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Notes

The Number of Labeled k-Dimensional Trees

L. W. BEINEKE AND R. E. PIPPERT

Purdue University at Fort Wayne, Indiana 46805

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ABSTRACT

The graphs known as trees have natural analogues in higher dimensional simplicial complexes. As an extension of Cayley's formula n^{n-2}, the number of these k-dimensional trees on n-labeled vertices is shown to be $\binom{n}{k}$ $(kn - k^2 + 1)^{n-k-2}$.

One view of graphs is that they are 1-dimensional simplicial complexes, so graphs can be studied in this more general setting. The most elementary connected graphs are of course trees, and these have direct analogs in higher dimensions. The main purpose of this paper is to obtain a formula for the number of these "k-dimensional trees" with n labeled vertices. Some interesting identities are obtained as side products. Cayley [3] first showed that the number of labeled (1-dimensional) trees on n vertices is n^{n-2}, and we earlier announced [2] that there are $\binom{n}{2}(2n - 3)^{n-4}$ labeled 2-dimensional trees. These are special cases of our general formula.

One definition of a tree is inductive: a single vertex is a tree, and a tree with $n + 1$ vertices is obtained when an $(n + 1)$-st vertex is added adjacent to one vertex in a tree with n vertices. An analogous definition is given for *k-dimensional trees* (henceforth called *k-trees*): a set of k mutually adjacent vertices is a k-tree, and a k-tree with $n + 1$ vertices is obtained when an $(n + 1)$-st vertex is added adjacent to each of k already mutually adjacent vertices in a k-tree with n vertices. (This definition treats k-trees as graphs. It is effectively equivalent to the following statement in terms of simplicial complexes. A $(k - 1)$-dimensional simplex is a k-tree, and a k-tree with $n + 1$ vertices is obtained when a k-dimensional simplex is added to a k-tree with n vertices and has precisely a $(k - 1)$-dimensional face in common with it. It is really of little consequence whether a k-tree is considered as a k-dimensional simplicial complex or as the 1-dimensional skeleton thereof; we choose the graph theory terminology.)

In line with the terminology of topology, we make the following definition: an *r-cell* of a k-tree is a complete subgraph of $r + 1$ vertices. Thus, vertices are 0-cells and edges are 1-cells. Furthermore, the building blocks of k-trees are k-cells, with joins being made along $(k - 1)$-cells. The number of r-cells in a k-tree with n vertices can readily be computed to be

$$\binom{k}{r+1} + (n-k)\binom{k}{r}$$

or equivalently

$$\binom{k}{r}\frac{n + r(n-k-1)}{r+1}.$$

Thus, there are $\frac{1}{2}k(2n - k - 1)$ edges, as well as $n - k$ of the k-cells and $nk - k^2 + 1$ of the $(k - 1)$-cells. These last two numbers appear in ormula (1) for the number $T_k(n)$ of labeled k-trees on n vertices.

A k-tree on n vertices is called *labeled* when the integers from 1 to n have been assigned to its vertices (one-to-one). Two labeled k-trees are considered different when there exist two integers which are assigned to adjacent vertices in one graph but not in the other.

THEOREM. *The number of labeled k-trees on n vertices is*

$$T_k(n) = \binom{n}{k}(kn - k^2 + 1)^{n-k-2}. \tag{1}$$

As mentioned earlier, this specializes to known results for the 1- and 2-dimensional cases. Our proof is based on that of Dziobek [4] for Cayley's result. A combinatorial identity will be utilized, the proof of which will be given in two lemmas following this proof.

PROOF: Let $R_k(n)$ denote the number of labeled k-trees on n labeled vertices which are rooted at a particular $(k - 1)$-cell (set of k mutually adjacent vertices). Because there are $\binom{n}{k}$ ways of selecting this cell, the number of labeled k-trees rooted at any $(k - 1)$-cell is $\binom{n}{k} R_k(n)$. Therefore, because the number of $(k - 1)$-cells in a k-tree with n vertices is $kn - k^2 + 1$,

$$(kn - k^2 + 1)T_k(n) = \binom{n}{k} R_k(n). \tag{2}$$

Now consider the following method for constructing labeled k-trees on n vertices (for $n > k$). Begin with a k-cell (that is, $k + 1$ vertices and all corresponding edges), and on each of its $(k - 1)$-cells put a rooted k-tree. How many such constructions are possible? First, the original k-cell can be chosen in $\binom{n}{k+1}$ ways, then there are $(k + 1)!$ possible orderings of its $(k - 1)$-cells. The number of ways of distributing the remaining $n - k - 1$ vertices among the $k + 1$ cells is the multinomial coefficient

$$\binom{n-k-1}{i_1\, i_2 \cdots i_{k+1}},$$

where $i_1 + i_2 + \cdots + i_{k+1}$ is a partition of $n - k - 1$ into non-negative terms. Finally, on the j-th $(k - 1)$-cell there are $R_k(k + i_j)$ possible rooted k-trees. Therefore, the number of constructions is

$$\binom{n}{k+1}(k+1)! \sum_{(n-k-1)} \binom{n-k-1}{i_1\, i_2 \cdots i_{k+1}} \prod_{j=1}^{k+1} R_k(k+i_j), \tag{3}$$

where the notation will indicate that the sum is to be taken over all partitions of $n - k - 1$ into $k + 1$ parts.

However, each labeled k-tree is constructed more than once via this procedure, in fact, in $(n - k)(k + 1)!$ ways. This is because there are $(n - k)$ possible k-cells on which it can be built and, from a given k-cell, there are the $(k + 1)!$ possible orderings of its $(k - 1)$-cells. Therefore, the quantity in (3) is equal to

$$(n-k)(k+1)!\, T_k(n). \tag{4}$$

Taken with (2), the equality of (3) and (4) implies

$$(k+1)R_k(n) = (kn - k^2 + 1) \sum_{(n-k-1)} \binom{n-k-1}{i_1\, i_2 \cdots i_{k+1}} \prod_{j=1}^{k+1} R_k(k+i_j).$$

This equation will be instrumental in an induction proof of the equality

$$R_k(m) = (mk - k^2 + 1)^{m-k-1} \qquad (m \geqslant k). \tag{5}$$

Statement (5) is certainly true for $m = k$ and $m = k + 1$. Assume $n > k$ and that (5) holds for all values of m less than n. Then, since $k + i_j < n$,

$$R_k(k+i_j) = (ki_j + 1)^{i_j - 1}$$

so that

$$(k+1)R_k(n) = (kn - k^2 + 1) \sum_{(n-k-1)} \binom{n-k-1}{i_1\, i_2 \cdots i_{k+1}} \prod_{j=1}^{k+1} (ki_j + 1)^{i_j - 1}.$$

By Lemma 2, this reduces to

$$(k+1)\, R_k(n) = (k+1)(kn - k^2 + 1)^{n-k-2},$$

which completes the induction proof.

The theorem now follows at once from a combination of equations (2) and (5).

The identity used in proving our main theorem requires another identity in its proof. The proof of this additional identity relies on a formula of Abel [1].

LEMMA 1. *For* $a \neq 0$,

$$a \sum_{h=0}^{m} \binom{m}{h} (bh+1)^{h-1}(bm - bh + a)^{m-h-1}$$
$$= (a+1)(bm + a + 1)^{m-1}. \qquad (6)$$

PROOF: Two equivalent forms of an identity due to Abel are the following. For $c \neq 0$,

$$(x+c)^{m-1} = c \sum_{h=0}^{m-1} \binom{m-1}{h} (bh+c)^{h-1}(x - bh)^{m-h-1} \qquad (7)$$

and

$$(x+c)^{m-1}$$
$$= c \sum_{j=0}^{m-1} \binom{m-1}{j} (b(m-j-1)+c)^{m-j-2}(x - b(m-j-1))^{j}. \qquad (8)$$

In (7) let $x = bm + a$ and $c = 1$; then

$$(bm + a + 1)^{m-1} = \sum_{h=0}^{m-1} \binom{m-1}{h} (bh+1)^{h-1}(bm - bh + a)^{m-h-1}. \qquad (9)$$

In (8) let $x = bm + 1$, and $c = a$, and replace j by $h - 1$; then

$$(bm + a + 1)^{m-1} = a \sum_{h=1}^{m} \binom{m-1}{h-1} (bm - bh + a)^{m-h-1}(bh+1)^{h-1}. \qquad (10)$$

The addition of a times (9) to (10) and the utilization of a well-known property of binomial coefficients yield

$$\begin{aligned}
&(a+1)(bm+a+1)^{m-1}\\
&\quad = a\sum_{h=1}^{m-1}\binom{m}{h}(bh+1)^{h-1}(bm-bh+a)^{m-h-1}+a(bm+a)^{m-1}\\
&\qquad + a(a)^{-1}(bm+1)^{m-1}\\
&\quad = a\sum_{h=0}^{m}\binom{m}{h}(bh+1)^{h-1}(bm-bh+a)^{m-h-1}.
\end{aligned}$$

This completes the proof of the lemma.

LEMMA 2.

$$\sum_{(n-k-1)}\binom{n-k-1}{i_1\, i_2\, \cdots\, i_{k+1}}\prod_{j=1}^{k+1}(ki_j+1)^{i_j-1}=(k+1)(kn-k^2+1)^{n-k-2},$$

where the sum is over all $(k+1)$-part partitions of $n-k-1$.

PROOF: This will be proved by a rather intricate induction on the number of terms in the partition, that is, the number of sums. Notation which will be used is the following. Let

$$I_r = n-k-1-\sum_{j=1}^{r-1} i_j \qquad \text{for} \quad r = 1, 2, \ldots, k+1.$$

Then in general

$$I_r = \sum_{j=r}^{k+1} i_j\,,$$

and in particular $I_1 = n-k-1$ and $I_{k+1} = i_{k+1}$. Further, let

$$S_{k+1} = (ki_{k+1}+1)^{i_{k+1}-1}.$$

Assuming that S_{p+1} has been given, let

$$S_p = \sum_{i_p=0}^{I_p}\binom{I_p}{i_p}(ki_p+1)^{i_p-1}S_{p+1}\,.$$

Then, because a multinomial coefficient can be expressed as a product of binomial coefficients, S_1 is the sum given in the statement of the lemma.

The induction will show (in reverse) that

$$S_r = (k - r + 2)(kI_r + k - r + 2)^{I_r - 1} \qquad \text{for} \quad r = 1, 2, \ldots, k + 1.$$

This is clearly true for $r = k + 1$. Assume that it is true for $r = p + 1$. Then

$$S_p = (k - p + 1) \sum_{i_p=0}^{I_p} \binom{I_p}{i_p} (ki_p + 1)^{i_p - 1} (kI_{p+1} + k - p + 1)^{I_{p+1} - 1}.$$

Now in the identity (6) of Lemma 1 let $a = k - p + 1$, $b = k$, $h = i_p$, and $m = I_p$ (so $m - h = I_{p+1}$). It follows that

$$S_p = (k - p + 2)(kI_p + k - p + 2)^{I_p - 1},$$

which completes the induction step. Therefore

$$S_1 = (k + 1)(nk - k^2 + 1)^{n-k-2}$$

and the lemma is proved.

References

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