ELSEVIER
SCIENCE DIRECT
PHYSICS LETTERS B
Physics Letters B 609 (2005) 173–179
www.elsevier.com/locate/physletb

Geometrical loci and CFTs via the Virasoro symmetry of the mKdV-SG hierarchy: an excursus

Davide Fioravanti

Department of Mathematics, University of York, Heslington, York YO10 5DD, UK

Received 13 October 2004; accepted 16 January 2005

Available online 25 January 2005

Editor: L. Alvarez-Gaumé

Abstract

We will describe the appearance of specific algebraic KdV potentials as a consequence of a requirement on an integro-differential expression. This expression belongs to a class generated by means of Virasoro vector fields acting on the KdV field. The "almost" rational KdV fields are described in terms of a geometrical locus of complex points. A class of solutions of this locus has recently appeared as a description of any conformal Verma module without degeneration.

PACS: 11.30.-j; 02.40.-k; 03.50.-z

Keywords: KdV vector fields; Virasoro vector fields; Airault–McKean–Moser geometrical locus; Duistermaat–Grünbaum geometrical locus; Conformal Verma module

1. Introductory remarks

A large well-known class of (Abelian) isospectral deformations of the Schrödinger operator $L = -\partial_x^2 + u$ takes the name of Korteweg–de Vries (KdV) hierarchy [1]; for the Korteweg–de Vries equation

$$\frac{\partial u}{\partial t_1} = -u_{xxx} + 6u_x u, \tag{1}$$

is the first non-trivial example of flow in this hierarchy (t_1 and x are time and space respectively, their appearance as indices means derivation). The entire KdV hierarchy can be derived from the Schrödinger operator as the only fundamental object [1], but it is also generated effectively by the pseudo-differential operator

$$N_u = -\partial_x^2 + 4u + 2u_x \partial_x^{-1}, \tag{2}$$

E-mail address: df14@york.ac.uk (D. Fioravanti).

0370-2693
doi:10.1016/j.physletb.2005.01.037

after successive actions on the trivial vector field $K_0[u] = u_x$

$$K_{j+1} = N_u K_j, \quad j \geqslant 0. \tag{3}$$

And the actions of the different vector fields K_j must be understood in the usual way, as given by the time derivatives

$$\frac{\partial u}{\partial t_j} = K_j[u]. \tag{4}$$

These flows do not change the spectrum of the Schrödinger operator $L = -\partial_x^2 + u$ [1] and are all compatible, in the sense that they commute with each other:

$$[K_i, K_j] = 0. \tag{5}$$

In this Letter we will not be self-contained and would like to address the attentive reader to the main Refs. [2,3] for definitions and theorems.

2. The Virasoro vector fields and their decaying rational solutions

In the same spirit, Zubelli and Magri [2] constructed recursively an algebra of vector fields[1] (also called, in another context, master-symmetries after Fokas and Fuchsteiner [5]) starting from the generator of the scaling transformation

$$V_0[u] = u + \frac{1}{2} x u_x, \qquad V_{j+1} = N_u V_j, \quad j \geqslant 0. \tag{6}$$

Albeit the flows of the KdV hierarchy commute with each other, the flows V_j were proved to close half centerless Virasoro algebra (only generators for $j \geqslant 0$ are present):

$$[V_i, V_j] = (j - i) V_{i+j}. \tag{7}$$

Moreover, the commutator with a higher KdV flow gives another KdV vector field according to

$$[V_i, K_j] = \left(j + \frac{1}{2}\right) K_{j+i}. \tag{8}$$

Since the right-hand side is not zero, the Virasoro vectors are not proper symmetries of the KdV hierarchy, but they nevertheless generate all the hierarchy by successive commutators starting only from the first flow (1). In contrast with the KdV flows, they also change the spectrum of the Schrödinger operator, though their action on the "energy levels" is simply realised by the polynomial vector fields in the complex plane [6]. Later on, how they act on the modified KdV field, ϕ such that

$$u = \phi_x^2 + \phi_{xx}, \tag{9}$$

has been studied in [7], showing that the algebra (6) can be completed to form an entire Virasoro algebra.[2] Furthermore, the Virasoro algebra has been given a geometrical origin and meaning, by which it has been also proved to enjoy a zero-curvature form (involving the differential first-order matrix operator associated to the second-order Schrödinger operator through the Miura transformation) [7] and to commute (genuine symmetry) with the (light-cone) sine-Gordon flows [8]. Going back to the paper by Zubelli and Magri [2], they proved that the half Virasoro

[1] These vector fields may also be derived from those of [4] putting formally all the KdV times to zero.

[2] Actually, we should apologise because at that time we did not know the content of [2]. Later, we also had a nice conversation with F. Magri about the algebraic structure of [7], though we have discovered the content of [2] only very recently.

algebra (6) is tangent to some previously well-known manifolds: these are spanned by the bispectral potentials $u(x)$ of the Schrödinger operator $L=-\partial_x^2+u$ [3]. In fact, Duistermaat and Grünbaum characterised these potentials to be the family of the pure angular momentum potentials $u(x)=\frac{l(l+1)}{x^2}$, with $l(l+1)$ an arbitrary constant, and two other classes: both are obtained by iterative actions of *rational* Darboux transformations starting from two very simple potentials, i.e., $u(x)=0$ and $u(x)=-\frac{1}{4x^2}$. Therefore, they concluded that all the bispectral potentials are *rational functions* decaying at $x=\infty$[3] (with a peculiar pattern of poles, cf. [3] and below). As a consequence of tangency it comes out that the (bounded) bispectral potentials stay rational while they evolve according to one of the Virasoro vector fields (6) [2]. Recently, Zubelli and Silva have shown the reverse statement: if $u(x)$ is a rational function decaying at infinity, which remains rational by each of the flows (6), then $u(x)$ is a (bounded) bispectral potential [9]. To prove it, they have almost used only the first vector field obtained from the scaling transformation $V_0[u]$, i.e.,

$$V_1[u]=-\frac{x}{2}(u_{xxx}-6uu_x)-2u_{xx}+4u^2+u_x\partial_x^{-1}u. \tag{10}$$

In fact, they have sought for decaying rational solutions, decomposed in partial fractions

$$u_r(x)=\sum_{p\in P}\sum_{m=1}^{m_p}\frac{c_{p,m}}{(x-p)^m}, \tag{11}$$

of the "equation of motion"

$$\frac{\partial u_r}{\partial\beta_1}=V_1[u_r], \tag{12}$$

where β_1 is the "time" associated with the flow under examination and the poles $p=p(\beta_1)$ and the coefficients $c_{p,m}=c_{p,m}(\beta_1)$ depend smoothly on it. Here they have assumed that the number of poles is constant (neither poles creation nor annihilation) and that the leading order coefficient $c_{p,m_p}(\beta_1)$ must not vanish. They have discovered that the "generic" rational function of the form (11), under these conditions, undergoes severe restrictions on its form as necessary conditions to satisfy the previous equation of motion (12) and takes the simpler double pole form

$$u_r(x)=\frac{l(l+1)}{x^2}+\sum_{p\in P}\frac{2}{(x-p)^2}, \tag{13}$$

where $l(l+1)$ is a constant, that may also be zero, and each $p=p(\beta_1)$ is a function of time. If $l(l+1)=0$ then $x=0$ is not a stationary pole and then it may belong to the set of complex non-stationary poles P; instead if $l(l+1)\neq 0$ then $x=0$ is a stationary pole and it does not belong to P. Now, the time evolution of u_r (13) comes out by equating the $1/(x-p)^3$ terms of both members of (12) and involves only the poles $p(\beta_1)$

$$\dot{p}=-2\left(\frac{l(l+1)}{p}+\sum_{q\in P_p}\frac{2p+q}{(p-q)^2}\right), \tag{14}$$

where the upper dot indicates the derivative with respect to the time β_1 and we have for short defined $P_p=P-\{p\}$. Finally, the poles in P must satisfy a static constraint which derives from equating to zero the $1/(x-p)^2$ terms of the right-hand side of (12):

$$\frac{l(l+1)}{p^3}+\sum_{q\in P_p}\frac{2}{(p-q)^3}=0. \tag{15}$$

[3] To be rigorous, they also found the only exception to zero-boundedness, i.e., the linear potential $u(x)=c_1x+c_0$. But it was excluded from the analysis in [2] because of its triviality.

When $l \in \mathbb{Z}_{\geqslant 0}$ or $l \in \mathbb{Z}_{\geqslant 0} - 1/2$ these relations give the celebrated *locus* of Duistermaat and Grünbaum [3]. As particular case, it reduces to the even more famous locus of Airault, McKean and Moser if $l(l+1) = 0$ [10]

$$\sum_{q \in P_p} \frac{1}{(p-q)^3} = 0, \tag{16}$$

where now P may contain the zero. The $u_r(x)$ satisfying the previous condition are again given by the form (13) with now $l(l+1) = 0$ and they are also the only rational solutions of the KdV flows [10] (in which case the poles would depend on the KdV times t_j (4)). Moreover, they were obtained in [3] by rational Darboux transformations from the initial potential $u_0 = 0$. Instead, if $l(l+1) \neq 0$ the potentials (13), (15) cannot be solutions of the KdV hierarchy, albeit they are yielded by rational Darboux transformations from $u_0(x) = \frac{-1/4}{x^2}$. Actually, the locus (15) seems to be more general than in the Duistermaat and Grünbaum context since no restriction on l comes out manifestly. In fact, this appears as a consequence of imposing that the rational potentials are also solutions of the higher Virasoro equations [9]

$$\frac{\partial u_r}{\partial \beta_n} = V_n[u_r], \quad n > 1. \tag{17}$$

Actually, the poles of $u_r(x)$ cannot be considered proper functions of all the times β_j $(j > 0)$ at once, since the different flows V_j do not commute. Nevertheless, the imposition of the rational invariance under the higher flows is geometrically meaningful and suggestive. But it does not easily imply that the restriction on l might be seen as a consequence of the only equation of motion (12), although it has been conjectured in [9]. Of course, there is no restriction on l, if the only pole in $u_r(x)$ (13) is $x = 0$ (bispectral potential $u_r(x) = l(l+1)/x^2$, which invariant: $V_0[u_r] = 0$).

3. Rational potentials perturbed by $x^{2\alpha}$: CFTs

So far, we have been dealing with rational functions $u_r(x)$ (11) decaying at infinity and now we would like to consider potentials which play some rôles in 1-dimensional quantum mechanics. Moreover, as the whole Virasoro algebra was derived in [7] by dressing the infinitesimal generators of diffeomorphisms in the spectral parameter, we expect it to have an interesting action on more general potential. Therefore, we consider a simple *perturbation* of (11)

$$u(x) = x^{2\alpha} + u_r(x), \tag{18}$$

where 2α is a non-negative real number. This anharmonic perturbation[4] clearly changes the behaviour at $x = \infty$, dominating the asymptotic expansion, and destroys any chance that u may be consistently a solution of Eq. (12). Nevertheless, thanks to their aforementioned origin in [7], we may think of the vector fields (6) as infinitesimal transformations acting on (18)

$$\delta_j u = \epsilon_j V_j[u], \quad n \geqslant 0, \tag{19}$$

where ϵ_j is an infinitesimal variation of β_j. In this perspective, it is natural to require that these transformations do not introduce new double poles; for this is the requirement which gives rise to (15). Besides, a (rational) Darboux transformation maps the solutions of the algebraic equations (15) into the solutions of a set of equations with the same form (but different l) [3].[5] In the first instance, we do not want double poles in the first non-trivial

[4] A coupling constant in front of $x^{2\alpha}$ can be considered as re-absorbed through a redefinition of x.

[5] We do not expect that the Darboux transformation will play exactly the same rôle here, since the addition (18) itself breaks the rationality.

transformation of (18)

$$\delta_1 u = V_1[u], \tag{20}$$

where we have omitted ϵ_1. Hence, if we limit $2\alpha = r/s$ to be rational, we realise rather easily that we need $u_r(x)$ in (18) to be more stringently a rational function of $y = x^{1/s}$, in order to equate the squared powers of the transformation (20) to zero. And this easily means that we can assume, without loss of generality, 2α to be a non-negative integer number. A similar but more refined balancing of partial fraction decomposition yields, as necessary condition, the restriction (13) and hence

$$u(x) = x^{2\alpha} + \frac{l(l+1)}{x^2} + \sum_{p\in P} \frac{2}{(x-p)^2}. \tag{21}$$

Eventually, plugging again this form into the right-hand side of (20) we equate to zero the coefficients of the powers $1/(x-p)^2$, obtaining $\forall p \in P$

$$l(l+1) + \sum_{q\in P_p} \frac{2p^3}{(p-q)^3} - \alpha p^{2\alpha+2} = 0. \tag{22}$$

These constraints can be thought of as a *deformation* of the Duistermaat and Grünbaum locus (15), in the sense that they reduce to that locus when $\alpha = 0$. Actually, we must emphasise that so far no restriction on the possible values of l has appeared. Nevertheless, we might think to deduce a restriction by using the higher Virasoro vector fields, likewise to what happens in the decaying potential case [9]. This possible restriction would be of crucial interest for what we are going to illustrate. Moreover, we would like to deliver a more detailed analysis of the geometrical interpretation of (22) in a future publication [11], where we also should shed light on the creation and annihilation properties of the negative and positive Virasoro generators of [7], respectively.

To gain some meaning from the algebraic equations (22), we will look for particular solutions. Thanks to the presence of the last term in the left-hand side, it would be very natural to have solutions p which gather in $(2\alpha+2)$th roots of another variable z. More precisely, we want that the whole set P is generated from another non-empty set of complex numbers, Z, in this way:

$$P = \{p\colon\ p^{2\alpha+2} = z,\ z \in Z\}. \tag{23}$$

Naturally, the sum in (22) splits into two parts: (1) the sum over the q which are not roots of $z = p^{2\alpha+2}$, i.e., over $P_p^{(1)} = \{q \in P_p\colon\ q^{2\alpha+2} \neq p^{2\alpha+2}\}$; (2) the sum over the roots q of $z = p^{2\alpha+2}$ different from p, i.e., over $P_p^{(2)} = \{q \in P_p\colon\ q^{2\alpha+2} = p^{2\alpha+2}\}$. As for the case (1) we easily have

$$\sum_{q\in P_p^{(1)}} \frac{2}{(p-q)^3} = \frac{\partial^3}{\partial p^3} \ln \prod_{q\in P_p^{(1)}} (p-q) = \frac{\partial^3}{\partial p^3} \sum_{w\in Z_z} \ln\bigl(p^{2\alpha+2} - w\bigr), \tag{24}$$

where again we have defined $Z_z = Z - \{z\}$. Therefore, after three derivatives we obtain

$$\sum_{q\in P_p^{(1)}} \frac{2}{(p-q)^3} = \frac{2\alpha+2}{p^3} \sum_{w\in Z_z} \frac{2z[z^2 + (1+2\alpha)(\alpha+3)wz + (1+2\alpha)w^2]}{(z-w)^3}. \tag{25}$$

On the other hand the sum (2) over all the roots of $z = p^{2\alpha+2}$ except p itself yields, after simple trigonometric manipulations,

$$\sum_{q\in P_p^{(2)}} \frac{2}{(p-q)^3} = \frac{2\alpha+1}{2p^3} - \frac{3}{4p^3} \sum_{k=1}^{2\alpha+1} \frac{1}{\sin^2 \frac{\pi}{2\alpha+2}k}. \tag{26}$$

Moreover, the last sum can be computed explicitly (cf., e.g., [12]) with the simple outcome

$$\sum_{q\in P_p^{(2)}} \frac{2}{(p-q)^3} = \frac{1-4\alpha^2}{4p^3}. \tag{27}$$

Eventually, we collect both contributions (25) and (27) into the locus equations (22)

$$\frac{4l(l+1)-4\alpha^2+1}{16(\alpha+1)} + \sum_{w\in Z_z} \frac{z[z^2+(1+2\alpha)(\alpha+3)wz+(1+2\alpha)w^2]}{(z-w)^3} - \frac{\alpha}{4(\alpha+1)}z = 0, \tag{28}$$

which are the constraints $\forall z \in Z$. In the end, we remark that this locus of complex points has been recently proposed by Bazhanov et al. [13] as describing the non-degenerate conformal Verma module of highest weight $\Delta = \frac{4l(l+1)-4\alpha^2+1}{16(\alpha+1)}$ with central charge $c = 1 - \frac{6\alpha^2}{\alpha+1}$, where the cardinality of Z gives the level. This sort of description comes out "naturally" within a surprising correspondence between the whole spectrum of a specific Schrödinger operator $L = -\partial_x^2 + u$ and the eigenvalue of the Baxter Q-operator [14] on a conformal state. As for the vacuum the authors of [15] have first furnished the potential (without the non-zero poles $p \in P$) $u(x) = x^{2\alpha} + l(l+1)/x^2$. Therefore, the locus (28) extends the correspondence to the excited states by means of exactly the rational potentials (21), (23) we have found. In addition, we have met a restriction on l in the previous case where we have obtained (22) with $\alpha = 0$, i.e., when the potential $u(x)$ is indeed bispectral, and we might expect a similar restriction from implementing the higher Virasoro flows [11].

4. Perspective

We have already stressed some of the relevant issues we would like to pursue in the next future. Now we are interested in delineating and summarise the main perspective. Essentially, we want to obtain information about the geometry of the locus (22) and any possible algebraic structure connected to it. Of course, an important analysis tool should be the Virasoro algebra of flows, since the KdV hierarchy played a crucial rôle in the study of the Airault–McKean–Moser locus [10]. Besides, the centerless character of the algebra can be emended looking at the action on the τ-function, in terms of which the potential reads $u = -2\partial_x^2 \ln \tau$ (cf., e.g., [6] and the second reference of [1] about the KP hierarchy). In this scenario the relation with a possible *generalised* bispectrality is still mysterious, though most intriguing.

In the end, we would like to emphasise the importance of considering non-rational forms of the potential $u(x)$, like for instance the soliton case (for which the action of the entire Virasoro algebra has been already described in [16]).

Acknowledgements

The author thanks: Leverhulme Trust (grant F/00224/G) for a fellowship, E. Corrigan and J. Suzuki for interesting discussions, the Asian Pacific Center for Theoretical Physics for an invitation during which part of this work has been performed, the EC FP5 Network EUCLID (contract number HPRN-CT-2002-00325) for partial financial support.

References

[1] F. Calogero, A. Degasperis, Spectral Transform and Solitons, North-Holland, Amsterdam, 1982;
L.A. Dickey, Soliton Equations and Hamiltonian Systems, World Scientific, Singapore, 1991.

[2] J.P. Zubelli, F. Magri, Commun. Math. Phys. 141 (1991) 329.
[3] J.J. Duistermaat, F.A. Grünbaum, Commun. Math. Phys. 103 (1986) 177.
[4] A. Orlov, M. Schulman, Lett. Math. Phys. 12 (1986) 171.
[5] A.S. Fokas, B. Fuchsteiner, Phys. Lett. A 86 (1981) 341.
[6] P. Grinevich, A. Orlov, in: Proceedings of Problems of Modern QFT, Alushta, URSS, 1989.
[7] D. Fioravanti, M. Stanishkov, Phys. Lett. B 447 (1999) 277.
[8] D. Fioravanti, M. Stanishkov, Nucl. Phys. B 577 (2000) 500.
[9] J.P. Zubelli, D.S.V. Silva, Commun. Math. Phys. 211 (2000) 85.
[10] H. Airault, H.P. McKean, J. Moser, Commun. Pure Appl. Math. 30 (1977) 95.
[11] D. Fioravanti, in preparation.
[12] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series and Products, fifth edition, Academic Press, San Diego, 1994.
[13] V.V. Bazhanov, S.L. Lukyanov, A.B. Zamolodchikov, Adv. Theor. Math. Phys. 7 (2003) 711.
[14] R.J. Baxter, Ann. Phys. 70 (1972) 193.
[15] P.E. Dorey, R. Tateo, J. Phys. A 32 (1999) L419, hep-th/9812211;
V. Bazhanov, S. Lukyanov, A. Zamolodchikov, J. Stat. Phys. 102 (2001) 567, hep-th/9812247.
[16] D. Fioravanti, M. Stanishkov, Nucl. Phys. B 591 (2000) 685.