. We note the following criterion for a triangular Hopf order of rank \( p^2 \) to be Larson.

Proposition 12. For \( j > 0 \), the Hopf order \( H = R[(\sigma - 1)/\pi^{i_1}, (a_v \tau - 1)/\pi^{j}] \) is a Larson order iff \( \text{ord}(v - 1) \geq i' + j \).

This criterion is more or less well known (certainly in the cyclic case), but for convenience we sketch a proof:

Proof. If \( H \) is Larson, \( H = R[(\sigma - 1)/\pi^{i_1}, (\tau - 1)/\pi^{v}] \) for some \( v \), then \( v = j \) by a discriminant argument. Also, if \( R[(\sigma - 1)/\pi^{i_1}, (\tau - 1)/\pi^{j}] \subseteq H \), then we have equality by a discriminant argument. We have

\[
\text{ord}(v - 1) \geq i' + j
\]
iff

\[ R \left[ \frac{a_v - 1}{\pi^j} \right] \subseteq R \left[ \frac{\sigma - 1}{\pi^i} \right] \]

by Proposition 1. Now

\[ \frac{a_v \tau - 1}{\pi^j} = a_v \left( \frac{\tau - 1}{\pi^j} \right) + \frac{a_v - 1}{\pi^j}, \]

so if \( \text{ord}(v - 1) \geq i' + j \), then \( (a_v - 1)/\pi^j \) is in \( R[(\sigma - 1)/\pi^i] \), hence \( a_v \) is a unit of \( R[(\sigma - 1)/\pi^i] \) and so \( (\tau - 1)/\pi^j \) is in \( H \). Hence \( H \) is Larson.

Conversely, if \( H = R[(\sigma - 1)/\pi^i, (\tau - 1)/\pi^j] \), then

\[ \tau^{-1} \left( \frac{a_v \tau - 1}{\pi^j} \right) - \tau^{-1} \left( \frac{\tau - 1}{\pi^j} \right) = \frac{a_v - 1}{\pi^j}, \]

and so \( (a_v - 1)/\pi^j \) is in \( H \cap K[\sigma] = R[(\sigma - 1)/\pi^i] \), and hence by Proposition 1 we have \( \text{ord}(v - 1) \geq i' + j \). □

**Polynomial formal group Hopf orders**

[4] constructed Hopf orders in \( KC_n^p \) starting from a suitable \( n \times n \) lower triangular matrix \( \Theta \) in \( M_n(R) \) that is used to deform the \( n \)-dimensional multiplicative formal group to a degree 2 polynomial formal group \( F_\Theta \) and to further deform the endomorphism \([p]_{F_\Theta}\) into an isogeny whose kernel is represented by the Hopf order

\[ H_\Theta = R[h_1, h_2, \ldots, h_n] \]

with

\[ h_r = u_{r,1}(\sigma_1 - 1) + u_{r,2}(\sigma_2 - 1) + \cdots + u_{r,r}(\sigma_r - 1). \]

Here \( U = (u_{r,s}) \) is the inverse of \( \Theta \).

If \( \Theta \) is diagonal, then \( H_\Theta \) is a Larson order.

One can show that given \( H_\Theta \) arising from the polynomial formal group associated to \( \Theta \), then \( H_\Theta \cap K[G_1 \times \cdots \times G_r] \) is the Hopf order arising from the \( r \times r \) principal submatrix \( \Theta_r \) of \( \Theta \). Thus if \( H = H_n \) is a Hopf order in \( KC_n^p \) arising from a polynomial formal group, then so is \( H_2 \). But since Hopf orders of rank \( p^2 \) are classified, \( H_2 \) must be isomorphic to a Hopf order of the form

\[ H(i, j, u) = R\left[ \frac{\sigma_1 - 1}{\pi^i}, \frac{a_u \sigma_2 - 1}{\pi^j} \right] \]

for some \( u \). Then [16, Theorem 4.3] gives
Proposition 13. If $H(i, j, u) = R[\frac{\sigma_1 - 1}{\pi^i}, \frac{a_2 \sigma_2 - 1}{\pi^j}]$ is constructed via a polynomial formal group as in [4], then
\[ \text{ord}(u - 1) \geq i' + \frac{j'}{2}. \]
(The lower bound $i' + j/2$ also appears in the cyclic case—see [5, Theorem 3.0].)

Truncated exponential Hopf orders

The truncated exponential construction of Hopf orders of [8] is essentially similar to that of Theorem 9, but with restrictions. The construction in [8] assumes that the valuation parameters are increasing: $i_1 \leq i_2 \leq \cdots \leq i_n$, and $i'_1 \leq p_i'^i$, but more significantly, requires that the matrix $Y$ in Theorem 9 is in $GL_n(R)$, hence $\text{ord}(u_{r,s} - 1) \geq e'$ for all $r, s$.

Raynaud orders

[8, Sections 4–6] gives the basic theory of the Raynaud orders constructed in [10], and shows that there is a bijection between Raynaud orders and $n$-tuples of numbers $j_1, j_2, \ldots, j_n$ with $0 \leq j_k \leq e'$, essentially valuation parameters associated with the Raynaud order. If those invariants satisfy $j_k \leq p_j^j$ for all $1 \leq k, l \leq n$, then [8, Corollary 6.2] shows that using certain special matrices $Y$ in $GL_n(R)$, the truncated exponential construction of [8] constructs the corresponding Raynaud order. Thus given suitable valuation parameters, Raynaud orders are a subset of truncated exponential Hopf orders.

Examples based on Theorem 3

For $G$ of order $p^2$, Theorem 3 reduces to Greither’s construction of [7], which yields all Hopf orders in $KC^2_p$. So to look at the possibilities allowed by Theorem 3, we need $G$ of order $p^n$, $n \geq 3$.

As a first example, we show that there are Hopf orders of rank $p^3$ constructible by Theorem 3 that are not polynomial formal group Hopf orders. Recall that $G = G_1 \times G_2 \times \cdots \times G_n$ with $G_r = \langle \sigma_r \rangle$.

Proposition 14. Let $n = 3$. If $i_2 \leq i_3 \leq e'/2$, then there exist Hopf orders $H$ in $KC^3_p$ constructed as in Theorem 3 that are not polynomial formal group Hopf orders.

Proof. If $H$ arises from a polynomial formal group, then
\[ H_2 = H \cap K(C_1 \times C_2) = R \left[ \frac{\sigma_1 - 1}{\pi^{i_1}}, \frac{a_{2,1} \sigma_2 - 1}{\pi^{i_2}} \right] \]
also arises from a polynomial formal group. Hence to show $H$ is not a polynomial formal group Hopf order, it suffices to find $u_{2,1}$ with $\text{ord}(u_{2,1} - 1) < i'_1 + i_2/2$.

With $n = 3$, the only Larson parameter is $\mu_2$ such that $i_3/p \leq \mu_2 \leq i_2$. Set
\[
\text{ord}(u_{2,1} - 1) = i_1' + i_2 - \varphi_{2,1}, \\
\text{ord}(u_{3,2} - 1) = i_2' + i_3 - \varphi_{3,2}.
\]

We wish to find examples with \( \varphi_{2,1} > i_2/2 \).

The four inequalities of Theorem 3 involving \( \text{ord}(u_{r,r-1} - 1) \) and \( \mu_2 \) are

\[
\begin{align*}
  i_2' + i_3 - \varphi_{3,2} &\geq \frac{\mu_2}{p} + i_3, \\
  i_2' + i_3 - \varphi_{3,2} &\geq \mu_2' + \frac{i_3}{p}, \\
  i_2' + i_3 - \varphi_{3,2} &\geq \mu_2' + pi_3 - e, \\
  i_1' + i_2 - \varphi_{2,1} &\geq i_1' + \mu_2
\end{align*}
\]

which respectively yield the inequalities

\[
\begin{align*}
  i_2 - \mu_2 \leq (p - 1)i_2' - p\varphi_{3,2}, \\
  i_2 - \mu_2 \leq \left( \frac{p - 1}{p} \right)i_3 - \varphi_{3,2}, \\
  i_2 - \mu_2 \leq (p - 1)i_3' - \varphi_{3,2}, \\
  i_2 - \mu_2 \geq \varphi_{2,1}.
\end{align*}
\]

Thus \( \varphi_{2,1}, \varphi_{3,2} \) are constrained by the inequalities

\[
\begin{align*}
  (p - 1)i_2' &\geq p\varphi_{3,2} + \varphi_{2,1}, \\
  \left( \frac{p - 1}{p} \right)i_3 &\geq \varphi_{3,2} + \varphi_{2,1}, \\
  (p - 1)i_3' &\geq p\varphi_{3,2} + \varphi_{2,1}.
\end{align*}
\]

For \( \varphi_{3,2} = 0 \) these yield

\[
\begin{align*}
  (p - 1)i_2' &\geq \varphi_{2,1}, \\
  \left( \frac{p - 1}{p} \right)i_3 &\geq \varphi_{2,1}, \\
  (p - 1)i_3' &\geq \varphi_{2,1}.
\end{align*}
\]

The hypotheses of the proposition imply that \(( (p - 1)/p )i_3 > i_2/2 + 1, \ i_3' > i_3 \) and \((p - 1)i_2' \geq i_2/2 + 1 \), so we can set \( \varphi_{2,1} > i_2/2 \) and satisfy the inequalities. □
In general, a Hopf order \( H \) in \( KG \) is realizable if there exists a Galois extension \( L \) of \( K \) with Galois group \( G \) so that the valuation ring \( S \) of \( L \) is \( H \)-Galois over \( R \). For any rank \( p^2 \) Hopf order \( H = R[(\sigma - 1)/\pi^i, (a_u\tau - 1)/\pi^j] \), the unit parameter \( u \) must satisfy

\[
\text{ord}(u - 1) \geq i' + \frac{j}{p}.
\]

If \( \text{ord}(u - 1) \geq i' + j \), then \( H \) is a Larson order. If \( \text{ord}(u - 1) = i' + j/p \), then \( H \) is realizable. Thus the condition that a Hopf order be realizable imposes the most stringent condition on \( \text{ord}(u - 1) \).

**Proposition 15.** If \( n = 3 \) and \( 0 < i_1, i_2, i_3 < e' \), then no Hopf order \( H_3 \) constructed by Theorem 3 is realizable.

**Proof.** Since \( i_1, i_2, i_3 < e' \), \( R[(\sigma_k - 1)/\pi^{i_k}] \) is a local ring for \( k = 1, 2, 3 \) (since modulo \( \pi \), \( \alpha_k = (\sigma_k - 1)/\pi^{i_k} \) is nilpotent), hence by [3, (29.1)], \( H_3 \) is a local ring. Suppose there exists a totally ramified Galois extension \( L \) of \( K \) with Galois group \( G = G_1 \times G_2 \times G_3 \) such that the valuation ring \( S \) of \( L \) is \( H_3 \)-Galois. If we set \( L_2 = L^{G_3} \), \( L_1 = L^{G_3 \times G_2} \), and \( S_2 = S^{G_1} \), \( S_1 = S^{G_3 \times G_2} \), then \( S \) is \( S_1 \otimes H_2 \)-Galois over \( S_1 \) and \( S_2 \) is \( \overline{H}_3 \)-Galois over \( R \) by [3, (28.1), (29.1) and (14.7)]. (Note: [3, (14.7), (1)] should read “\( S \) is a tame \( H \)-extension of \( R \).”)

Now

\[
\overline{H}_3 \cong R \left[ \frac{\sigma_2 - 1}{\pi^{i_2}}, \frac{a_{3,2}\sigma_3 - 1}{\pi^{i_3}} \right]
\]

and

\[
H_2 \cong R \left[ \frac{\sigma_1 - 1}{\pi^{i_1}}, \frac{a_{2,1}\sigma_2 - 1}{\pi^{i_2}} \right],
\]

hence

\[
S \otimes H_2 \cong S_1 \left[ \frac{\sigma_1 - 1}{\pi^{i_1}}, \frac{a_{2,1}\sigma_2 - 1}{\pi^{i_2}} \right],
\]

where \( \pi_1 \) is a parameter for \( S_1 \). If \( S \) is \( S_1 \otimes H_2 \)-Galois, then by [2], \( \text{ord}_{L_1}(u_{2,1} - 1) = pi_1 + i_2 \), hence \( \text{ord}(u_{2,1} - 1) = i_1 + i_2/p \).

If \( S_2 \) is \( \overline{H}_3 \)-Galois, then \( \text{ord}(u_{3,2} - 1) = i_2' + i_3/p \). But then, in the proof of Proposition 14, we have

\[
\varphi_{3,2} = \left( \frac{p - 1}{p} \right) i_3,
\]

so \( \varphi_{2,1} = 0 \), hence \( \text{ord}(u_{2,1} - 1) = i'_1 + i_2 \) (so \( H_2 \) is Larson). Thus \( S \) cannot be \( S_1 \otimes H_2 \)-Galois, and so \( S \) is not \( H_3 \)-Galois. Hence \( H_3 \) is not realizable. \( \square \)
Grand Larsony

As observed in Proposition 12, if
\[ \text{ord}(u_{r,r-1} - 1) \geq i'_{r-1} + i_r, \]
then
\[ R \left[ \frac{\sigma_{r-1} - 1}{\pi^{i_{r-1}}}, \frac{a_{r,r-1} \sigma_r - 1}{\pi^{i_r}} \right] \]
is Larson. The proof of Proposition 15 suggests that one way to show the non-triviality of the constructions of Theorems 5 and 9 is to show that we may construct examples without rank \( p^2 \) subHopf algebras or quotient Hopf algebras that are Larson. We now proceed to do this.

We begin by identifying the rank \( p^2 \) subquotients we will examine.

Recall from Theorems 3 and 4 that \( H(U) = H_n, J(W) = J_n \) where \( H_r = R[t_1, \ldots, t_r], J_s = R[q_n, \ldots, q_s] \) with
\[ t_r = \frac{a_{r,1}a_{r,2} \cdots a_{r,r-1} \sigma_r - 1}{\pi^{i_r}}, \]
\[ q_s = \frac{b_{s,n}b_{s,n-1} \cdots b_{s,s+1}y_s - 1}{\pi^{i_s}}. \]

For \( 1 \leq k < r \) let \( \overline{H}_{k,r} \) be the image of \( H_r \) under the map from \( K(G_1 \times \cdots \times G_r) \) to \( K(G_k \times G_r) \) given by sending \( \sigma_{\mu} \) to 1 for \( \mu < r, \mu \neq k \). Then
\[ t_1, \ldots, t_{k-1} \mapsto 0, \]
\[ t_k \mapsto \frac{\sigma_k - 1}{\pi^{i_k}}, \]
\[ t_r \mapsto \frac{a_{r,k} \sigma_r - 1}{\pi^{i_r}}, \]
and for \( k < \mu < r \),
\[ t_\mu \mapsto \frac{a_{\mu,k} - 1}{\pi^{i_\mu}}, \]
and so
\[ \overline{H}_{k,r} \cong R \left[ \frac{\sigma_k - 1}{\pi^{i_k}}, \frac{a_{k+1,k} - 1}{\pi^{i_k+1}}, \ldots, \frac{a_{r-1,k} - 1}{\pi^{i_{r-1}}}, \frac{a_{r,k} \sigma_r - 1}{\pi^{i_r}} \right]. \]
Similarly, for $n \geq l > s$ we have $\overline{J}_{l,s}$ obtained from $J_s$ by sending $\gamma_v$ to 1 for $v \neq l, s$, and so

$$\overline{J}_{l,s} \cong R \left[ \frac{\gamma_l - 1}{\pi^{i_l}}, \frac{b_{l,s} - 1}{\pi^{i_l}}, \ldots, \frac{b_{l,s} \gamma_s - 1}{\pi^{i_s}} \right].$$

If

$$v_{l,k} = \text{ord}(u_{l,k} - 1) < i_l + e',$$

then by [15, Corollary 2.2],

$$\overline{H}_{k,r} \cong R \left[ \frac{\sigma_k - 1}{\pi^{u_{k,r}}}, \frac{a_{r,k} \sigma_r - 1}{\pi^{i_r}} \right],$$

with

$$\alpha_{k,r} = \max \{i_k, i_{k+1} + v_{k+1,k}, \ldots, i_r - 1 + v_{r-1,k} \},$$

that is,

$$\alpha'_{k,r} = \min \{i'_k, v_{k+1,k} - i_{k+1}, \ldots, v_{r-1,k} - i_{r-1} \}. $$

Similarly,

$$\overline{J}_{l,s} \cong R \left[ \frac{\gamma_l - 1}{\pi^{i'_l}}, \frac{b_{l,s} \gamma_s - 1}{\pi^{i'_s}} \right]$$

with

$$\beta'_{l,s} = \max \{i'_l, i'_{l-1} + v'_{l-1,l}, \ldots, i'_{s+1} + v'_{l,s+1} \},$$

that is,

$$\beta_{l,s} = \min \{i_l, v_{l-1,l} - i'_{l-1}, \ldots, v_{l,s+1} - i'_{s+1} \}. $$

We’ll say that $H(U)$, $J(W)$ are Larsonless if none of the rank $p^2$ subquotients $\overline{H}_{k,r}$, $\overline{J}_{l,s}$ of $H(U)$, $J(W)$ are Larson orders. Then $H(U)$ and $J(W)$ are Larsonless iff the inequalities:

$$v_{r,k} < i_r + \alpha'_{k,r},$$

$$v_{s,l} < i'_s + \beta'_{l,s}$$

hold for all $n \geq s > l \geq 1$ and all $1 \leq k < r \leq n$. In particular, for $k = r - 1$, $l = s + 1$ we require

$$v_{r,r-1} < i_{r-1} + i'_r.$$
for $H$, and

$$v_{s+1,s} < i_s + i'_{s+1}$$

for $J$ (which is the same condition as for $H$).

To construct Larsonless examples, it is convenient to rewrite the inequalities of Theorem 5.

Let $v_{r,s} = \text{ord}(u_{r,s} - 1)$. For all $r > s$ and all $r > k > s$, Theorem 5 requires the following inequalities on the $i_j, \mu_j$ and $\delta'_j$:

$$i_r \leq p \mu_s, \quad \mu_r \leq i_r, \quad i'_s \leq p \delta'_r, \quad \delta'_s \leq i'_s;$$

(1)

also the following inequalities involving the $v_{r,s}$:

$$v_{r,s} \geq \begin{cases} 1, \\ i_r + \mu'_s, \\ \frac{i_r}{p} + \mu'_s, \\ \frac{p i_r + \mu'_s - e}{p}, \\ \frac{\delta_r + i'_s}{p}, \\ \frac{\delta_r + i'_s}{p} - e, \\ \delta_r + p i'_s - e, \\ \mu_r + \mu'_s, \\ \delta_r + \delta'_s \end{cases}$$

(2)

and

$$v_{r,k} + v_{k,s} > e' + v_{r,s}$$

(4)

(omit inequalities involving $\delta'_1$ and $\mu_n$; note also, $\delta'_n = i'_n, \mu_1 = i_1$).

Set

$$\lambda_s = i_s - \mu_s, \quad \theta_s = i'_s - \delta'_s$$

for $r > s$. Then inequalities (1) require that $\lambda_s \geq 0$ and $\theta_s \geq 0$ for $2 \leq s \leq n - 1$, and also

$$\lambda_s \leq i_s - \frac{i_r}{p}, \quad \theta_r \leq i'_r - \frac{i'_s}{p}$$

for $r > s$. 
Also, set
\[ v_{r,s} = i_r + i'_s - \varphi_{r,s} \]
as in the proof of Proposition 14.

We rewrite the inequalities (2) and (3) of Theorem 5 in terms of the \( \varphi_{r,s}, \lambda_s \) and \( \theta_s \) for all \( r > s \) as follows:

\[
\begin{align*}
\lambda_s &\leq p \left( \frac{p-1}{p} i'_s - \varphi_{r,s} \right), \\
\lambda_s &\leq \left( \frac{p-1}{p} i_r - \varphi_{r,s} \right), \\
\lambda_s &\leq (p-1)i'_r - \varphi_{r,s}, \\
\theta_r &\leq p \left( \frac{p-1}{p} i_r - \varphi_{r,s} \right), \\
\theta_r &\leq \left( \frac{p-1}{p} i'_r - \varphi_{r,s} \right), \\
\theta_r &\leq (p-1)i_s - \varphi_{r,s}
\end{align*}
\]

and

\[
\begin{align*}
\lambda_r &\geq \lambda_s + \varphi_{r,s}, \\
\theta_s &\geq \theta_r + \varphi_{r,s}.
\end{align*}
\]

Inequalities (4) become
\[
i_r + i'_k - \varphi_{r,k} + i_k + i'_s - \varphi_{k,s} > e' + i_r + i'_s - \varphi_{r,s},
\]
which is the same as
\[
\varphi_{r,k} + \varphi_{k,s} < \varphi_{r,s}
\]
for \( r > k > s \). We wish to satisfy these inequalities with \( \lambda_s > 0, \theta_s > 0 \) and \( \varphi_{r,r-1} > 0 \), and we wish to satisfy the Larsonless inequalities (5) as well.

**Example 16.** Let \( p \geq 3 \), let
\[
\frac{e'}{3} < i_1, \ldots, i_n < \frac{2e'}{3},
\]
then
\[
\frac{e'}{3} < i'_1, \ldots, i'_n < \frac{2e'}{3}.
\]
Choose
\[ \varphi_{r,s} = h(r - s) - 1 \]
for \( r > s \), a minimal solution to \((4')\) with all \( \varphi_{r,s} > 0 \). Let \( \lambda_2 = \lambda, \theta_{n-1} = \theta \), and for \( r \geq 3, s \leq n - 2 \), let
\begin{align*}
\lambda_s &= \lambda_2 + \varphi_{s,2} = \lambda + h(s - 2) - 1, \\
\theta_r &= \theta_{n-1} + \varphi_{n-1,r} = \theta + h(n - 1 - r) - 1.
\end{align*}
Then \( \lambda_s, \theta_r \) satisfy inequalities \((3')\) for all \( \lambda, \theta \).

Suppose
\[ h(n - 2) - 1 < \frac{e'}{18} \]
and
\[ \lambda_s, \theta_r \leq \frac{e'}{9}. \]

Then inequalities \((2')\) follow because for all \( r, s \), \( \varphi_{r,s} \leq h(n - 2) - 1 < e'/18 \), hence
\begin{align*}
\left( \frac{p-1}{p} \right) i_r - \varphi_{r,s} &\geq \frac{2e'}{3} - \frac{e'}{18} = \frac{e'}{6}, \\
(p-1)i'_s - \varphi_{r,s} &\geq \frac{2e'}{3} - \frac{e'}{18} = \frac{11e'}{18},
\end{align*}
and
\[ \left( \frac{p-1}{p} \right) i'_s - \varphi_{r,s} \geq \frac{e'}{6}. \]
Since \( \mu_s = i_s - \lambda_s, \delta'_r = i'_r - \theta_r \), the inequalities \( i_r \leq p\mu_s, i'_s \leq p\delta'_r \) of \((1)\) become
\[ i_r \leq p(i_s - \lambda_s), \quad i'_s \leq p(i'_r - \theta_r) \]
for \( r > s \). In the worst case we have
\[ i_r = \frac{2e'}{3}, \quad i_s = \frac{e'}{3}, \quad \lambda_s = \theta_r = \frac{e'}{9}, \]
hence
\[ \frac{2e'}{3} \leq p \left( \frac{e'}{3} - \frac{e'}{9} \right) = \frac{2pe'}{9}, \]
which holds for \( p \geq 3 \).
To satisfy Theorem 9, we need to satisfy the additional inequalities:

\[
\begin{align*}
    i_r + i_k' - \varphi_{r,k} &\geq \frac{p-1}{2p-1} e' + \frac{i_r + i\prime_s}{2p-1}, \\
    i_k + i_s' - \varphi_{k,s} &\geq \frac{p-1}{2p-1} e' + \frac{i_r + i\prime_s}{2p-1}
\end{align*}
\]  

(6)

for all \( r > k > s \). But if \( e'/3 \leq i_r, i\prime_s \) for all \( r, s \) and \( \varphi_{k,s} = 2(k-s) - 1 < e'/18 \), then the left sides of Eqs. (6) are

\[
\geq \frac{2e'}{3} - \frac{e'}{18} = \frac{11e'}{18}
\]

while the right side of Eqs. (6) is

\[
\leq \frac{p-1}{2p-1} e' + \frac{4/3}{2p-1} e' = \frac{p+1/3}{2p-1} e' < \frac{11}{18} e'
\]

for \( p \geq 5 \). Hence for \( p \geq 5 \) these examples satisfy the conditions of Theorem 9 as well.

Now \( \varphi_{r,r-1} > 0 \) for all \( r > 1 \), and so \( \overline{H}_{r,r-1} \) is not Larson. In general, \( \overline{H}_{r,s} \) is not Larson if the inequalities (5) hold. Thus we need

\[ v_{r,s} < i_r + i\prime_s, \]

equivalent to \( \varphi_{r,s} > 0 \), and for all \( r > k > s \),

\[ v_{r,s} - i_r < v_{k,s} - i_s. \]

But

\[ v_{k,s} - i_k = i\prime_s - \varphi_{k,s}, \]

so this becomes

\[ i\prime_s - \varphi_{r,s} < i\prime_s - \varphi_{k,s} \]

or

\[ \varphi_{k,s} < \varphi_{r,s} \]

for \( r > k > s \), which follows from \((4')\). Similarly, for \( r > s \), \( \overline{J}_{s,r} \) is not Larson if

\[ v_{r,s} < i\prime_s + \beta_{r,s}, \]

which is equivalent to \( v_{r,s} < i_r + i\prime_s \), that is, \( \varphi_{r,s} > 0 \) for \( r > s \), and

\[ v_{r,s} - i\prime_s < v_{r,k} - i\prime_k. \]
for all $r > k > s$. But again this last inequality is equivalent to

$$i_r - \varphi_{r,s} < i_r - \varphi_{r,k},$$

which is equivalent to $\varphi_{r,k} < \varphi_{r,s}$ for $r > k > s$, and this again follows from $(4')$. Thus these examples are all Larsonless.

**Remark 17.** The examples in [8] look like those in Theorem 9 with the extra conditions:

$$i_1 \leq i_2 \leq \cdots \leq i_n \leq pi_1, \quad i'_n \leq p' i_1$$

and $Y, Z$ in $M_n(R)$. For $p \geq 5$ we can choose the parameters of Example 16 to yield examples with $Y, Z$ not in $M_n(R)$ as follows.

As in Example 16, let

$$v_{r,s} = i'_s + i_r - \varphi_{r,s}$$

for all $r > s$, where

$$\varphi_{r,s} = h(r - s) - 1$$

for $h$ satisfying $h(n - 2) < e'/18$. Assume also that

$$\frac{e'}{3} < i_1, \quad i_n \leq \frac{2e'}{3}$$

and for all $r$,

$$0 < i_r - i_{r-1} \leq h - 2$$

(so that $h \geq 3$). Then

$$v_{r,s} = i'_s + i_r - \varphi_{r,s}$$

$$\leq i'_s + i_s + (h - 2)(r - s) - (h(r - s) - 1)$$

$$\leq e' + 1 - 2(r - s)$$

$$\leq e' - 1$$

for all $s < r$. Hence these examples satisfy the inequalities of Theorem 9 and none of the off-diagonal entries of $Y$ and $Z$ have entries in $R$. This shows that the construction of Theorem 9 is a non-trivial extension of [8].
References