(Z_2)^k-actions with fixed point set of constant codimension 2^k + 2v + 1✩

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ABSTRACT

By using the mathematical induction and constructing indecomposable classes, the ideal J_{2^k+2v+1} of cobordism classes in the unoriented cobordism ring MO_\ast containing a representative admitting a (Z_2)^k-action with fixed point set of constant codimension 2^k + 2v + 1 is determined.

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1. Introduction

Let \( \phi : (Z_2)^k \times M^n \rightarrow M^n \) be a smooth action of the group \((Z_2)^k\) on a closed \(n\)-dimensional manifold. Here \((Z_2)^k\) is considered as the group generated by \(k\) commuting involutions. The fixed point set of the action of \((Z_2)^k\) on \(M^n\), i.e.

\[ F = \{ x \in M^n \mid \phi(g, x) = x, \forall g \in (Z_2)^k \}, \]

is a disjoint union of closed submanifolds of \(M^n\). If each component of \(F\) is \((n-r)\)-dimensional, then \(F\) has codimension \(r\). Let \(J'_{n,k}\) be the set of \(n\)-dimensional unoriented cobordism classes containing a representative \(M^n\) admitting a \((Z_2)^k\)-action with fixed point set of constant codimension \(r\) and \(J'_{n,k} = \sum_{n \geq r} J'_{n,k} \). By [1] we know that \(J'_{n,k}\) is a subgroup of the unoriented cobordism group \(MO_n\) and \(J'_{n,k}\) is an ideal in the unoriented cobordism ring \(MO_\ast = \sum_{n \geq 0} MO_n\). It is immediate that \(J'_{n,k} \subset J'_{n,k+1}\) and \(J'_{n,k} \cdot J'_{n,k} \subset J'_{n,k} \cdot J'_{n,k} \), where \(J'_{n,k} \cdot J'_{n,k} = \{ xy \mid x \in J'_{n,k}, \ y \in J'_{n,k} \}. \) It is well known that \(MO_\ast\) is a \(Z_2\)-polynomial algebra on a single generator \(x_i\) of each dimension not of the form \(2^u - 1\) [1]. If a cobordism class \([M^n]\) of a smooth closed manifold \(M^n\) can be expressed as a sum of products of lower-dimensional cobordism classes, then \([M^n]\) is called decomposable; otherwise it is indecomposable. Indecomposable classes can be used as generators of \(MO_\ast\).

In 1973, R.E. Stong raised a question of determining the ideal \(J'_{n,1}\) in terms of Stiefel–Whitney number [2], probably inspired in the fact, previously proved by P.E. Conner and E.E. Floyd [3], that if a manifold admits an involution whose fixed point set has codimension one, then it bounds. In the paper, Stong solved the case \(r = 2\).

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In 1992, P.L.Q. Pergher introduced this same question by replacing involutions by \((\mathbb{Z}_2)^k\)-actions \([4]\). Also, he computed \(J^1_{s,k}, J^2_{s,k}\) \((k > 1)\) and made a lot of characteristic numbers calculations to determine certain sub-ideals of \(J^3_{s,2}\), defined in terms of certain restrictions on the fixed data of the actions.

In 1994, R.J. Shaker got the most important advance in the direction of computing \(J^r_{s,k}\) \([5]\). He completely solved the problem for the case \(0 < r < 2^k\). The crucial point was the discovery of a new method for constructing models of \((\mathbb{Z}_2)^k\)-actions on certain manifolds so that the indecomposability of the manifold and the codimension of the fixed point set of the action can be controlled. Later, R.J. Shaker used the Dold manifolds to find new indecomposable manifolds equipped with actions and solved the case \(r = 2^k\) \([6]\).

In 1999, Y. Wang et al. made an important advance for \(r > 2^k\) in \([7]\). The authors showed that, for a given \(r > 2^k\), there exists a number \(g(r)\) much bigger than \(r\) so that, if \(n > g(r)\), then \(J^r_{n,k}\) contains all possible classes. For \(r > 2^k\), this completely solves the question of computing \(J^r_{n,k}\) for all sufficiently large \(n\). In other words, for \(r > 2^k\), this reduces the computation of \(J^r_{n,k}\) to a finite list of values of \(n\), \(n = r, r + 1, \ldots, g(r)\). As a consequence the authors calculated \(J^r_{n,k}\) for \(r = 2^k + 3, k > 4\) and \(n \geq 2^k + 4\).

By using Shaker scheme and the additional stuff, some particular cases \(r = 2^k + t\), where \(t\) has fixed small values \((t = 1, 2, 3, 4, 5, 6, \ldots)\), were solved \([7-12]\).

In this paper, by using the mathematical induction and constructing indecomposable classes we determine the ideal \(J^r_{n,k}\) \((k > 3)\), \(r = 2^k + t\), where \(1 \leq t \leq 2^{k-2} - 1\) and \(t\) is odd. The difference between this paper and previous papers is that if \(k\) increases, then the values of \(t\) \((r = 2^k + t)\) possible of computing also increase, depending on \(k\). For example, for \(2^k = 2^{10}\), the value \(t = 255\).

In Section 4 we also make a statement of the result for some \(r = 2^k + 2v\). In the end we pose a conjecture about the structure of the ideal \(J^r_{n,k}\) for a general \(r\).

Let \(\chi : MO = \sum_{n \geq 0} MO_n \to \mathbb{Z}_2\) denote the mod 2 Euler characteristic. The main results are

**Theorem 1.1.** For \(k \geq 3\), let \(2^k + 2v + 1 = 2^k + 2v_i + \cdots + 2^n\) with \(k - 3 \geq r_s > \cdots > r_0 = 0\) and \(\{x_i\} (i \neq 2^k - 1)\) be a system of generators of \(MO_n\). Then we have

\[
J_{n,k}^{2k+2v+1} = \begin{cases} 
MO_n \cap \text{Ker} \chi, & n \geq 2^k + 2v + 2, \\
\{x_{i_1} \cdots x_{i_m} | 2 \leq i_1 \leq \cdots \leq i_m < 2^k, m \geq 2\}, & n = 2^k + 2v + 1.
\end{cases}
\]

**Theorem 1.2.** For \(k \geq 2\), let \(2^k + 2v + 1 = 2^k + 2v_i + \cdots + 2^n\) with \(k - 2 = r_s > \cdots > r_0 = 0\). Then \(J_{n,k}^{2k+2v+1} = MO_n \cap \text{Ker} \chi\) for \(n \geq 2^k+1 + 2k-1 - 2\).

**Theorem 1.3.** For \(k \geq 2\), let \(2^k + 2v + 1 = 2^k + 2v_i + \cdots + 2^n\) with \(k - 1 = r_s > \cdots > r_0 = 0\). Then \(J_{n,k}^{2k+2v+1} = MO_n \cap \text{Ker} \chi\) for \(n \geq 2^{k+2} + 2k - 2\).

**Remark 1.** Theorem 1.1 determines the ideal \(J_{s,k}^{2k+2v+1}\) for \(k \geq 3\) and \(k - 3 \geq r_s\). For \(k \geq 2\) and \(r_s = k - 2\) or \(k - 1\), it is hard to find indecomposable classes in \(J_{s,k}^{2k+2v+1}\) on dimensions close to \(2^k + 2v + 1\), so the complete determination of the ideal is difficult. In this case Theorem 1.2 and Theorem 1.3 determine some groups \(J_{n,k}^{2k+2v+1}\) and improve the result of \([7]\).

2. Background

Let \(\text{RP}(n)\) be the real projective space of dimension \(n\). Binomial coefficients are \(\binom{m}{n}\) \(= m!/(n!(m-n)!).\) For our purpose, we will exhibit indecomposable classes as generators of \(MO_n\). They will come from the following source:

**Lemma 2.1.** \(([2])\) Let \(\text{RP}(n_1, n_2, \ldots, n_l)\) be the projective space bundle of \(\lambda_1 \oplus \lambda_2 \oplus \cdots \oplus \lambda_l\) over \(\text{RP}(n_1) \times \text{RP}(n_2) \times \cdots \times \text{RP}(n_l)\), where \(\lambda_i\) is the pullback of the canonical line bundle over the \(i\)-th factor. Then for \(l > 1\), \([\text{RP}(n_1, n_2, \ldots, n_l)]\) is indecomposable in \(MO_n\) if and only if

\[
S = \left(\frac{n + l - 2}{n_1}\right) + \left(\frac{n + l - 2}{n_2}\right) + \cdots + \left(\frac{n + l - 2}{n_l}\right) \equiv 1 \mod 2,
\]

where \(n = n_1 + n_2 + \cdots + n_l\).

The manifold \(\text{RP}(n_1, n_2, \ldots, n_l)\) has dimension \(n + l - 1\). If \(n_{i+1} = n_{i+2} = \cdots = n_l = 0\), then \(\text{RP}(n_1, n_2, \ldots, n_l)\) will sometimes be written as \(\text{RP}(n_1, n_2, \ldots, n_l, 1)\).

To calculate binomial coefficients mod 2 the following lemma is extremely useful.
Lemma 2.2. ([5]) If \( m = \sum_{i=0}^{l} m_i 2^i \) and \( n = \sum_{i=0}^{l} n_i 2^i \), with \( 0 \leq m_i, n_i \leq 1 \), then \( \left( \frac{m}{n} \right) \equiv 1 \mod 2 \) if and only if \( n_i \leq m_i \) for every \( i \).

Lemma 2.3. ([5]) Let \( \lambda_i \to X_i \) be line bundles, and let \((\mathbb{Z}_2)^k \) act on \( \lambda_i \) as bundle maps with fixed point set \( F_i \) on \( X_i \) for all \( 1 \leq l \leq l \) with \( \sum_{l=0}^{l} 2^l \leq 2^k \). Then \( RP(\lambda_1 \oplus \lambda_2 \oplus \cdots \oplus \lambda_l) \) admits a \((\mathbb{Z}_2)^k\)-action with fixed point set \( F_1 \times F_2 \times \cdots \times F_l \times E \), where \( E \) is a set of \( l \) points.

Lemma 2.4. ([7]) For \( l + 1 \leq 2^k \), \( \{RP(2m + 1, n_2, n_3, \ldots, n_l)\} \in \mathcal{J}_{s,k} \).

Lemma 2.5. ([7]) Let \( k \geq 2 \), \( r = 2^{m_1} + 2^{m_2} - 1 + \cdots + 2^{m_l} > 2^k \), \( r_m > r_{m-1} > \cdots > r_1 \geq 0 \), then for \( n \geq 2^{m+1} \) and \( n \neq 2^u - 1 \), there exist indecomposable generators \( x_0 \) in \( \mathcal{J}_{n,k} \).

The generalized Dold manifold was introduced in [13]. Let \( S^m \) be an \( m \)-dimensional sphere. For a space \( X \) and a positive integer \( m \), let \( P(m, X) \) be formed from \( S^m \times X \times X \) by identifying \((u,x,y)\) with \((-u,y,x)\). If \( X \) is an \( n \)-manifold, then \( P(m, X) \) is an \((m+2n)\)-manifold which is called generalized Dold manifold. The next lemma was proved in [13]:

Lemma 2.6. ([13]) \( P(m, M^m) \) is indecomposable in \( MO_* \) if and only if \( [M^m] \) is indecomposable and
\[
\left( \frac{m + n - 1}{m - 1} \right) \equiv 1 \mod 2.
\]

Lemma 2.7. ([5]) If \([M^m] \) is indecomposable and \( n - r < [n/2^k] \), then \([M^m] \notin \mathcal{J}_{s,k} \).

Lemma 2.8. ([14]) Let \((\mathbb{Z}_2)^k \) act on a manifold \( M^m \) and characteristic number \( s_{(\lambda_1, \ldots, \lambda_k)}[M] \neq 0 \mod 2 \), then the dimension of some component of the fixed point set of \( M \) is no less than \([\lambda_1/2^k] + \cdots + [\lambda_k/2^k] \).

Lemma 2.9. ([15]) If \( |x_i| \) \( (i \neq 2^u - 1) \) are generators of \( MO_* \), then characteristic number
\[
s_\omega(x_1, x_2, \ldots, x_{m_{ij}}) = \begin{cases} 
0 & \text{if } \omega \text{ is not the refinement of } (i_1, i_2, \ldots, i_m), \\
1 & \text{if } \omega = (i_1, i_2, \ldots, i_m).
\end{cases}
\]

If \( w(M^n) = \prod_{i=1}^{l} (1 + a_i) \), where \( w \) denote the Stiefel–Whitney class, then
\[
s_\omega(M^n) = \begin{cases} 
s_\omega(a_1, a_2, \ldots, a_j) & \text{if } j \geq n, \\
s_\omega(a_1, a_2, \ldots, a_j, 0, \ldots, 0) & \text{if } j < n.
\end{cases}
\]

Lemma 2.10. ([5]) For \( \mathcal{J}_{s,k} \) we have the following result
\[
\mathcal{J}_{s,k}^r \leq \bigoplus_{n=r}^{\infty} MO_n \quad \text{r even,} \\
\mathcal{J}_{s,k}^r \leq \bigoplus_{n=r}^{\infty} MO_n \cap \text{Ker } \chi \quad \text{r odd, } r > 1.
\]

Lemma 2.11. There exist indecomposable classes \( x_0 \in \mathcal{J}_{n,k}^{2^k+2l} \) for \( n \neq 2^u - 1, n > 2^k + 2l \) and \( 2^k + 2l > 2l \) \( \geq 0 \).

Proof. The proof is divided into two cases \( l = 0 \) and \( l > 0 \).

1. \( l = 0 \). By [6], the lemma is established.
2. \( l > 0 \). Since \( n \) is not of the form \( 2^u - 1 \), \( \frac{n-1}{2} \) is also not of the form \( 2^u - 1 \). For \( n > 2^k + 2l \), take an indecomposable class \( X \in \mathcal{J}_{s,k}^{2^k+2l} \) by [5]. Let \( M \) be a representative of \( X \) admitting a \((\mathbb{Z}_2)^k\)-action with fixed point set \( F' \) of dimension \( \frac{n-1}{2} - 2^{k-1} - l \). Let \( T'_i \) \( (i = 1, 2, \ldots, k) \) be the \((\mathbb{Z}_2)^k\)-action on \( M \). Define commuting involutions \( T_1, T_2, \ldots, T_k \) on \( S^1 \times M \times M \) as follows:
\[
T_1(u, x, y) = (u, T'_1(x), T'_1(y)), \\
T_2(u, x, y) = (u, T'_2(x), T'_2(y)), \\
\vdots \\
T_k(u, x, y) = (u, T'_k(x), T'_k(y)).
\]
These involutions commute with $T$ and they induce a $(\mathbb{Z}_2)^k$-action on $P(1, M)$. The fixed point set of the action on $P(1, M)$ is $F = S^1 \times F' / T$ with dimension $n - 2k - 2l$. Since
\[
\left( 1 + \frac{n-1}{1 - 1} \right) \equiv 1 \text{ mod } 2.
\]
by Lemma 2.6 $[P(1, M)]$ is indecomposable. Therefore, there exist indecomposable classes $x_n = [P(1, M)] \in J_{n,k}^{2k+2l}$.

\section{Existence of indecomposable classes}

The main task in this section is to prove the existence of indecomposable classes in $J_{n,k}^{2^k+2v+1}$. We will complete it by the mathematical induction.

\textbf{Lemma 3.1.} For $k \geq 3$, let $2^k + 2v + 1 = 2^k + 2^{r_1} + \cdots + 2^{r_k}$ with $k - 3 \geq r_3 \geq \cdots > r_0 = 0$. Then there exist indecomposable classes $x_n \in J_{n,k}^{2^k+2v+1}$ for $n \geq 2^k + 2v + 2$ and $n \not\equiv 2^u - 1$.

\textbf{Proof.} When $v = 0$, $J_{n,k}^{2^k+2v+1} = J_{n,k}^{2^k+1}$. From [8,9], there exist indecomposable classes $x_n \in J_{n,k}^{2^k+2v+1}$ for $n \geq 2^k + 2v + 2$.

Assume that for $0, 1, 2, \ldots, v - 1$, the lemma is established.

We prove that the lemma holds for $v \geq 1$.

\textbf{Case 1.} $n$ odd.

In this case, we consider the following dimensions respectively.

- Dimension $2^k + 2v + 3 \leq n < 2^{k+1} - 1$.
- Dimension $2^{k+1} + 1 \leq n < 2^{k+2} - 1$.
- Dimension $n \geq 2^{k+2} + 1$.

Dimension $2^k + 2v + 3 \leq n < 2^{k+1} - 1$.

Suppose $n = 2^k + 2v + 1 + 2p$, $1 \leq p \leq 2^{k-1} - v - 2$, $p \in \mathbb{Z}$ (the set of integers).

In order to exhibit indecomposable classes $x_n \in J_{n,k}^{2^k+2v+1}$ we show that there exist $(2^k-1 + v + p)$-dimensional indecomposable classes $[M] \in J_{n,k}^{2^k-1+v+p+1}$.

1. $p \leq v$.
   (a) $v - p$ even.
   Since $2^k-1 + v - p + 1 < 2^{k-1} + 2v + 1$, $r_3 \geq 1$ and $k - 1 \geq (r_3 - 1) + 3$, by the indictional assumption there exist $(2^k-1 + v + p)$-dimensional indecomposable classes $[M] \in J_{n,k}^{2^k-1+v+p+1}$.

   (b) $v - p$ odd.
   Since $v + p$ is odd and $k \geq r_3 + 1$, by Lemma 2.11 there exist $(2^k-1 + v + p)$-dimensional indecomposable classes $[M] \in J_{n,k}^{2^k-1+v+p+1}$.

2. $p > v$.
   Since $2v + 2 \leq 2^k-1 + v - p + 1 \leq 2^{k-1}$, by [5,6] there exist $(2^k-1 + v + p)$-dimensional indecomposable classes $[M] \in J_{n,k}^{2^k-1+v+p+1}$.

Therefore, there exist $(2^k-1 + v + p)$-dimensional indecomposable classes $[M] \in J_{n,k}^{2^k-1+v+p+1}$ where $M$ admits commuting involutions $T'_1, T'_2, \ldots, T'_{k-1}$ with fixed point set $F'$ of dimension $2p - 1$. Consider the generalized Dold manifold $P(1, M) = S^1 \times M \times M / T$ which is of dimension $n$. Define commuting involutions $T_1, T_2, \ldots, T_k$ on $S^1 \times M \times M$ as follows:

\[
T_1(u, x, y) = (-u, x, y),
T_2(u, x, y) = (-u, T'_1(x), T'_1(y)),
\]
\[
\vdots
\]
\[
T_k(u, x, y) = (-u, T'_{k-1}(x), T'_{k-1}(y)).
\]

Then each $T_1$ commutes with $T$ and they induce a $(\mathbb{Z}_2)^k$-action on $P(1, M)$. The fixed point set of the action on $P(1, M)$ is $F = S^1 \times \Delta(F' \times F') / T$ with dimension $2p$, where $\Delta(F' \times F') = \{(x, x) \mid x \in F'\}$. Since $[M]$ is indecomposable and
by Lemma 2.6 \([P(1, M)]\) is indecomposable. Therefore, there exist indecomposable classes \([P(1, M)] \in F_{n,k}^{2k+2v+1}\).

**Dimension** \(2k+1 + 1 \leq n < 2k+2 - 1\).

Take \(x_n = [RP(2k+1 - 4v - 3, n - 2k+1 + 1; 4v + 3)]\). Since

\[
\left( \frac{n - 1}{2k+1 - 4v - 3} \right) + \left( \frac{n - 1}{n - 2k+1 + 1} \right) + (4v + 1) \left( \frac{n - 1}{0} \right) \equiv 1 \text{ mod } 2,
\]

by Lemma 2.1 \(x_n\) is indecomposable.

By Lemma 2.4, \(l + 1 = 4v + 3 + 1 \leq 2k, 2m + 1 = 2k+1 - 4v - 3, \) so \(m = 2k - 2v - 2, m + l = 2k + 2v + 1, x_n \in F_{n,k}^{2k+2v+1}\).

**Dimension** \(n \geq 2k+2 + 1\).

There exist indecomposable classes \(x_n \in F_{n,k}^{2k+2v+1}\) by Lemma 2.5. Therefore, the lemma holds for \(n\) odd.

**Case 2.** \(n\) even.

We consider the following dimensions respectively.

- **Dimension** \(n = 2k + 2v + 2\).
- **Dimension** \(2k + 2v + 4 \leq n \leq 2k+1 - 2\).
- **Dimension** \(n = 2k+1\).
- **Dimension** \(2k+1 + 2 \leq n \leq 2k+2 - 4v + 2\).
- **Dimension** \(2k+2 - 4v + 4 \leq n \leq 2k+2 - 2\).
- **Dimension** \(n \geq 2k+2\).

**Dimension** \(n = 2k + 2v + 2\).

Take \(x_n = [RP(2^r + 3 - 1; 2k + 2v - 2^r + 3 + 4)]\).

Since \(n - 1 = 2k + 2v + 1 = 2k + 2^r + \ldots + 2^0, 2^r + 3 - 1 = 2^{r+2} + 2^r + 1 + \ldots + 1, \) so

\[
\left( \frac{n - 1}{2^{r+3} - 1} \right) + \left( \frac{2k + 2v - 2^{r+3} + 4 - 1}{2^r} \right) \left( \frac{n - 1}{0} \right) \equiv 1 \text{ mod } 2.
\]

By Lemma 2.1, \(x_n\) is indecomposable.

In order to prove \(x_n \in F_{n,k}^{2k+2v+1}\), let \(y_1, \ldots, y_{2n+1}\) be in \(RP(2^r + 3 - 1)\) and \(T_j (1 \leq j \leq r_1 + 2)\) act on \(RP(2^r + 3 - 1)\) as multiplication by \(-1\) in \(y_1\) if \(i \equiv 1, 2, \ldots, 2^j \mod 2^{j+1}\). Then \((T_1, T_2, \ldots, T_{r_1+2})\) defines a \((Z_2)^{r_1+2}\)-action on \(RP(2^r + 3 - 1)\) with fixed point set of \(2^r+2\) copies of \(RP(1)\). Let \((Z_2)^0\) act as the identity on the rest of the base. Since \(2^{r+2} + 2k + 2v - 2^{r+3} + 4 - 1 \leq 2k\), by Lemma 2.3 \(x_n\) is as required.

**Dimension** \(2k + 2v + 4 \leq n \leq 2k+1 - 2\).

In this part, suppose \(n = 2k + 2v + 2 + 2p, 1 \leq p \leq 2k-1 - v - 2, p \in \mathbb{Z}.

1. \(p + v\) even.
   (a) \(p \leq v\).
      Since \(2k-1 + 1 \leq 2k-1 + v - p + 1 \leq 2k-1 + 2v - 1\) and \(k - 1 \geq (r_1 - 1) + 3\), by the inductional assumption, there exist \((2k-1 + v + p)\)-dimensional indecomposable classes \([M] \in F_{2k-1 + v + p, k-1}^{2k-1 + v + p, k-1}\).
   (b) \(p > v\).
      Since \(2k-1 + v - p + 1 \leq 2k-1\), by [5] there exist \((2k-1 + v + p)\)-dimensional indecomposable classes \([M] \in F_{2k-1 + v + p, k-1}^{2k-1 + v + p, k-1}\).

So, when \(p + v\) is even, there exist \((2k-1 + v + p)\)-dimensional indecomposable classes \([M] \in F_{2k-1 + v + p, k-1}^{2k-1 + v + p, k-1}\), where \(M\) admits commuting involutions \(T_1, T_2, \ldots, T_{k-1}\) with fixed point set \(F'\) of dimension \(2p - 1\). Consider the generalized Dold manifold \(P(2, M) = S^2 \times M \times M / T\) which is of dimension \(n\). Define commuting involutions \(T_1, T_2, \ldots, T_k\) on \(S^2 \times M \times M\) as follows:

\[T_1(u, x, y) = (-u, x, y),\]
\[ T_2(u, x, y) = (-u, T'_1(x), T'_1(y)), \]

\[ T_k(u, x, y) = (-u, T'_{k-1}(x), T'_{k-1}(y)). \]

Then each \( T_i \) commutes with \( T \) and they induce a \( (\mathbb{Z}_2)^k \)-action on \( P(2, M) \). The fixed point set of the action on \( P(2, M) \) is \( F = S^{2^k} \times \Delta(F \times F')/T \) with dimension \( 2p + 1 \). Since \( [M] \) is indecomposable and

\[
\left( 2 + 2^{k-1} + v + p - 1 \right) \equiv 1 \mod 2
\]

by Lemma 2.6 \([P(2, M)]\) is as required.

2. \( p + v \) odd.

Suppose \( n = 2^k + 2^{n_1} + 2^{n_2} + \cdots + 2^{n_1}, k - 1 \geq n_1 > \cdots > n_0 \geq 2. \)

(a) \( p \geq 2^{n_0} - 1.\)

We have \( \frac{n + 2^{n_1}}{2} - p - 1 = 2^{k-1} + v - p + 2^{n_1} - 1 \leq 2^{k-1} + 2v - 1. \)

By the inductional assumption, there exist \( \frac{n}{2} - 2^{n_1} \)-dimensional indecomposable classes \([M] \in J_{\frac{n}{2} - 2^{n_1} - 1}. \)

(ii) \( \frac{n + 2^{n_0}}{2} - 2p - 1 \leq 2^{k-1} \)

By [5,6] there exist \( \frac{n}{2} - 2^{n_1} \)-dimensional indecomposable classes \([M] \in J_{\frac{n}{2} - 2^{n_1} - 1}. \)

Therefore, when \( p + v \) is odd and \( p \geq 2^{n_1} - 1 \), there exist \( \frac{n}{2} - 2^{n_1} \)-dimensional irreducible classes \([M], \) where \( M \)

admits commuting involutions \( T_1, T_2, \ldots, T_k \) with fixed point set \( F' \) of dimension \( 2p + 1 \) and \( 2^{n_0}. \) Consider the generalized Dold manifold \( P(2^{n_0}, M) = S^{2^{n_0}} \times M \times M/T \) with dimension \( n. \) Define commuting involutions \( T_1, T_2, \ldots, T_k \)

on \( S^{2^{n_0}} \times M \times M \) as follows:

\[ T_1(u, x, y) = (-u, x, y), \]
\[ T_2(u, x, y) = (-u, T'_1(x), T'_1(y)), \]
\[ \vdots \]
\[ T_k(u, x, y) = (-u, T'_{k-1}(x), T'_{k-1}(y)). \]

Then each \( T_i \) commutes with \( T \) and they induce a \( (\mathbb{Z}_2)^k \)-action on \( P(2^{n_0}, M). \) The fixed point set of the action on \( P(2^{n_0}, M) \) is \( F = S^{2^{n_0}} \times \Delta(F \times F')/T \) with dimension \( 2p + 1. \) Since \( [M] \) is indecomposable and

\[
\left( 2^{n_0} + \frac{n}{2} - 2^{n_1} - 1 \right) \equiv 1 \mod 2
\]

by Lemma 2.6 \([P(2^{n_0}, M)]\) is as required.

(b) \( 1 \leq p < 2^{n_0} - 1.\)

Suppose \( 2p + 2 \leq 2^{n_1} + 2^{n_1} - 2 \cdots + 2^{n_0} \), \( 2^{k} > g_1 \geq g_2 \geq \cdots > g_{k} \geq 1, \) \( 2 \leq g_{k} \leq n_0. \)

Take \( x_0 = \left[ \text{RP}(2^{n_1} - 2^{n_1} - 2^{k} + 2p + 1) \right] \).

By Lemma 2.1, \( x_0 \) is indecomposable.

In order to prove \( x_0 \in J_{2^{k+2}+2^{k+1}+1} \), let \( y_1, \ldots, y_{2^{k+2}+2^{k+1}+1} \) be in \( \text{RP}(2^{n_1} + 2^{k} + 2^{k} + 2^{2k} + \cdots + 2^{n_0} \equiv 1 \mod (2p + 2)) \). Then \( (T_1, T_2, \ldots, T_k) \) defines a \( (\mathbb{Z}_2)^{2k} \)-action on \( \text{RP}(2^{n_1} + 2^{k} + 2^{k} + 2p + 2) \) with fixed point set of \( 2^{n_1} + 2^{k} + 2p + 2 \) copies of \( \text{RP}(2^{n_1} + 2^{k} + 2p + 2) \).

So there exist indecomposable classes \( x_0 \in J_{2^{k+2}+2^{k+1}+1} \) for \( 2^{k} + 2^{k} + 4 \leq n \leq 2^{k+1} - 2. \)

Dimension \( n = 2^{k+1}. \)

Take \( x_0 = \left[ \text{RP}(2^{k} + 1, 2^{k} - 2 - 2^{k+1} + 2^{k+1} + 1) \right]. \) By Lemma 2.1 and Lemma 2.4, \( x_0 \) is indecomposable and \( x_0 \in J_{2^{k+2}+2^{k+1}+1}. \)

Dimension \( 2^{k+1} + 2 \leq n \leq 2^{k+2} - 4^{v} + 2. \)

Take \( x_0 = \left[ \text{RP}(2^{k+1} - 2^{k} + 3, n - 2^{k+1} - 2; 2^{4v}) \right]. \) By Lemma 2.1 and Lemma 2.4, \( x_0 \) is indecomposable and \( x_0 \in J_{2^{k+2}+2^{k+1}+1}. \)
Dimension $2^{k+2} - 4v + 4 \leq n \leq 2^{k+2} - 2$.

Let $n = 2^{k+2} - (2^{r_i+1} + \cdots + 2^{r_0+1}) + 6 + 2p, 0 \leq p \leq 2v - 3, p \in \mathbb{Z}$.

Then $n = (2^{k+1} + \cdots + 1) - (2^{r_1+1} + \cdots + 2^{r_0+1} - 6 - 2p)$.

Suppose $2^{r_1+1} + \cdots + 2^{r_0+1} - 6 - 2p = 2^{h_1} + \cdots + 2^{h_p}, g_1 > g_i - 1 > \cdots > g_0 \geq 1, 2^{r_1+1} + \cdots + 2^{r_0+1} - 6 = 2^{h_j} + \cdots + 2^{h_p}$, $h_j > h_{j-1} > \cdots > h_0 \geq 2$.

1. $(g_0, \ldots, g_1) \not\subseteq (h_0, \ldots, h_j)$.

Take $x_n = [RP(2^{k+1} - 4v + 3, n - 2^{k+1} - 2; 4v)]$. By Lemma 2.1 and Lemma 2.4, $x_n$ is indecomposable and $x_n \in J_{n,k}^{2^{k+2}+v+1}$.

2. $(g_0, \ldots, g_1) \subseteq (h_0, \ldots, h_j)$.

(a) $g_0 = h_i$ for some $i \geq 1$.

Take $x_n = [RP(2^{k+1} - 4v + 3 + 2^{h_i} + \cdots + 2^{h_0} + 2^2, n - 2^{k+1} - (2^{h_i-1} + \cdots + 2^{h_0-1} + 2^2); 4v - (2^{h_i-1} + \cdots + 2^{h_0-1} + 2))].$ By Lemma 2.1 and Lemma 2.4, $x_n$ is indecomposable and $x_n \in J_{n,k}^{2^{k+2}+v+1}$.

(b) $g_0 = h_0$.

Take $x_n = [RP(2^{k+1} - 4v + 7, n - 2^{k+1} - 4; 4v - 2)].$ By Lemma 2.1 and Lemma 2.4, $x_n$ is indecomposable and $x_n \in J_{n,k}^{2^{k+2}+v+1}$.

Dimension $n \geq 2^{k+2}$.

There exist indecomposable classes $x_n \in J_{n,k}^{2^{k+2}+v+1}$ by Lemma 2.5.

The lemma holds for $n$ even.

Therefore the proof is completed.

\[\square\]

**Lemma 3.2.** For $k \geq 2$, let $2^{k} + 2v + 1 = 2^{k} + 2^{i_1} + \cdots + 2^{i_m}$ with $k - 2 = i_1 > \cdots > i_0 = 0$. Then there exist indecomposable classes $x_n \in J_{n,k}^{2^{k+2}+v+1}$ for $n \geq 2^{k+1}$ and $n \neq 2^{u} - 1$.

**Proof.** For $n \geq 2^{k+1}$, take $x_n$ as in Lemma 3.1. Then $x_n$ is as required. $\square$

### 4. Proof of theorems

We prove Theorem 1.1 by the mathematical induction.

**Proof of Theorem 1.1.** When $v = 0$, $J_{n,k}^{2^{k+2}+v+1} = J_{n,k}^{2^{k+1}}$. Theorem 1.1 is established by [8].

Assume that Theorem 1.1 is established for $0, 1, 2, \ldots, v - 1$.

By choosing a system of generators of $MO_\ast$, we prove that Theorem 1.1 holds for $v \geq 1$. From Lemma 2.10 we have $J_{n,k}^{2^{k+2}+v+1} \subseteq MO_\ast \cap \text{Ker } \chi$. Take a system of generators of $MO_\ast$ as follows:

(i) Let $x_2 = [RP(2)]$, hence $\chi(x_2) = 1$. By [5], $x_2 \in J_{n,k}^{2}$ and $x_2 \not\subseteq J_{n,k}^{2^{k+2}+v+1}$.

(ii) For $3 < n < 2^k$, $n \neq 2^{u} - 1$, we choose indecomposable classes $x_n$ such that $\chi(x_n) = 0$. From [5], $x_n \in J_{n,k}^{2}$ for $2 \leq r \leq \min(n, 2^{k} - 1)$.

(iii) For $2^k < n \leq 2^{k} + 2v - 1$, when $n$ is even, by the inductional assumption and [5,6,8] there exist indecomposable classes $x_n \in J_{n,k}^{2^{k+2}+v+1}$ such that $\chi(x_n) = 0$; when $n$ is odd, by [6] and Lemma 2.11 there also exist the indecomposable classes $x_n \in J_{n,k}^{2^{k+2}+v+1}$ such that $\chi(x_n) = 0$.

(iv) For $n = 2^{k} + 2v$ or $n = 2^{k} + 2v + 1$, by inductional assumption there exist indecomposable classes $x_n \in J_{n,k}^{2^{k+2}+v+1}$ such that $\chi(x_n) = 0$. By Lemma 2.7, $x_{2^{k}+2v+1} \not\subseteq J_{n,k}^{2^{k+2}+v+1}$.

(v) For $n \geq 2^{k} + 2v + 2$ and $n \neq 2^{u} - 1$, let $x_n$ be the classes in Lemma 3.1, hence $x_n \in J_{n,k}^{2^{k+2}+v+1}$.

We have chosen a system of generators of $MO_\ast$. Since $J_{n,k}^{2^{k+2}+v+1}$ is an ideal in $MO_\ast$, to complete the proof of the theorem, it is necessary only to show that $J_{n,k}^{2^{k+2}+v+1}$ contains the decomposable classes $x_{i_1}x_{i_2}\cdots x_{i_m}(\neq x_2^m)$, where $m \geq 2$. $\sum_{j=1}^{m} i_j \geq 2^{k} + 2v + 1$, $2 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq 2^{k} + 2v + 1$ and $i_j \neq 2^{u} - 1$.

**Case 1.** $i_1 + i_2 + \cdots + i_m \geq 2^{k} + 2v + 2$.

1. $i_m \geq 2^{k} + 2v$.

By (iv) $x_{i_m} \in J_{n,k}^{2^{k+2}+v+1}$ and $\chi(x_{i_m}) = 0$, so $\chi(x_{i_1}\cdots x_{i_{m-1}}) = 0$. Since $m \geq 2$, then $i_1 + \cdots + i_{m-1} \geq 2$. By (i) and (ii) $x_{i_1}x_{i_2}\cdots x_{i_{m-1}} \in J_{n,k}^{2^{k+2}+v+1}$, so $x_{i_1}x_{i_2}\cdots x_{i_{m}} \in J_{n,k}^{2^{k+2}+v+1} \subseteq J_{n,k}^{2^{k+2}+v+1}$. 

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2. \(2^k \leq i_m \leq 2^k + 2v - 1\).

By (iii) \(\chi(x_{i_m}) = 0\), so \(\chi(x_{i_1}x_{i_2} \cdots x_{i_m}) = 0\).

(a) \(i_m\) is even, or \(i_m\) is odd and there exists some \(i_j > 3\) for \(1 \leq j \leq m - 1\).

Then \(i_1 + i_2 + \cdots + i_{m-1} \geq 2^k + 2v + 2 - i_m\). Since \(2^k + 2v + 2 - i_m < 2^k\), by (i) and (ii) \(x_1x_2 \cdots x_{i_{m-1}} \in J_{s,k}^{2^k+2v+2-i_m}\).

By (iii) \(x_m \in J_{s,k}^{i_m-1}\), so \(x_1x_2 \cdots x_{i_m} \in J_{s,k}^{2^k+2v+2-i_m} J_{s,k}^{i_m-1} \subset J_{s,k}^{2^k+2v+1}\).

(b) \(i_m\) is odd and each \(2^k\) for \(1 \leq j \leq m - 1\).

Then \(i_1 + i_2 + \cdots + i_{m-1} \geq 2^k + 2v + 3 - i_m\). Since \(2^k + 2v + 3 - i_m < 2^k\), by (i) and (ii) \(x_1x_2 \cdots x_{i_{m-1}} \in J_{s,k}^{2^k+2v+3-i_m}\).

By \([5,8]\) and the inductive assumption, \(x_m \in J_{s,k}^{i_m-2}\), so \(x_1x_2 \cdots x_{i_m} \in J_{s,k}^{2^k+2v+2-i_m} J_{s,k}^{i_m-2} \subset J_{s,k}^{2^k+2v+1}\).

3. \(i_m < 2^k - 1\).

By (ii) \(\chi(x_{i_m}) = 0\), so \(\chi(x_{i_1}x_{i_2} \cdots x_{i_m}) = 0\). There exists some \(l\) such that \(2v + 1 < i_1 + i_2 + \cdots + i_l < 2^k + 2v\) and \(i_1 + i_2 + \cdots + i_l + i_{l+1} \geq 2^k + 2v\). Since \(1 < 2^k + 2v + 1 - (i_1 + i_2 + \cdots + i_l) < 2^k\) and \(i_{l+1} + \cdots + i_m > 2^k + 2v + 1 - (i_1 + \cdots + i_l)\), by (i) and (ii) \(x_{i_1}x_{i_2} \cdots x_{i_{l+1}} \in J_{s,k}^{2^k+2v+1-(i_1+\cdots+i_l)}\). From \(x_{i_1} \cdots x_{i_l} \in J_{s,k}^{i_l+1} \cdots J_{s,k}^{i_1+\cdots+i_l}\), we have \(x_{i_1}x_{i_2} \cdots x_{i_m} \in J_{s,k}^{2^k+2v+1-(i_1+\cdots+i_l)} J_{s,k}^{i_1+\cdots+i_l} \subset J_{s,k}^{2^k+2v+1}\).

**Case 2.** \(i_1 + i_2 + \cdots + i_m = 2^k + 2v + 1\).

From \(i_j \geq 2\) and \(m \geq 2\), we have \(i_m \leq 2^k + 2v - 1\).

1. \(2^k < i_m < 2^k + 2v - 1\).

By Lemma 2.7 \(x_1x_2 \cdots x_{i_m} \notin J_{s,k}^{2^k+2v+1}\). Consider cobordism classes in the form of \(\sum_{j=1}^{q} x_{i_1}x_{i_2} \cdots x_{i_{m_j}}\), where \(m_j \geq 2, 2 \leq i_1 < \cdots < i_m < 2^k + 2v + 1\) and \(i_1 + i_2 + \cdots + i_l + i_{l+1} = 2^k + 2v + 1\). Suppose \(\omega_j = (i_1, i_2, \ldots, i_m)\) and \(m_1 \leq m_2 \leq \cdots \leq m_q\).

By Lemma 2.9 \(s_{\omega_j}(\sum_{j=1}^{q} x_{i_1}x_{i_2} \cdots x_{i_{m_j}}) = s_{\omega_j}(x_{i_1}x_{i_2} \cdots x_{i_{m_j}}) + \sum_{j=2}^{q} s_{\omega_j}(x_{i_1}x_{i_2} \cdots x_{i_{m_j}}) = 1 + 0 = 1\) mod 2. By Lemma 2.8 \(\sum_{j=1}^{q} x_{i_1}x_{i_2} \cdots x_{i_{m_j}} \notin J_{s,k}^{2^k+2v+1}\).

2. \(i_m < 2^k\).

By (ii) \(x_j \in J_{s,k}^{i_j}\), \(j = 1, 2, \ldots, m\), so

\[
x_1x_2 \cdots x_{i_m} \in J_{s,k}^{i_1} J_{s,k}^{i_2} \cdots J_{s,k}^{i_m} \subset J_{s,k}^{2^k+2v+1}.
\]

**Theorem 1.1** is independent of selection of generators of \(M_{O_n}\). Thus the proof is completed. \(\square\)

**Proof of Theorem 1.2.** By Lemma 2.10 \(J_{s,k}^{2^k+2v+1} \subseteq M_{O_n} \cap \ker \chi\). By choosing a system of generators of \(M_{O_n}\), we prove \(M_{O_n} \cap \ker \chi \subseteq J_{s,k}^{2^k+2v+1}\) for \(k = r_s + 2\) and \(n \geq 2^k + 2v - 1\).

(i) Let \(x_2 = [RP(2)]\), hence \(\chi(x_2) = 1\). By [5] \(x_2 \notin J_{s,k}^{2^k+2v+1}\).

(ii) For \(3 < n < 2^k+1\), \(n \neq 2^v - 1\), we choose indecomposable classes \(x_n\) such that \(\chi(x_n) = 0\). From [5,6], \(x_n \in J_{s,k}^{i_n}\) for \(2 \leq r \leq \min(n, 2^k - 1)\).

(iii) For \(n \geq 2^k+1\) and \(n \neq 2^v - 1\), let \(x_n\) be the classes in Lemma 3.2, then \(x_n \in J_{s,k}^{2^k+2v+1}\).

We have chosen a system of generators of \(M_{O_n}\). Since \(J_{s,k}^{2^k+2v+1}\) is an ideal in \(M_{O_n}\), to complete the proof of the theorem, it is necessary only to show that \(J_{s,k}^{2^k+2v+1}\) contains the decomposable classes \(x_1x_2 \cdots x_{i_m} (\neq x_{i_1}^{p_1})\), where \(m \geq 2, \sum_{j=1}^{m} i_j \geq 2^k+1 + 2^{k-1} - 2, 2 \leq i_1 \leq i_2 \leq \cdots \leq i_m < 2^k - 1\) and \(i_j \neq 2^v - 1\).

1. \(2^k - 2^v + 2^v + 1 \leq i_m < 2^k + 2^{k-1} - 2\).

By (ii) \(x_m \in J_{s,k}^{2^k-2^v+1} J_{s,k}^{2^v+1}\) and \(x_{i_m} = 0\), so \(\chi(x_1x_2 \cdots x_{i_m}) = 0\). Since \(m \geq 2\), then \(i_1 + \cdots + i_m - 1 \geq 2^k + 2^{k-1} - 2 - i_m = 2^k - 1\). By (i) and (ii) \(x_1x_2 \cdots x_{i_{m-1}} \in J_{s,k}^{2^k-1}\), so \(x_1x_2 \cdots x_{i_m} \in J_{s,k}^{2^k-1} J_{s,k}^{2^v+1} \subset J_{s,k}^{2^k+2v+1}\).

2. \(2 < i_m < 2^k - 1 + 2^{k-1} + 2^v + 1\).

By (ii) \(\chi(x_{i_m}) = 0\), so \(\chi(x_1x_2 \cdots x_{i_m}) = 0\). There exists some \(l\) such that \(2v + 1 < i_1 + i_2 + \cdots + i_l < 2^k + 2v\). Since \(1 < 2^k + 2v + 1 - (i_1 + i_2 + \cdots + i_l) < 2^k\) and \(i_{l+1} + \cdots + i_m \geq (2^k + 2^{k-1} - 2) - (i_1 + \cdots + i_l) > (2^k + 2v + 1) - (i_1 + i_2 + \cdots + i_l) > 2^k - 2^v + 2^v + 1\), we have \(x_1x_2 \cdots x_{i_m} \in J_{s,k}^{2^k-2^v+1} J_{s,k}^{2^v+1}\).
(i₁ + ⋅⋅⋅ + iₘ), by (i) and (ii) \(x_{i₁}x_{i₂}⋯x_{iₘ} \in J_{n,k}^{2k+2v+1-(i₁+⋯+iₘ)}\). From \(x₁⋯xₘ \in J_{n,k}^{i₁}⋯J_{n,k}^{iₘ} \subset J_{n,k}^{i₁+⋯+iₘ}\), we have \(x₁x₂⋯xₘ \in J_{n,k}^{2k+2v+1-(i₁+⋯+iₘ)}\). \(J_{n,k}^{i₁+⋯+iₘ} \subset J_{n,k}^{2k+2v+1}\). □

Proof of Theorem 1.3. By Lemma 2.10 \(J_{n,k}^{2k+2v+1} \leq MO_n \cap \ker \chi\). By choosing a system of generators of \(MO_n\), we prove \(MO_n \cap \ker \chi \subset J_{n,k}^{2k+2v+1}\) for \(k = r₅ + 1\) and \(n \geq 2k² + 2k² - 2\).

(i) Let \(x₂ = [R(2)]\), hence \(\chi(x₂) = 1\). By \([5]\) \(x₂ \in J_{n,k}^{2k+1}\) and \(x₂ \notin J_{n,k}^{2k+2v+1}\).

(ii) For \(3 < n < 2k²\), we choose indecomposable classes \(xₙ\) such that \(\chi(xₙ) = 0\). From \([5,6]\), \(xₙ \in J_{n,k}^{i} \) for \(2 \leq r \leq \min(n,2k)\).

(iii) For \(n \geq 2k² + 2v - 2\) and \(n \neq 2v - 1\), let \(xᵢ\) be the generators in Lemma 2.5, then \(xᵢ \in J_{n,k}^{2k+2v+1}\).

We have chosen a system of generators of \(MO_n\). Since \(J_{n,k}^{2k+2v+1}\) is an ideal in \(MO_n\), to complete the proof of the theorem, it is necessary only to show that \(J_{n,k}^{2k+2v+1}\) contains the decomposable classes \(xᵢxᵢ₂⋯xᵢₘ(\neq xᵢₘ)\), where \(m \geq 2\), \(\sum_{j=1}^{m} i_j \geq 2k² + 2v - 2\), \(2 \leq i₁ \leq i₂ \leq ⋅⋅⋅ \leq iₘ \leq 2k² - 2\) and \(i_j \neq 2v - 1\).

1. \(iₘ = 2k² + 2v - 2\).
   From \(\sum_{j=1}^{m} i_j \geq 2k² + 2v - 2\), we have \(i₁ + ⋅⋅⋅ + i_{m-1} \geq 2k² + 2v - 2 - iₘ \geq 2k²\).
   (a) \(i₁ + ⋅⋅⋅ + i_{m-1} > 2k²\).
   By (ii) \(xₘ \in J_{n,k}^{2k+1}\) and \(\chi(xₘ) = 0\), so \(\chi(x₁x₂⋯xₘ) = 0\). Since \(x₁x₂⋯x_{m-1} \in J_{n,k}^{2k}\), then \(x₁x₂⋯xₘ \in J_{n,k}^{2k}J_{n,k}^{2k+1} \subset J_{n,k}^{2k+2v+1}\).

(b) \(i₁ + ⋅⋅⋅ + i_{m-1} = 2k²\).
   By (ii) \(xₘ \in J_{n,k}^{2k}\) and \(\chi(xₘ) = 0\), so \(\chi(x₁x₂⋯xₘ) = 0\). Since \(x₁⋯x_{m-1} \in J_{n,k}^{2k+1}\), then \(x₁⋯xₘ \in J_{n,k}^{2k}J_{n,k}^{2k+1} \subset J_{n,k}^{2k+2v+1}\).

2. \(2v + 1 \leq iₘ < 2k² + 2v - 2\).
   By (ii) \(xₘ \in J_{n,k}^{2k+1}\) and \(\chi(xₘ) = 0\), so \(\chi(x₁x₂⋯xₘ) = 0\). Since \(m \geq 2\), \(i₁ + ⋅⋅⋅ + i_{m-1} \geq 2k² + 2v - 2 - iₘ > 2k²\).
   By (i) and (ii) \(x₁x₂⋯x_{m-1} \in J_{n,k}^{2k}\), so \(x₁x₂⋯xₘ \in J_{n,k}^{2k}J_{n,k}^{2k+1} \subset J_{n,k}^{2k+2v+1}\).

3. \(2 < iₘ < 2v + 1\).
   By (ii) \(\chi(xₘ) = 0\), so \(\chi(x₁x₂⋯xₘ) = 0\). There exists some \(l\) such that \(2v + 1 < i₁ + i₂ + ⋅⋅⋅ + iₗ < 2k² + 2v\). Since \(1 < 2k² + 2v + 1 - (i₁ + i₂ + ⋅⋅⋅ + iₗ) \leq 2k²\) and \(i₁+ ⋅⋅⋅ + iₗ \geq (2k² + 2v + 1 - 2v + 1) - (i₁ + ⋅⋅⋅ + iₗ) \geq (2k² + 2v + 1) - (i₁ + ⋅⋅⋅ + iₗ)\), by (i) and (ii) \(x₁x₂⋯xₗ \in J_{n,k}^{2k+1+(i₁+⋯+iₗ)}\). From \(x₁⋯xₗ \in J_{n,k}^{2k+1+(i₁+⋯+iₗ)}\), \(x₁x₂⋯xₘ \in J_{n,k}^{2k+2v+1-(i₁+⋯+iₗ)}\), \(J_{n,k}^{i₁+⋯+iₗ} \subset J_{n,k}^{2k+2v+1}\). □

Remark 2. Similarly, for \(r₅ > 3\) and \(2k² + 2v = 2k² + 2v + ⋅⋅⋅ + 2vₖ \) with \(k \geq 3\), \(r₅ > ⋅⋅⋅ > r₀ = 3\) and \(rₐ+₁ = rₐ+1(i = 0, ⋅⋅⋅, r₅₋₁)\), the ideal \(J_{n,k}^{2k² + 2v}\) can be determined. We have

\[
J_{n,k}^{2k² + 2v} = \begin{cases} 
MO_n, & n \geq 2k² + 2v + 1, \\
\{ \sum x₁, ⋅⋅⋅, xₘ \mid 2 \leq i₁ \leq ⋅⋅⋅ \leq iₘ < 2k², m \geq 2 \}, & n = 2k² + 2v,
\end{cases}
\]

where \(\{xₖ\}(i \neq 2v - 1)\) is a system of generators of \(MO_n\).

Remark 3. For a general given \(r > 2k²\) and sufficiently large \(k\), we pose the following conjecture

\[
J_{n,k}^{r} = \begin{cases} 
MO_n \cap \ker \chi, & n > r \text{ odd, } r > 1, \\
MO_n, & n > r \text{ even,}
\end{cases}
\]

where \(\{xₖ\}(i \neq 2v - 1)\) is a system of generators of \(MO_n\).

Note that the conjecture does not hold for small \(k\) (see [8]).

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