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Computation of convex bounds for present value functions with random payments

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Abstract

In this contribution we study the distribution of the present value function of a series of random payments in a stochastic financial environment. Such distributions occur naturally in a wide range of applications within fields of insurance and finance. We obtain accurate approximations by developing upper and lower bounds in the convex-order sense for present value functions. Technically speaking, our methodology is an extension of the results of Dhaene et al. [Insur. Math. Econom. 31(1) (2002) 3–33, Insur. Math. Econom. 31(2) (2002) 133–161] to the case of scalar products of mutually independent random vectors.

Keywords: Convex order; Comonotonicity; Present value functions; Black and Scholes model

1. Introduction

Within the fields of finance and actuarial science one is often confronted with the problem of determining the distribution function of a scalar product of two random vectors of the form

$$S = \sum_{i=1}^{n} X_i V_i. \tag{1}$$

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In this contribution we will interpret the random variables X_i as future payments/liabilities due at times $i = t_1, t_2, \ldots, t_n$ and V_i as random discount factors equal to $\mathrm{e}^{-Y(t_i)}$, where the process $Y(t)$ represents the return on investment in period $(0, t)$. Notice that here the random vector $\vec{X} = (X_1, X_2, \ldots, X_n)$ may reflect e.g. the insurance or credit risk while the vector $\vec{V} = (V_1, V_2, \ldots, V_n)$ represents the financial/investment risk. In general we assume that these vectors are mutually independent.

In practical applications the independence assumption may be often violated, e.g. due to an inflation factor which strongly influences both payments and investment results. One can however tackle this problem by considering sums of the form

$$S = \sum_{i=1}^{n} \tilde{X}_i \tilde{V}_i,$$

where $\tilde{X}_i = X_i / Z_i$ and $\tilde{V}_i = V_i Z_i$ are the adjusted values expressed in real terms (Z_i denotes here an inflation factor over period $(0, t_i)$). For this reason the assumption of independence between the insurance risk and the financial risk is in most cases realistic and can be efficiently deployed to obtain various quantities describing risk within financial institutions, e.g. discounted insurance claims or the embedded/appraisal value of a company.

Distributions of sums of form (1) are often encountered in practice and need to be analyzed thoroughly by actuaries and other practitioners involved in the risk management process. Not only the basic summary measures (like the first few moments) have to be computed, but also more sophisticated risk measures which require much deeper knowledge about the underlying distributions (e.g. the Value-at-Risk).

Unfortunately there are no analytical methods to compute distribution functions for random variables of this form. That's why usually one has to rely on volatile and time consuming Monte Carlo simulations. Despite the enormous increase in computational power observed within last few years, the computational time remains a serious drawback of Monte Carlo simulations, especially when one is interested in estimating very high values of quantiles (note that a solvency capital of an insurance company may be determined e.g. as the 99.95%-quantile, which is extremely difficult to estimate within reasonable time by simulation methods).

In this contribution we propose an alternative solution. By extending the methodology of Dhaene et al. [3,4] to the case of scalar products of independent random vectors, we obtain convex upper and lower bounds for sums of form (1). As we demonstrate by means of a series of numerical illustrations, the methodology provides an excellent framework to get accurate and easily obtainable approximations of distribution functions for random variables of form (1).

The structure of the paper is as follows. In Section 2 we briefly revise the theoretical concepts on which our methodology is based. Next, we demonstrate in Section 3 how to obtain the bounds for (1) in the convex order sense in case when $\vec{V}$ follows the log-normal law. Section 4 contains several applications for discounted claim processes under the Black and Scholes setting. Finally, we conclude the paper in Section 5.

2. Methodology

2.1. Convex order and comonotonicity

In this subsection we briefly recapitulate some theoretical results of Dhaene et al. [3].

Definition 1. A random variable X is said to precede a random variable Y in the convex order sense, notation $X \leqslant_{cx} Y$, if and only if $E[X] = E[Y]$ and $E[(X-d)_+] \leqslant E[(Y-d)_+]$ for any retention d.

Roughly speaking, the convex order corresponds to the intuition of riskiness. Indeed, $X \leqslant_{cx} Y$ means that Y is more likely to take on extreme values than X. Note that Definition 1 is equivalent to the statement that X is preferred by all risk-averse decision makers in the framework of utility theory. It can be also proved that the same holds for the dual theory of choice under risk of Yaari [14]—see e.g. [3]. Thus from the viewpoint of an insurer it will be always a prudent strategy to replace a random variable X by a riskier random variable Y.

Definition 2. Let $\vec{X} = (X_1, X_2, \ldots, X_n)$ be a random vector with marginal distributions given by $F_{X_i}(t) = \Pr[X_i \leqslant t]$. Then $\vec{X}$ is said to be comonotonic if there exist a random variable Z and non-decreasing (nonincreasing) functions $g_1, g_2, \ldots, g_n : \mathbb{R} \to \mathbb{R}$ such that

$$\vec{X} \stackrel{\mathrm{d}}{=} (g_1(Z), g_2(Z), \ldots, g_n(Z)),$$

where $\stackrel{\mathrm{d}}{=}$ means equality in distribution.

If a random variable S consists of a sum of random variables $(X_1, \ldots, X_n)$, replacing the copula of $(X_1, \ldots, X_n)$ by the comonotonic copula yields an upper bound for S in the convex order. On the other hand, applying Jensen's inequality to S provides us a lower bound. This is formalized in the following theorem, which is taken from [3,10].

Theorem 1. *Consider a sum of random variables* $S = X_1 + X_2 + \cdots + X_n$ *and define the following related random variables*:

$$S^u = F_{X_1}^{-1}(U) + F_{X_2}^{-1}(U) + \cdots + F_{X_n}^{-1}(U), \tag{2}$$

$$S^l = E[X_1|\Lambda] + E[X_2|\Lambda] + \cdots + E[X_n|\Lambda], \tag{3}$$

with U a Uniform$(0,1)$ *random variable and Λ an arbitrary random variable. Then the following relations hold*:

$$S^l \leqslant_{cx} S \leqslant_{cx} S^u.$$

Proof. See e.g. [3]. □

The comonotonic upper bound changes the original copula, but keeps the marginal distributions unchanged. The comonotonic lower bound on the other hand, changes both the copula and the marginals involved. Intuitively, one can expect that an appropriate choice of the conditioning variable Λ will lead to much better approximations compared to the upper bound. This observation has been confirmed empirically in numerous illustrations (see e.g. [4,5]).

2.2. Convex upper and lower bounds for scalar products of random vectors

As mentioned in the beginning we want to find accurate approximations for sums of the following form:

$$S = \sum_{i=1}^{n} X_i V_i, \tag{4}$$

where the random vectors $\vec{X} = (X_1, X_2, \ldots, X_n)$ and $\vec{V} = (V_1, V_2, \ldots, V_n)$ are assumed to be mutually independent. In deriving lower and upper bounds for sums of the form (4) we recall a helpful lemma.

Lemma 1. *Let $\vec{X} = (X_1, X_2, \ldots, X_n)$, $\vec{V} = (V_1, V_2, \ldots, V_n)$ and $\vec{W} = (W_1, W_2, \ldots, W_n)$ be nonnegative random vectors and assume that $\vec{X}$ is mutually independent of the vectors $\vec{V}$ and $\vec{W}$. If for all possible outcomes $x_1, x_2, \ldots, x_n$* of *$\vec{X}$ one has*

$$\sum_{i=1}^{n} x_i V_i \leqslant_{cx} \sum_{i=1}^{n} x_i W_i,$$

then the corresponding scalar products are ordered in the convex order sense, i.e.

$$\sum_{i=1}^{n} X_i V_i \leqslant_{cx} \sum_{i=1}^{n} X_i W_i.$$

Proof. See [7]. □

Theorem 2. *Consider a sum of random variables of form* (4). *Define the following quantities*:

$$S^u = \sum_{i=1}^{n} F_{X_i}^{-1}(U_1) F_{V_i}^{-1}(U_2), \tag{5}$$

$$S^l = \sum_{i=1}^{n} E[X_i|\Theta] E[V_i|\Lambda], \tag{6}$$

where U_1 and U_2 are independent standard Uniform random variables, Θ is a random variable independent of $\vec{V}$ and Λ and the second conditioning random variable Λ is independent of $\vec{X}$ and Θ. Then, the following relation holds:

$$S^l \leqslant_{cx} S \leqslant_{cx} S^u.$$

Proof. The proof is based on a multiple application of Lemma 1.

1. First, we prove that $\sum_{i=1}^{n} X_i V_i \leqslant_{cx} \sum_{i=1}^{n} F_{X_i}^{-1}(U_1) F_{V_i}^{-1}(U_2)$.
   From Theorem 1 it follows that for all possible outcomes $(x_1, x_2, \ldots, x_n)$ of $\vec{X}$ the following inequality holds:

$$\sum_{i=1}^{n} x_i V_i \leqslant_{cx} \sum_{i=1}^{n} F_{x_i V_i}^{-1}(U_2) = \sum_{i=1}^{n} x_i F_{V_i}^{-1}(U_2).$$

Thus from Lemma 1 it follows that $\sum_{i=1}^n X_i V_i \leqslant_{cx} \sum_{i=1}^n X_i F_{V_i}^{-1}(U_2)$. The same reasoning can be applied to show that

$$\sum_{i=1}^n X_i F_{V_i}^{-1}(U_2) \leqslant_{cx} \sum_{i=1}^n F_{X_i}^{-1}(U_1) F_{V_i}^{-1}(U_2).$$

2. In a similar way, one can show—using Theorem 1—that

$$\sum_{i=1}^n E[X_i|\Theta]E[V_i|\Lambda] \leqslant_{cx} \sum_{i=1}^n X_i E[V_i|\Lambda] \leqslant_{cx} \sum_{i=1}^n X_i V_i. \qquad \square$$

Remark 1. Notice that $\sum_{i=1}^n F_{X_i}^{-1}(U_1) F_{V_i}^{-1}(U_2) \leqslant_{cx} \sum_{i=1}^n F_{X_i Y_i}^{-1}(U)$. Therefore the upper bound (5) improved compared to the comonotonic upper bound (2). It takes efficiently into account information that the vectors $\vec{X}$ and $\vec{V}$ are mutually independent.

We remark also that having obtained the convex upper and lower bounds one can construct a new approximation, called the moments-based approximation S^m defined by the distribution function as follows:

$$F_{S^m}(t) = z F_{S^l}(t) + (1-z) F_{S^u}(t), \tag{7}$$

where

$$z = \frac{\mathrm{Var}[S^u] - \mathrm{Var}[S]}{\mathrm{Var}[S^u] - \mathrm{Var}[S^l]}. \tag{8}$$

This approximation results in $E[S^m] = E[S]$ and $\mathrm{Var}[S^m] = \mathrm{Var}[S]$. For more details we refer to [13].

3. Convex bounds for log-normal discount factors

In a lot of financial and actuarial problems one encounters sums of the form

$$S = \sum_{i=1}^n X_i \mathrm{e}^{Z_i}, \tag{9}$$

with $\vec{Z} = (Z_1, Z_2, \ldots, Z_n)$ following the multivariate normal law. In this section we use the following notations:

$$\mu_i = E[Z_i], \quad \sigma_i^2 = \mathrm{Var}[Z_i] \quad \text{and} \quad \sigma_{ij} = \mathrm{Cov}(Z_i, Z_j).$$

Further we assume that the random vectors $\vec{X}$ and $\vec{Z}$ are mutually independent.

In this section we consider the problem in general, without imposing any conditions on the random variables X_i. In particular we do not discuss the choice of conditioning variable Θ—we will demonstrate it by means of some special cases in the next two sections. The upper and lower bound can be calculated by means of a three step approach which is described in the following two subsections.

3.1. The upper bound

From Theorem 2 it follows that for the case of log-normally distributed discount factors the upper bound can be expressed as

$$S^u = \sum_{i=1}^{n} F_{X_i}^{-1}(U_1) F_{\mathrm{e}^{Z_i}}^{-1}(U_2) = \sum_{i=1}^{n} F_{X_i}^{-1}(U_1)\mathrm{e}^{\mu_i + \sigma_i \Phi^{-1}(U_2)}, \tag{10}$$

where U_1 and U_2 are independent standard Uniform random variables.

The cumulative distribution function of S^u is computed in three steps:

1. Suppose that $U_1 = u_1$ is fixed. Then from (10) it follows that conditional quantiles can be computed as

$$F_{S^u|U_1=u_1}^{-1}(p) = \sum_{i=1}^{n} F_{X_i}^{-1}(u_1)\mathrm{e}^{\mu_i + \sigma_i \Phi^{-1}(p)}; \tag{11}$$

2. Obviously for any u_1 the function given by (11) is continuous and strictly increasing. Thus for any $y \geqslant 0$ one can compute the value of the conditional distribution function using one of the well-known numerical methods (e.g. Newton–Raphson) as a solution of

$$\sum_{i=1}^{n} F_{X_i}^{-1}(u_1)\mathrm{e}^{\mu_i + \sigma_i \Phi^{-1}(F_{S^u|U_1=u_1}(y))} = y; \tag{12}$$

3. The cumulative distribution function of S^u can now be derived as

$$F_{S^u}(y) = \int_0^1 F_{S^u|U_1=u_1}(y)\,\mathrm{d}u_1.$$

3.2. The lower bound

Although the computations for the lower bound are performed in a similar way as in the case of the upper bound, one should note that the quality of the bound heavily depends on the choice of the conditioning random variables.

Recall that from Theorem 2 one has that

$$S^l = \sum_{i=1}^{n} E[X_i|\Theta]E[\mathrm{e}^{Z_i}|\Lambda], \tag{13}$$

where the first conditioning variable Θ is independent of Λ and $\vec{Z}$ and where the second conditioning variable Λ is independent of Θ and $\vec{X}$. In this section the choice of Θ will not be discussed, whereas the choice of Λ is given by the following equation:

$$\Lambda = \sum_{i=1}^{n} E[X_i]\mathrm{e}^{\mu_i + (1/2)\sigma_i^2} Z_i. \tag{14}$$

Then the lower bound (13) can be written out as

$$S^l = \sum_{i=1}^{n} E[X_i|\Theta]E[\mathrm{e}^{Z_i}|\Lambda] = \sum_{i=1}^{n} E[X_i|\Theta]\mathrm{e}^{\mu_i+(1/2)\sigma_i^2(1-r_i^2)+\sigma_i r_i \Phi^{-1}(U_2)}, \tag{15}$$

with U_2 a standard uniform random variable and correlation r_i given by

$$r_i = \mathrm{Corr}(Z_i, \Lambda) = \frac{\mathrm{Cov}(Z_i, \Lambda)}{\sqrt{\mathrm{Var}[Z_i]}\sqrt{\mathrm{Var}[\Lambda]}} \\ = \frac{\sum_{j=1}^{n} E[X_i]\mathrm{e}^{\mu_j+(1/2)\sigma_j^2}\sigma_{ij}}{\sigma_i\sqrt{\sum_{1\leqslant k,l\leqslant n} E[X_k]E[X_l]\mathrm{e}^{\mu_k+\mu_l+(1/2)(\sigma_k^2+\sigma_l^2)}\sigma_{kl}}}. \tag{16}$$

Note that in case $\vec{X}$ is nonnegative and $\vec{Z}$ has nonnegative correlations, the random variable S^l is (given a value $\Theta=\theta$) a sum of the components of a comonotonic vector. Thus the cumulative distribution function of the lower bound S^l can be computed as for the case of the upper bound S^u, in three steps:

1. From (15) it follows that the conditional quantiles (given $\Theta = \theta$) can be computed as

$$F^{-1}_{S^l|\Theta=\theta}(p) = \sum_{i=1}^{n} E[X_i|\Theta = \theta]\mathrm{e}^{\mu_i+(1/2)\sigma_i^2(1-r_i^2)+\sigma_i r_i \Phi^{-1}(p)}; \tag{17}$$

2. The conditional distribution function is computed as the solution of

$$\sum_{i=1}^{n} E[X_i|\Theta = \theta]\mathrm{e}^{\mu_i+(1/2)\sigma_i^2(1-r_i^2)+\sigma_i r_i \Phi^{-1}(F_{S^l|\Theta=\theta}(y))} = y; \tag{18}$$

3. Finally, the cumulative distribution function of S^l can be derived as

$$F_{S^l}(y) = \int_0^1 F_{S^l|\Theta=F_\Theta^{-1}(u_1)}(y)\,\mathrm{d}u_1. \tag{19}$$

4. Present value of stochastic cash flows

In this section we derive convex upper and lower bounds for general discounted cash flows S of the form

$$S = \sum_{i=1}^{n} X_i \mathrm{e}^{-Y(i)},$$

where the random variables X_i denote future (nonnegative) payments due at time i. We model the returns in this paper by means of a Brownian motion (the Black and Scholes model; see [2]) described by the following equation:

$$Y(t) = \mu t + \sigma B_t,$$

where B_t denotes a standard Brownian motion.

Note that the mean and variance functions are given by

$$E[Y(i)] = \mu i,$$

$$\mathrm{Cov}(Y(i), Y(j)) = \sigma^2 \min(i, j) \stackrel{\text{not}}{=} \sigma_{ij}.$$

We use the notation $\sigma_i^2 = \sigma_{ii}$ and give explicit results in three specific cases:

1. The vector $\ln(\vec{X}) = (\ln X_1, \ln X_2, \ldots, \ln X_n)$ has a multivariate normal distribution and hence the losses are log-normally distributed;
2. The vector $\vec{X} = (X_1, X_2, \ldots, X_n)$ has a multivariate elliptical distribution where $E[X_i]/\sqrt{\mathrm{Var}[X_i]} \geqslant 0$. Formally the described methodology is valid only in the case when $X_i > 0$. However if we assure that the probabilities $\Pr[X_i < 0]$ are very small then the influence of the negative outcomes of $\vec{X}$ on the overall distribution will be negligible;
3. The yearly payments X_i are independent and identically distributed.

4.1. *Log-normally distributed payments*

4.1.1. *Convex upper and lower bounds*

Consider a sum of the form

$$S_{\mathscr{LN}} = \sum_{i=1}^{n} \mathrm{e}^{N_i} \mathrm{e}^{-Y(i)}, \tag{20}$$

where $\vec{N} = (N_1, N_2, \ldots, N_n) = (\ln X_1, \ln X_2, \ldots, \ln X_n)$ is a normally distributed random vector with mean $\vec{\mu}_{\vec{N}} = (\mu_{N_1}, \mu_{N_2}, \ldots, \mu_{N_n})$ and covariance matrix $\mathbf{\Sigma}_{\vec{N}} = [\sigma_{ij}^{\vec{N}}]_{1 \leqslant i,j \leqslant n}$; we denote $\sigma_{ii}^{\vec{N}}$ by $\sigma_{N_i}^2$.

There are two different approaches to derive convex upper and lower bounds for $S_{\mathscr{LN}}$ as defined in (20). In the first approach independent parts of the scalar product are treated separately. In the second approach we treat $S_{\mathscr{LN}}$ unidimensionally, by noticing that it can be written as

$$S_{\mathscr{LN}} = \sum_{i=1}^{n} \hat{X}_i = \sum_{i=1}^{n} \mathrm{e}^{\hat{N}_i}, \tag{21}$$

where $\vec{\hat{N}} = (\hat{N}_1, \hat{N}_2, \ldots, \hat{N}_n) = (N_1 - Y(1), N_2 - Y(2), \ldots, N_n - Y(n))$ has a multivariate normal distribution with

$$\vec{\mu}_{\vec{\hat{N}}} = (\mu_{\hat{N}_1}, \mu_{\hat{N}_2}, \ldots, \mu_{\hat{N}_n}) \quad \text{and} \quad \mathbf{\Sigma}_{\vec{\hat{N}}} = [\sigma_{ij}^{\vec{\hat{N}}}]_{1 \leqslant i,j \leqslant n} \quad \left(\sigma_{ii}^{\vec{\hat{N}}} \stackrel{\text{not}}{=} \sigma_{\hat{N}_i}^2 \right), \tag{22}$$

with

$$\mu_{\hat{N}_i} = \mu_{N_i} - \mathrm{i}\mu \quad \text{and} \quad \sigma_{ij}^{\vec{\hat{N}}} = \sigma_{ij}^{\vec{N}} + \sigma_{ij}. \tag{23}$$

Thus one can derive convex upper and lower bounds of (20) just by adapting the methodology described in Section 3. Below we work out both approaches explicitly. Note that the second method is much less time-consuming because of unidimensionality.

(i) In the first approach the upper bound can be written as

$$S^u_{\mathscr{L}\mathscr{N}} = \sum_{i=1}^{n} \mathrm{e}^{\mu_{N_i}+\sigma_{N_i}\Phi^{-1}(U_1)-\mathrm{i}\mu+\sigma_i\Phi^{-1}(U_2)}$$

and its distribution function computed as described in Section 3.1.
To compute the lower bound we propose to define a conditioning random variable Θ analogously to the conditioning variable Λ, i.e.

$$\Theta = \sum_{i=1}^{n} E[\mathrm{e}^{-Y(i)}]\mathrm{e}^{\mu_{N_i}+(1/2)\sigma^2_{N_i}} N_i = \sum_{i=1}^{n} \mathrm{e}^{\mu_{N_i}-\mathrm{i}\mu+(1/2)(\sigma^2_{N_i}+\sigma^2_i)} N_i. \tag{24}$$

The conditioning variable Λ is chosen as in (14), which gives after the obvious substitution

$$\Lambda = -\sum_{i=1}^{n} \mathrm{e}^{\mu_{N_i}-\mathrm{i}\mu+(1/2)(\sigma^2_{N_i}+\sigma^2_i)} Y(i). \tag{25}$$

Now the corresponding lower bound can be written as

$$S^{l1}_{\mathscr{L}\mathscr{N}} = \sum_{i=1}^{n} \mathrm{e}^{\mu_{N_i}-\mathrm{i}\mu+(1/2)\sigma^2_{N_i}(1-r^2_{N_i})+(1/2)\sigma^2_i(1-r^2_i)+\sigma_{N_i} r_{N_i}\Phi^{-1}(U_1)+\sigma_i r_i\Phi^{-1}(U_2)},$$

where correlations $r_i = \mathrm{Corr}(-Y(i), \Lambda)$ are defined as in (16) and

$$r_{N_i} = \mathrm{Corr}(N_i, \Theta) = \frac{\sum_{j=1}^{n} \mathrm{e}^{\mu_{N_j}-\mathrm{j}\mu+(1/2)(\sigma^2_{N_j}+\sigma^2_j)} \sigma^{\vec{N}}_{ij}}{\sigma_{N_i}\sqrt{\sum_{k,l=1}^{n} \mathrm{e}^{\mu_{N_k}+\mu_{N_l}-k\mu-l\mu+(1/2)(\sigma^2_{N_k}+\sigma^2_{N_l}+\sigma^2_k+\sigma^2_l)}\sigma^{\vec{N}}_{kl}}}.$$

Its distribution function can be computed by conditioning on U_1 as described in Section 3.2.

(ii) From Remark 1 it follows that

$$S^u \leqslant_{cx} \sum_{i=1}^{n} F^{-1}_{\mathrm{e}^{\hat{N}_i}}(U),$$

and thus we do not consider the comonotonic upper bound for (21). To compute the lower bound we take as conditioning random variable

$$\hat{\Lambda} = \sum_{i=1}^{n} \mathrm{e}^{\mu_{\hat{N}_i}+(1/2)\sigma^2_{\hat{N}_i}} \hat{N}_i. \tag{26}$$

Then the lower bound is given explicitly by the following formula:

$$S^{l2}_{\mathscr{L}\mathscr{N}} = \sum_{i=1}^{n} \mathrm{e}^{\mu_{\hat{N}_i}+(1/2)\sigma^2_{\hat{N}_i}(1-r^2_{\hat{N}_i})+\sigma_{\hat{N}_i} r_{\hat{N}_i}\Phi^{-1}(U)},$$

where

$$r_{\hat{N}_i} = \mathrm{Corr}(\hat{N}_i, \tilde{\Lambda}) = \frac{\sum_{j=1}^{n} \mathrm{e}^{\mu_{\hat{N}_j} + (1/2)\sigma^2_{\hat{N}_j}} \sigma^{\vec{\hat{N}}}_{ij}}{\sigma_{\hat{N}_i} \sqrt{\sum_{k,l=1}^{n} \mathrm{e}^{\mu_{\hat{N}_k} + \mu_{\hat{N}_l} + (1/2)(\sigma^2_{\hat{N}_k} + \sigma^2_{\hat{N}_l})} \sigma^{\vec{\hat{N}}}_{kl}}}.$$

Note that to obtain a comonotonic lower bound one has to assure additionally that $r_{\hat{N}_i} > 0$ for all i.

Thus quantiles of this (comonotonic) lower bound are given by the following closed-form expression:

$$F^{-1}_{\mathscr{L}\mathscr{N}^{l2}}(p) = \sum_{i=1}^{n} \mathrm{e}^{\mu_{\hat{N}_i} + (1/2)\sigma^2_{\hat{N}_i}(1 - r^2_{\hat{N}_i}) + \sigma_{\hat{N}_i} r_{\hat{N}_i} \Phi^{-1}(p)},$$

from which one can easily find values of the corresponding cumulative distribution function e.g. by means of the Newton–Raphson method.

4.1.2. A numerical illustration

In this subsection we study the performance of the derived approximations for a cash flow with log-normally distributed payments. For purpose of this numerical illustration we chose parameters $\mu_{N_i} = -\ln(1.01)/2$ and $\sigma^2_{N_i} = \ln(1.01)$ (note that under this choice one has $E[X_i] = 1$ and $\mathrm{Var}[X_i] = 0.01$). Moreover, we allow for some level of dependency between the payments by imposing correlations between the normal exponents given by

$$r(N_i, N_j) = \begin{cases} 1 & \text{if } i = j, \\ 0.5 & \text{if } |i - j| = 1, \\ 0.2 & \text{if } |i - j| = 2, \\ 0 & \text{if } |i - j| > 2. \end{cases}$$

Regarding discounting factors, we assume that the returns follow the Black and Scholes model with drift parameter $\mu = 0.05$ and volatility $\sigma = 0.1$.

We compare the distribution functions of the upper bound $S^{u}_{\mathscr{L}\mathscr{N}}$ and the lower bounds $S^{l1}_{\mathscr{L}\mathscr{N}}$ and $S^{l2}_{\mathscr{L}\mathscr{N}}$ to the empirical distribution function of $S_{\mathscr{L}\mathscr{N}}$ obtained through a Monte Carlo (MC) simulation based on generating $500 \times 100\,000$ sample paths.

The performance of the derived approximations is illustrated in Fig. 1. One can see that the upper bound $S^{u}_{\mathscr{L}\mathscr{N}}$ gives quite poor approximation. The main reason for that is a relatively weak dependence between payments, for which the comonotonic approximation significantly overestimates the tails (it is very clear both from the plot of cdf's and from the QQ-plot). On the other hand, both lower bounds $S^{l1}_{\mathscr{L}\mathscr{N}}$ and $S^{l2}_{\mathscr{L}\mathscr{N}}$ give excellent approximations (the corresponding QQ-plots form almost a perfect diagonal). One may be surprised especially with the performance of the second lower bound—it turns out that the results are not less accurate for 1 conditioning random variable than in case of 2 conditioning random variables. The latter lower bound has even slightly higher variance—10.2450 compared to 10.2230 computed for the first distribution.

These visual observations are confirmed by the numerical values of some upper quantiles displayed in Table 1 (in the table we include also two moment-based approximations, which also perform excellent).

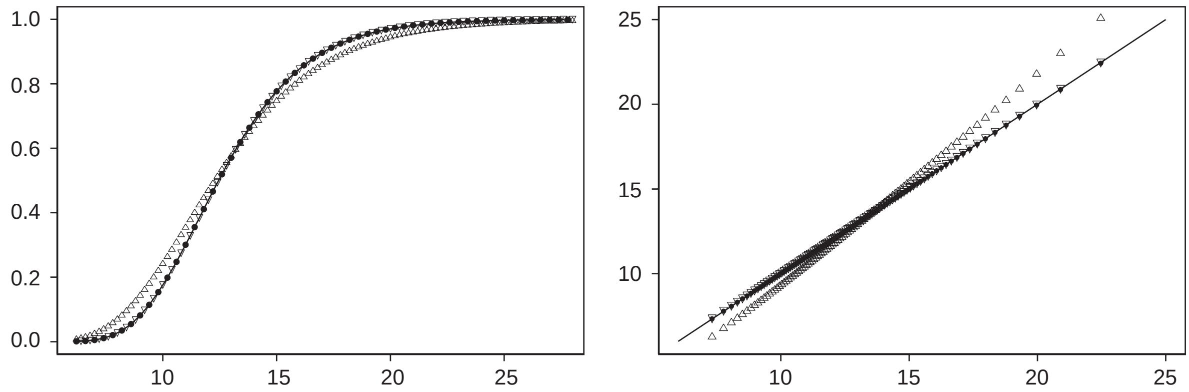

Fig. 1. The convex upper bound $S^u_{\mathscr{LN}}$ (triangles) and the lower bounds $S^{l1}_{\mathscr{LN}}$ (solid circles) and $S^{l2}_{\mathscr{LN}}$ (inverse triangles) versus the simulated distribution of $S_{\mathscr{LN}}$ (solid line)—the cdf's and the QQ-plot.

Table 1
Approximations of upper quantiles of $S_{\mathscr{LN}}$ for some probability levels p

p	$S_{\mathscr{LN}}$	(s.e.$\times 10^3$)	$S^{l1}_{\mathscr{LN}}$	$S^{l2}_{\mathscr{LN}}$	$S^{m1}_{\mathscr{LN}}$	$S^{m2}_{\mathscr{LN}}$	$S^{u}_{\mathscr{LN}}$
0.75	14.6795	(0.71)	14.6818	14.6822	14.6847	14.6839	15.0295
0.90	17.1019	(1.06)	17.0976	17.1024	17.1067	17.1078	18.0976
0.95	18.7769	(1.45)	18.7642	18.7723	18.7788	18.7815	20.2580
0.975	20.3881	(2.08)	20.3631	20.3753	20.3843	20.3882	22.3610
0.995	24.0237	(4.59)	23.9603	23.9823	24.0032	24.0082	27.1914

4.2. Elliptically distributed payments

4.2.1. Definition

The class of elliptical distributions is a natural extension of the normal law. We say that a random vector $\vec{X} = (X_1, X_2, \ldots, X_n)$ has an n-dimensional elliptical distribution with parameters $\vec{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)$, $\mathbf{\Sigma} = [\sigma_{ij}]_{1 \leqslant i,j \leqslant n}$ (symmetric and positive definite matrix) and characteristic generator $\phi(\cdot)$ if the characteristic function of $\vec{X}$ is given by

$$\varphi_{\vec{X}}(\vec{t}) = \mathrm{e}^{\mathrm{i}\vec{t}^{\mathrm{T}}\vec{\mu}}\phi(\vec{t}^{\mathrm{T}}\Sigma\vec{t}).$$

We write $\vec{X} \sim \mathscr{E}_n(\vec{\mu}, \mathbf{\Sigma}, \phi)$. Obviously the normal distribution satisfies this definition, with $\phi(y) = \mathrm{e}^{-(1/2)y}$. Elliptical distributions are very useful for several reasons. First of all they are very easy to manipulate because they inherit surprisingly many properties from the normal law. On the other hand the normal distribution is not very flexible in modelling tails (in practice we often encounter much heavier tails than the Gaussian ones). The class of elliptical laws offers a full variety of random distributions, from very heavy-tailed ones (like Cauchy or stable distributions), distributions with tails of the polynomial-type (t-Student), through the exponentially tailed Laplace and logistic distributions to the light-tailed Gaussian distribution.

Below we give a brief overview of the properties of elliptical distributions. More information about elliptical distributions can be found e.g. in [12].

1. $E[X_i] = \mu_i$, $\mathrm{Var}[X_i] = -2\phi'(0)\sigma_{ii}$ and $\mathrm{Cov}(X_i, X_j) = -2\phi'(0)\sigma_{ii}$ if only the corresponding moments exist;
2. Let $\vec{Y} = \mathbf{A}\vec{X} + \vec{b}$, where $\mathbf{A}$ denote $m \times n$-matrix and $\vec{b}$ is a vector in $\mathbb{R}^n$. Then $\vec{Y} \sim \mathscr{E}_m(\mathbf{A}\vec{\mu} + \vec{b}, \mathbf{A\Sigma A}^{\mathrm{T}}, \phi)$;
3. If the density function $f_{\vec{X}}(\cdot)$ exists, it is given by the formula

$$f_{\vec{X}}(\vec{x}) = \frac{c}{\sqrt{\det[\Sigma]}}\, g((\vec{x} - \vec{\mu})^{\mathrm{T}} \Sigma^{-1} (\vec{x} - \vec{\mu}))$$

for any nonnegative function g satisfying

$$0 < \int_0^\infty z^{(1/2)d-1} g(z)\, \mathrm{d}z < \infty$$

and c being a normalizing constant. The function $g(\cdot)$ is called the density generator of the distribution $\mathscr{E}_m(\vec{\mu}, \mathbf{\Sigma}, \phi)$;
4. Let $\vec{X} = (\vec{X}_1, \vec{X}_2)$ denote a $\mathscr{E}_{n+m}(\vec{\mu}, \mathbf{\Sigma}, \phi)$-random vector, where $\vec{\mu} = (\vec{\mu}_1, \vec{\mu}_2)$ and

$$\mathbf{\Sigma} = \begin{pmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} \end{pmatrix}.$$

Then, given conditionally that $\vec{X}_2 = \vec{x}_2$, the vector $\vec{X}_1$ has the $\mathscr{E}_n(\vec{\mu}_{1|2}, \mathbf{\Sigma}_{11|2}, \phi_{x_2})$-distribution with parameters given by

$$\vec{\mu}_{1|2} = \vec{\mu}_1 + \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}(\vec{x}_2 - \vec{\mu}_2)$$

and

$$\mathbf{\Sigma}_{11|2} = \mathbf{\Sigma}_{11} - \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21}.$$

Notice that in general (unlike in the normal case) the characteristic generator of the conditional distribution is not known explicitly and depends on the value of x_2.

4.2.2. *Convex upper and lower bounds*

Consider a sum of the form

$$S_{\mathrm{el}} = \sum_{i=1}^{n} X_i \mathrm{e}^{-Y(i)},$$

where the return process $Y(t)$ is, like in the previous example, described by the Black and Scholes model and $\vec{X} = (X_1, X_2, \ldots, X_n)$ is elliptically distributed with parameters $\vec{\mu}_{\vec{X}} = (\mu_{X_1}, \mu_{X_2}, \ldots, \mu_{X_n})$, $\mathbf{\Sigma}_{\vec{X}} = [\sigma_{ij}^{\vec{X}}]_{1 \leqslant i,j \leqslant n}$ and characteristic generator $\phi(\cdot)$. Here we note only that for $\phi(u) = \mathrm{e}^{-u/2}$ one gets a multivariate normal distribution with mean parameter equal to $\vec{\mu}_{\vec{X}}$ and the covariance matrix $\mathbf{\Sigma}_{\vec{X}}$.

Note that elliptical random variables take both positive and negative values and therefore one cannot apply immediately Theorem 2. Therefore we propose to consider pragmatically only the cases where the probability $\Pr[X_i < 0]$ is very small. This can be achieved by chosing the parameters in such a way that $\mu_{X_i}/\sigma_{X_i} \gg 0$. ($\sigma_{X_i}^2 \overset{\text{not}}{=} \sigma_{ii}^{\vec{X}}$).

The upper bound. The computation of the upper bound is straightforward if the inverse distribution function for the specific elliptical distribution is available in the used software package. We take

$$S_{\text{el}}^{u} = \sum_{i=1}^{n} F^{-1}_{\mathscr{E}_n(\mu_{X_i},\sigma^2_{X_i},\phi)}(U_1)\mathrm{e}^{-\mathrm{i}\mu+\sigma_i \Phi^{-1}(U_2)}. \tag{27}$$

Note that for the most interesting case of a multivariate normal distribution, one gets

$$S_{\mathscr{N}}^{u} = \sum_{i=1}^{n} (\mu_{X_i} + \sigma_{X_i}\Phi^{-1}(U_1))\mathrm{e}^{-\mathrm{i}\mu+\sigma_i \Phi^{-1}(U_2)}.$$

The lower bound. To compute the lower bound, we define the conditioning random variable Θ as follows:

$$\Theta = \sum_{j=1}^{n} E[\mathrm{e}^{-Y(j)}]X_j = \sum_{j=1}^{n} \mathrm{e}^{-\mathrm{j}\mu+(1/2)\sigma_j^2} X_j.$$

Then a random vector (X_i, Θ) has a bivariate elliptical random variable, with parameters $\vec{\mu}^{\Theta,i} = (\mu_{X_i}, \mu_\Theta)$ and $\mathbf{\Sigma}^{\Theta,i} = [\sigma_{kl}^{\Theta,i}]_{1\leqslant k,l\leqslant 2}$, where

$$\mu_\Theta = \sum_{j=1}^{n} \mathrm{e}^{-\mathrm{j}\mu+(1/2)\sigma_j^2}\mu_{X_j},$$

$$\sigma_{11}^{\Theta,i} = \sigma_{X_i}^2, \quad \sigma_{12}^{\Theta,i} = \sigma_{21}^{\Theta,i} = \sum_{j=1}^{n} \mathrm{e}^{-\mathrm{j}\mu+(1/2)\sigma_j^2}\sigma_{ij}^{\vec{X}}$$

and

$$\sigma_\Theta^2 = \sigma_{22}^{\Theta,i} = \sum_{j=1}^{n}\sum_{k=1}^{n} \mathrm{e}^{-\mathrm{j}\mu-k\mu+(1/2)(\sigma_j^2+\sigma_k^2)}\sigma_{jk}^{\vec{X}}.$$

From Section 4.2.1, item (4), it follows that, given $\Theta = \theta$, X_i has a univariate elliptical distribution with parameters

$$\mu_{X_i,\theta} = \mu_{X_i} + \frac{\sigma_{12}^{\Theta,i}}{\sigma_\Theta^2}(\theta - \mu_\Theta), \quad \sigma_{X_i,\theta}^2 = \sigma_{X_i}^2 - \frac{(\sigma_{12}^{\Theta,i})^2}{\sigma_\Theta^2} \tag{28}$$

and unknown characteristic generator $\phi_a(\cdot)$ depending on $a = (\theta - \mu_\Theta)^2/\sigma_\Theta^2$.

Note that for the multivariate normal case the conditional distribution remains normal. In our application it does not really matter that the characteristic generator $\phi_a(\cdot)$ is not known—it suffices to notice that

$$E[X_i \mid \Theta] = \mu_{X_i,\Theta} = \mu_{X_i} + \frac{\sigma_{12}^{\Theta,i}}{\sigma_\Theta^2}(\Theta - \mu_\Theta).$$

The second conditioning random variable is chosen analogously as in (25), i.e.

$$\Lambda = -\sum_{i=1}^{n} E[X_i]\mathrm{e}^{-\mathrm{i}\mu+(1/2)\sigma_i^2}Y(i) = -\sum_{i=1}^{n} \mu_{X_i}\mathrm{e}^{-\mathrm{i}\mu+(1/2)\sigma_i^2}Y(i).$$

Applying the results of Section 3.2, the lower bound is given by the following expression:

$$S_{\mathrm{el}}^{l} = \sum_{i=1}^{n}\left(\mu_{X_i} + \frac{\sigma_{12}^{\Theta,i}}{\sigma_{\Theta}^{2}}(F_{\Theta}^{-1}(U_1) - \mu_{\Theta})\right)\mathrm{e}^{-\mathrm{i}\mu+(1/2)\sigma_i^2(1-r_i^2)+r_i\sigma_i\Phi^{-1}(U_2)}, \tag{29}$$

where the correlations $r_i = \mathrm{Corr}(-Y(i), \Lambda)$ are defined as in (16) (with $E[X_i]$ substituted by μ_{X_i}). Note that expression (29) simplifies in the normal case to

$$S_{\mathcal{N}}^{l} = \sum_{i=1}^{n}(\mu_{X_i} + r_{X_i}\sigma_{X_i}\Phi^{-1}(U_1))\mathrm{e}^{-\mathrm{i}\mu+(1/2)\sigma_i^2(1-r_i^2)+r_i\sigma_i\Phi^{-1}(U_2)},$$

where

$$r_{X_i} = \mathrm{Corr}(X_i, \Theta) = \frac{\sum_{j=1}^{n}\mu_{X_j}\mathrm{e}^{-\mathrm{j}\mu+(1/2)\sigma_j^2}\sigma_{ij}^{\vec{X}}}{\sigma_{X_i}\sqrt{\sum_{k,l=1}^{n}\mu_{X_k}\mu_{X_l}\mathrm{e}^{-k\mu-\mathrm{l}\mu+(1/2)(\sigma_k^2+\sigma_l^2)}\sigma_{kl}^{\vec{X}}}}.$$

4.2.3. A numerical illustration

Now we evaluate numerically the case when future payments are normally distributed, with mean parameter $\mu_{X_i} = 1$ and variance $\sigma_{X_i}^2 = 0.01$ (note that mean and variance are the same as in the log-normal case, see Section 4.1.2). Like in the log-normal case, we also impose some positive dependencies between payments, given by

$$r(N_i, N_j) = \begin{cases} 1 & \text{if } i = j, \\ 0.5 & \text{if } |i-j| = 1, \\ 0.2 & \text{if } |i-j| = 2, \\ 0 & \text{if } |i-j| > 2. \end{cases}$$

As in Section 4.1.2, we work in the framework of the Black and Scholes model with drift parameter $\mu = 0.05$ and volatility $\sigma = 0.1$. We compare the distributions of the lower bound $S_{\mathcal{N}}^{l}$, the upper bound $S_{\mathcal{N}}^{u}$ and the moment-based approximation $S_{\mathcal{N}}^{m}$ to the empirical distribution of $S_{\mathcal{N}}$ obtained by means of a Monte Carlo simulation based on $500 \times 100\,000$ simulated paths.

The performance of the approximations is illustrated in Fig. 2. Note that the graphs look almost exactly the same as in the log-normal case—the upper bound $S_{\mathcal{N}}^{u}$ gives a quite poor approximation, while the lower bound $S_{\mathcal{N}}^{l}$ and the moments-based approximation perform excellent. These visual observations are confirmed by the numerical values obtained for some upper quantiles displayed in Table 2.

4.3. Independent and identically distributed payments

Finally we consider the case where the payments X_i are independent and identically distributed. The independence assumption accounts for more flexibility in modelling the underlying marginal distributions,

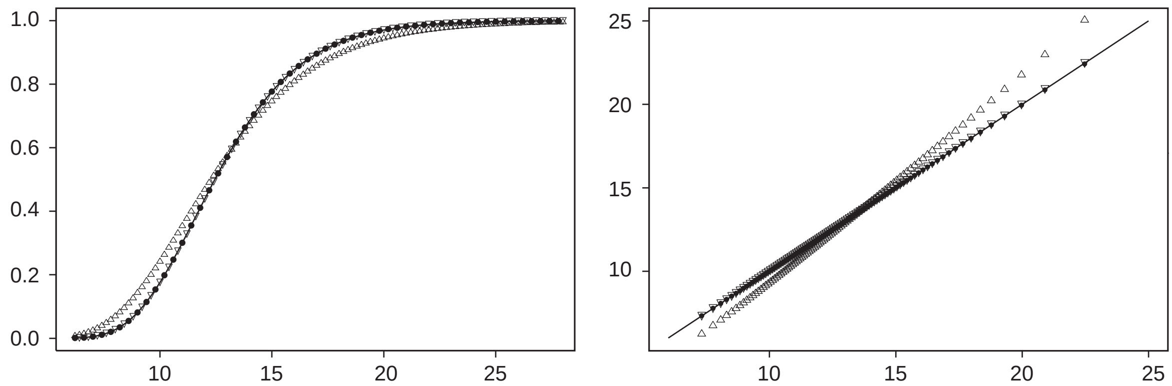

Fig. 2. The convex upper bound $S^u_{\mathcal{N}}$ (triangles), the lower bound $S^l_{\mathcal{N}}$ (inverse triangles) and the moment-based approximation $S^m_{\mathcal{N}}$ (solid circles) versus the simulated distribution of $S_{\mathcal{N}}$ (solid line)—the cdf's and the QQ-plot.

Table 2
Approximations of upper quantiles of $S_{\mathcal{N}}$ for some probability levels p

p	$S_{\mathcal{N}}$	(s.e.$\times 10^3$)	$S^l_{\mathcal{N}}$	$S^m_{\mathcal{N}}$	$S^u_{\mathcal{N}}$
0.75	14.6820	(0.70)	14.6820	14.6849	15.0368
0.90	17.1025	(1.02)	17.0978	17.1068	18.0992
0.95	18.7789	(1.46)	18.7642	18.7787	20.2522
0.975	20.3895	(2.11)	20.3630	20.3840	22.3456
0.995	24.0354	(4.61)	23.9599	24.0020	27.1468

however—unlike in the log-normal and elliptical cases—it imposes a rigid condition on the dependence structure. We start with defining the class of tempered stable distributions for which the methodology works particularly efficient.

4.3.1. Tempered stable distributions

The Tempered Stable law $\mathcal{TS}(\delta, a, b)$ for $a, b > 0$ and $0 < \delta < 1$ is a one-dimensional distribution given by the characteristic function:

$$\varphi_{\mathcal{TS}}(t; \delta, a, b) = \mathrm{e}^{ab - a(b^{1/\delta} - 2\mathrm{i}t)^\delta}. \tag{30}$$

(See e.g. [11].) This distribution has one very special property, i.e. namely one has that

$$(\varphi_{\mathcal{TS}}(t; \delta, a, b))^n = \varphi_{\mathcal{TS}}(t; \delta, na, b).$$

Therefore, a sum of n independent and identically distributed tempered stable random variables is again tempered stable, with the only difference that the parameter a is transformed to na.

The first two moments of a random variable $X \sim \mathcal{TS}(\delta, a, b)$ are given by $E[X] = 2a\delta b^{(\delta-1)/\delta}$ and $\mathrm{Var}[X] = 4a\delta(1-\delta)b^{(\delta-2)/\delta}$.

In the sequel we provide more details about two well-known special cases: the gamma distribution and the inverse Gaussian distribution.

The gamma distribution. The gamma distribution $\Gamma(a, b)$ corresponds to the limiting case when $\delta \to 0$. Therefore, the characteristic function of the Γ-distribution is given by

$$\varphi_{\Gamma}(t; a, b) = \left(1 - \frac{\mathrm{i}t}{b}\right)^{-a},$$

what corresponds to the density function

$$f_{\Gamma}(x; a, b) = \frac{b^a}{\Gamma(a)} x^{a-1} \mathrm{e}^{-bx}, \quad x > 0.$$

Note that $X \sim \Gamma(a, b)$ one has $E[X] = a/b$ and $\mathrm{Var}[X] = a/b^2$.

The inverse gaussian distribution. The inverse Gaussian distribution is a member of the class of Tempered Stable distributions with $\delta = \frac{1}{2}$. Thus, the characteristic function is given by

$$\varphi_{\mathscr{IG}}(t; a, b) = \mathrm{e}^{-a\left(\sqrt{-2\mathrm{i}t + b^2} - b\right)},$$

what corresponds to the density function

$$f_{\mathscr{IG}}(x; a, b) = \frac{a}{\sqrt{2\pi}} x^{-3/2} \mathrm{e}^{ab - (1/2)(a^2/x + b^2 x)}, \quad x > 0.$$

Moreover the mean and variance of $X \sim \mathscr{IG}(a, b)$ are given by $E[X] = a/b$ and $\mathrm{Var}[X] = a/b^3$.

Tempered stable random variables are very useful in our application because of the following result:

Lemma 2. *If X_i are i.i.d. random variables $\mathscr{TS}(\kappa, a, b)$-distributed for $i = 1, 2, \ldots, n$, then their sum $X_1 + X_2 + \cdots + X_n$ is $\mathscr{TS}(\kappa, na, b)$-distributed.*

Proof. Consider the corresponding characteristic functions. We get

$$\varphi_{X_1+X_2+\cdots+X_n}(t) = (\varphi_{\mathscr{TS}}(t; \kappa, a, b))^n = \mathrm{e}^{(na)b - (na)(b^{1/\kappa} - 2\mathrm{i}t)^{\kappa}} = \varphi_{\mathscr{TS}}(t; \kappa, na, b). \qquad \square$$

4.3.2. Convex upper and lower bounds

We consider sums of the form

$$S_{\mathrm{ind}} = \sum_{i=1}^{n} X_i \mathrm{e}^{-Y(i)},$$

where the process $Y(i)$ is defined like in the previous examples whereas payments X_i are independent and follow the law defined by the cdf $F_X(\cdot)$.

The upper bound. The computation of the upper bound is straightforward:

$$S_{\mathrm{ind}}^{u} = F_X^{-1}(U_1) \sum_{i=1}^{n} \mathrm{e}^{-\mathrm{i}\mu + \sigma_i \Phi^{-1}(U_2)}$$

as described in Section 3.1.

The lower bound. We begin with defining conditioning random variables Θ and Λ to compute the lower bound. Let

$$\Theta = X_1 + X_2 + \cdots + X_n.$$

It is well-known that if we know distributions of X_i, the distribution of Θ is also known. Indeed, it can be defined e.g. by a characteristic function as

$$\varphi_\Theta(t) = (\varphi_X(t))^n.$$

Note that under some integrability conditions the distribution function can be expressed by means of a characteristic function (see e.g. [6] for details). However if X_i are tempered stable random variables with known distribution functions then the distribution function of Θ is of the same type and a time-consuming procedure of transforming the characteristic function can be avoided. In particular, for X_i Γ-distributed the sum Θ remains Γ-distributed and for X_i $\mathscr{IG}$-distributed the random variable Θ remains $\mathscr{IG}$-distributed.

Next, the conditional random variable Λ is chosen, like in previous examples, as

$$\Lambda = -\sum_{i=1}^{n} E[X_i]\mathrm{e}^{-\mathrm{i}\mu+(1/2)\sigma_i^2} Y(i). \tag{31}$$

Then the lower bound can be written as

$$S^l_{\mathrm{ind}} = \frac{1}{n} F_\Theta^{-1}(U_1) \sum_{i=1}^{n} \mathrm{e}^{-\mathrm{i}\mu+(1/2)(1-r_i^2)\sigma_i^2+r_i\sigma_i\Phi^{-1}(U_2)},$$

where the correlations $r_i = \mathrm{Corr}(-Y(i), \Lambda)$ are defined as in (16).

Cumulative distribution functions. In this case there is a more efficient method to compute the distribution functions than the one described in Sections 3.1 and 3.2. We use the following result.

Lemma 3. *Let W be a random variable of the form $W = \tilde{X}\tilde{V}$, where $\tilde{X}$ and $\tilde{V}$ are independent. Then the distribution function of W can be derived as*

$$F_W(y) = \int_{-\infty}^{\infty} F_{\tilde{X}}\left(\frac{y}{v}\right) \mathrm{d}F_{\tilde{V}}(v) = \int_0^1 F_{\tilde{X}}\left(\frac{y}{F_{\tilde{V}}^{-1}(u_2)}\right) \mathrm{d}u_2. \tag{32}$$

Proof. See Appendix B in [7]. □

Therefore one can compute the cumulative distribution functions of the upper and the lower bound as

$$F_{S^u_{\mathrm{ind}}}(y) = \int_0^1 F_X\left(\frac{y}{F_{\tilde{S}^u}^{-1}(u_2)}\right) \mathrm{d}u_2$$

and

$$F_{S^l_{\mathrm{ind}}}(y) = \int_0^1 F_{\frac{1}{n}\Theta}\left(\frac{y}{F_{\tilde{S}^l}^{-1}(u_2)}\right) \mathrm{d}u_2,$$

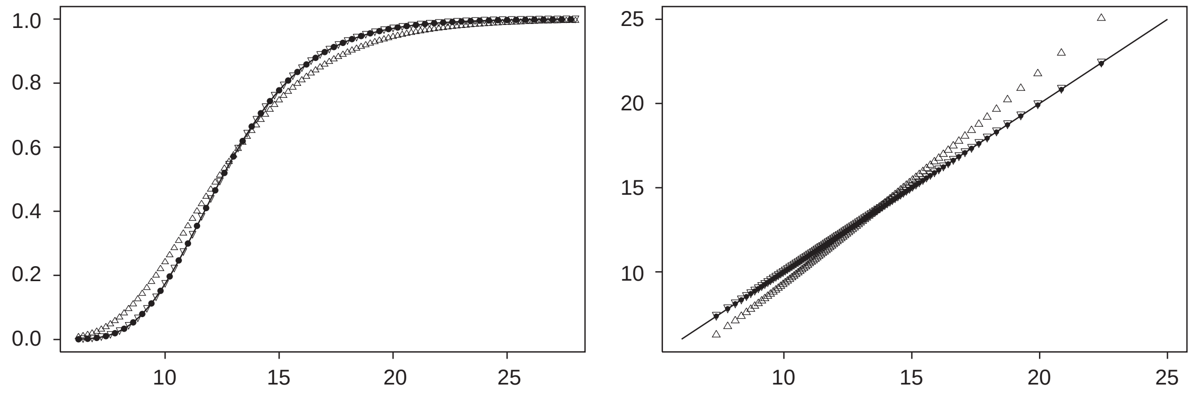

Fig. 3. The convex upper bound S_Γ^u (triangles), the lower bound S_Γ^l (inverse triangles) and the moment-based approximation S_Γ^m (solid circles) versus the simulated distribution of S_Γ (solid line)—the cdf's and the QQ-plot.

where

$$\tilde{S}^u = \sum_{i=1}^{n} \mathrm{e}^{-\mathrm{i}\mu+\sigma_i \Phi^{-1}(U_2)}, \quad \tilde{S}^l = \sum_{i=1}^{n} \mathrm{e}^{-\mathrm{i}\mu+(1/2)(1-r_i^2)\sigma_i^2+r_i\sigma_i\Phi^{-1}(U_2)}$$

and

$$F_{\tilde{S}^u}^{-1}(u_2) = \sum_{i=1}^{n} \mathrm{e}^{-\mathrm{i}\mu+\sigma_i \Phi^{-1}(u_2)}, \quad F_{\tilde{S}^l}^{-1}(u_2) = \sum_{i=1}^{n} \mathrm{e}^{-\mathrm{i}\mu+(1/2)(1-r_i^2)\sigma_i^2+r_i\sigma_i\Phi^{-1}(u_2)}.$$

4.3.3. A numerical illustration

Now we present a numerical application of the method to the case when future payments are independent, Γ-distributed, with parameters $a = b = 100$. Note that this choice of parameters implies that $E[X] = 1$ and $\mathrm{Var}[X] = 0.01$—i.e. we take the same mean and variance of liabilities as in the log-normal (Section 4.1.2) and normal (Section 4.2.3) cases. As before we work in the Black and Scholes setting with drift $\mu = 0.05$ and volatility $\sigma = 0.1$. We compare the obtained distributions of S_Γ^l, S_Γ^u and S_Γ^m to the empirical distribution of value S_Γ obtained in the same fashion as in previous cases.

The results are very similar to the normal and log-normal case. It is worth noticing that the variance of S_Γ (10.1489) s a bit lower that in the log-normal case (10.2789) and in the normal case (10.2792). It is a consequence of independence between consecutive Γ-payments while before we imposed a slight positive dependence.

The quality of the approximations is illustrated in Fig. 3. One can see that the fit of the upper bound is quite poor. The lower bound S_Γ^l and the moments based approximation S_Γ^m perform very well, but a bit poorer than in the log-normal and normal cases (probably because the conditioning random variable Θ does not take discounting factors into account). These visual observations are confirmed by the numerical values of some upper quantiles, contained in Table 3.

Table 3
Approximations of upper quantiles of S_Γ for some probability levels p

p	S_Γ	(s.e.$\times 10^3$)	S^l_Γ	S^m_Γ	S^u_Γ
0.75	14.6820	(0.70)	14.6709	14.6723	15.0320
0.90	17.1025	(1.02)	17.0767	17.0810	18.0984
0.95	18.7789	(1.46)	18.7372	18.7443	20.2563
0.975	20.3895	(2.11)	20.3309	20.3412	22.3560
0.995	24.0354	(4.61)	23.9183	23.9390	27.1762

5. Conclusion

In this paper we present a methodology that allows us to obtain accurate approximations for distribution functions of scalar products of independent random vectors for which no direct analytical expression exist. The approach is based on deriving upper and lower bounds in a sense of convex order for the underlying distribution, which has a very natural economical interpretation in terms of the utility theory or Yaari's dual theory of choice under risk. Our methodology is an extension of results obtained in [3,4,7,8].

As demonstrated in a series of numerical examples, the technique provides a very useful tool to evaluate cash flows of future stochastic payments. The distributions of the lower bound and the moment-based approximation are almost indistinguishable from the empirical distribution, obtained by means of a Monte Carlo simulation. It should be noted however that a Monte Carlo simulation is much more time consuming than our approximations, and despite that the simulated values of upper quantiles are still quite volatile.

The methodology finds much wider range of applications than the ones presented in the paper. In [9] a similar approach is employed to find an approximate distribution of a life annuity. The same technique is also applied in [1] to find an optimal asset mix in the multi-period portfolio selection problem.

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