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Relative extremal projectors

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Abstract

This paper proves the existence of relative extremal projectors. An infinite factorization is given as well as a summation formula.

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1. Introduction

Generalizing work of Ašerova et al. [1], Zhelobenko developed the notion of an *extremal projector* [9,10]. Roughly speaking, the extremal projector is the operator on the universal Verma module that projects onto the highest weight space along all other weight spaces. It admits very nontrivial and powerful factorization theorems. Moreover, it has a wide variety of applications such as the study of K-types of irreducible admissible representations (via the Mickelsson step algebra), the study of branching rules, the description of homorphisms between Verma modules, the construction of special bases of representations (e.g., generalized Gelfand–Tsetlin bases), the calculation of Clebsch–Gordan coefficients, the construction of generalized harmonic polynomials, and others [2–6,8–10].

This paper develops the notion of a *relative extremal projector*. Roughly speaking, if $\mathfrak{l} \subseteq \mathfrak{g}$ is a regular reductive subalgebra of a complex reductive Lie algebra, then the

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relative extremal projector is the operator on the universal Verma module that projects onto the highest $\mathfrak{l}$-subrepresentation of $\mathfrak{g}$. An existence theorem (Theorem 5), an infinite commutative factorization theorem (Theorem 7), and a summation formula (Theorem 8) for relative extremal projectors are proved in this paper. These theorems also shed additional light on the original extremal projector (Lemma 3) and provide new factorizations for it (Theorem 6). In a future paper we will study finite noncommutative factorizations of relative extremal projectors generalizing the original work of Ašerova et al. [1].

2. Extremal projectors

We begin by summarizing some of the relevant facts about extremal projectors needed later in this paper. The theorems in this section are all due to Ašerova et al. [1] and Zhelobenko [9,10]. The reader is referred to their work for details.

We begin with the usual notation. Let $\mathfrak{g}$ be a reductive Lie algebra over $\mathbb{C}$ and fix a Cartan subalgebra $\mathfrak{h}$. Write $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ for the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$, fix $\Delta^+ = \Delta^+(\mathfrak{g}, \mathfrak{h})$ a positive root system, and write $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ for the corresponding triangular decomposition of $\mathfrak{g}$. If $\alpha \in \Delta^+$, choose a standard $\mathfrak{sl}(2, \mathbb{C})$ basis $e_{-\alpha}, h_\alpha, e_\alpha$ in $\mathfrak{g}$ where $e_{\pm\alpha}$ are weight vectors corresponding to the roots $\pm\alpha$. Thus $[e_\alpha, e_{-\alpha}] = h_\alpha$ and $\alpha(h_\alpha) = 2$. Write $\mathfrak{U}(\mathfrak{g})$ for the universal enveloping algebra of $\mathfrak{g}$.

Recall $\mathfrak{U}(\mathfrak{h})$ is naturally isomorphic to the symmetric algebra on $\mathfrak{h}$ which is isomorphic to the set of polynomials on the dual space $\mathfrak{h}^*$. Define $\bar{\mathfrak{U}}(\mathfrak{h})$ to be the fraction field of $\mathfrak{U}(\mathfrak{h})$,

$$\bar{\mathfrak{U}}(\mathfrak{h}) = \mathrm{Frac}\,\mathfrak{U}(\mathfrak{h}).$$

$\bar{\mathfrak{U}}(\mathfrak{h})$ is isomorphic to the field of rational functions on $\mathfrak{h}^*$. Write $\bar{\mathfrak{U}}(\mathfrak{g})$ for the extension of $\mathfrak{U}(\mathfrak{g})$ by $\bar{\mathfrak{U}}(\mathfrak{h})$,

$$\bar{\mathfrak{U}}(\mathfrak{g}) = \mathfrak{U}(\mathfrak{g}) \underset{\mathfrak{U}(\mathfrak{h})}{\otimes} \bar{\mathfrak{U}}(\mathfrak{h}).$$

For $\lambda \in \mathfrak{h}^*$, write $\bar{\mathfrak{U}}(\mathfrak{g})_\lambda$ for the λ weight space of $\bar{\mathfrak{U}}(\mathfrak{g})$. If $\Delta^+ = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$, the Poincaré–Birkhoff–Witt theorem implies $\bar{\mathfrak{U}}(\mathfrak{g})_\lambda$ is spanned over $\bar{\mathfrak{U}}(\mathfrak{h})$ by monomials of the form

$$e_{-\alpha_m}^{r_m} \cdots e_{-\alpha_2}^{r_2} e_{-\alpha_1}^{r_1} e_{\alpha_1}^{s_1} e_{\alpha_2}^{s_2} \cdots e_{\alpha_m}^{s_m}, \tag{2.1}$$

where $\lambda = \sum_{i=1}^{m} (-r_i + s_i)\alpha_i$. Define $\bar{\mathfrak{F}}(\mathfrak{g})_\lambda$ to be the vector space of all *formal series* over $\bar{\mathfrak{U}}(\mathfrak{h})$ in these monomials (with fixed weight λ). Let

$$\bar{\mathfrak{F}}(\mathfrak{g}) = \bigoplus_{\lambda} \bar{\mathfrak{F}}(\mathfrak{g})_\lambda.$$

It is a theorem that $\bar{\mathfrak{F}}(\mathfrak{g})$ is an algebra with respect to the multiplication of formal series. We will see that one way of looking at the extremal projector places it as an element of $\bar{\mathfrak{F}}(\mathfrak{g})$. If $P \in \bar{\mathfrak{F}}(\mathfrak{g})_0$, we say that its *constant term* is its summand coming from $\bar{\mathfrak{U}}(\mathfrak{h})$.

Another important piece of this story is the *universal Verma module*. It is defined as

$$M(\mathfrak{g}) = \mathfrak{U}(\mathfrak{g})/\mathfrak{U}(\mathfrak{g})\mathfrak{n}^+$$

which is both a left $\mathfrak{g}$-module and a two-sided $\mathfrak{h}$-module. Write $\bar{M}(\mathfrak{g})$ for its extension by $\bar{\mathfrak{U}}(\mathfrak{h})$,

$$\bar{M}(\mathfrak{g}) = M(\mathfrak{g}) \underset{\mathfrak{U}(\mathfrak{h})}{\otimes} \bar{\mathfrak{U}}(\mathfrak{h}),$$

and $\bar{M}(\mathfrak{g})_\lambda$ for the λ weight space of $\bar{M}(\mathfrak{g})$. It is a critical theorem that elements of $\bar{\mathfrak{F}}(\mathfrak{g})$ act by left multiplication on $\bar{M}(\mathfrak{g})$. In general, let $\mathrm{End}_\lambda \bar{M}(\mathfrak{g}) = \{T \in \mathrm{End}_{\bar{\mathfrak{U}}(\mathfrak{h})} \bar{M}(\mathfrak{g}) \mid [h, T] = \lambda(h)T \text{ for all } h \in \mathfrak{h}\}$, where the subscript $\bar{\mathfrak{U}}(\mathfrak{h})$ denotes right $\bar{\mathfrak{U}}(\mathfrak{h})$-linearity, and let

$$\mathrm{End}_{\mathfrak{h}} \bar{M}(\mathfrak{g}) = \bigoplus_{\lambda} \mathrm{End}_\lambda \bar{M}(\mathfrak{g}).$$

In other words, $\mathrm{End}_{\mathfrak{h}} \bar{M}(\mathfrak{g})$ is the span of the right $\bar{\mathfrak{U}}(\mathfrak{h})$-linear endomorphisms of $\bar{M}(\mathfrak{g})$ with well-defined weights under the adjoint $\mathfrak{h}$-action. We will see that another way of looking at the extremal projector is as an element of $\mathrm{End}_{\mathfrak{h}} \bar{M}(\mathfrak{g})$. The relevant theorem follows.

Theorem 1 (Zhelobenko [9]). $\bar{\mathfrak{F}}(\mathfrak{g})$ *is isomorphic to* $\mathrm{End}_{\mathfrak{h}} \bar{M}(\mathfrak{g})$. *The isomorphism maps* $f \in \bar{\mathfrak{F}}(\mathfrak{g})$ *to the operator on* $\bar{M}(\mathfrak{g})$ *given by left multiplication by* f.

Definition 1. The *extremal projector*, $P(\mathfrak{g}, \mathfrak{h})$, is the element of $\mathrm{End}_{\mathfrak{h}} \bar{M}(\mathfrak{g})$ projecting $\bar{M}(\mathfrak{g})$ to its highest weight space, $\bar{M}(\mathfrak{g})_0 \cong \bar{\mathfrak{U}}(\mathfrak{h})$, along its lower weight spaces.

Also crucial to this discussion is the *Shapovalov form*. Let $(\cdot)^*$ be the Hermitian anti-involution of $\mathfrak{U}(\mathfrak{g})$ that is -1 times the Cartan involution on $\mathfrak{g}$. $(\cdot)^*$ clearly extends to a Hermitian anti-involution of $\bar{\mathfrak{U}}(\mathfrak{g})$ and $\bar{\mathfrak{F}}(\mathfrak{g})$ acting trivially on $\bar{\mathfrak{U}}(\mathfrak{h})$. The Shapovalov form on $\mathfrak{U}(\mathfrak{g})$ is the right $\mathfrak{h}$-bilinear $\mathfrak{U}(\mathfrak{h})$-valued form

$$\langle x, y \rangle = \mathrm{HC}_{\mathfrak{g},\mathfrak{h}}\, x^* y,$$

where $\mathrm{HC}_{\mathfrak{g},\mathfrak{h}}$ is the Harish-Chandra projection from $\mathfrak{U}(\mathfrak{g})$ to $\mathfrak{U}(\mathfrak{h})$ along $\mathfrak{n}^-\mathfrak{U}(\mathfrak{g}) + \mathfrak{U}(\mathfrak{g})\mathfrak{n}^+$. $\langle \cdot, \cdot \rangle$ clearly extends to a form on $\bar{\mathfrak{U}}(\mathfrak{g})$. By (Shapovalov) [7], this form descends to a *nondegenerate* right $\bar{\mathfrak{U}}(\mathfrak{h})$-bilinear $\bar{\mathfrak{U}}(\mathfrak{h})$-valued form on $\bar{M}(\mathfrak{g})$.

In light of Theorem 1, we may alternately view $P(\mathfrak{g}, \mathfrak{h})$ as an element of $\bar{\mathfrak{F}}(\mathfrak{g})$.

Theorem 2 (Ašerova et al. [1]; Zhelobenko [9]). *$P(\mathfrak{g},\mathfrak{h})^* = P(\mathfrak{g},\mathfrak{h}) = P(\mathfrak{g},\mathfrak{h})^2$ so that $P(\mathfrak{g},\mathfrak{h})$ is a Hermitian projector with respect to the Shapovalov form. Moreover, $P(\mathfrak{g},\mathfrak{h})$ is the unique element P in $\bar{\bar{\mathfrak{F}}}(\mathfrak{g})_0$ with constant term 1 satisfying*

$$e_\alpha P = 0$$

for all $\alpha \in \Delta^+$. Alternately, $P(\mathfrak{g},\mathfrak{h})$ is the unique element P in $\bar{\bar{\mathfrak{F}}}(\mathfrak{g})$ with constant term 1 satisfying

$$e_\alpha P = 0 = P e_{-\alpha}$$

for all $\alpha \in \Delta^+$.

Viewed as an element of $\bar{\bar{\mathfrak{F}}}(\mathfrak{g})$, there are two relevant formulas for $P(\mathfrak{g},\mathfrak{h})$. The first is a remarkable *noncommutative* finite factorization. In the case of $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$, a formula for $P(\mathfrak{g},\mathfrak{h})$ is easy. Namely let F, H, E be the standard basis of $\mathfrak{sl}(2,\mathbb{C})$. It is straightforward to check

$$P(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{h}) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} F^k E^k \frac{1}{(H+2)(H+3)\cdots(H+k+1)}.$$

For general $\mathfrak{g}$, fix any *normal ordering* $\alpha_1, \alpha_2, \dots, \alpha_m$ of Δ^+; i.e., whenever $\alpha_i + \alpha_j$ is a root α_k for $i<j$, then $i<k<j$. For $t \in \mathbb{C}$ define

$$P_t(\alpha_i) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} e_{-\alpha_i}^k e_{\alpha_i}^k \frac{1}{(h_{\alpha_i}+t+1)(h_{\alpha_i}+t+2)\cdots(h_{\alpha_i}+t+k)}.$$

Writing $\rho_\mathfrak{g}$ for the semisum of positive roots, a formula for $P(\mathfrak{g},\mathfrak{h})$ is now possible.

Theorem 3 (Ašerova et al. [1]; Zhelobenko [9]).

$$P(\mathfrak{g},\mathfrak{h}) = \prod_{i=1}^{m} P_{\rho_\mathfrak{g}(h_{\alpha_i})}(\alpha_i).$$

The second formula for $P(\mathfrak{g},\mathfrak{h})$ is an infinite factorization. Write $\Omega(\mathfrak{g})$ for the Casimir element of $\mathfrak{g}$ and define

$$\Omega(\mathfrak{g},\mathfrak{h}) = \mathrm{HC}_{\mathfrak{g},\mathfrak{h}}\Omega(\mathfrak{g}).$$

Given any $\nu \in \mathfrak{h}^*$ and $Q \in \bar{\mathfrak{U}}(\mathfrak{h})$, view Q as a rational function on $\mathfrak{h}^*$ to define $Q^\nu \in \bar{\mathfrak{U}}(\mathfrak{h})$ by

$$Q^\nu(\lambda) = Q(\lambda + \nu)$$

for all $\lambda \in \mathfrak{h}^*$. In other words, if $u \in \bar{\mathfrak{U}}(\mathfrak{g})_\nu$, then $Qu = uQ^\nu$. Alternately, this is the algebra isomorphism induced by the map $h \to h + \nu(h)$ for $h \in \mathfrak{h}$. We employ the notation $\Delta(\mathfrak{U}(\mathfrak{n}^+)\mathfrak{n}^+)$ for the set of weights of $\mathfrak{U}(\mathfrak{n}^+)\mathfrak{n}^+$ with respect to $\mathfrak{h}$. In general, given a representation V of $\mathfrak{h}$, we write $\Delta(V)$ for the set of weights of V with respect to $\mathfrak{h}$.

Theorem 4 (Zhelobenko [10]).

$$P(\mathfrak{g}, \mathfrak{h}) = \prod_{\nu \in \Delta(\mathfrak{U}(\mathfrak{n}^+)\mathfrak{n}^+)} \left(1 - \frac{\Omega(\mathfrak{g}) - \Omega(\mathfrak{g}, \mathfrak{h})}{\Omega(\mathfrak{g}, \mathfrak{h})^\nu - \Omega(\mathfrak{g}, \mathfrak{h})}\right).$$

3. The relative extremal projector

Let $\mathfrak{l} \supseteq \mathfrak{h}$ be a *regular* reductive subalgebra of $\mathfrak{g}$. Thus $\mathfrak{l} = \mathfrak{l}^- \oplus \mathfrak{h} \oplus \mathfrak{l}^+$ is a triangular decomposition of $\mathfrak{l}$ with $\mathfrak{l}^\pm = \mathfrak{n}^\pm \cap \mathfrak{l}$, $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{l} \oplus \mathfrak{u}^+$ with $\mathfrak{n}^\pm = \mathfrak{l}^\pm \oplus \mathfrak{u}^\pm$, and $\mathfrak{q}^\pm = \mathfrak{l} \oplus \mathfrak{u}^\pm$ is the Levi decomposition of a parabolic subalgebra of $\mathfrak{g}$. Given $u \in \bar{\mathfrak{U}}(\mathfrak{g})$, write $\bar{u}$ for the image of u in $\bar{M}(\mathfrak{g})$. Let $M_\mathfrak{l} = \bar{\mathfrak{U}}(\mathfrak{l})\bar{1} \subseteq \bar{M}(\mathfrak{g})$. $M_\mathfrak{l}$ is an $\mathfrak{l}$-invariant subspace of $\bar{M}(\mathfrak{g})$ isomorphic to $\bar{M}(\mathfrak{l})$. We say $M_\mathfrak{l}$ is the *highest* $\mathfrak{l}$-*subrepresentation* of $\bar{M}(\mathfrak{g})$.

Definition 2. The relative extremal projector of $\mathfrak{g}$ to $\mathfrak{l}$, $P(\mathfrak{g}, \mathfrak{l})$, is the Hermitian projector in $\mathrm{End}_\mathfrak{h} \bar{M}(\mathfrak{g})$ whose image is the highest $\mathfrak{l}$-subrepresentation. Thus $P(\mathfrak{g}, \mathfrak{l})^* = P(\mathfrak{g}, \mathfrak{l}) = P(\mathfrak{g}, \mathfrak{l})^2$.

Theorem 5. $P(\mathfrak{g}, \mathfrak{l})$ *exists and commutes with* $\mathfrak{l}$. *Moreover*, $P(\mathfrak{g}, \mathfrak{l})$ *is the unique Hermitian element* P *in* $\bar{\mathfrak{F}}(\mathfrak{g})_0$ *with constant term* 1 *satisfying*

$$e_\beta P = 0 \quad \textit{and} \quad e_\alpha P = P e_\alpha$$

for all $\beta \in \Delta(\mathfrak{u}^+)$ *and all simple* $\alpha \in \Delta^+(\mathfrak{l})$. *Alternately*, $P(\mathfrak{g}, \mathfrak{l})$ *is the unique element* P *in* $\bar{\mathfrak{F}}(\mathfrak{g})_0$ *with constant term* 1 *satisfying*

$$P e_{-\beta} = 0 \quad \textit{and} \quad e_{\pm\alpha} P = P e_{\pm\alpha}$$

for all $\beta \in \Delta(\mathfrak{u}^+)$ *and all simple* $\alpha \in \Delta^+(\mathfrak{l})$.

Proof. Let $M_\mathfrak{l}^\perp = \{u \in \bar{M}(\mathfrak{g}) \mid \langle u, m \rangle = 0 \text{ for all } m \in M_\mathfrak{l}\}$. Since $\langle \cdot, \cdot \rangle$ is nondegenerate on both $\bar{M}(\mathfrak{g})$ and $M_\mathfrak{l}$, $\bar{M}(\mathfrak{g}) = M_\mathfrak{l} \oplus M_\mathfrak{l}^\perp$. Consider the operator in $\mathrm{End}_\mathfrak{h} \bar{M}(\mathfrak{g})$ given by projection onto the $M_\mathfrak{l}$ component with respect to this decomposition. It is clearly Hermitian and so $P(\mathfrak{g}, \mathfrak{l})$ exists and is clearly uniquely defined.

Now suppose $P \in \bar{\mathfrak{F}}(\mathfrak{g})_0$ satisfies the hypothesis of the theorem. The weight zero condition and the constant term condition imply $P\bar{1} = \bar{1}$. The weight zero condition, the Hermitian condition (in the first case), and the $\Delta^+(\mathfrak{l})$ condition imply P commutes with $\mathfrak{l}$. Hence P acts trivially on $M_\mathfrak{l}$. We simply need to show $PM_\mathfrak{l}^\perp = 0$ to finish the proof.

For this, recall the Poincaré–Birkhoff–Witt theorem implies $\mathfrak{U}(\mathfrak{g}) = \mathfrak{U}(\mathfrak{u}^-) \otimes \mathfrak{U}(\mathfrak{l}) \otimes \mathfrak{U}(\mathfrak{u}^+)$. Since $\bar{M}(\mathfrak{g}) = \bar{\mathfrak{U}}(\mathfrak{g})\bar{1}$ and $M_\mathfrak{l} = \bar{\mathfrak{U}}(\mathfrak{l})\bar{1}$, it is clear $\mathfrak{U}(\mathfrak{u}^-) \otimes \mathfrak{U}(\mathfrak{l}^-) \otimes \bar{\mathfrak{U}}(\mathfrak{h})$ maps bijectively onto $\bar{M}(\mathfrak{g})$ and $\mathfrak{U}(\mathfrak{l}^-) \otimes \bar{\mathfrak{U}}(\mathfrak{h})$ maps bijectively onto $M_\mathfrak{l}$. Thus $\bar{M}(\mathfrak{g}) = M_\mathfrak{l} \oplus [\mathfrak{u}^-\mathfrak{U}(\mathfrak{u}^-) \otimes \mathfrak{U}(\mathfrak{l}^-) \otimes \bar{\mathfrak{U}}(\mathfrak{h})]\bar{1}$. We claim $M_\mathfrak{l}^\perp = [\mathfrak{u}^-\mathfrak{U}(\mathfrak{u}^-) \otimes \mathfrak{U}(\mathfrak{l}^-) \otimes \bar{\mathfrak{U}}(\mathfrak{h})]\bar{1}$. In the first case, P is Hermitian so that the condition $e_\beta P = 0$ implies $Pe_{-\beta} = 0$. Thus in either case, assuming the previous claim, we see $PM_\mathfrak{l}^\perp = 0$. This finishes the proof that $P = P(\mathfrak{g}, \mathfrak{l})$.

To prove the claim that $M_\mathfrak{l}^\perp = [\mathfrak{u}^-\mathfrak{U}(\mathfrak{u}^-) \otimes \mathfrak{U}(\mathfrak{l}^-) \otimes \bar{\mathfrak{U}}(\mathfrak{h})]\bar{1}$, let $u^- \in \mathfrak{u}^-\mathfrak{U}(\mathfrak{u}^-)$ and $l^\pm \in \mathfrak{U}(\mathfrak{l}^\pm)$. It suffices to show $\mathrm{HC}_{\mathfrak{g},\mathfrak{h}}(l^+u^-l^-) = 0$. But since $[\mathfrak{l}, \mathfrak{u}^-] \subseteq \mathfrak{u}^-$, it is possible to write $l^+u^-l^- = \sum u_i^- l_i^- l_i^+ h_i$ where $u_i^- \in \mathfrak{u}^-\mathfrak{U}(\mathfrak{u}^-)$, $l^\pm \in \mathfrak{U}(\mathfrak{l}^\pm)$, and $h_i \in \mathfrak{U}(\mathfrak{h})$. □

The existence of relative extremal projectors gives rise to many finite *commutative* factorizations of the original extremal projector. They all follow from the next theorem.

Theorem 6. *If* $\mathfrak{h} \subseteq \mathfrak{l}_1 \subseteq \mathfrak{l}_2 \subseteq \mathfrak{g}$ *is a chain of regular reductive subalgebras, then*

$$P(\mathfrak{g}, \mathfrak{l}_1) = P(\mathfrak{g}, \mathfrak{l}_2)\, P(\mathfrak{l}_2, \mathfrak{l}_1) = P(\mathfrak{l}_2, \mathfrak{l}_1)\, P(\mathfrak{g}, \mathfrak{l}_2).$$

Proof. Since every $P(\mathfrak{g}, \mathfrak{l})$ is Hermitian, it suffices to prove $P(\mathfrak{g}, \mathfrak{l}_1) = P(\mathfrak{l}_2, \mathfrak{l}_1)\, P(\mathfrak{g}, \mathfrak{l}_2)$. For this use Theorem 5 and let $\beta \in \Delta(\mathfrak{u}_1^+)$ and $\alpha \in \Delta^+(\mathfrak{l}_1)$. Then $P(\mathfrak{l}_2, \mathfrak{l}_1)\, P(\mathfrak{g}, \mathfrak{l}_2)e_{-\beta} = P(\mathfrak{l}_2, \mathfrak{l}_1)0 = 0$ since $\Delta(\mathfrak{u}_1^+) \subseteq \Delta(\mathfrak{u}_2^+)$. Also $e_\alpha P(\mathfrak{l}_2, \mathfrak{l}_1)\, P(\mathfrak{g}, \mathfrak{l}_2) = P(\mathfrak{l}_2, \mathfrak{l}_1)e_\alpha P(\mathfrak{g}, \mathfrak{l}_2) = P(\mathfrak{l}_2, \mathfrak{l}_1)\, P(\mathfrak{g}, \mathfrak{l}_2)e_\alpha$ since $\Delta^+(\mathfrak{l}_1) \subseteq \Delta^+(\mathfrak{l}_2)$. The argument for $e_{-\alpha}$ is similar. Finally, $P(\mathfrak{l}_2, \mathfrak{l}_1)\, P(\mathfrak{g}, \mathfrak{l}_2)$ is clearly weight 0 and has constant term 1. □

4. Infinite factorization of the relative extremal projector

Let $\mathrm{HC}_{\mathfrak{g},\mathfrak{l}}$ be the Harish-Chandra projection from $\mathfrak{U}(\mathfrak{g})$ to $\mathfrak{U}(\mathfrak{l})$ along $\mathfrak{u}^-\mathfrak{U}(\mathfrak{g}) + \mathfrak{U}(\mathfrak{g})\mathfrak{u}^+$ and let

$$\Omega(\mathfrak{g}, \mathfrak{l}) = \mathrm{HC}_{\mathfrak{g},\mathfrak{l}}\Omega(\mathfrak{g}).$$

It is an element of $\mathfrak{Z}(\mathfrak{l})$, the center of $\mathfrak{U}(\mathfrak{l})$, since it is easy to verify that left and right multiplication by elements of $\mathfrak{l}$ commute with $\mathrm{HC}_{\mathfrak{g},\mathfrak{l}}$.

Theorem 7 gives an infinite factorization of $P(\mathfrak{g}, \mathfrak{l})$ as a commutative product of elements in $\bar{\mathfrak{F}}(\mathfrak{g})^\mathfrak{l}$. To properly interpret it, recall $\mathrm{HC}_{\mathfrak{l},\mathfrak{h}}$ defines the Harish-Chandra isomorphism from $\mathfrak{Z}(\mathfrak{l})$ to $\mathfrak{U}(\mathfrak{h})^{W(\mathfrak{l}),\cdot}$, the $\mathfrak{l}$-Weyl group invariants of $\mathfrak{U}(\mathfrak{h})$ under the *dot* action of $W(\mathfrak{l})$, where the dot action is given by

$$w \cdot Q = w(Q^{w^{-1}\rho_{\mathfrak{l}}-\rho_{\mathfrak{l}}})$$

for $w \in W(\mathfrak{l})$ and $Q \in \mathfrak{U}(\mathfrak{h})$. Note also $w(Q^v) = (wQ)^{wv}$. In Theorem 7, the notation

$$\mathrm{HC}_{\mathfrak{l},\mathfrak{h}}^{-1} \prod_{w \in W(\mathfrak{l})} \left(1 - \frac{\Omega(\mathfrak{g}) - \Omega(\mathfrak{g},\mathfrak{l})}{\Omega(\mathfrak{g},\mathfrak{h})^{wv} - \Omega(\mathfrak{g},\mathfrak{h})} \right) \tag{4.1}$$

appears and is understood as follows. Write

$$\prod_{w \in W(\mathfrak{l})} \left(1 - \frac{\Omega(\mathfrak{g}) - \Omega(\mathfrak{g},\mathfrak{l})}{\Omega(\mathfrak{g},\mathfrak{h})^{wv} - \Omega(\mathfrak{g},\mathfrak{h})} \right) = \sum_{k=0}^{|W(\mathfrak{l})|} \frac{p_k}{q} [\Omega(\mathfrak{g}) - \Omega(\mathfrak{g},\mathfrak{l})]^k \tag{4.2}$$

for uniquely determined $p_k \in \mathfrak{U}(\mathfrak{h})$ with $q = \prod_{w \in W(\mathfrak{l})} (\Omega(\mathfrak{g},\mathfrak{h})^{wv} - \Omega(\mathfrak{g},\mathfrak{h}))$. Eq. (4.1) is then defined to be

$$\sum_{k=0}^{|W(\mathfrak{l})|} \frac{\mathrm{HC}_{\mathfrak{l},\mathfrak{h}}^{-1}\, p_k}{\mathrm{HC}_{\mathfrak{l},\mathfrak{h}}^{-1}\, q} [\Omega(\mathfrak{g}) - \Omega(\mathfrak{g},\mathfrak{l})]^k. \tag{4.3}$$

It will be shown in the proof of Theorem 7 that this is well defined in the sense that $p_k, q \in \mathfrak{U}(\mathfrak{h})^{W(\mathfrak{l}),\cdot}$.

Before proving Theorem 7, we must show elements of $\mathfrak{Z}(\mathfrak{l})$ are invertible as elements of $\mathrm{End}_{\mathfrak{h}}\, \bar{M}(\mathfrak{g})$. Towards this end, we need the following lemma. It also justifies the terminology that refers to $M_{\mathfrak{l}}$ as the highest $\mathfrak{l}$-subrepresentation.

Lemma 1. *As an* $\mathfrak{l}$*-representation,* $\bar{M}(\mathfrak{g})$ *is a direct sum of copies of* $\bar{M}(\mathfrak{l})$*.* $P(\mathfrak{g},\mathfrak{l})$ *projects* $\bar{M}(\mathfrak{g})$ *onto the unique copy of* $\bar{M}(\mathfrak{l})$ *generated by the highest weight space,* $\bar{\mathfrak{U}}(\mathfrak{h})\bar{1}$*. In particular, if* $\{u_{\gamma,i}\}_{i=1}^{n_\gamma}$ *is a basis of* $\mathfrak{U}(\mathfrak{u}^-)_\gamma$ *then*

$$\bar{M}(\mathfrak{g}) = \bigoplus_{\substack{\gamma \in \Delta(\mathfrak{U}(\mathfrak{u}^-)),\\ 1 \leqslant i \leqslant n_\gamma}} \bar{\mathfrak{U}}(\mathfrak{l}^-) P(\mathfrak{l},\mathfrak{h}) \bar{u}_{\gamma,i}.$$

Each subspace $\bar{\mathfrak{U}}(\mathfrak{l}^-) P(\mathfrak{l},\mathfrak{h}) \bar{u}_{\gamma,i}$ *is* $\mathfrak{l}$*-invariant and isomorphic to* $\bar{M}(\mathfrak{l})$ *as an* $\mathfrak{l}$*-module.*

Proof. First observe $\mathfrak{U}(\mathfrak{l}^-) P(\mathfrak{l},\mathfrak{h}) \bar{\mathfrak{U}}(\mathfrak{h}) \bar{u}_{\gamma,i}$ is clearly an $\mathfrak{l}$-invariant subspace of $\bar{M}(\mathfrak{g})$ since $\mathfrak{U}(\mathfrak{l}) = \mathfrak{U}(\mathfrak{l}^-) \otimes \mathfrak{U}(\mathfrak{h}) \otimes \mathfrak{U}(\mathfrak{l}^+)$ and $\mathfrak{l}^+ P(\mathfrak{l},\mathfrak{h}) = 0$. Also recall there is a bijection $\phi : \bar{\mathfrak{U}}(\mathfrak{l}^-) \otimes \mathfrak{U}(\mathfrak{u}^-) \to \bar{M}(\mathfrak{g})$ induced by $\phi(l \otimes u) = lu\bar{1}$ for $l \in \bar{\mathfrak{U}}(\mathfrak{l}^-)$ and $u \in \mathfrak{U}(\mathfrak{u}^-)$. Viewing $P(\mathfrak{l},\mathfrak{h}) \in \bar{\mathfrak{F}}(\mathfrak{l})_0$ and using the fact that $[\mathfrak{l}^+, \mathfrak{u}^-] \subseteq \mathfrak{u}^-$, it is easy to verify $\phi^{-1}(P(\mathfrak{l},\mathfrak{h})\bar{u}_{\gamma,i}) = 1 \otimes u_{\gamma,i} + X$ where $X \in \mathfrak{l}^- \bar{\mathfrak{U}}(\mathfrak{l}^-) \otimes \mathfrak{U}(\mathfrak{u}^-)$ and the $\mathfrak{U}(\mathfrak{u}^-)$-components of X have *strictly* higher weight than $u_{\gamma,i}$.

In particular, $P(\mathfrak{l},\mathfrak{h})\bar{u}_{\gamma,i}$ is nonzero. Moreover, left multiplication by a nonzero element of $\mathfrak{U}(\mathfrak{n}^-)$ on $\mathfrak{U}(\mathfrak{n}^-)$ is injective (for instance by looking at top filtration degrees and using the Poincaré–Birkhoff–Witt theorem). Since $P(\mathfrak{l},\mathfrak{h})\bar{u}_{\gamma,i}$ is nonzero and $\mathfrak{l}^+P(\mathfrak{l},\mathfrak{h})=0$, it is therefore clear the mapping of $\bar{\mathfrak{U}}(\mathfrak{l}^-)$ to $\bar{\mathfrak{U}}(\mathfrak{l}^-)P(\mathfrak{l},\mathfrak{h})\bar{u}_{\gamma,i}$ given by $l\mapsto lP(\mathfrak{l},\mathfrak{h})\bar{u}_{\gamma,i}$ induces an $\mathfrak{l}$-isomorphism from $\bar{M}(\mathfrak{l})$ to $\bar{\mathfrak{U}}(\mathfrak{l}^-)P(\mathfrak{l},\mathfrak{h})\bar{u}_{\gamma,i}$.

Now let $l_1,l_2\in\bar{\mathfrak{U}}(\mathfrak{l}^-)$. Then

$$\begin{aligned}\langle l_1P(\mathfrak{l},\mathfrak{h})\bar{u}_{\gamma_1,i_1}, l_2P(\mathfrak{l},\mathfrak{h})\bar{u}_{\gamma_2,i_2}\rangle &= \mathrm{HC}_{\mathfrak{g},\mathfrak{h}}(u^*_{\gamma_1,i_1}P(\mathfrak{l},\mathfrak{h})l_1^*l_2P(\mathfrak{l},\mathfrak{h})u_{\gamma_2,i_2})\\ &= \mathrm{HC}_{\mathfrak{g},\mathfrak{h}}(u^*_{\gamma_1,i_1}P(\mathfrak{l},\mathfrak{h})\mathrm{HC}_{\mathfrak{l},\mathfrak{h}}(l_1^*l_2)P(\mathfrak{l},\mathfrak{h})u_{\gamma_2,i_2})\\ &= \mathrm{HC}_{\mathfrak{g},\mathfrak{h}}(u^*_{\gamma_1,i_1}P(\mathfrak{l},\mathfrak{h})u_{\gamma_2,i_2})\mathrm{HC}_{\mathfrak{l},\mathfrak{h}}(l_1^*l_2).\end{aligned}$$

In particular, $l_1P(\mathfrak{l},\mathfrak{h})\bar{u}_{\gamma_1,i_1}\perp l_2P(\mathfrak{l},\mathfrak{h})\bar{u}_{\gamma_2,i_2}$ unless perhaps $\langle l_1,l_2\rangle\neq 0$ and $\gamma_1=\gamma_2$. Thus to show the modules $\bar{\mathfrak{U}}(\mathfrak{l}^-)P(\mathfrak{l},\mathfrak{h})\bar{u}_{\gamma,i}$ form a direct sum, it is enough to show they are direct for $\{u_{\gamma,i}\}_{i=1}^{n_\gamma}$ for a fixed γ. So suppose $\bar{0}=\sum_{i=1}^{n_\gamma} l_iP(\mathfrak{l},\mathfrak{h})\bar{u}_{\gamma,i}$. Apply ϕ^{-1} to see $0=\sum_{i=1}^{n_\gamma} l_i\otimes u_{\gamma,i}+X_i$ where $X_i\in(l_i\mathfrak{l}^-\bar{\mathfrak{U}}(\mathfrak{l}^-)\otimes\mathfrak{U}(\mathfrak{u}^-))_\gamma$ and the $\mathfrak{U}(\mathfrak{u}^-)$-components of X_i have strictly higher weight than γ. In particular, looking only at the terms with $\mathfrak{U}(\mathfrak{u}^-)$-components in $\mathfrak{U}(\mathfrak{u}^-)_\gamma$, $0=\sum_{i=1}^{n_\gamma} l_i\otimes u_{\gamma,i}$. Since $\{u_{\gamma,i}\}_{i=1}^{n_\gamma}$ is independent, it is easy to see $l_i=0$.

To finish the proof, we show the modules $\bar{\mathfrak{U}}(\mathfrak{l}^-)P(\mathfrak{l},\mathfrak{h})\bar{u}_{\gamma,i}$ span $\bar{M}(\mathfrak{g})$. As $\bar{M}(\mathfrak{g})=(\mathfrak{U}(\mathfrak{l}^-)\otimes\mathfrak{U}(\mathfrak{u}^-)\otimes\bar{\mathfrak{U}}(\mathfrak{h}))\bar{1}$, it suffices to show each $lu_{\gamma,i}$ is a contained in a finite sum of modules of the form $\bar{\mathfrak{U}}(\mathfrak{l}^-)P(\mathfrak{l},\mathfrak{h})\bar{u}_{\gamma',i'}$. The above discussion shows $lP(\mathfrak{l},\mathfrak{h})\bar{u}_{\gamma,i}$ matches $l\bar{u}_{\gamma,i}$ up to lower weight spaces in the first factor of $\bar{\mathfrak{U}}(\mathfrak{l}^-)\otimes\mathfrak{U}(\mathfrak{u}^-)$ and higher weight spaces in the second. As the weights of $\mathfrak{U}(\mathfrak{u}^-)$ are bounded from above, an inductive procedure finishes the proof. □

Lemma 2. *If* $z\in\mathfrak{Z}(\mathfrak{l})$, *then* z *is invertible viewed as an element of* $\mathrm{End}_{\mathfrak{h}}\,\bar{M}(\mathfrak{g})$. *In particular, the element* z^{-1} *acts on* $\bar{\mathfrak{U}}(\mathfrak{l}^-)_\lambda P(\mathfrak{l},\mathfrak{h})\bar{u}_{\gamma,i}$ *as multiplication by*

$$\frac{1}{(\mathrm{HC}_{\mathfrak{l},\mathfrak{h}}z)^{-\lambda}}.$$

Proof. Decompose $\bar{M}(\mathfrak{g})$ as in Lemma 1. For $z\in\mathfrak{Z}(\mathfrak{l})$ and $l\in\mathfrak{U}(\mathfrak{l}^-)_\lambda$,

$$\begin{aligned}zlP(\mathfrak{l},\mathfrak{h})\bar{u}_{\gamma,i} &= l(\mathrm{HC}_{\mathfrak{l},\mathfrak{h}}z)P(\mathfrak{l},\mathfrak{h})\bar{u}_{\gamma,i}\\ &= (\mathrm{HC}_{\mathfrak{l},\mathfrak{h}}z)^{-\lambda}lP(\mathfrak{l},\mathfrak{h})\bar{u}_{\gamma,i}.\end{aligned}$$

Therefore, for $z\in\mathfrak{Z}(\mathfrak{l})$, define z^{-1} as the element of $\mathrm{End}_{\mathfrak{h}}\,\bar{M}(\mathfrak{g})$ that acts by

$$\frac{1}{(\mathrm{HC}_{\mathfrak{l},\mathfrak{h}}z)^{-\lambda}}$$

on $\bar{\mathfrak{U}}(\mathfrak{l}^-)_\lambda P(\mathfrak{l},\mathfrak{h})\bar{u}_{\gamma,i}$. It is straightforward to check this is a well defined element of $\mathrm{End}_\mathfrak{h}\,\bar{M}(\mathfrak{g})$ and $zz^{-1}=z^{-1}z=\mathrm{Id}_{\bar{M}(\mathfrak{g})}$. $\square$

The next theorem gives an infinite commutative factorization of the relative extremal projector analogous to Theorem 4. Notice the factors are polynomials in $\Omega(\mathfrak{g})$ over $\mathrm{Frac}\,\mathfrak{Z}(\mathfrak{l})$ and so each term commutes with $\mathfrak{l}$.

Theorem 7.

$$P(\mathfrak{g},\mathfrak{l})=\prod_{\substack{v\in\Delta(\mathfrak{u}^+\mathfrak{U}(\mathfrak{u}^+)),\\ \mathfrak{l}\text{-dominant}}}\mathrm{HC}_{\mathfrak{l},\mathfrak{h}}^{-1}\prod_{w\in W(\mathfrak{l})}\left(1-\frac{\Omega(\mathfrak{g})-\Omega(\mathfrak{g},\mathfrak{l})}{\Omega(\mathfrak{g},\mathfrak{h})^{wv}-\Omega(\mathfrak{g},\mathfrak{h})}\right). \tag{4.4}$$

Proof. Denote by P the right-hand side of the formula of Eq. (4.4). To see P is well defined and commutes with $\mathfrak{l}$, we show the p_k, q from Eq. (4.3) are in $\mathfrak{U}(\mathfrak{h})^{W(\mathfrak{l}),\cdot}$. We already know $\Omega(\mathfrak{g},\mathfrak{h})\in\mathfrak{U}(\mathfrak{h})^{W(\mathfrak{g}),\cdot}$ so

$$\Omega(\mathfrak{g},\mathfrak{h})=w(\Omega(\mathfrak{g},\mathfrak{h})^{w^{-1}\rho_\mathfrak{g}-\rho_\mathfrak{g}})$$

for all $w\in W(\mathfrak{g})$. But $w^{-1}\rho_\mathfrak{g}-\rho_\mathfrak{g}=[w^{-1}(\rho_\mathfrak{l})-\rho_\mathfrak{l}]+[w^{-1}(\rho_{\mathfrak{u}^+})-\rho_{\mathfrak{u}^+}]$ so that if $w\in W(\mathfrak{l})$

$$\Omega(\mathfrak{g},\mathfrak{h})=w(\Omega(\mathfrak{g},\mathfrak{h})^{w^{-1}\rho_\mathfrak{l}-\rho_\mathfrak{l}})$$

since $w(\rho_{\mathfrak{u}^+})=\rho_{\mathfrak{u}^+}$. In particular, $\Omega(\mathfrak{g},\mathfrak{h})\in\mathfrak{U}(\mathfrak{h})^{W(\mathfrak{l}),\cdot}$. More generally,

$$\begin{aligned}w\cdot\Omega(\mathfrak{g},\mathfrak{h})^v&=w(\Omega(\mathfrak{g},\mathfrak{h})^{v+w^{-1}\rho_\mathfrak{l}-\rho_\mathfrak{l}})\\&=[w(\Omega(\mathfrak{g},\mathfrak{h})^{w^{-1}\rho_\mathfrak{l}-\rho_\mathfrak{l}})]^{wv}=\Omega(\mathfrak{g},\mathfrak{h})^{wv}.\end{aligned}$$

Letting $r_w=\Omega(\mathfrak{g},\mathfrak{h})^{wv}-\Omega(\mathfrak{g},\mathfrak{h})$, we see $w\cdot r_{w'}=r_{ww'}$ for $w,w'\in W(\mathfrak{l})$. Since $q=\prod_{w\in W(\mathfrak{l})}r_w$, we see $w\cdot q=q$ so that $q\in\mathfrak{U}(\mathfrak{h})^{W(\mathfrak{l}),\cdot}$. Finally, enumerate $W(\mathfrak{l})=\{w_1,w_2,\ldots,w_s\}$ so that

$$p_k=(-1)^k\sum_{k=0}^{s}\prod_{1\leqslant i_1<i_2<\cdots<i_{s-k}\leqslant s}r_{w_{i_1}}r_{w_{i_2}}\cdots r_{w_{i_{s-k}}}.$$

Thus $w\cdot p_k=p_k$ so $p_k\in\mathfrak{U}(\mathfrak{h})^{W(\mathfrak{l}),\cdot}$.

It now suffices to show P acts by 1 on $M_\mathfrak{l}$ and by 0 on $M_\mathfrak{l}^\perp$. Note $\Omega(\mathfrak{g})$ acts on $\bar{M}(\mathfrak{g})_\gamma$ as $\Omega(\mathfrak{g},\mathfrak{h})^{-\gamma}$ and that Lemma 2 implies $\Omega(\mathfrak{g},\mathfrak{l})P(\mathfrak{l},\mathfrak{h})=\mathrm{HC}_{\mathfrak{l},\mathfrak{h}}\Omega(\mathfrak{g},\mathfrak{l})P(\mathfrak{l},\mathfrak{h})=$

$\Omega(\mathfrak{g},\mathfrak{h})P(\mathfrak{l},\mathfrak{h})$. So if $l \in \mathfrak{U}(\mathfrak{l}^-)$,

$$\begin{aligned}
&\mathrm{HC}^{-1}_{\mathfrak{l},\mathfrak{h}}\left[\prod_{w\in W(\mathfrak{l})}\left(1-\frac{\Omega(\mathfrak{g})-\Omega(\mathfrak{g},\mathfrak{l})}{\Omega(\mathfrak{g},\mathfrak{h})^{wv}-\Omega(\mathfrak{g},\mathfrak{h})}\right)\right] lP(\mathfrak{l},\mathfrak{h})\bar{u}_{\gamma,i}\\
&\quad= l\mathrm{HC}^{-1}_{\mathfrak{l},\mathfrak{h}}\left[\prod_{w\in W(\mathfrak{l})}\left(1-\frac{\Omega(\mathfrak{g})-\Omega(\mathfrak{g},\mathfrak{l})}{\Omega(\mathfrak{g},\mathfrak{h})^{wv}-\Omega(\mathfrak{g},\mathfrak{h})}\right)\right] P(\mathfrak{l},\mathfrak{h})\bar{u}_{\gamma,i}\\
&\quad= l\prod_{w\in W(\mathfrak{l})}\left(1-\frac{\Omega(\mathfrak{g})-\Omega(\mathfrak{g},\mathfrak{l})}{\Omega(\mathfrak{g},\mathfrak{h})^{wv}-\Omega(\mathfrak{g},\mathfrak{h})}\right) P(\mathfrak{l},\mathfrak{h})\bar{u}_{\gamma,i}\\
&\quad= l\prod_{w\in W(\mathfrak{l})}\left(1-\frac{\Omega(\mathfrak{g},\mathfrak{h})^{-\gamma}-\Omega(\mathfrak{g},\mathfrak{h})}{\Omega(\mathfrak{g},\mathfrak{h})^{wv}-\Omega(\mathfrak{g},\mathfrak{h})}\right) P(\mathfrak{l},\mathfrak{h})\bar{u}_{\gamma,i}.
\end{aligned}$$

On $M_{\mathfrak{l}}$, $\gamma = 0$ so P acts by 1. On $M_{\mathfrak{l}}^{\perp}$, $\gamma \in \Delta(\mathfrak{u}^-\mathfrak{U}(\mathfrak{u}^-))$ so there exists $w \in W(\mathfrak{l})$ and $v \in \Delta(\mathfrak{u}^+\mathfrak{U}(\mathfrak{u}^+))$ that is $\mathfrak{l}$-dominant with $-\gamma = wv$. In particular, P acts by 0 on $M_{\mathfrak{l}}^{\perp}$. □

5. Summation formulas for relative extremal projectors

This section gives a summation formula for $P(\mathfrak{g},\mathfrak{l})$ in terms of $P(\mathfrak{g},\mathfrak{h})$ which is in turn given by Theorems 3 and 4.

Recall $\Delta^+ = \{\alpha_1, \alpha_2, \dots, \alpha_m\}$. Given sequences of nonnegative integers $r = (r_1, r_2, \dots, r_m)$ and $s = (s_1, s_2, \dots, s_m)$, define

$$E_r = \prod_{i=1}^{m} e_{\alpha_i}^{r_i}, \quad F_r = E_r^*, \quad \text{and} \quad \mathrm{wt}(F_rE_s) = \sum_{i=1}^{m}(-r_i + s_i)\alpha_i.$$

Lemma 3. *For* $\mathrm{wt}(F_rE_s) = 0$, *there exist unique* $p_{r,s} \in \bar{\mathfrak{U}}(\mathfrak{h})$ *so*

$$1 = \sum_{\mathrm{wt}(F_rE_s)=0} F_r\, P(\mathfrak{g},\mathfrak{h})\, E_s\, p_{r,s}.$$

The coefficient of $P(\mathfrak{g},\mathfrak{h})$ *is* 1 *and the coefficients are symmetric*, i.e., $p_{r,s} = p_{s,r}$. *When* α *is* simple, *the coefficient of* $e_{-\alpha}^k\, P(\mathfrak{g},\mathfrak{h})\, e_{\alpha}^k$ *is*

$$\frac{1}{k!\,(h_\alpha + k + 1)(h_\alpha + k + 2)\cdots(h_\alpha + 2k)}$$

for all $k \in \mathbb{N}$. *Finally, if* $\{u_i\}_{i=1}^{\infty}$ *is a weight basis of* $\mathfrak{U}(\mathfrak{n}^-)$ *and* $\{\tilde{u}_i\}_{i=1}^{\infty}$ *is the dual basis under the Shapovalov form*

$$1 = \sum_{i=1}^{\infty} u_i \, P(\mathfrak{g}, \mathfrak{h}) \, \tilde{u}_i^*.$$

Proof. Fix λ a weight of $\mathfrak{U}(\mathfrak{n}^-)$. Let $u_1, \ldots, u_N$ be any basis for $\mathfrak{U}(\mathfrak{n}^-)_\lambda$ where N is the value of the Kostant partition function at $-\lambda$. By the nondegeneracy of the Shapovalov form, choose $\tilde{u}_1, \ldots, \tilde{u}_N$ in $\bar{\mathfrak{U}}(\mathfrak{n}^-)_\lambda$ to be its dual basis. Consider the expression

$$X = \sum_{i,j=1}^{N} u_i \, P(\mathfrak{g}, \mathfrak{h}) \, \tilde{u}_j^* \, p_{i,j},$$

where $p_{i,j} \in \bar{\mathfrak{U}}(\mathfrak{h})$. Then X acts by zero on any weight space $\bar{M}(\mathfrak{g})_\mu$ when $\mu \neq \lambda$. Furthermore, X acts as the identity on $\bar{M}(\mathfrak{g})_\lambda$ if and only if $X \, u_k = u_k$ for all k, regarded as an equality in $\bar{M}(\mathfrak{g})$. However, $u_i \, P(\mathfrak{g}, \mathfrak{h}) \, \tilde{u}_j^* \, p_{i,j} \, u_k = u_i \, \langle \tilde{u}_j, u_k \rangle \, p_{i,j}^{\lambda}$ and so $X \, u_k = \sum_{i=1}^{N} u_i \, p_{i,k}^{\lambda}$. Therefore X acts as the identity on $\bar{M}(\mathfrak{g})_\lambda$ if and only if $p_{i,j} = \delta_{i,j}$, the Dirac delta function. In other words, if and only if

$$X = \sum_{i=1}^{N} u_i \, P(\mathfrak{g}, \mathfrak{h}) \, \tilde{u}_i^*.$$

Of course u_i and $\tilde{u}_i^*$ may be expanded uniquely in terms of any particular basis of $\mathfrak{U}(\mathfrak{n}^-)_\lambda$ and $\mathfrak{U}(\mathfrak{n}^+)_{-\lambda}$, respectively. This finishes the existence and uniqueness parts of the lemma. The statement regarding the coefficient of $P(\mathfrak{g}, \mathfrak{h})$ is obvious. The statement regarding $e_{-\alpha}^k \, P(\mathfrak{g}, \mathfrak{h}) \, e_\alpha^k$ follows by a simple calculation of $\langle e_{-\alpha}^k, e_{-\alpha}^k \rangle$ which may be easily carried out in a copy of $\mathfrak{sl}(2, \mathbb{C})$. The statement regarding symmetry follows by uniqueness. □

It is easy to see $P(\mathfrak{l}, \mathfrak{h}) = P(\mathfrak{l}_{ss}, \mathfrak{h} \cap \mathfrak{l}_{ss})$, where $\mathfrak{l}_{ss}$ is the semisimple part of $\mathfrak{l}$. Then Lemma 3 can be applied to the special case of $\mathfrak{g} = \mathfrak{l}$ to write

$$1 = \sum_{\mathrm{wt}(F_r^{\mathfrak{l}} E_s^{\mathfrak{l}})=0} F_r^{\mathfrak{l}} \, P(\mathfrak{l}, \mathfrak{h}) \, E_s^{\mathfrak{l}} p_{r,s}^{\mathfrak{l}}, \tag{5.1}$$

where now the $F_r^{\mathfrak{l}}$ and the $E_s^{\mathfrak{l}}$ are the corresponding basis elements for $\mathfrak{U}(\mathfrak{l}^-)$ and $\mathfrak{U}(\mathfrak{l}^+)$, respectively, and $p_{r,s}^{\mathfrak{l}} \in \bar{\mathfrak{U}}(\mathfrak{h} \cap \mathfrak{l}_{ss})$.

Theorem 8. *With the notation from Eq.* (5.1),

$$P(\mathfrak{g}, \mathfrak{l}) = \sum_{\mathrm{wt}(F_r^{\mathfrak{l}} E_s^{\mathfrak{l}})=0} F_r^{\mathfrak{l}} P(\mathfrak{g}, \mathfrak{h}) E_s^{\mathfrak{l}} p_{r,s}^{\mathfrak{l}}.$$

Proof. Apply $P(\mathfrak{g}, \mathfrak{l})$ to both sides of Eq. (5.1) and use Theorems 5 and 6. $\square$

Corollary 1. *If* $\mathfrak{l}_{ss} \simeq \mathfrak{sl}(2, \mathbb{C})$ *and corresponds to the simple root* α, *then*

$$P(\mathfrak{g}, \mathfrak{l}) = \sum_{k=0}^{\infty} e_{-\alpha}^{k} P(\mathfrak{g}, \mathfrak{h}) e_{\alpha}^{k} \frac{1}{k!(h_\alpha + k + 1)(h_\alpha + k + 2) \cdots (h_\alpha + 2k)}.$$

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