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# Linear Programming with Fuzzy Coefficients in Constraints

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**Abstract**—This paper presents a new method for solving linear programming problems with fuzzy coefficients in constraints. It is shown that such problems can be reduced to a linear semi-infinite programming problem. The relations between optimal solutions and extreme points of the linear semi-infinite program are established. A cutting plane algorithm is introduced with a convergence proof, and a numerical example is included to illustrate the solution procedure. © 1999 Elsevier Science Ltd. All rights reserved.

**Keywords**—Fuzzy mathematical programming, Linear semi-infinite programming.

## 1. INTRODUCTION

This paper studies a linear programming problem with fuzzy coefficients in constraints. To describe this problem, we consider the following linear program in the conventional form [1]:

$$\begin{aligned} \text{maximize} \quad & \mathbf{c}^{\mathrm{T}}\mathbf{x}, \\ \text{s.t.} \quad & A\mathbf{x} \le \mathbf{b}, \\ & \mathbf{x} \ge \mathbf{0}, \end{aligned} \tag{1}$$

where $\mathbf{c}$ and $\mathbf{x}$ are $n$-dimensional column vectors, $A$ is an $m \times n$ $(m \le n)$ matrix, $\mathbf{b}$ is an $m$-dimensional column vector, and $\mathbf{0}$ is the $n$-dimensional zero vector. Note that in this model,

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all coefficients of $A, \mathbf{b}$, and $\mathbf{c}$ are crisp numbers, and each constraint must be satisfied strictly. However, in the real-world decision problems, a decision maker does not always know the exact values of the coefficients taking part in the problem, and that vagueness in the coefficients may not be of a probabilistic type. In this situation, the decision maker can model the inexactness by means of fuzzy parameters [2]. Many papers have appeared in the literature on this subject [3]. An early contribution was made by Tanaka et al. [4]. They considered the following problem:

$$\begin{array}{lll} \text{maximize} & \displaystyle\sum_{j=1}^{n} \tilde{c}_j x_j, & \\ \text{s.t.} & \displaystyle\sum_{j=1}^{n} \tilde{a}_{ij} x_j \leq \tilde{b}_i, & i = 1, \dots, m, \\ & x_j \geq 0, & j = 1, \dots, n, \end{array} \tag{2}$$

where $\tilde{a}_{ij}$, $\tilde{b}_i$, and $\tilde{c}_j, i = 1, \dots, m, j = 1, \dots, n$, are fuzzy coefficients in terms of fuzzy sets. Ramík and Rímánek [5] also dealt with problem (1) with fuzzy parameters in the constraints. Later, Delgado, Verdegay and Vila [6] studied a "general model" for fuzzy linear programming problems which involve fuzziness both in the coefficients and in the accomplishment of the constraints. In order to convert the fuzzy constraint $\sum_{j=1}^{n} \tilde{a}_{ij} x_j \leq \tilde{b}_i$ in (2) into a more tractable one, all the papers mentioned above assumed that there exists a comparison relation " $\leq$ " between two fuzzy numbers $\tilde{r}$ and $\tilde{s}$, for ranking purpose. In this way, $\tilde{r} \leq \tilde{s}$ means the fuzzy number $\tilde{r}$ is less than or equal to fuzzy number $\tilde{s}$ under this ranking. Because there is no universal ranking in fuzzy set theory, depending on the comparison relation adopted, different auxiliary models and solution methods can be established [3].

In this paper, we focus on the linear programming problem (1) with fuzzy coefficients in both $A$ and $\mathbf{b}$. We show such problems can be reduced to a Linear Semi-Infinite Programming (LSIP) problem. The optimality conditions of solutions to (LSIP) are investigated. An efficient algorithm for solving the original fuzzy linear program in terms of (LSIP) is also developed. The organization of the rest of this paper is as follows. Section 2 models a linear program with fuzzy coefficients as a linear semi-infinite programming problem. Section 3 presents basic analysis of (LSIP). In Section 4, some relationships between the optimal solutions and extreme points of (LSIP) are established. A solution algorithm with a convergence proof is proposed in Section 5. Section 6 concludes this paper by making some remarks.

## 2. THE MODEL

To specify the fuzzy coefficients in the constraint set, we use convex fuzzy numbers [2].

DEFINITION 1. *A convex fuzzy number $\tilde{N}$ is a fuzzy set defined on the real line $R$ with a membership function $\mu_{\tilde{N}}(\cdot)$ such that its $\alpha$-level set $\tilde{N}_\alpha \triangleq \{x \in R \mid \mu_{\tilde{N}}(x) \geq \alpha\}$ forms an interval $[L_{\tilde{N}}(\alpha), R_{\tilde{N}}(\alpha)]$ where*

$$L_{\tilde{N}}(\alpha) \triangleq \min\{x \in R \mid \mu_{\tilde{N}}(x) \geq \alpha\}$$

*and*

$$R_{\tilde{N}}(\alpha) \triangleq \max\{x \in R \mid \mu_{\tilde{N}}(x) \geq \alpha\}$$

*are real-valued continuous functions in $\alpha \in [0, 1]$.*

Figure 1 shows the membership function of the convex fuzzy number $\tilde{N}$.

Let $F(\tilde{N})$ be the set of all convex fuzzy numbers. Based upon the Extension Principle [2], we have the following results.

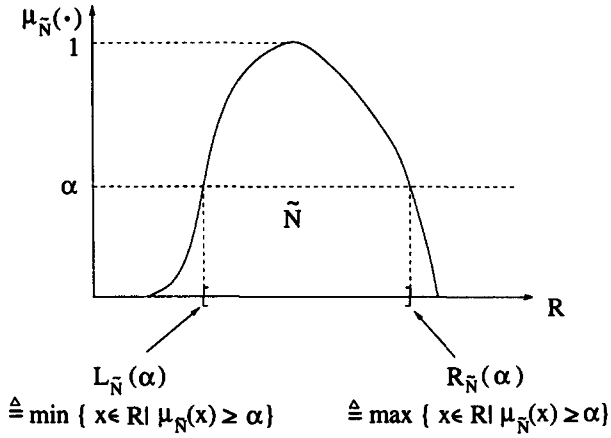


Figure 1. The membership function of a convex fuzzy number $\tilde{N}$.

PROPERTY 1. If $\tilde{N}_1, \tilde{N}_2 \in F(\tilde{N})$, then $\tilde{M} \triangleq \tilde{N}_1 + \tilde{N}_2 \in F(\tilde{N})$ and

$$\left.\begin{aligned} L_{\tilde{M}}(\alpha) &= L_{\tilde{N}_1}(\alpha) + L_{\tilde{N}_2}(\alpha), \\ R_{\tilde{M}}(\alpha) &= R_{\tilde{N}_1}(\alpha) + R_{\tilde{N}_2}(\alpha), \end{aligned}\right. \qquad \forall \alpha \in [0,1].$$

PROPERTY 2. If $\tilde{N} \in F(\tilde{N})$ and $k$ is a positive real number, then $\tilde{M} \triangleq k \cdot \tilde{N} \in F(\tilde{N})$ and

$$\left.\begin{aligned} L_{\tilde{M}}(\alpha) &= k \cdot L_{\tilde{N}}(\alpha), \\ R_{\tilde{M}}(\alpha) &= k \cdot R_{\tilde{N}}(\alpha), \end{aligned}\right. \qquad \forall \alpha \in [0,1].$$

PROPERTY 3. If $\tilde{N} \in F(\tilde{N})$ and $k = 0$, then $k \cdot \tilde{N} \triangleq 0$.

After introducing the concept of fuzzy numbers with their properties, we have to discuss the issue of ranking fuzzy numbers. There are many ranking methods available for the comparison relation between two fuzzy numbers [3,7]. Here we adopt the commonly used concept of $\alpha$-preference [8,9] and provide the following ranking method.

DEFINITION 2. *For $\tilde{N}_1, \tilde{N}_2 \in F(\tilde{N})$ and $\alpha \in [0,1]$, $\tilde{N}_1 \geq_\alpha \tilde{N}_2$ if and only if*

$$\left.\begin{aligned} L_{\tilde{N}_1}(t) &\geq L_{\tilde{N}_2}(t), \\ R_{\tilde{N}_1}(t) &\geq R_{\tilde{N}_2}(t), \end{aligned}\right. \qquad \forall t \in [\alpha,1].$$

Figure 2 illustrates such a relation of $\tilde{N}_1 \geq_\alpha \tilde{N}_2$ for some $\alpha \in [0,1]$.

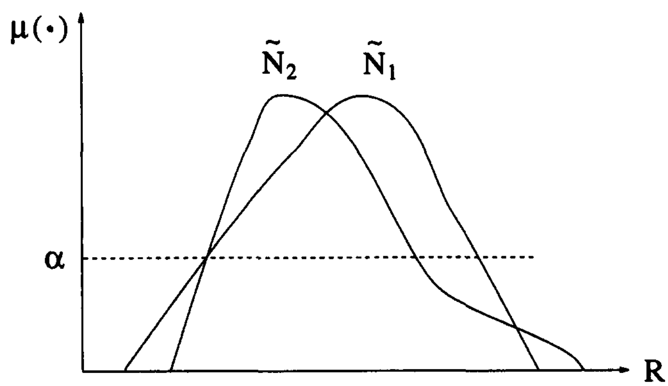


Figure 2. $\tilde{N}_1 \geq_\alpha \tilde{N}_2$.

According to the fuzzy ranking method provided above, given $\alpha \in [0,1]$, the Fuzzy Linear Programming (FLP) problem we considered can be described as follows:

$$
\text{(FLP)} \qquad
\begin{aligned}
\text{minimize} \quad & \sum_{j=1}^{n} c_j x_j, \\
\text{s.t.} \quad & \sum_{j=1}^{n} \tilde{a}_{ij} x_j \geq_\alpha \tilde{b}_i, \qquad i = 1,2,\dots,q, \\
& x_j \geq 0, \qquad j = 1,2,\dots,n,
\end{aligned}
\tag{3}
$$

where $\tilde{a}_{ij}, \tilde{b}_i \in F(\tilde{N})$, for $i = 1,2,\dots,q$, $j = 1,2,\dots,n$. Here, $\sum_{j=1}^{n} \tilde{a}_{ij} x_j \geq_\alpha \tilde{b}_i$, $i = 1,2,\dots,q$, means that

$$
\left.
\begin{aligned}
& \sum_{j=1}^{n} L_{\tilde{a}_{ij}}(t) \cdot x_j \geq L_{\tilde{b}_i}(t), \\
& \sum_{j=1}^{n} R_{\tilde{a}_{ij}}(t) \cdot x_j \geq R_{\tilde{b}_i}(t),
\end{aligned}
\right\} \quad \forall t \in [\alpha, 1].
\tag{4}
$$

Substituting expression (4) into problem (3) yields the following problem:

$$
\begin{aligned}
\text{minimize} \quad & \sum_{j=1}^{n} c_j x_j, \\
\text{s.t.} \quad & \sum_{j=1}^{n} L_{\tilde{a}_{ij}}(t) \cdot x_j \geq L_{\tilde{b}_i}(t), \qquad \forall t \in [\alpha, 1], \qquad i = 1,2,\dots,q, \\
& \sum_{j=1}^{n} R_{\tilde{a}_{ij}}(t) \cdot x_j \geq R_{\tilde{b}_i}(t), \qquad \forall t \in [\alpha, 1], \qquad i = 1,2,\dots,q, \\
& x_j \geq 0, \qquad j = 1,2,\dots,n.
\end{aligned}
$$

Let $f_{ij}(t) \triangleq L_{\tilde{a}_{ij}}(t)$, $b_i(t) \triangleq L_{\tilde{b}_i}(t)$, for $i = 1,2,\dots,q$, $j = 1,2,\dots,n$, $f_{ij}(t) \triangleq R_{\tilde{a}_{i-q,j}}(t)$, $b_i(t) \triangleq R_{\tilde{b}_{i-q}}(t)$, for $i = q+1,\dots,2q$, $j = 1,2,\dots,n$, $m \triangleq 2q$, and $T \triangleq [\alpha, 1]$. Then we have the following equivalent problem:

$$
\text{(LSIP)} \qquad
\begin{aligned}
\text{minimize} \quad & \sum_{j=1}^{n} c_j x_j, \\
\text{s.t.} \quad & \begin{pmatrix} f_{11}(t) & \cdots & f_{1n}(t) \\ \vdots & \ddots & \vdots \\ f_{m1}(t) & \cdots & f_{mn}(t) \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \geq \begin{pmatrix} b_1(t) \\ \vdots \\ b_m(t) \end{pmatrix}, \qquad \forall t \in T, \\
& x_j \geq 0, \qquad j = 1,2,\dots,n,
\end{aligned}
\tag{5}
$$

where $T$ is a compact metric space, $f_{ij}(t)$ and $b_i(t)$, $i = 1,\dots,m$, $j = 1,\dots,n$, are real-valued continuous functions on $T$. Notice that problem (5) is a linear semi-infinite programming problem with $n$ variables and infinitely many constraints. Its feasible region and the optimal objective value are denoted by FP and v(LSIP), respectively, in this paper.

To investigate the optimality conditions of solutions to (LSIP), some basic analysis for (LSIP) are presented in the next section.

## 3. BASIC ANALYSIS

Let $T$ be a compact metric space, $C(T)$ be the space of all real-valued continuous functions on $T$, $M(T)$ be the space of bounded regular Borel measures on $T$, $C^+(T) \triangleq \{h \in C(T) \mid h(t) \geq 0, \forall\, t \in T\}$, and $M^+(T) \triangleq \{\mu \in M(T) \mid \mu(B) \geq 0, \forall\, B \in B(T)$, where $B(T)$ is the set of all Borel set in $T\}$. Consider the dual problem of (LSIP) [10]:

$$\begin{array}{llll} & \text{maximize} & \displaystyle\sum_{i=1}^{m} \int_T b_i(t)\, d\mu_i, & \\ \text{(DLSIP)} & \text{s.t.} & \displaystyle\sum_{i=1}^{m} \int_T f_{ij}(t)\, d\mu_i \leq c_j, & j = 1, \dots, n, \\ & & \mu_i \in M^+(T), & i = 1, 2, \dots, m. \end{array} \tag{6}$$

Let FD be the feasible region of (DLSIP) and v(DLSIP) the optimal objective value of (DLSIP). From a result of [11], it follows that if the optimal value of (DLSIP) has finite value and there is a $\mu^0 = (\mu_1^0, \mu_2^0, \dots, \mu_m^0) \in (M^+(T))^m$ such that $\sum_{i=1}^m \int_T f_{ij}(t)\, d\mu_i^0 < c_j$, $j = 1, \dots, n$, then the strong duality holds for (DLSIP). This is stated in Theorem 1.

THEOREM 1. *Assume that* FD $\neq \emptyset$ *and* $-\infty <$ v(DLSIP) $< \infty$. *If there exists* $\mu^0 = (\mu_1^0, \mu_2^0, \dots, \mu_m^0) \in (M^+(T))^m$ *such that* $\sum_{i=1}^m \int_T f_{ij}(t)\, d\mu_i^0 < c_j$, $j = 1, \dots, n$, *then* FP $\neq \emptyset$ *and* v(LSIP) $=$ v(DLSIP).

Applying Theorem 1, we have the following result.

THEOREM 2. *Assume that* v(LSIP) =v(DLSIP), *then* $\mathbf{x}^* \in$ FP *solves* (LSIP) *and* $\mu^* \in$ FD *solves* (DLSIP) *if and only if*

$$\sum_{j=1}^{n} f_{ij}(t) x_j^* - b_i(t) = 0, \qquad \forall\, t \in \text{Supp}\,(\mu_i^*), \quad i = 1, \dots, m$$

*and*

$$c_j - \sum_{i=1}^{m} \int_T f_{ij}(t)\, d\mu_i^* = 0, \qquad \forall\, j \in \{k \mid x_k^* \neq 0\}.$$

PROOF. Let $\mathbf{x}^* \in$ FP and $\mu^* \in$ FD. Then we have

$$c_j \geq \sum_{i=1}^{m} \int_T f_{ij}(t)\, d\mu_i^*, \qquad j = 1, \dots, n$$

and

$$\sum_{j=1}^{n} f_{ij}(t) x_j^* \geq b_i(t), \qquad i = 1, \dots, m.$$

Thus,

$$\begin{aligned} \sum_{j=1}^{n} c_j x_j^* &\geq \sum_{j=1}^{n} \left( \sum_{i=1}^{m} \int_T f_{ij}(t)\, d\mu_i^* \right) x_j^* \\ &= \sum_{i=1}^{m} \int_T \sum_{j=1}^{n} f_{ij}(t) x_j^*\, d\mu_i^* \geq \sum_{i=1}^{m} \int_T b_i(t)\, d\mu_i^*. \end{aligned} \tag{7}$$

If $\mathbf{x}^*$ and $\mu^*$ are optimal solutions to (LSIP) and (DLSIP), respectively, then by the assumption that v(LSIP) = v(DLSIP), we have

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} \int_T b_i(t)\, d\mu_i^*. \tag{8}$$

Therefore, by (7), the identity (8) holds if and only if

$$\sum_{j=1}^{n} f_{ij}(t) x_j^* - b_i(t) = 0, \qquad \forall t \in \text{ Supp } (\mu_i^*), \quad i = 1, \dots, m$$

and

$$c_j - \sum_{i=1}^{m} \int_T f_{ij}(t)\, d\mu_i^* = 0, \qquad \forall j \in \{k \mid x_k^* \neq 0\}.$$

This proves the theorem. ∎

Next we discuss the existence theorem for (LSIP).

THEOREM 3. *If* FP *is bounded, then* (LSIP) *has an optimal solution which is an extreme point of* FP.

PROOF. It is obvious that the feasible set FP is bounded and closed, and hence, is a compact set in $\mathbf{R}^n$. Since the objective function of (LSIP) is a continuous linear function on the compact set FP $\subset \mathbf{R}^n$, it will attain its minimum at an extreme point of FP. ∎

From Theorem 3, we see that the extreme points of the feasible set FP play an important role for optimal solutions of (LSIP). We will discuss the relationship between the optimal solutions and extreme points of the feasible region of (LSIP) in the next section.

## 4. EXTREME POINTS

To study the extreme points of the feasible region of (LSIP), we recall some useful definitions for general linear programming. Let $E$ and $F$ be real linear spaces, and $A : E \to F$ a linear operator. Consider the following linear program (LP):

$$\text{(LP)} \qquad \begin{array}{ll} \text{minimize} & \langle \mathbf{c}^*, \mathbf{x} \rangle, \\ \text{s.t.} & A\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \in P, \end{array} \tag{9}$$

where $\mathbf{c}^*$ is a linear functional in $E$, $\mathbf{b} \in F$, and $P$ is a positive convex cone in $E$. For $\mathbf{x}^0 \in P$, we define

$$B\left(\mathbf{x}^0\right) = \left\{\mathbf{x} \in E \mid \mathbf{x}^0 \pm \lambda \mathbf{x} \in P \text{ for some real } \lambda > 0\right\}.$$

Reference [12] showed that $\mathbf{x}^0$ is an extreme point of the feasible region for (LP) if and only if $B(\mathbf{x}^0) \bigcap \aleph(A) = \{\mathbf{0}\}$ where $\mathbf{0}$ denotes the zero vector and $\aleph(A) = \{\mathbf{x} \in E \mid A\mathbf{x} = 0\}$, the null space of $A$.

In order to investigate the conditions under which a feasible solution becomes an extreme point, the inequality constraints of (LSIP) are transformed to equality constraints. Let $\mathbf{g} = (g_1, \dots, g_m) \in (C^+(T))^m$ be the vector of "slack variables" of (LSIP), and consider a new semi-infinite programming problem with equality constraints $(\text{LSIP})_e$:

$$(\text{LSIP})_e \qquad \begin{array}{lll} \text{minimize} & \displaystyle\sum_{j=1}^{n} c_j x_j, & \\ \text{s.t.} & \displaystyle\sum_{j=1}^{n} f_{ij}(t) x_j - g_i(t) = b_i(t), \quad \forall t \in T, & i = 1, \dots, m, \\ & x_j \geq 0, & j = 1, \dots, n, \\ & g_i(t) \in C^+(T), & i = 1, \dots, m. \end{array} \tag{10}$$

Let $\mathrm{FP}_e$ be the feasible region of $(\mathrm{LSIP})_e$ and $(\mathbf{x}, \mathbf{g}) \in \mathrm{FP}_e$. Suppose that exactly $p$ components of the variable $\mathbf{x}$ are greater than zero, and, without loss of generality, we assume that the first $p$ components of $\mathbf{x}$ are positive, i.e., $\mathbf{x} = (x_1, \ldots, x_p, 0, \ldots, 0)^\top$. Let $t_1^{(i)}, t_2^{(i)}, \ldots, t_{\ell_i}^{(i)} \in T$ such that $g_i(t_1^{(i)}) = g_i(t_2^{(i)}) = \cdots = g_i(t_{\ell_i}^{(i)}) = 0$. Define the $\ell_i \times p$ matrices

$$K_i \triangleq \begin{pmatrix} f_{i1}\left(t_1^{(i)}\right) & f_{i2}\left(t_1^{(i)}\right) & \cdots & f_{ip}\left(t_1^{(i)}\right) \\ \vdots & \vdots & \ddots & \vdots \\ f_{i1}\left(t_{\ell_i}^{(i)}\right) & f_{i2}\left(t_{\ell_i}^{(i)}\right) & \cdots & f_{ip}\left(t_{\ell_i}^{(i)}\right) \end{pmatrix}, \qquad \text{for } i = 1, \ldots, m \tag{11}$$

and

$$\mathbf{K} \triangleq \begin{pmatrix} K_1 \\ K_2 \\ \vdots \\ K_m \end{pmatrix}. \tag{12}$$

Then we have the following theorem.

THEOREM 4. *Let* $\mathbf{K}$ *and* $(\mathbf{x}, \mathbf{g}) \in \mathrm{FP}_e$ *be defined as above. If* $\operatorname{rank}(\mathbf{K}) = p$, *then* $(\mathbf{x}, \mathbf{g})$ *is an extreme point of* $\mathrm{FP}_e$.

PROOF. By definition, it is easy to see that if the point $(\tilde{\mathbf{x}}, \tilde{\mathbf{g}})$ is in $B(\mathbf{x}, \mathbf{g})$, then

$$\operatorname{Supp}(\tilde{\mathbf{x}}) \subseteq \operatorname{Supp}(\mathbf{x}),$$

and $g_i(t_k^{(i)}) = 0$ implies $\tilde{g}_i(t_k^{(i)}) = 0$, for $i = 1, \ldots, m$, $k = 1, \ldots, \ell_i$. Define a constraint map $\Phi_i : \mathbf{R}^n \times C^+(T) \to C(T)$ by

$$\Phi_i(\mathbf{x}, \mathbf{g})(t) = \sum_{j=1}^{n} f_{ij}(t) x_j - g_i(t), \qquad \text{for } t \in T$$

and

$$\Phi \triangleq \begin{pmatrix} \Phi_1 \\ \vdots \\ \Phi_m \end{pmatrix}.$$

Then $\Phi$ is a linear operator. Thus, for a point $(\mathbf{x}', \mathbf{g}') \in B(\mathbf{x}, \mathbf{g}) \bigcap \aleph(\Phi)$, we have

$$\operatorname{Supp}(\mathbf{x}') \subseteq \operatorname{Supp}(\mathbf{x})$$

and

$$g_i'\left(t_k^{(i)}\right) = 0, \qquad \text{for } i = 1, \ldots, m, \quad k = 1, \ldots, \ell_i.$$

It follows that $g_i'(t) = \sum_{j=1}^{n} f_{ij}(t) x_j'$, for $i = 1, \ldots, m$, where $\mathbf{x}' = (x_1', \ldots, x_n')^\top \in R^n$ (since $(\mathbf{x}', \mathbf{g}') \in \aleph(\Phi)$). Let $\hat{g}_i = (g_i'(t_1^{(i)}), g_i'(t_2^{(i)}), \ldots, g_i'(t_{\ell_i}^{(i)}))^\top$, for $i = 1, \ldots, m$ and $\hat{\mathbf{x}} = (x_1', \ldots, x_p')^\top$. Then $\hat{\mathbf{g}} \triangleq = (\hat{g}_1, \ldots, \hat{g}_m)^\top = (\mathbf{0}, \ldots, \mathbf{0})^\top$ and $\mathbf{K}\hat{\mathbf{x}} = \hat{\mathbf{g}} = \mathbf{0}$. Since $\mathbf{K}$ has rank $p$, $\mathbf{K}\hat{\mathbf{x}} = \mathbf{0}$ implies $\hat{\mathbf{x}} = \mathbf{0}$. It then follows that $B(\mathbf{x}, \mathbf{g}) \bigcap \aleph(\Phi) = \{\mathbf{0}\}$. This shows that $(\mathbf{x}, \mathbf{g})$ is an extreme point of the feasible region of $(\mathrm{LSIP})_e$. ∎

Next we check the conditions for an extreme point $(\mathbf{x}, \mathbf{g})$ to be an optimal solution for $(\mathrm{LSIP})_e$.

Let $(\mathbf{x}, \mathbf{g})$ be an extreme point of $\mathrm{FP}_e$ and $\{s_k^{(1)}\}_{k=1,\dots,r_1} \subset \{t_k^{(1)}\}_{k=1,\dots,\ell_1}, \dots, \{s_k^{(m)}\}_{k=1,\dots,r_m} \subset \{t_k^{(m)}\}_{k=1,\dots,\ell_m}$. Suppose that exactly $p$ components of $\mathbf{x}$ are greater than zero, and without loss of generality, the first $p$ components of $\mathbf{x}$ are positive. Define

$$\bar{K}_i \triangleq \begin{pmatrix} f_{i1}\left(s_1^{(i)}\right) & \cdots & f_{ip}\left(s_1^{(i)}\right) \\ \vdots & \ddots & \vdots \\ f_{i1}\left(s_{r_i}^{(i)}\right) & \cdots & f_{ip}\left(s_{r_i}^{(i)}\right) \end{pmatrix}, \qquad \text{for } i = 1, 2, \dots, m$$

and

$$\bar{\mathbf{K}} \triangleq \begin{pmatrix} \bar{K}_1 \\ \bar{K}_2 \\ \vdots \\ \bar{K}_m \end{pmatrix}.$$

Let $\underline{\mathbf{x}} = (x_1, \dots, x_p)^\top$ and $\underline{\mathbf{c}} = (c_1, \dots, c_p)^\top$. We have the following theorem.

THEOREM 5. *Suppose that $(\mathbf{x}, \mathbf{g})$ and $\bar{\mathbf{K}}$ are defined as above. If we find $\bar{\mathbf{K}}$ such that*

(i) *$\bar{\mathbf{K}}$ is invertible,*

(ii)

$$\left(\bar{\mathbf{K}}^\top\right)^{-1} \underline{\mathbf{c}} \triangleq \begin{pmatrix} u_1^1 \\ \vdots \\ u_{r_1}^1 \\ \vdots \\ u_1^m \\ \vdots \\ u_{r_m}^m \end{pmatrix} \geq 0,$$

(iii)

$$\sum_{i=1}^{m} \sum_{k=1}^{r_i} f_{ij}\left(s_k^{(i)}\right) u_k^i - c_j \leq 0, \qquad \textit{for } j = p+1, \dots, n,$$

*then $(\mathbf{x}, \mathbf{g})$ is an optimal solution of $(\mathrm{LSIP})_e$.*

PROOF. Since $g_i(s_k^{(i)}) = 0$, for $i = 1, \dots, m$, $k = 1, \dots, r_i$, and by the definition of $\underline{\mathbf{x}} = (x_1, \dots, x_p)^\top$ and $\bar{\mathbf{K}}$, we have

$$\bar{\mathbf{K}}\underline{\mathbf{x}} = \begin{pmatrix} f_{11}\left(s_1^{(1)}\right) & \cdots & f_{1p}\left(s_1^{(1)}\right) \\ \vdots & \cdots & \vdots \\ f_{11}\left(s_{r_1}^{(1)}\right) & \cdots & f_{1p}\left(s_{r_1}^{(1)}\right) \\ \vdots & \cdots & \vdots \\ f_{m1}\left(s_1^{(m)}\right) & \cdots & f_{mp}\left(s_1^{(m)}\right) \\ \vdots & \cdots & \vdots \\ f_{m1}\left(s_{r_m}^{(m)}\right) & \cdots & f_{mp}\left(s_{r_m}^{(m)}\right) \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ x_p \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{j=1}^{p} f_{1j}\left(s_1^{(1)}\right) x_j \\ \vdots \\ \sum_{j=1}^{p} f_{1j}\left(s_{r_1}^{(1)}\right) x_j \\ \vdots \\ \sum_{j=1}^{p} f_{mj}\left(s_1^{(m)}\right) x_j \\ \vdots \\ \sum_{j=1}^{p} f_{mj}\left(s_{r_m}^{(m)}\right) x_j \end{pmatrix} = \begin{pmatrix} b_1\left(s_1^{(1)}\right) \\ \vdots \\ b_1\left(s_{r_1}^{(1)}\right) \\ \vdots \\ b_m\left(s_1^{(m)}\right) \\ \vdots \\ b_m\left(s_{r_m}^{(m)}\right) \end{pmatrix}.$$

It follows that

$$\underline{\mathbf{x}} = \bar{\mathbf{K}}^{-1} \begin{pmatrix} b_1\left(s_1^{(1)}\right) \\ \vdots \\ b_1\left(s_{r_1}^{(1)}\right) \\ \vdots \\ b_m\left(s_1^{(m)}\right) \\ \vdots \\ b_m\left(s_{r_m}^{(m)}\right) \end{pmatrix}.$$

$$\sum_{j=1}^{n} c_j x_j = \sum_{j=1}^{p} c_j x_j = \underline{\mathbf{x}}^\top \underline{\mathbf{c}}$$

$$= \left(b_1\left(s_1^{(1)}\right), \dots, b_1\left(s_{r_1}^{(1)}\right), \dots, b_m\left(s_1^{(m)}\right), \dots, b_m\left(s_{r_m}^{(m)}\right)\right) \left(\bar{\mathbf{K}}^\top\right)^{-1} \begin{pmatrix} c_1 \\ \vdots \\ c_p \end{pmatrix}$$

$$= \left(b_1\left(s_1^{(1)}\right), \dots, b_1\left(s_{r_1}^{(1)}\right), \dots, b_m\left(s_1^{(m)}\right), \dots, b_m\left(s_{r_m}^{(m)}\right)\right) \begin{pmatrix} u_1^1 \\ \vdots \\ u_{r_1}^1 \\ \vdots \\ u_1^m \\ \vdots \\ u_{r_m}^m \end{pmatrix}$$

$$= \sum_{i=1}^{m} \sum_{k=1}^{r_i} b_i\left(s_k^{(i)}\right) u_k^i.$$

Let $\mathbf{u}^* = (u_1^1, \dots, u_{r_1}^1, \dots, u_1^m, \dots, u_{r_m}^m)^\top$. From (ii) and (iii), we see that $\mathbf{u}^*$ is a feasible solution for (DLSIP). Hence, Theorem 2 asserts that $(\mathbf{x}, \mathbf{g})$ is an optimal solution for $(\text{LSIP})_e$. ∎

## 5. SOLUTION PROCEDURE

There are many semi-infinite programming algorithms [13,14] available for solving linear semi-infinite programming problems. The difficulty lies in how to effectively deal with the infinite number of constraints. Based on a recent review [13], the "cutting plane approach" is an effective one for such application.

Following the basic concept of the cutting plane approach, we can easily design an iterative algorithm which adds $m$ constraints at a time until an optimal solution is identified. To be more specific, at the $k^{\text{th}}$ iteration, given $T_k = \{\mathbf{t}^1, \mathbf{t}^2, \ldots, \mathbf{t}^k\}$, where $\mathbf{t}^k = (t_1^k, t_2^k, \ldots, t_m^k) \in T^m$, and $k \geq 1$, we consider the following linear programming problem (LP$^k$):

$$
\begin{array}{lll}
 & \text{minimize} & \displaystyle\sum_{j=1}^{n} c_j x_j, \\
(\mathrm{LP}^k) & \text{s.t.} & \left(\begin{array}{ccc}
f_{11}(t_1^1) & \cdots & f_{1n}(t_1^1) \\
\vdots & \ddots & \vdots \\
f_{m1}(t_m^1) & \cdots & f_{mn}(t_m^1) \\
\hline
\vdots & \ddots & \vdots \\
\hline
f_{11}(t_1^k) & \cdots & f_{1n}(t_1^k) \\
\vdots & \ddots & \vdots \\
f_{m1}(t_m^k) & \cdots & f_{mn}(t_m^k)
\end{array}\right)
\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \geq
\left(\begin{array}{c}
b_1(t_1^1) \\ \vdots \\ b_m(t_m^1) \\ \hline \vdots \\ \hline b_1(t_1^k) \\ \vdots \\ b_m(t_m^k)
\end{array}\right), \\
 & & x_j \geq 0, \qquad j = 1, 2, \ldots, n.
\end{array}
$$

Let $F^k$ be the feasible region of (LP$^k$). Suppose that $\mathbf{x}^k = (x_1^k, \ldots, x_n^k)$ is an optimal solution of (LP$^k$). We define the "constraint violation functions" as follows:

$$
v_i^{k+1}(t) \triangleq \sum_{j=1}^{n} f_{ij}(t) x_j^k - b_i(t), \qquad \forall\, t \in T, \quad i = 1, \ldots, m.
$$

Since $f_{ij}(t)$ and $b_i(t)$ are continuous over $T$ and $T$ is compact, the function $v_i^{k+1}(t)$ achieves its minimum over $T$, for $i = 1, \ldots, m$. Let $t_i^{k+1}$ be such a minimizer and consider the value of $v_i^{k+1}(t_i^{k+1})$, for $i = 1, \ldots, m$. If the value is greater than or equal to zero, for $i = 1, \ldots, m$, then $\mathbf{x}^k$ becomes a feasible solution of (LSIP), and hence, $\mathbf{x}^k$ is optimal for (LSIP) (because the feasible region $F^k$ of (LP$^k$) is no smaller than the feasible region $FP$ of (LSIP)). Otherwise, $\mathbf{x}^k$ is not optimal and $\mathbf{t}^{k+1} = (t_1^{k+1}, \ldots, t_m^{k+1}) \notin T_k$. We then augment $T_k$ to a larger set $T_{k+1} = \{\mathbf{t}^1, \ldots, \mathbf{t}^k, \mathbf{t}^{k+1}\}$. By repeating this process, $\mathbf{x}^k$ will converge to the optimal solution of (LSIP). This background provides a foundation for us to outline a cutting plane algorithm for solving (LSIP).

CPLSIP ALGORITHM.

Step 0. [Initialization] Set $k = 1$; Choose any $t_i^1 \in T$; Set $T_1 = \{\mathbf{t}^1\}$.

Step 1. Solve $(\mathrm{LP})^k$ and obtain an optimal solution $\mathbf{x}^k$.

Step 2. Find a minimizer $t_i^{k+1}$ of $v_i^{k+1}(t)$ over $T$, for $i = 1, \ldots, m$.

Step 3. If $v_i^{k+1}(t_i^{k+1}) \geq 0$, for $i = 1, \ldots, m$, then stop with $\mathbf{x}^k$ being an optimal solution of (LSIP). Otherwise, set $T_{k+1} = T_k \bigcup \{\mathbf{t}^{k+1}\}$ and $k \leftarrow k + 1$; go to Step 1.

When (LSIP) has at least one feasible solution, i.e., FP $\neq \emptyset$, it is easy to see that the CPLSIP algorithm either terminates in a finite number of iterations with an optimal solution or generates a sequence of points $\{\mathbf{x}^k \mid k = 1, 2, \ldots\}$. Our objective for the remaining part of this section is to show that if the CPLSIP algorithm does not terminate in finite iterations, then $\{\mathbf{x}^k\}$ has a subsequence which converges to an optimal solution of (LSIP). We now show a convergence proof for the CPLSIP algorithm.

THEOREM 6. *Let $\{\mathbf{x}^k\}$ be a sequence generated by the* CPLSIP *algorithm. If there exists an $M > 0$ such that $\|\mathbf{x}^k\| \leq M$, $\forall\, k$, then there is a subsequence of $\{\mathbf{x}^k\}$ which converges to an optimal solution of* (LSIP).

PROOF. It is easy to see that the feasible domain of ($\mathrm{LP}^k$) contains that of ($\mathrm{LP}^{k+1}$), for $k = 1, 2, \ldots$, and

$$\sum_{j=1}^{n} c_j x_j^1 \le \sum_{j=1}^{n} c_j x_j^2 \le \cdots \le \sum_{j=1}^{n} c_j x_j^*, \tag{13}$$

where $\mathbf{x}^* = (x_1^*, \ldots, x_n^*)^\top$ is an optimal solution of (LSIP). Due to the boundedness of $\{\mathbf{x}^k\}$, we know that there exists a subsequence $\{\mathbf{x}^{k_\ell}\}$ of $\{\mathbf{x}^k\}$ with a limit $\bar{\mathbf{x}}$. It is obvious that $\sum_{j=1}^{n} c_j x_j^{k_\ell} \to \sum_{j=1}^{n} c_j \bar{x}_j$. From (13), we see that

$$\sum_{j=1}^{n} c_j \bar{x}_j \le \sum_{j=1}^{n} c_j x_j^*. \tag{14}$$

To show that $\bar{\mathbf{x}} \in \mathrm{FP}$, consider

$$v_i(t) \triangleq \sum_{j=1}^{n} f_{ij}(t)\bar{x}_j - b_i(t), \qquad t \in T, \quad i = 1, \ldots, m, \tag{15}$$

and let $\bar{t}_i \in T$ be a minimizer of $v_i(t)$ over $T$, for $i = 1, \ldots, m$. By the definition of $\bar{\mathbf{x}}$, we know that

$$v_i\left(t_i^{k_\ell+1}\right) \ge 0, \qquad i = 1, \ldots, m, \tag{16}$$

where $t_i^{k_\ell+1} \in T$ is generated by the CPLSIP algorithm for minimizing $v_i^{k_\ell+1}(t)$ over $T$. Since $T$ is a compact metric space, there exists a subsequence $\{\mathbf{x}^{s_\ell}\}$ of $\{\mathbf{x}^{k_\ell}\}$ such that $\{t_i^{s_\ell+1}\}$ converges to a limit point $t_i^* \in T$. Consequently, by (16), $v_i(t_i^*) \ge 0$, for $i = 1, \ldots, m$. Remember that $t_i^{s_\ell+1}$ is the minimizer of $v_i^{s_\ell+1}(t)$ over $T$, hence,

$$\sum_{j=1}^{n} f_{ij}\left(t_i^{s_\ell+1}\right) x_j^{s_\ell} - b_i\left(t_i^{s_\ell+1}\right) \le \sum_{j=1}^{n} f_{ij}\left(\bar{t}_i\right) x_j^{s_\ell} - b_i\left(\bar{t}_i\right), \qquad i = 1, \ldots, m. \tag{17}$$

Moreover, since $\mathbf{x}^{s_\ell}$ also converges to $\bar{\mathbf{x}}$, we have

$$0 \le v_i\left(t_i^*\right) \le v_i\left(\bar{t}_i\right), \qquad i = 1, \ldots, m. \tag{18}$$

It follows that $\bar{\mathbf{x}} \in \mathrm{FP}$, and hence,

$$\sum_{j=1}^{n} c_j \bar{x}_j \ge \sum_{j=1}^{n} c_j x_j^*. \tag{19}$$

Combining (14) and (19), we see that $\bar{\mathbf{x}} \in \mathrm{FP}$ and $\sum_{j=1}^{n} c_j \bar{x}_j = \sum_{j=1}^{n} c_j x_j^*$. Therefore, we know that $\{\mathbf{x}^k\}$ has a subsequence which converges to an optimal solution of (LSIP). ∎

## 6. NUMERICAL EXAMPLE

In this section, we use one simple example to illustrate the proposed theory and solution procedures. Let us consider the following fuzzy linear programming problem:

$$\text{(FLP)} \qquad \begin{aligned} \min \quad & -x_1 - 2x_2 - 2x_3, \\ \text{s.t.} \quad & -\tilde{2}x_1 - \tilde{1}x_2 \ge_\alpha -\tilde{8}, \\ & -\tilde{1}x_3 \ge_\alpha -\tilde{10}, \\ & x_1, x_2, x_3 \ge 0, \end{aligned} \tag{20}$$

where $\alpha \in [0.1]$ and the membership function of each fuzzy coefficient is specified below.

$$\mu_{-\tilde{1}}(x) = \begin{cases} x+2, & \text{if } -2 \le x \le -1, \\ -x, & \text{if } -1 \le x \le 0, \\ 0, & \text{otherwise.} \end{cases} \qquad \mu_{-\tilde{2}}(x) = \begin{cases} \dfrac{x+4}{2}, & \text{if } -4 \le x \le -2, \\ \dfrac{-x}{2}, & \text{if } -2 \le x \le 0, \\ 0, & \text{otherwise.} \end{cases}$$

$$\mu_{-\tilde{8}}(x) = \begin{cases} x+9, & \text{if } -9 \le x \le -8, \\ \dfrac{-x-5}{3}, & \text{if } -8 \le x \le -5, \\ 0, & \text{otherwise.} \end{cases} \quad \mu_{-\tilde{10}}(x) = \begin{cases} x+11, & \text{if } -11 \le x \le -10, \\ \dfrac{-x-8}{2}, & \text{if } -10 \le x \le -8, \\ 0, & \text{otherwise.} \end{cases} \tag{21}$$

Substituting expression (21) in problem (20) results in the following problem:

$$\begin{aligned} \min \quad & -x_1 - 2x_2 - 2x_3, \\ \text{s.t.} \quad & [-2,2,2]x_1 + [-1,1,1]x_2 \ge [-8,1,3], \\ & [-1,1,1]x_3 \ge [-10,1,2], \\ & x_1, x_2, x_3 \ge 0, \end{aligned}$$

which is equivalent to the following linear semi-infinite programming problem:

$$\text{(LSIP)} \quad \begin{aligned} \min \quad & -x_1 - 2x_2 - 2x_3, \\ \text{s.t.} \quad & \begin{pmatrix} -4+2t_1 & -2+t_1 & 0 \\ 0 & 0 & -2+t_2 \\ -2t_3 & -t_3 & 0 \\ 0 & 0 & -t_4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \ge \begin{pmatrix} -9+t_1 \\ -11+t_2 \\ -5-3t_3 \\ -8-2t_4 \end{pmatrix}, \quad \forall t_i \in [\alpha, 1], \\ & x_1, x_2, x_3 \ge 0. \end{aligned}$$

Given any $\alpha \in [0,1]$, say $\alpha = 0.6$ in this example and an arbitrary starting point, say $\mathbf{t}^1 = (t_1^1, t_2^1, t_3^1, t_4^1) = (0.7, 0.8, 0.7, 0.8)$, we have a regular linear program

$$(\text{LP}^1) \quad \begin{aligned} \min \quad & -x_1 - 2x_2 - 2x_3, \\ \text{s.t.} \quad & \begin{pmatrix} -4+2t_1^1 & -2+t_1^1 & 0 \\ 0 & 0 & -2+t_2^1 \\ -2t_3^1 & -t_3^1 & 0 \\ 0 & 0 & -t_4^1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \ge \begin{pmatrix} -9+t_1^1 \\ -11+t_2^1 \\ -5-3t_3^1 \\ -8-2t_4^1 \end{pmatrix}, \\ & x_1, x_2, x_3 \ge 0. \end{aligned}$$

Solving $(\text{LP}^1)$ results in an optimal solution $\mathbf{x}^1 = (x_1^1, x_2^1, x_3^1) = (0, 6.3846, 8.5)$.

$$\begin{aligned} \text{Define} \quad & v_1^2(t_1) = (-4+2t_1)x_1^1 + (-2+t_1)x_2^1 - (-9+t_1) = 5.3846t_1 - 3.7692, \\ & v_2^2(t_2) = (-2+t_2)x_3^1 - (-11+t_2) = 7.5t_2 - 6, \\ & v_3^2(t_3) = -2t_3x_1^1 - t_3x_2^1 - (-5-3t_3) = -3.3846t_3 + 5, \\ & v_4^2(t_4) = -t_4x_3^1 - (-8-2t_4) = -6.5t_4 + 8. \end{aligned}$$

The minimizers of $v_1^2(t_1)$, $v_2^2(t_2)$, $v_3^2(t_3)$, $v_4^2(t_4)$ over $[\alpha, 1] = [0.6, 1]$ are 0.6,0.6,1,1, respectively. Hence, we choose $\mathbf{t}_2 = (t_1^2, t_2^2, t_3^2, t_4^2) = (0.6, 0.6, 1, 1)$.

Since $v_1^2(t_1^2) \le 0$, $v_2^2(t_2^2) \le 0$, $v_3^2(t_3^2) \ge 0$, $v_4^2(t_4^2) \ge 0$, the CPLSIP algorithm iterates with a new linear program

$$
(\mathrm{LP}^2) \qquad \begin{aligned}
\min \quad & -x_1 - 2x_2 - 2x_3, \\
\text{s.t.} \quad & \begin{pmatrix} -4+2t_1^1 & -2+t_1^1 & 0 \\ 0 & 0 & -2+t_2^1 \\ -2t_3^1 & -t_3^1 & 0 \\ 0 & 0 & -t_4^1 \\ \cdots & \cdots & \cdots \\ -4+2t_1^2 & -2+t_1^2 & 0 \\ 0 & 0 & -2+t_2^2 \\ -2t_3^2 & -t_3^2 & 0 \\ 0 & 0 & -t_4^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \ge \begin{pmatrix} -9+t_1^1 \\ -11+t_2^1 \\ -5-3t_3^1 \\ -8-2t_4^1 \\ \cdots \\ -9+t_1^2 \\ -11+t_2^2 \\ -5-3t_3^2 \\ -8-2t_4^2 \end{pmatrix}, \\
& x_1, x_2, x_3 \ge 0.
\end{aligned}
$$

Solving $(\mathrm{LP}^2)$ results in an optimal solution $\mathbf{x}^2 = (x_1^2, x_2^2, x_3^2) = (0, 6.0001, 7.4287)$.

$$
\begin{aligned}
\text{Define} \quad v_1^3(t_1) &= (-4+2t_1)\,x_1^3 + (-2+t_1)\,x_2^3 - (-9+t_1) = 5.0001t_1 - 3.0003, \\
v_2^3(t_2) &= (-2+t_2)\,x_3^2 - (-11+t_2) = 6.4287t_2 - 3.8575, \\
v_3^3(t_3) &= -2t_3x_1^2 - t_3x_2^2 - (-5-3t_3) = -3.0001t_3 + 5, \\
v_4^3(t_4) &= -t_4x_3^2 - (-8-2t_4) = -5.4287t_4 + 8.
\end{aligned}
$$

The minimizers of $v_1^3(t_1)$, $v_2^3(t_2)$, $v_3^3(t_3)$, $v_4^3(t_4)$ over [0.6,1] are 0.6,0.6,1,1, respectively. Hence, we choose $\mathbf{t} = (t_1^3, t_2^3, t_3^3, t_4^3) = (0.6, 0.6, 1, 1)$.

Now, since $v_1^3(t_1^3) \ge 0$, $v_2^3(t_2^3) \ge 0$, $v_3^3(t_3^3) \ge 0$, $v_4^3(t_4^3) \ge 0$, the algorithm stops and returns an optimal solution $\mathbf{x}^* = \mathbf{x}^2 = (0, 6.0001, 7.4287)$ to the fuzzy linear program (20) with $\alpha = 0.6$.

## 7. CONCLUDING REMARKS

In this paper, a linear programming problem with fuzzy coefficients in $A$ and $\mathbf{b}$ is studied. Based on the specific ranking of fuzzy numbers, we have shown that such problems can be reduced to a linear semi-infinite programming problem. The relationship between the optimal solutions and extreme points of (LSIP) are established. A cutting plane algorithm is proposed for solving a linear programming problem with fuzzy coefficients in terms of linear semi-infinite programming. Only those constraints which become binding are generated and used in the solution procedure.

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