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On a generalization of M-group

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ABSTRACT

In this paper, we show that if for every nonlinear complex irreducible character χ of a finite group G, some multiple of χ is induced from an irreducible character of some proper subgroup of G, then G is solvable. This is a generalization of Taketa's theorem on the solvability of M-group.

1. Introduction and notation

All groups in this paper are finite and all characters are complex characters. For a group G, let $\mathrm{Irr}(G)$ denote the set of all irreducible characters of G. An irreducible character χ of a group G is *monomial* if it is induced from a linear character of a subgroup of G, that is $\chi = \lambda^G$, where $\lambda \in \mathrm{Irr}(U)$ with $\lambda(1) = 1$ and $U \leqslant G$. A group G is called an *M-group* if every irreducible character of G is monomial. A well known theorem of Taketa says that all M-groups are solvable (see [10, Theorem 5.12]). There have been many generalizations of Taketa's theorem in the literature. Observe that if $\chi \in \mathrm{Irr}(G)$ is a monomial character induced from the subgroup $U \leqslant G$ and the linear character $\lambda \in \mathrm{Irr}(U)$, then $U/Ker(\lambda)$ is cyclic, in particular $U/Ker(\lambda)$ is solvable. With this observation, Dornhoff showed in [7] that a group G is solvable provided that every irreducible character of G is induced from an

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irreducible character of a solvable section of G. More generally, Isaacs proved in [11] that if every irreducible character $\chi \in \mathrm{Irr}(G)$ is induced from an irreducible character λ of a subgroup H such that $H/Ker(\lambda) \in \mathfrak{F}$, then G is in $\mathfrak{F}$, where $\mathfrak{F}$ is a class of groups closed under isomorphisms, subgroups and extensions. If we choose $\mathfrak{F}$ to be the class of solvable groups, then we obtain the result of Dornhoff mentioned above. A group G is called a *Quasi-Solvable Induced* (QSI) group if every irreducible character χ of G has some multiple which is induced from a character λ of a subgroup U with $U/Ker\,\lambda$ solvable. Recently, König [15] showed that every QSI group is solvable. Obviously, this is a generalization of both Dornhoff's and Taketa's theorems. The main purpose of this paper is to remove the solvability assumption on the quotient $U/Ker(\lambda)$.

Recall that an irreducible character of a group is *imprimitive* if it is induced from an irreducible character of some proper subgroup and it is *primitive* if it is not induced by any character of any proper subgroups. For convenience reason, we make the following definitions. A nonlinear character $\chi \in \mathrm{Irr}(G)$ is called a *multiply imprimitive character* (or m.i *character* for short) induced from the pair (U, λ) if there exist a proper subgroup U of G and an irreducible character $\lambda \in \mathrm{Irr}(U)$ such that $\lambda^G = m\chi$ for some nonnegative integer m. Moreover, a group G is said to be an MI-*group* if every nonlinear irreducible character of G is an m.i character.

Let $N \trianglelefteq G$. We write $\mathrm{Irr}(G|N) = \mathrm{Irr}(G) - \mathrm{Irr}(G/N)$. If N is a normal subgroup of G and every nonlinear irreducible character in $\mathrm{Irr}(G|N)$ is an m.i character, then G is called an MI-*group relative to* N. We now state our main result.

Theorem 1.1. *Let N be a normal subgroup of a group G. If G is an* MI-*group relative to N, then N is solvable.*

If we take $N = G'$, then the set $\mathrm{Irr}(G|N)$ is exactly the set of all nonlinear irreducible characters of G. Now assume that G is an MI-group. Then G is an MI-group relative to G' and thus by applying Theorem 1.1, we deduce that G' is solvable and so G is solvable. Therefore, we have proved the following corollary.

Corollary 1.2. *Every* MI-*group is solvable.*

This gives a positive answer to [2, Problem 162]. We also obtain an answer to [2, Problem 123] as follows.

Corollary 1.3. *Let $H \leqslant G$ be a proper subgroup of a group G. Suppose that for any $\lambda \in \mathrm{Irr}(H)$ with $\lambda \neq 1_H$, we have $\lambda^G = m\chi$ for some $\chi \in \mathrm{Irr}(G)$ and some integer $m \geqslant 1$. Then the normal closure of H in G is solvable. In particular, H is solvable.*

For the proof of Theorem 1.1, in Section 2 we present some results needed for reducing the problem to a question concerning the existence of a special m.i character in nonabelian simple groups. Using the classification of nonabelian simple groups, we obtain the answer to this question which is stated as Theorem 1.4 below. This theorem will be verified in Sections 3, 4 and 5. Finally, the proofs of Theorem 1.1 and Corollary 1.3 will be carried out in the last section.

Theorem 1.4. *If S is a nonabelian simple group, then S has a nonlinear irreducible character which is extendible to* $\mathrm{Aut}(S)$ *but it is not an* m.i *character.*

Notation. If G is a group, then we write $\pi(G)$ to denote the set of all prime divisors of the order of G. For a normal subgroup N of G, if $\theta \in \mathrm{Irr}(N)$, then the set of all irreducible constituents of θ^G is denoted by $\mathrm{Irr}(G|\theta)$. If n is a positive integer and p is a prime then n_p and $n_{p'}$ are the largest p-part and p'-part of n, respectively. The greatest common divisor of two integers a and b is denoted by $\gcd(a, b)$. We follow [5] for notation of simple groups. Other notation is standard.

2. Reduction to simple groups

The following lemma is a modification of Lemma 2.1 in [15].

Lemma 2.1. *Let K and N be normal subgroups of a group G. Suppose that G is an* MI*-group relative to N. Then the following hold.*

(i) *G/K is an* MI*-group relative to NK/K.*
(ii) *G is an* MI*-group relative to K provided that $K \leqslant N$.*

Proof. Assume that $\hat{\chi} \in \operatorname{Irr}(G/K|NK/K)$. Then $\hat{\chi}$ can be considered as a character χ of G with $K \leqslant Ker(\chi) = Ker(\hat{\chi})$. As $NK/K \nsubseteq Ker(\hat{\chi})$ but $K \leqslant Ker(\hat{\chi})$, we deduce that $N \nleqslant Ker(\chi)$ so $\chi \in \operatorname{Irr}(G|N)$ with $K \leqslant Ker(\chi)$. Since G is an MI-group relative to N, we deduce that $m\chi = \lambda^G$, where $U \lneqq G$, $\lambda \in \operatorname{Irr}(U)$ and $m \geqslant 1$. We have

$$K \leqslant Ker\big(\lambda^G\big) = \bigcap_{g \in G} \big(Ker(\lambda)\big)^g$$

and hence $K \leqslant Ker(\lambda) \trianglelefteq U$. Thus λ can be considered as a character $\hat{\lambda}$ of U/K. For $x \in G$, we have

$$\begin{aligned} \hat{\lambda}^{G/K}(xK) &= \frac{1}{|U/K|} \sum_{\substack{yK \in G/K \\ (xK)^{yK} \in U/K}} \hat{\lambda}\big((xK)^{yK}\big) \\ &= \frac{1}{|U|} \sum_{\substack{y \in G \\ x^y \in U}} \lambda\big(x^y\big) = \lambda^G(x) = m\chi(x) = m\hat{\chi}(xK). \end{aligned}$$

Therefore $\hat{\chi} \in \operatorname{Irr}(G/K|NK/K)$ is an m.i character induced from $(U/K, \hat{\lambda})$. This proves (i). If $K \leqslant N \leqslant G$, then (ii) is obvious since $\operatorname{Irr}(G|K) \subseteq \operatorname{Irr}(G|N)$. $\square$

Lemma 2.2. *Let $\chi \in \operatorname{Irr}(G)$ be an* m.i *character induced from a subgroup $U \lneqq G$ and $\lambda \in \operatorname{Irr}(U)$ with $\lambda^G = m\chi$. Then:*

(i) *If $\chi(g) \neq 0$ for some $g \in G$, then $g^G \cap U \neq \emptyset$.*
(ii) *We have $|G:U|\lambda(1) = m\chi(1)$, $\chi(1) \geqslant m\lambda(1)$ and $|G:U| \geqslant m^2$.*

Proof. As $\lambda^G = m\chi$, if $g \in G$ with $\chi(g) \neq 0$, then $\lambda^G(g) = m\chi(g) \neq 0$. By the definition of induced characters, we have that $xgx^{-1} \in U$ for some $x \in G$, which proves (i). For (ii), by comparing the degrees, we have $\lambda^G(1) = m\chi(1)$, which implies that $|G:U|\lambda(1) = m\chi(1)$. By the Frobenius reciprocity, we have $m = (\lambda^G, \chi) = (\lambda, \chi_U)$ so $\chi_U = m\lambda + \psi$ for some character ψ of U. Hence $\chi(1) = m\lambda(1) + \psi(1) \geqslant m\lambda(1)$, which proves the second statement of (ii). Finally, we have $|G:U|\lambda(1) = m\chi(1) \geqslant m^2\lambda(1)$, which deduces that $|G:U| \geqslant m^2$. $\square$

Let $\chi \in \operatorname{Irr}(G)$ be an m.i character induced from (U, λ), that is, $m\chi = \lambda^G$ for some $m \geqslant 1$. We will show that U could be chosen to be a maximal subgroup of G. By definition, U is a proper subgroup of G, and thus there is a maximal subgroup H of G that contains U. Let $\mu \in \operatorname{Irr}(H)$ be an irreducible constituent of λ^H. Write $\lambda^H = \mu + \psi$, where ψ is a character of H. By the transitivity of character induction, we have that $(\lambda^H)^G = \lambda^G = m\chi$ so $\mu^G + \psi^G = m\chi$. Thus $\mu^G = e\chi$ for some $e \geqslant 1$, which means that χ is an m.i character induced from (H, μ), where H is maximal in G and $\mu \in \operatorname{Irr}(H)$.

The next result is similar to Lemma 2.8 in [18].

Lemma 2.3. *Let N be a normal subgroup of a group G and let $\theta \in \mathrm{Irr}(N)$ be a nonlinear character of N. Suppose that θ extends to $\chi \in \mathrm{Irr}(G)$. If χ is an* m.i *character of G, then θ is also an* m.i *character of N.*

Proof. Assume that χ is an m.i character of G. Then there exist a proper subgroup $U \lneq G$, $\lambda \in \mathrm{Irr}(U)$ and $m \geqslant 1$ such that $m\chi = \lambda^G$. Assume that $T = \{r_1, r_2, \ldots, r_t\}$ is a set of representatives for the double cosets of U and N in G. As χ is an extension of θ, we have that $\chi_N = \theta$. By the discussion above, we can and will assume that U is maximal in G. As $\lambda^G = m\chi$ and $\chi_N = \theta$, we deduce that $m\theta = (\lambda^G)_N$. By Mackey's Lemma, we have that

$$\left(\lambda^G\right)_N = \sum_{j=1}^{t} \left(\left(\lambda^{r_j}\right)_{U^{r_j} \cap N}\right)^N$$

so

$$m\theta = \sum_{j=1}^{t} \left(\left(\lambda^{r_j}\right)_{U^{r_j} \cap N}\right)^N.$$

It follows that for each j, we obtain that $((\lambda^{r_j})_{U^{r_j} \cap N})^N$ is a multiple of θ. In particular, we have that $(\lambda_{U\cap N})^N = k\theta$ for some $k \geqslant 1$.

Assume first that $N \leqslant U$. We then have that $\lambda_N = k\theta$. By Lemma 2.2(ii), we obtain that $\chi(1) = \theta(1) \geqslant m\lambda(1) = mk\theta(1)$, which implies that $mk = 1$, hence $m = k = 1$. By Lemma 2.2(ii) again, we have $|G : U|\lambda(1) = m\chi(1)$ and thus $|G : U|k\theta(1) = m\theta(1)$, which implies that $|G : U| = 1$, a contradiction.

Assume next that $N \nleqslant U$. As U is maximal in G, we obtain that $G = UN$. Hence by Mackey's Lemma, we have that

$$\left(\lambda^G\right)_N = (\lambda_{U\cap N})^N = m\theta.$$

Since $N \nleqslant U$, we deduce that $U \cap N \lneq N$. Let $\mu \in \mathrm{Irr}(U \cap N)$ be an irreducible constituent of $\lambda_{U\cap N}$. It follows that $\mu^N = l\theta$ for some $l \geqslant 1$. Hence θ is an m.i character of N induced from $(U \cap N, \mu)$ as required. $\square$

Lemma 2.4. *Suppose that N is a unique minimal normal nonabelian subgroup of a group G. Assume that $N = R_1 \times R_2 \times \cdots \times R_k$, where each R_i is isomorphic to a nonabelian simple group R, and $k \geqslant 1$. Let θ be a nonlinear irreducible character of R such that θ extends to $\mathrm{Aut}(R)$. Let $\varphi = \theta^k \in \mathrm{Irr}(N)$. If $\chi \in \mathrm{Irr}(G)$ is any extension of φ to G and χ is an* m.i *character of G, then θ is an* m.i *character of R.*

Proof. Since $N \cong R^k$ is the unique minimal normal subgroup of G, we deduce that G embeds into $\mathrm{Aut}(N) \cong \mathrm{Aut}(R) \wr S_k$, where S_k denotes the symmetric group of degree k. As $\theta \in \mathrm{Irr}(R)$ extends to $\mathrm{Aut}(R)$, by [3, Lemma 2.5] we deduce that $\varphi = \theta^k \in \mathrm{Irr}(N)$ extends to G. Assume that $\chi \in \mathrm{Irr}(G)$ is an extension of φ and that χ is an m.i character of G. By Lemma 2.3, we deduce that φ is an m.i character of N induced from (U, λ), where U is a maximal subgroup of N, and $\lambda \in \mathrm{Irr}(U)$. Then $m\varphi = \lambda^N$ for some $m \geqslant 1$. As $N = R_1 \times R_2 \times \cdots \times R_k$ and U is maximal in N, there exists $1 \leqslant i \leqslant k$ such that $R_i \nleqslant U$. Without loss of generality, we assume that $R_1 \nleqslant U$. Since $R_1 \trianglelefteq N$, we obtain that $N = R_1U$, here we identify R_1 with $R_1 \times 1 \times \cdots \times 1 \trianglelefteq N$. Observe that $\varphi_{R_1} = \theta(1)^{k-1}\theta_1$, where $\theta_1 \in \mathrm{Irr}(R_1)$ is N-invariant. Since $N = R_1U$, by Mackey's Lemma, we obtain that $(\lambda^N)_{R_1} = \lambda_{U_1}^{R_1}$, where $U_1 := R_1 \cap U \lneq R_1$. Then it follows from $m\varphi_{R_1} = (\lambda^N)_{R_1}$ that $m\theta(1)^{k-1}\theta_1 = \lambda_{U_1}^{R_1}$. Let $\lambda_1 \in \mathrm{Irr}(U_1)$ be any irreducible constituent of λ_{U_1}, we then have that $\lambda_1^{R_1} = m_1\theta_1$ for some $m_1 \geqslant 1$. Therefore we conclude that $\theta_1 \in \mathrm{Irr}(R_1)$ is an m.i character of R_1. Hence θ is an m.i character of R as wanted. $\square$

3. Finite simple groups of Lie type

In this section, we aim to prove Theorem 1.4 for simple groups of Lie type. Note that we will consider the Tits group as a sporadic simple group rather than a simple group of Lie type and exclude it from consideration in this section. Now it is well known that every simple group of Lie type S in characteristic p possesses an irreducible character of degree $|S|_p$, the size of the Sylow p-subgroup of S, which is called the Steinberg character of S and is denoted by St_S. (See [4, Chapter 6].) Moreover the Steinberg character of S is always extendible to the full automorphism group $\mathrm{Aut}(S)$. (See for instance [3, Theorem 2].) Using the information on the character values of the Steinberg character given in [4] and also the classification of the maximal subgroups of simple groups of Lie type satisfying certain properties given in [14,17], we will prove that apart from some exceptions, the Steinberg character cannot be an m.i character. This is achieved in Lemma 3.8. Finally, for these exceptions, using [5] we will find another nonlinear irreducible character of S which extends to $\mathrm{Aut}(S)$ but it is not an m.i character.

We first draw some consequences under the assumption that the Steinberg character is an m.i character. Recall that if G is a group and $p \in \pi(G)$, then an element $g \in G$ is called *p-semisimple* (or just *semisimple* when p is understood) whenever the order of g is coprime to p.

Lemma 3.1. *Let S be a simple group of Lie type in characteristic p. Suppose that $m\,\mathrm{St}_S = \lambda^S$, where $\lambda \in \mathrm{Irr}(H)$, $m \geqslant 1$ and H is a maximal subgroup of S. Then the following hold.*

(1) *$g^S \cap H \neq \emptyset$ for any p-semisimple element $g \in S$.*
(2) *$p \nmid m$ and $\lambda(1)_p = |H|_p$.*
(3) *$|S:H|_p \geqslant m \geqslant |S:H|_{p'}$.*

Proof. By [4, Theorem 6.5.9], we have $\mathrm{St}_S(g) = \pm|C_S(g)|_p$ for any p-semisimple element $g \in S$. Thus for any p-semisimple element $g \in S$, we obtain that $\lambda^S(g) = m\,\mathrm{St}_S(g) \neq 0$. By the definition of induced characters, we obtain (1). For $g = 1 \in S$, we have that $m\,\mathrm{St}_S(1) = |S:H|\lambda(1)$. As $\mathrm{St}_S(1) = |S|_p$ and $\lambda(1)_p \mid |H|_p$, we deduce that $|S:H|_p\lambda(1)_p$ divides $|S|_p$ and so $|S:H|_p\lambda(1)_p = |S|_p$ as it is divisible by $\mathrm{St}_S(1)$. This implies that $p \nmid m$ and $\lambda(1)_p = |H|_p$, which proves (2). Finally as $m\,\mathrm{St}_S(1) = |S:H|\lambda(1)$, by applying (2) we have

$$m = |S:H|_{p'}\lambda(1)_{p'} \geqslant |S:H|_{p'}.$$

By Lemma 2.2(ii), we obtain that $m^2 \leqslant |S:H|$ and so

$$m^2 \leqslant |S:H|_p|S:H|_{p'} \leqslant m|S:H|_p,$$

which implies that $|S:H|_p \geqslant m \geqslant |S:H|_{p'}$ as required. □

The following result is a well known theorem due to Zsigmondy.

Lemma 3.2. *(See [13, Theorems 5.2.14, 5.2.15].) Let q and n be integers with $q \geqslant 2$ and $n \geqslant 3$. Assume that $(q,n) \neq (2,6)$. Then $q^n - 1$ has a prime divisor ℓ such that:*

- *ℓ does not divide $q^m - 1$ for $m < n$.*
- *If $\ell \mid q^k - 1$ then $n \mid k$.*
- *$\ell \equiv 1 \pmod n$.*

Such an ℓ is called a *primitive prime divisor*. We denote by $\ell_n(q)$ the smallest primitive prime divisor of $q^n - 1$ for fixed q and n. When n is odd and $(q,n) \neq (2,3)$ then there is a primitive prime divisor of $q^{2n} - 1$ which we denote by $\ell_{-n}(q)$.

Let S be a nonabelian simple group. If $S \trianglelefteq G \leqslant \mathrm{Aut}(S)$, then G is said to be an *almost simple group* with socle S and we write $\mathrm{soc}(G)$ to denote the socle S of G. We refer to [13, Chapter 4] for the detailed descriptions and definitions of the geometric classes $\mathcal{C}_i(S)$ $(1 \leqslant i \leqslant 8)$ and the class $\mathcal{S}(S)$ of subgroups of simple classical groups S.

Suppose that S is a simple group of Lie type in characteristic p and that St_S is an m.i character of S induced from the pair (H, λ), where H is a maximal subgroup of S and $\lambda \in \mathrm{Irr}(H)$. It follows from Lemma 3.1(1) that $g^S \cap H \neq \emptyset$ for any semisimple element $g \in S$ and thus $\pi_{p'}(S) \subseteq \pi(H)$, where $\pi_{p'}(S)$ is the set of all primes divisors of $|S|$ different from the characteristic p. Hence two cases can occur, either $\pi(H) = \pi(S)$ or $\pi(H) = \pi_{p'}(S)$. We first consider the case $\pi(H) = \pi_{p'}(S)$. This implies that $|H|$ is prime to p and that H possesses some semisimple element of certain maximal torus of S. Hence we can apply [17] for classical groups and [14] for exceptional groups of Lie type to obtain the possibilities for the pairs (S, H).

Lemma 3.3. *Let S be a nonabelian simple group of Lie type in characteristic p and let H be a maximal subgroup of S. Suppose that $p \nmid |H|$ and that for any p-semisimple element $g \in S$, we have that $g^S \cap H \neq \emptyset$. Then*

$$(S, H) \in \big\{\big(\mathrm{L}_2(5), \mathrm{A}_4\big), \big(\mathrm{L}_2(5), \mathrm{S}_3\big), \big(\mathrm{L}_2(7), \mathrm{S}_4\big), \big(\mathrm{L}_3(2), 7:3\big)\big\}.$$

Proof. It follows from the hypotheses that $\pi_{p'}(S) = \pi(H)$.

Case 1. S is a simple classical group in characteristic p defined over a field of size $q = p^f$. For each p-semisimple element $g \in S$, by the hypotheses some conjugate of g belongs to H. In particular H possesses some element of the maximal torus of S with order given in [17, Table I]. By [17, Theorem 1.1], the following cases hold.

(A) $H \in \bigcup_{i=1}^{8} \mathcal{C}_i(S)$. Then one of the following cases holds.

(A_1) $H \in \mathcal{C}_1(S)$ and $S \in \{\mathrm{L}_n^\epsilon(q), \mathrm{O}_{2n+1}(q), \mathrm{O}_{2n}^+(q)\}$, where n is at least 3, 3, and 4, respectively. By inspecting the orders of maximal subgroups in class $\mathcal{C}_1(S)$ in [13, §4.1] and using the restriction on n, we see that p always divides $|H|$. Hence this subcase cannot happen.

(A_2) $H \in \mathcal{C}_8(S)$ and $S \cong \mathrm{S}_{2n}(q)$, where $n \geqslant 2$. By [13, Proposition 4.8.16] we have that $H \cong \mathrm{O}_{2n}^\epsilon(q)$ with q even. Obviously p always divides $|H|$.

(A_3) $H \in \mathcal{C}_3(S)$ and $S \in \{\mathrm{L}_n^\epsilon(q)\ (n \geqslant 3 \text{ odd}), \mathrm{S}_{2n}(q)\ (n \geqslant 2), \mathrm{O}_{2n}^\epsilon(q)\ (n \geqslant 4)\}$. By inspecting the orders of maximal subgroups in class $\mathcal{C}_3(S)$ in [13, §4.3] and using the restriction on n, we see that p always divides $|H|$ unless $S \cong \mathrm{L}_n^\epsilon(q)$ where n is an odd prime and H is of type $\mathrm{GL}_1^\epsilon(q^n) \cdot n$.

Assume that $n = 3$. By [13, Proposition 4.3.6], we have $|H| = 3(q^2 + \epsilon q + 1)/d$, where $d = \gcd(3, q - \epsilon 1)$. In this case, S has an element of order $(q^2 - 1)/d$. It follows that $(q^2 - 1)/d$ must divide $3(q^2 + \epsilon q + 1)/d$, and so $(q^2 - 1) = (q - \epsilon 1)(q + \epsilon 1)$ divides $3(q^2 + \epsilon q + 1)$. As $\gcd(q + \epsilon 1, q^2 + \epsilon q + 1) = 1$, we deduce that $q + \epsilon 1 \mid 3$, which implies that $q + \epsilon 1 = 1$ or $q + \epsilon 1 = 3$ since $q + \epsilon 1 > 0$. Solving these equations, we obtain that $q = 2$ and $\epsilon = \pm$ or $q = 4$ and $\epsilon = -$. Since $\mathrm{U}_3(2)$ is not simple and $4^2 - 1 \nmid 3(4^2 - 4 + 1)$, we deduce that $S \cong \mathrm{L}_3(2)$ and $H \cong 7:3$.

Now suppose that $n \geqslant 5$ is odd prime. Assume first that both $\ell_{n-1}(q)$ and $\ell_{\epsilon(n-2)}(q)$ exist. Observe that these two primes are distinct. Then S has elements of orders $\ell_{n-1}(q)$ and $\ell_{\epsilon(n-2)}(q)$, respectively. Thus $\ell_{n-1}(q)$ and $\ell_{\epsilon(n-2)}(q)$ divides $|H|$. By Lemma 3.2 neither $\ell_{n-1}(q)$ nor $\ell_{\epsilon(n-2)}(q)$ can divide $|\mathrm{GL}_1^\epsilon(q^n)| = q^n - \epsilon 1$ since both $n - 1$ and $n - 2$ cannot divide $2n$ as $n \geqslant 5$ is odd prime. As a result, both $\ell_{n-1}(q)$ and $\ell_{\epsilon(n-2)}(q)$ must be equal to the prime n as $|H| \mid n(q^n - \epsilon 1)$, which is impossible. We now consider the case when either $\ell_{n-1}(q)$ or $\ell_{\epsilon(n-2)}(q)$ does not exist. By Lemma 3.2, we deduce that $S \cong \mathrm{L}_7^\epsilon(2)$ or $S \cong \mathrm{U}_5(2)$. If the first case holds, then $\ell_{\epsilon 5}(2) \in \pi(S)$ exists. However we see that $\ell_{\epsilon 5}(2)$ cannot divide $7(2^7 - \epsilon 1)$ so $\ell_{\epsilon 5}(2)$ cannot divide $|H|$, a contradiction. For the latter case, we have $H \cong 11:5$. But $\pi_{2'}(\mathrm{U}_5(2)) = \{3, 5, 11\} \neq \pi(H)$.

(B) $H \in \mathcal{S}(S)$. Then the following cases hold.

(B_1) $(S, H) \in \{(\mathrm{L}_4(2), \mathrm{A}_7), (\mathrm{U}_3(3), \mathrm{L}_2(7)), (\mathrm{U}_3(5), \mathrm{A}_7), (\mathrm{U}_4(3), \mathrm{A}_7), (\mathrm{U}_4(3), \mathrm{L}_3(4)), (\mathrm{U}_5(2), \mathrm{L}_2(11)), (\mathrm{U}_6(2), \mathrm{M}_{22}), (\mathrm{O}_7(3), \mathrm{S}_9), (\mathrm{S}_8(2), \mathrm{L}_2(17))\}$.

(B_2) $(S, \mathrm{soc}(H)) = (\mathrm{O}_8^+(q), \mathrm{O}_7(q))$ with q odd, or $(\mathrm{O}_8^+(q), \mathrm{S}_6(q))$ with q even.

For these cases, we see that the characteristic p of S divides the order of H.
(C) $S \cong \mathrm{L}_2(q)$, $\mathrm{U}_4(2)$ or $(S, H) \in \{(\mathrm{L}_3(4), \mathrm{L}_3(2)), (\mathrm{S}_4(3), 2^4 \cdot \mathrm{A}_5), (\mathrm{O}_8^+(2), \mathrm{A}_9)\}$.

(C_1) $S \cong \mathrm{U}_4(2)$. As $\mathrm{U}_4(2) \cong \mathrm{S}_4(3)$, the characteristic p of S is either 2 or 3. Assume first that $p = 2$. In this case, S contains 2-semisimple elements of order 5 and 9. However by using [5], no maximal subgroup of S possesses two such elements simultaneously. Now assume that $p = 3$. In this case, by using [5] again we can check that p divides the order of every maximal subgroup of S.
(C_2) $(S, H) \in \{(\mathrm{L}_3(4), \mathrm{L}_3(2)), (\mathrm{S}_4(3), 2^4 \cdot \mathrm{A}_5), (\mathrm{O}_8^+(2), \mathrm{A}_9)\}$. For these cases, the characteristic p of S divides the order of $|H|$.
(C_3) $S \cong \mathrm{L}_2(q)$, where $q \geqslant 4$.

Assume first that $S \cong \mathrm{L}_2(4) \cong \mathrm{L}_2(5)$. By [5], every maximal subgroup of S is of even order so we can assume that $p = 5$. In this case, using [5] again, we deduce that $H \cong \mathrm{S}_3$ or A_4. Assume next that $S \cong \mathrm{L}_2(7) \cong \mathrm{L}_3(2)$. By [5], we can see that if $p = 2$, then $H \cong 7:3$ and if $p = 7$, then $H \cong \mathrm{S}_4$. Assume that $S \cong \mathrm{L}_2(9) \cong \mathrm{A}_6$ or $\mathrm{L}_2(8)$. By [5], the order of every maximal subgroup of S is divisible by p. Assume that $S \cong \mathrm{L}_2(q)$ where $q \in \{11, 13\}$. By [5], S possesses p-semisimple elements of order $(q+1)/2$ and $(q-1)/2$ respectively. However no maximal subgroups of S can possess both such elements simultaneously.

Thus we can assume that $q \geqslant 16$. Since H is a maximal subgroup of S and $p \nmid |H|$, inspecting the list of maximal subgroups of $\mathrm{L}_2(q)$ in [12], the following cases hold.

(i) H is a dihedral group of order $q+1$, with q odd.
(ii) H is a dihedral group of order $q-1$, with q odd.
(iii) $H \cong \mathrm{S}_4$ and $q \equiv \pm 1 \pmod 8$, q prime or $q = p^2$ and $3 < p \equiv \pm 3 \pmod 8$.
(iv) $H \cong \mathrm{A}_4$ and $q \equiv \pm 3 \pmod 8$ with $q > 3$ prime.
(v) $H \cong \mathrm{A}_5$ and $q \equiv \pm 1 \pmod{10}$, q prime or $q = p^2$ and $p \equiv \pm 3 \pmod{10}$.

It follows that $q \geqslant 17$ is odd and thus S has two p-semisimple elements of order $(q \pm 1)/2$ so H possesses elements of such orders. As $\gcd((q-1)/2, (q+1)/2) = 1$, we deduce that $(q^2-1)/4$ divides $|H|$. Since $q \geqslant 17$, we can see that $(q^2-1)/4 > 60 = |A_5|$ and that $(q^2-1)/4 > q+1$ and hence H cannot be one of the groups given in (i)–(v) above.

Case 2. S is a simple exceptional group of Lie type in characteristic p with $S \neq {}^2\mathrm{F}_4(2)'$. As $\pi_{p'}(S) = \pi(H)$, $|H|$ is divisible by all the primes in the second column of [14, Table 10.5] so it follows from the proof of [14, Theorem 4] and [14, Table 10.5] that $S \cong \mathrm{G}_2(q)$ with $q > 2$ odd and $H \cong \mathrm{L}_2(13)$ where $\{\ell_3(q), \ell_6(q)\} = \{7, 13\}$ and $p \neq 13$. By [1, 15.1], S possesses a cyclic maximal torus of order q^2-1. Now if $q = 3$ then $q^2 - 1 = 8$. But then $\mathrm{L}_2(13)$ has no element of order 8. Thus $q \geqslant 4$ and hence $q^2 - 1 \geqslant 15$ which is strictly larger than any element orders in $\mathrm{L}_2(13)$. Hence H contains no element of order q^2-1, a contradiction. $\square$

We now consider the case $\pi(S) = \pi(H)$. In this case we can apply [14, Corollary 5] to obtain the possibilities for the pairs (S, H).

Lemma 3.4. *Let S be a nonabelian simple group of Lie type in characteristic p and let H be a maximal subgroup of S. Suppose that $p \mid |H|$ and that for any p-semisimple element $g \in S$, we have that $g^S \cap H \neq \emptyset$. Then one of the following cases holds.*

(1) $S \cong \mathrm{S}_4(3)$ *and* $H \cong 2^4 : \mathrm{A}_5$.
(2) $S \cong \mathrm{S}_{2n}(q)$ *and* $H \cong \Omega_{2n}^-(q) \cdot 2 \cong \mathrm{SO}_{2n}^-(q)$, *with q, n even.*
(3) $S \cong \Omega_{2n+1}(q)$ *and* $H \cong \Omega_{2n}^-(q) \cdot 2$, *with $n \geqslant 2$ even and q odd.*
(4) $S \cong \mathrm{O}_{2n}^+(q)$ *and* $H \cong \Omega_{2n-1}(q)$ *with $n \geqslant 4$ even.*
(5) $S \cong \mathrm{S}_4(q)$ *and* $H \cong \mathrm{L}_2(q^2) \cdot 2$ *with $q \geqslant 4$ even.*

Table 1
Simple groups of Lie type.

S	H	Element order	S	H	Element order
$\mathrm{L}_2(9)$	$\mathrm{L}_2(5)$	4	$\mathrm{S}_4(7)$	A_7	8
$\mathrm{U}_3(3)$	$\mathrm{L}_2(7)$	8	$\mathrm{G}_2(3)$	$\mathrm{L}_2(13)$	4
$\mathrm{U}_3(5)$	A_7	8	$\mathrm{U}_4(2)$	S_6	9
$\mathrm{U}_5(2)$	$\mathrm{L}_2(11)$	9	$\mathrm{S}_6(2)$	S_8	9
$\mathrm{U}_6(2)$	M_{22}	9	$\mathrm{U}_4(3)$	$\mathrm{L}_3(4), \mathrm{A}_7$	8

Proof. It follows from the hypotheses that $\pi(S) = \pi(H)$ and thus by [14, Corollary 5], one of the following cases holds.

(i) $S \cong \mathrm{U}_4(2) \cong \mathrm{S}_4(3)$ and $H \cong 2^4 : \mathrm{A}_5$. In this case, the characteristic of S is either 2 or 3. If $p = 3$, then the pair $(S, H) = (\mathrm{S}_4(3), 2^4 : \mathrm{A}_5)$ satisfies the hypotheses of the lemma. If $p = 2$, then S has a 2-semisimple element of order 9 but H has no such element.

(ii) $S \cong \mathrm{L}_6(2)$ and $H \cong \mathrm{L}_5(2)$, P_1 or P_5. As H is maximal in S, we deduce that H is isomorphic to the maximal parabolic subgroup P_1 or P_5. Hence $H \cong 2^5 : \mathrm{L}_5(2)$. In this case, S possesses an element of order 63 but H contains no such element.

(iii) $S \cong \mathrm{O}_8^+(2)$ and $H \cong \mathrm{A}_9$ or P_i, $i = 1, 3, 4$. Note that $P_i \cong 2^6 : \mathrm{A}_8$ for $i = 1, 3, 4$. In any cases, H has only one conjugacy class of elements of order 5 while S has 3 conjugacy classes of elements of order 5. Therefore these cases cannot happen.

(iv) $S \cong \mathrm{S}_{2n}(q)$ with $n \geqslant 2$, n, q even and $\Omega_{2n}^-(q) \trianglelefteq H$. In this case, $H \in \mathcal{C}_8(S)$ and by [13, Proposition 4.8.6], we have that $H \cong \mathrm{SO}_{2n}^-(q) \cong \Omega_{2n}^-(q) \cdot 2$.

(v) $S \cong \mathrm{O}_{2n+1}(q)$ with $n \geqslant 3$ even, q odd and $\Omega_{2n}^-(q) \trianglelefteq H$. In this case, $H \in \mathcal{C}_1(S)$ and by [13, Proposition 4.1.6], we have that $H \cong \Omega_{2n}^-(q) \cdot 2$.

(vi) $S \cong \mathrm{O}_{2n}^+(q)$ with $n \geqslant 4$ even and $\Omega_{2n-1}(q) \trianglelefteq H$. In this case, $H \in \mathcal{C}_1(S)$ and by [13, Proposition 4.1.6], we have that $H \cong \Omega_{2n-1}(q)$.

(vii) $S \cong \mathrm{S}_4(q)$ and $\mathrm{L}_2(q^2) \trianglelefteq H$. As $\mathrm{S}_4(2)$ is not simple, we assume that $q \geqslant 3$. If $q = 3$, then $\mathrm{S}_4(3) \cong \Omega_5(3) \cong \mathrm{U}_4(2)$ and $H \cong \mathrm{S}_6$. This case will be handled in (viii) so we assume that $q > 3$. Assume first that $q > 3$ is odd. Using the isomorphism $\mathrm{S}_4(q) \cong \Omega_5(q)$, it follows from [17, Theorem 1.1] that $H \in \mathcal{C}_1(\Omega_5(q))$ and by [13, Proposition 4.1.6] we have that $H \cong \Omega_4^-(q) \cdot 2 \cong \mathrm{L}_2(q^2) \cdot 2$. This possibility is included in case (3). Assume now that $q \geqslant 4$ is even. Then by [17, Theorem 1.1] again, $H \in \mathcal{C}_3(S) \cup \mathcal{C}_8(S)$ and hence by [13] we obtain that $H \cong \Omega_4^-(q) \cdot 2 \cong \mathrm{L}_2(q^2) \cdot 2$. This is case (5) in the lemma.

(viii) The pair (S, H) appears in Table 1. For these cases, the conjugacy class of p-semisimple elements with order given in the column 'element order' in Table 1 does not intersect H. $\square$

As we will see shortly, only in case (1) of the previous lemma is the Steinberg character an m.i character. Cases (2), (4) and (5) can be ruled out easily by using Lemma 3.1 and [6]. For case (3) we will need more work.

We refer to [13,5] for the basic definitions and properties of orthogonal groups and their associated geometries. Let p be an odd prime. Let q be a power of p and let $\mathbb{F}_q$ be a finite field of size q. Let $(V, \mathbb{F}_q, Q)$ be a classical orthogonal geometry with $\dim V = 2n+1$, $n \geqslant 2$ and Q a non-degenerate quadratic form on V. For $x \in V - \{0\}$, a one-space with representative x is called a *point* in V and is denoted by $\langle x \rangle$. The vector $x \in V - \{0\}$ is said to be *non-singular* provided that $Q(x) \neq 0$. Recall that a non-singular point $\langle x \rangle$ with representative $x \in V - \{0\}$ is called a *plus point* (or *minus point*) if $sgn(x^{\perp})$ is $+$ (or $-$), respectively (cf. [5, p. xi]). In this situation, we also say that the non-singular point $\langle x \rangle$ is of plus or minus type if $\langle x \rangle$ is a plus or minus point, respectively. Note that the group $\Omega(V)$ as defined in [13, (2.1.14)] is isomorphic to $\Omega_{2n+1}(q) \cong \mathrm{O}_{2n+1}(q)$. In this situation, V is called the natural module for $\Omega(V)$. For $\xi \in \{\pm\}$, we define $\mathfrak{E}_\xi(V)$ to be the set of all non-singular points of type ξ in V. For $\tau \in \mathbb{F}_q$, we define

$$V_\tau = \big\{ v \in V - \{0\} \bigm| Q(v) = \tau \big\}.$$

Lemma 3.5. *Assume the set up above. Then the following hold.*

(i) *Two non-singular points $\langle x\rangle$ and $\langle y\rangle$ have the same type if and only if $Q(x) \equiv Q(y) \pmod{(\mathbb{F}_q^*)^2}$. Indeed, for any non-singular point $\langle z\rangle$ with type ζ, we have*

$$\mathfrak{E}_\zeta(V) = \{\langle v\rangle \subseteq V \mid Q(v) \equiv Q(z) \pmod{(\mathbb{F}_q^*)^2}\}.$$

(ii) *For $\xi \in \{\pm\}$, $\Omega(V)$ acts transitively on $\mathfrak{E}_\xi(V)$.*

(iii) *The stabilizers in $\Omega(V)$ of minus points form a unique conjugacy class of subgroups of $\Omega(V)$.*

Proof. As (iii) is a direct consequence of (ii), we only need to prove (i) and (ii).

For (i), assume that $Q(x) \equiv Q(y) \pmod{(\mathbb{F}_q^*)^2}$. By [13, Proposition 2.5.4(*ii*)], we have that $\langle x\rangle$ and $\langle y\rangle$ are isometric. By Witt's lemma [13, Proposition 2.1.6], this isometry extends to an isometry g of V such that $\langle x\rangle g = \langle y\rangle$. As $\langle x\rangle$, $\langle y\rangle$ are non-degenerate, we obtain $x^\perp g = y^\perp$. It follows that $x^\perp$ and $y^\perp$ are isometric, and hence $sgn(x^\perp) = sgn(y^\perp)$, so x and y have the same type. Now assume that x, y have the same type. By Witt's lemma and [13, Proposition 2.5.4(*i*)], there exists an isometry between $x^\perp$ and $y^\perp$. This isometry can extend to an isometry g of V such that $(x^\perp)g = y^\perp$. Since $(x^\perp)^\perp = \langle x\rangle$, and $(y^\perp)^\perp = \langle y\rangle$, we deduce that $\langle x\rangle g = \langle y\rangle$. Thus $xg = \mu y$ for some $\mu \in \mathbb{F}_q^*$. Therefore, $Q(x) = Q(xg) = Q(\mu y) = \mu^2 Q(y)$. The other statements are obvious. This proves (i).

For (ii), since $Q(\mu x) = \mu^2 Q(x)$ for $x \in V$, $\mu \in \mathbb{F}_q^*$ and $Q(xg) = Q(x)$ for all $x \in V$, $g \in \Omega(V)$, we see that $\Omega(V)$ acts on $\mathfrak{E}_\xi(V)$. Now fix a non-singular point $\langle x\rangle$ of type ξ. Let $\langle v\rangle$ be any non-singular point of the same type as that of $\langle x\rangle$. By (i), we have that $Q(v) = \mu^2 Q(x)$ for some $\mu \in \mathbb{F}_q^*$. Let $y = \mu^{-1} v \in \langle v\rangle$. Then $Q(y) = Q(x) =: \tau$ and $\langle x\rangle = \langle y\rangle = \langle v\rangle$. It follows that $x, y \in V_\tau$ and hence by [13, Lemma 2.10.5], $\Omega(V)$ acts transitively on V_τ so there exists $g \in \Omega(V)$ such that $xg = y$. Therefore, we obtain that $\langle x\rangle g = \langle y\rangle = \langle v\rangle$, which means that $\Omega(V)$ is transitive on $\mathfrak{E}_\xi(V)$ as wanted. $\square$

Remark 3.6. In case (3) of Lemma 3.4, the maximal subgroup H is exactly the stabilizer in $S \cong \Omega_{2n+1}(q)$ of a minus point in the natural module for S. By Lemma 3.5(iii), there is only one class of such maximal subgroups in S.

We now consider the following set up. Let $n \geqslant 2$ be even and let q be an odd prime power. Let $S \cong \Omega_{2n+1}(q)$ with $S \neq \Omega_5(3)$ and let H be the stabilizer of a minus point in the natural module for S. We deduce that $K := H' \cong \Omega_{2n}^-(q) \leqslant S$ which is a normal subgroup of index 2 in H. Let $\tilde{S} \cong \mathrm{Spin}_{2n+1}(q)$ and $\tilde{K} \cong \mathrm{Spin}_{2n}^-(q)$. By the assumption on n and q, we deduce that the centers of both $\tilde{S}$ and $\tilde{K}$ are cyclic of order 2. We know that $\tilde{K}$ possesses a cyclic maximal torus $\tilde{T}$ of order $q^n + 1$ which contains the center of $\tilde{K}$ and then by factoring out the center $Z(\tilde{K})$, we obtain a maximal torus T of K with order $k := (q^n + 1)/2$ (cf. [9]). Let $\tilde{g}$ and g be generators for $\tilde{K}$ and K, respectively.

We know that the conjugacy classes of maximal tori of $\tilde{S}$ are parametrized by pairs of partitions (α, β) of n, that is, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_r)$ and $\beta = (\beta_1, \beta_2, \dots, \beta_s)$ and $\sum \alpha_i + \sum \beta_j = n$. The order of the maximal torus parametrized by such pair (α, β) is given by

$$\prod_{\alpha_i} (q^{\alpha_i} - 1) \prod_{\beta_j} (q^{\beta_j} + 1).$$

The conjugacy classes of the maximal tori of $\tilde{K}$ are also parametrized by pairs of partitions (α, β) of n, where β has an odd number of parts, and the order of the maximal torus parametrized by (α, β) is the same as in case $\tilde{S}$ (cf. [16]). It follows that $\tilde{T}$ is a maximal torus of $\tilde{K}$ parametrized by the pair of partition $(\emptyset, (n))$. Now by applying Lemma 3.2 and the order formula of maximal tori of $\tilde{K}$, we can easily see that the conjugacy class of $\tilde{K}$ containing $\tilde{T}$ is the unique class whose order is divisible by $\ell_{2n}(q)$. Since $Z(\tilde{K}) \leqslant \tilde{T}^x$ for all $x \in \tilde{K}$, we deduce that the conjugacy class of T in K is the unique conjugacy class of maximal torus whose order is divisible by $\ell_{2n}(q)$. Since $\tilde{g} \in \tilde{K} \leqslant \tilde{S}$

is semisimple, it lies in some maximal torus of $\tilde{S}$. Using the order formula of the maximal tori of $\tilde{S}$ given above, we deduce that $\tilde{g}$ must lie in the Coxeter torus of $\tilde{S}$ (see [9]). Comparing the orders, we obtain that $\tilde{T}$ is the Coxeter torus of $\tilde{S}$ so T is also a maximal torus of S. We also know that $\tilde{T}$ has a regular element $\tilde{y}$, i.e., $C_{\tilde{S}}(\tilde{y}) = \tilde{T}$ which implies that $\tilde{T} \leqslant C_{\tilde{S}}(\tilde{T}) \leqslant C_{\tilde{S}}(\tilde{y}) = \tilde{T}$ and hence $C_{\tilde{S}}(\tilde{T}) = \tilde{T}$. As a consequence, we obtain that $C_S(T) = T$ which in turn implies that $C_S(g) = \langle g \rangle$ and $C_H(g) = \langle g \rangle$ as $g \in K \leqslant H \leqslant S$.

Lemma 3.7. *Assume the set up above. Then the following hold.*

(1) *If $T \leqslant U \leqslant S$, where U is maximal in S, then U is conjugate to H in S.*
(2) *If $T^x \leqslant H$ for some $x \in S$, then $T^x = T^u$ for some $u \in H$.*
(3) *We have $N_S(T) \leqslant H$ and $C_S(g) = \langle g \rangle = C_H(g)$.*
(4) *We have that $g^S \cap H = g^H$.*

Proof. Let U be any maximal subgroup of S containing T. As $n \geqslant 2$ is even, we consider the case $n = 2$ and $n \geqslant 4$ separately. Assume first that $n \geqslant 4$. By [17, Theorem 1.1], we have that $U \in \mathcal{C}_1(S)$. Since $n \geqslant 4$ and q is odd, we deduce that $\ell_{2n}(q)$ exists and divides $|T|$ so it divides $|U|$. Using the descriptions of the subgroups in class $\mathcal{C}_1(S)$ given in Propositions 4.1.6 and 4.1.20 in [13], U must be the stabilizer of a minus point so U is conjugate to H in S by Lemma 3.5(iii). The remaining case can be argued similarly or we can check directly using the list of maximal subgroups of S given in [12]. This proves (1).

Assume that $T^x \leqslant H$ for some $x \in S$. It follows that $g^x \in H$ where $x \in S$, $T = \langle g \rangle$ and the order of g and g^x is k. Since q is odd and n is even, we have that k is always odd. As $K \trianglelefteq H$ is of index 2, we deduce that $g^x \in K$. As k is prime to the characteristic of $K \cong \Omega_{2n}^-(q)$, we obtain that $g^x \in K$ is semisimple and hence it must lie in some maximal torus of K whose order is divisible by $\ell_{2n}(q)$. Using the discussion above, the conjugacy class of maximal torus of K containing T is the only class of maximal torus of K whose order is divisible by $\ell_{2n}(q)$ so $T^x = \langle g^x \rangle$ is conjugate to T in K and hence in H. This proves (2).

We next show that $N_S(T) \leqslant H$. Indeed if $T \leqslant N_S(T) \nleqslant H$, then $N_S(T)$ must lie in some maximal subgroup of S containing T since $N_S(T) \neq S$. By (1) we have $T \trianglelefteq N_S(T) \leqslant H^x$ for some $x \in S$. It follows that $T^{x^{-1}} \leqslant H$ and hence $T^{x^{-1}} = T^u$ for some $u \in H$ by (2). Thus $T^{ux} = T$ or equivalently $ux \in N_S(T) \leqslant H^x$, which implies that $ux = h^x$, where $h \in H$. Thus we conclude that $x = hu^{-1} \in H$ and so $N_S(T) \leqslant H^x = H$. This proves the first statement of (3). The other statement has already been proved in the discussion above.

Finally, since $g \in H$, we obtain that $g^H \subseteq g^S \cap H$. To prove the equality, it suffices to show that if $g^x \in H$, where $x \in S$, then $x \in H$. Suppose that $g^x \in H$, where $x \in S$. Then $T^x = \langle g \rangle^x = \langle g^x \rangle \subseteq H$ and thus by (2) we have that $T^x = T^u$ for some $u \in H$. It follows that $xu^{-1} \in N_S(T)$ and hence by (3) we have $N_S(T) \leqslant H$ so $xu^{-1} \in H$, which implies that $x \in H$ as $u \in H$. The proof is now complete. □

We now classify all simple groups of Lie type in which the Steinberg character is an m.i character.

Lemma 3.8. *Let S be a simple group of Lie type in characteristic p. If H is a maximal subgroup of S such that $m\,\mathrm{St}_S = \lambda^S$, where $\lambda \in \mathrm{Irr}(H)$ and $m \geqslant 1$, then*

$$(S, H) \in \big\{\big(\mathrm{L}_2(5), \mathrm{A}_4\big), \big(\mathrm{L}_2(7), \mathrm{S}_4\big), \big(\mathrm{L}_3(2), 7:3\big), \big(\mathrm{S}_4(3), 2^4 : \mathrm{A}_5\big)\big\}.$$

In the first three cases, $\lambda \in \mathrm{Irr}(H) - \{1_H\}$ are chosen with $\lambda(1) = 1$. In the last case, $\lambda \in \mathrm{Irr}(H)$ is chosen with $\lambda(1) = 3$. Furthermore, $m = 1$ in all cases.

Proof. Assume that $m\,\mathrm{St} = \lambda^S$, where $\lambda \in \mathrm{Irr}(H)$, $m \geqslant 1$ and H is a maximal subgroup of S. By Lemma 3.1(1) the hypotheses of Lemmas 3.3 and 3.4 are satisfied so one of the following cases holds.

(i) $(S, H) \in \{(\mathrm{L}_2(5), \mathrm{A}_4), (\mathrm{L}_2(5), \mathrm{S}_3), (\mathrm{L}_2(7), \mathrm{S}_4), (\mathrm{L}_3(2), 7:3), (\mathrm{S}_4(3), 2^4 : \mathrm{A}_5)\}$. Apart from the pair $(\mathrm{L}_2(5), \mathrm{S}_3)$, the remaining pairs satisfy the conclusion of the lemma. This is done by using [8].

(ii) $(S,H)=(\mathrm{S}_{2n}(q),\Omega^-_{2n}(q)\cdot 2)$ with $n\geqslant 2$ and n, q even. Let $K=H'\cong\Omega^-_{2n}(q)$. Then K is a normal subgroup of index 2 in H. Let $\mu\in\mathrm{Irr}(K)$ be an irreducible constituent of λ_K. Since $\lambda(1)_2=|H|_2=2|K|_2$, we have that λ_K is not irreducible so $\mu(1)=\lambda(1)/2$ since $|H:K|=2$, and thus $\mu(1)=|K|_2$. It follows that $\mu\in\mathrm{Irr}(K)$ is of 2-defect zero and then by [6] the only irreducible character of 2-defect zero of $\Omega^-_{2n}(q)$ with even q is exactly the Steinberg character, we deduce that $\mu=\mathrm{St}_K$, where $K\cong\Omega^-_{2n}(q)$. By [3, Theorem 2], μ extends to $\mu_0\in\mathrm{Irr}(H)$ and hence by Gallagher's theorem [10, Corollary 6.17], $\psi\mu_0$ are all the irreducible constituents of μ^H, where $\psi\in\mathrm{Irr}(H/K)$. Since H/K is abelian of order 2, we obtain that all irreducible constituents of μ^H are of the same degree $\mu(1)$. However this is a contradiction as $\lambda(1)=2\mu(1)$ and λ is also an irreducible constituent of μ^H by the Frobenius reciprocity.

(iii) $S=\mathrm{O}^+_{2n}(q)$ with $n\geqslant 4$ even, and $H\cong\Omega_{2n-1}(q)$ when q is odd or $H\cong\mathrm{S}_{2n-2}(q)$ when q is even. We have

$$|S:H|=\frac{q^{n-1}(q^n-1)\gcd(2,q-1)}{\gcd(4,q^n-1)}.$$

It follows that

$$|S:H|_p=q^{n-1}\quad\text{and}\quad |S:H|_{p'}=\frac{(q^n-1)\gcd(2,q-1)}{\gcd(4,q^n-1)}.$$

As $q\geqslant 3$, we can check that $|S:H|_{p'}>|S:H|_p$, contradicting Lemma 3.1(3).

(iv) $(S,H)=(\mathrm{O}_{2n+1}(q),\Omega^-_{2n}(q)\cdot 2)$ with $n\geqslant 2$ even, q odd and $(n,q)\neq(2,3)$. Let $K=H'\cong\Omega^-_{2n}(q)$. Then K is a normal subgroup of index 2 in H. Since p is odd, we deduce that $\lambda(1)_p=|H|_p=|K|_p$ as $|H:K|=2$. Let $\mu\in\mathrm{Irr}(K)$ be an irreducible constituent of λ_K. We know that $\lambda(1)/\mu(1)$ divides $|H:K|=2$ and thus $\lambda(1)_p=\mu(1)_p=|K|_p$, which means that μ is an irreducible character of K with p-defect zero. By applying the same argument as in case (ii), we deduce that $\mu=\mathrm{St}_K$ and μ extends to $\mu_0\in\mathrm{Irr}(H)$. By Gallagher's theorem, $\psi\mu_0$ are all the irreducible constituents of μ^H, where $\psi\in\mathrm{Irr}(H/K)$. As H/K is abelian and λ is an irreducible constituent of μ^H, we obtain that $\lambda=\mu_0\tau$ for some $\tau\in\mathrm{Irr}(H/K)$. It follows that $\lambda(1)=\mu_0(1)\tau(1)=\mu(1)$ so λ is an extension of μ to H. In particular $\lambda(1)=|H|_p$.

By the discussion before Lemma 3.7, K possesses a cyclic maximal torus T with generator g whose order k is prime to p so g is semisimple. By Lemma 3.7(3) we have that $C_S(g)=C_H(g)=\langle g\rangle$ is a p'-group and hence by [4, Theorem 6.5.9] we have that

$$\mathrm{St}_S(g)=\pm\big|C_S(g)\big|_p=\pm 1$$

and

$$\lambda(g)=\mu(g)=\mathrm{St}_K(g)=\pm\big|C_K(g)\big|_p=\pm 1.$$

By Lemma 3.7(4), we have that $g^S\cap H=g^H$ and thus by [10, p. 64] we have

$$\lambda^S(g)=\frac{|C_S(g)|}{|C_H(g)|}\lambda(g)=\mu(g)=\pm 1.$$

As $m\,\mathrm{St}_S=\lambda^S$, we obtain that $m\,\mathrm{St}_S(g)=\lambda^S(g)$ and hence $m=1$. By Lemma 3.1(3), we have that $m\geqslant|S:H|_{p'}=(q^n-1)/2$. As $q\geqslant 3$ and $n\geqslant 2$, it is obvious that $m>1$, which contradicts our previous claim that $m=1$. $\square$

Comparing the previous lemma with [18, Lemma 2.4(2)] and [15, Lemma 6.3], we can see that the Steinberg character of a simple group of Lie type is m.i if and only if it is imprimitive; if and only if it is QSI, i.e., it is induced from a character φ of a subgroup U such that $U/Ker(\varphi)$ is solvable.

We are now ready to prove the main result of this section.

Proposition 3.9. *Theorem 1.4 holds for simple groups of Lie type.*

Proof. Let S be a simple group of Lie type in characteristic p. By way of contradiction, suppose that every nonlinear irreducible character of S which is extendible to $\mathrm{Aut}(S)$ is an m.i character. As the Steinberg character of S is extendible to $\mathrm{Aut}(S)$, it is an m.i character. By Lemma 3.8, one of the following cases holds.

(1) $S \cong \mathrm{L}_2(5)$ with $p = 5$. Since $\mathrm{L}_2(5) \cong \mathrm{L}_2(4)$, the Steinberg character of S with degree $|S|_2 = 4$ extends to $\mathrm{Aut}(S)$ but it is not an m.i character by Lemma 3.8.

(2) $S \cong \mathrm{L}_2(7) \cong \mathrm{L}_3(2)$ with $p = 2$ or $p = 7$. In this case, both irreducible characters of degrees 7 and 8 are m.i characters. Using [5], the irreducible character χ labeled by the symbol χ_4 with degree 6 of S is extendible to $\mathrm{Aut}(S)$. We will show that χ is not an m.i character. Suppose that χ is an m.i character of S. Then $m\chi = \lambda^S$, where H is a maximal subgroup of S, $\lambda \in \mathrm{Irr}(H)$ and $m \geqslant 1$ is an integer. Let a and b be elements in S with order 2 and 7 respectively. By [5], we have that $\chi(a) \neq 0 \neq \chi(b)$. Hence $a^S \cap H \neq \emptyset \neq b^S \cap H$. Thus H possesses elements of orders 2 and 7 which implies that $\{2, 7\} \subseteq \pi(H)$. However by inspecting the list of maximal subgroups of S in [5], we see that no maximal subgroups of H satisfies this property. This contradiction shows that χ is not an m.i character.

(3) $S \cong \mathrm{S}_4(3)$ with $p = 3$. In this case, we have that $S \cong \mathrm{S}_4(3) \cong \mathrm{U}_4(2)$, the Steinberg character of S with degree $|S|_2 = 2^6$ is extendible to $\mathrm{Aut}(S)$ but it is not an m.i character by Lemma 3.8. The proof is now complete. □

4. Alternating groups

The main purpose of this section is to prove the following result.

Proposition 4.1. *Theorem 1.4 holds for alternating groups of degree at least* 7.

Proof. Let A_n act on the set $\Omega = \{1, 2, \dots, n\}$ of size n, where $n \geqslant 7$. We follow the argument in [15, Lemma 3.1]. Let $\chi_n \in \mathrm{Irr}(\mathrm{A}_n)$ be an irreducible character of A_n which is extendible to $\mathrm{Aut}(S) \cong \mathrm{S}_n$ with degree $n - 1$. In fact, we can choose $\chi_n = \pi_n - 1$, where π_n is the permutation character of the natural action of A_n on Ω. As $n \geqslant 7$, the 2-point stabilizer $\mathrm{A}_{n-2} = \mathrm{Stab}_{\mathrm{A}_n}(\{1, 2\})$ is doubly transitive on $\Omega - \{1, 2\}$. As χ_n is an m.i character, we have that $m\chi_n = \lambda^{\mathrm{A}_n}$ for some $\lambda \in \mathrm{Irr}(U)$ and $U \lneq \mathrm{A}_n$. We have that

$$(\chi_n)_{\mathrm{A}_{n-2}} = \chi_{n-2} + 2 \cdot 1_{A_{n-2}}.$$

By Mackey's Lemma, we obtain that

$$m\chi_{n-2} + 2m1_{\mathrm{A}_{n-2}} = \sum_{x \in T} \left(\lambda^x_{U^x \cap \mathrm{A}_{n-2}}\right)^{\mathrm{A}_{n-2}},$$

where T is a representative set of the double cosets of A_{n-2} and U in A_n. Hence

$$\left(\lambda_{U \cap \mathrm{A}_{n-2}}\right)^{\mathrm{A}_{n-2}} = m_1\chi_{n-2} + m_2 1_{\mathrm{A}_{n-2}},$$

where $m_1, m_2 \geqslant 1$. By replacing U with its conjugate, we can assume that $m_2 > m_1$ so $\mathrm{A}_{n-2} \leqslant U$ as $(\lambda_{U \cap \mathrm{A}_{n-2}})^{\mathrm{A}_{n-2}}$ takes only positive values. As U is maximal in A_n and $\mathrm{A}_{n-2} \leqslant U$, we deduce that

Table 2
Sporadic simple groups and the Tits group.

S	Primes	χ	$\chi(1)$	Possible H	Conjugacy class
M_{11}	11	χ_9	45	$L_2(11)$	$8A$
M_{12}	11	χ_7	54	$L_2(11)$, M_{11}	$10A$
J_1	11, 19	χ_2	56	–	
M_{22}	7, 11	χ_3	45	–	
J_2	5, 7	χ_6	36	–	
M_{23}	7, 23	χ_2	22	–	
HS	7, 11	χ_{24}	3200	M_{22}	$15A$
J_3	17, 19	χ_6	324	–	
M_{24}	23	χ_{22}	3520	$L_2(23)$, M_{23}	$21A$
McL	7, 11	χ_{12}	4500	M_{22}	$12A$
He	7, 17	χ_{12}	1920	–	
Ru	13, 29	χ_2	378	–	
Suz	11, 13	χ_{43}	248 832	–	
O'N	19, 31	χ_{23}	175 616	–	
Co_3	7, 23	χ_5	275	M_{23}	$24A$
Co_2	7, 23	χ_4	275	M_{23}	$30A$
Fi_{22}	11, 13	χ_{54}	1 360 800	–	
HN	11, 19	χ_{44}	2 985 984	–	
Ly	37, 67	χ_2	2480	–	
Th	19, 31	χ_4	27 000	–	
Fi_{23}	17, 23	χ_6	30 888	–	
Co_1	13, 23	χ_{40}	21 049 875	–	
J_4	37, 43	χ_{15}	32 307 363	–	
Fi_{24}'	23, 29	χ_6	1 603 525	–	
B	31, 47	χ_{119}	2 642 676 197 359 616	–	
M	59, 71	χ_{16}	8 980 616 927 734 375	–	
${}^2F_4(2)'$	5, 13	χ_{20}	1728	$L_2(25)$	$10A$

$U \cong A_{n-1}$ or $U \cong S_{n-2}$. Assume that the latter case holds. Then the conjugacy class of A_n with representative $g \in A_n$, where $g = (1, 2, \ldots, n)$ or $(1, 2 \cdots, n-3)(n-2, n-1, n)$ depending on whether n is odd or even, respectively, will intersect U as $\chi_n(g) \neq 0$, which is impossible. Thus $U \cong A_{n-1}$. As $m\chi_n = \lambda^{A_n}$ and $|A_n : U| = n$, by Lemma 2.2(ii), we have that $m(n-1) = n\lambda(1)$ and $|A_n : U| = n \geqslant m^2$. As $m(n-1) = n\lambda(1)$ and $\gcd(n, n-1) = 1$, we deduce that $n \mid m$, hence $n \leqslant m$, which is impossible as $n \geqslant m^2$ and $n \geqslant 7$. $\square$

5. Sporadic simple groups and the Tits group

In this section, we will prove the following result.

Proposition 5.1. *Theorem 1.4 holds for sporadic simple groups and the Tits group.*

Proof. Let S be a simple sporadic group or the Tits group. All information that we need for the proof of Proposition 5.1 is presented in Table 2. For each sporadic simple group or the Tits group S, let π_S be the set of primes in the second column of Table 2. In the fifth column, we list all the possibilities for the maximal subgroups H of S such that $\pi_S \subseteq \pi(H)$. This is taken from [14, Table 10.6] for sporadic simple groups and from [14, Table 10.5] for the Tits group. Let $\chi \in \mathrm{Irr}(S)$ be the irreducible character of S labeled by the symbol in the third column of Table 2. The corresponding degree of χ is listed in the next column. The character χ is chosen to satisfy the following properties.

(i) χ is extendible to $\mathrm{Aut}(S)$.
(ii) For each prime $p \in \pi_S$ and any element $g_p \in S$ with order p, we have that $\chi(g_p) \neq 0$.
(iii) For any element $x \in S$ lying in the conjugacy class in the last column of Table 2, we also have that $\chi(x) \neq 0$.

We now show that χ is not an m.i character. By way of contradiction, assume that χ is an m.i character of S. Then there exist a maximal subgroup $H \leqslant S$ and $\lambda \in \mathrm{Irr}(H)$ such that $m\chi = \lambda^S$ for some nonnegative integer m. For each prime $p \in \pi_S$ and $g_p \in S$, by (ii) we have $\chi(g_p) \neq 0$, so $\lambda^S(g_p) = m\chi(g_p) \neq 0$, thus $g_p^S \cap H \neq \emptyset$. In particular H possesses an element of order p and thus $\pi_S \subseteq \pi(H)$. The possibilities for H are given Table 2. In these cases, with the same argument, we obtain that $x^S \cap H \neq \emptyset$ as $\chi(x) \neq 0$ by (iii) and thus H has an element of order equal to that of x. However we can see that H has no elements with such orders by checking [5] directly. □

6. Proofs of the main results

Proof of Theorem 1.4. This follows from Propositions 3.9, 4.1, 5.1 and the classification of finite simple groups. □

It would be interesting if one could classify all m.i characters of simple groups.

Proof of Theorem 1.1. Assume that $N \trianglelefteq G$ and that G is an MI-group relative to N. We show that N is solvable. Let $N \trianglelefteq G$ be a counterexample such that $|G| + |N|$ is minimal. It follows that N is nonsolvable.

We show that N is the unique minimal normal subgroup of G. We first show that N is a minimal normal subgroup of G. Suppose not. Let $K \leqslant N$ be a minimal normal subgroup of G. Then $K \lneq N$. By Lemma 2.1(ii), we obtain that G is an MI-group relative to K. Since $|G| + |K| < |G| + |N|$, by induction hypotheses we deduce that K is solvable and hence N/K is a nonsolvable normal subgroup of G/K. Now by Lemma 2.1(i), we have that G/K is an MI-group relative to N/K. Thus by induction hypothesis again, we deduce that N/K is solvable. Combining with the previous claim, we obtain that N is solvable, which is a contradiction. We have proved that N is a minimal normal subgroup of G. Let $C = C_G(N)$. In order to show that N is the unique minimal normal subgroup of G, it suffices to show that C is trivial. Observe that $C \trianglelefteq G$ and thus by Lemma 2.1(i), we have that G/C is an MI-group relative to NC/C. As N is nonsolvable and is a minimal normal subgroup of G, we have that N is isomorphic to a direct product of some nonabelian isomorphic simple groups so $N \cap C = 1$. Therefore, $NC/C \cong N/N \cap C \cong N$ is nonsolvable. If C is nontrivial, then since G/C is an MI-group relative to NC/C, by induction hypothesis we deduce that $NC/C \cong N$ is solvable, which is impossible. Thus C is trivial. Hence N is the unique minimal normal subgroup of G as required.

Let R be a nonabelian simple group such that $N = R_1 \times R_2 \times \cdots \times R_k$, where $R_i \cong R$ for all i. By Lemma 2.4, every $\mathrm{Aut}(R)$-invariant nonlinear irreducible character of R is an m.i character of R. Now Theorem 1.4 provides a contradiction. The proof is now complete. □

Proof of Corollary 1.3. Let $L = \langle H^G \rangle$ be the normal closure of H in G. Then $H \leqslant L \trianglelefteq G$. By definition, L is the smallest normal subgroup of G containing H. We show that L is solvable. The remaining statement is clear.

We first assume that $L = G$. We show that G is an MI-group and then the result follows from Corollary 1.2. Let $\chi \in \mathrm{Irr}(G)$ be any nonlinear irreducible character of G. If 1_H is the only irreducible constituent of χ_H, then obviously $\chi_H = \chi(1)1_H$ and hence $H \leqslant \mathit{Ker}(\chi) \trianglelefteq G$. Since χ is a nonlinear irreducible character of G, we deduce that $\mathit{Ker}(\chi)$ is a proper normal subgroup of G containing H, which is a contradiction as $\langle H^G \rangle = G$. It follows that χ_H has an irreducible constituent $\lambda \in \mathrm{Irr}(H)$ with $\lambda \neq 1_H$. By the hypotheses, we know that λ^G is a multiple of some irreducible character of G. Since $(\lambda^G, \chi)_G = (\lambda, \chi_H) \neq 0$ by the Frobenius reciprocity, we must have that $\lambda^G = m\chi$ for some nonnegative integer m. Thus χ is an m.i character. Hence G is an MI-group as wanted.

Now assume that $L \neq G$. We claim that G is an MI-group relative to L. Let $\chi \in \mathrm{Irr}(G|L)$ and let θ be any irreducible constituent of χ upon restriction to L. Since $L \nleqslant \mathit{Ker}(\chi)$, we can choose $\theta \neq 1_L$. If θ is not G-invariant, then by the Clifford theory, we know that $\chi = \psi^G$ for some $\psi \in \mathrm{Irr}(I_G(\theta)|\theta)$ and hence χ is an m.i character. Therefore, we can assume that θ is G-invariant. It follows that $\mathit{Ker}(\theta) \leqslant L$ is a proper normal subgroup of L since $\theta \neq 1_L$. As in the previous case, if 1_H is the only irreducible constituent of θ_H, then $H \leqslant \mathit{Ker}(\theta) \lneq L \trianglelefteq G$, which is a contradiction as L is the smallest normal subgroup of G containing H. Hence we conclude that θ_H possesses an irreducible constituent $\lambda \in \mathrm{Irr}(H)$ with $\lambda \neq 1_H$. By the transitivity of induction, we obtain that χ is an irreducible constituent of λ^G, and so by the hypotheses we deduce that $\lambda^G = m\chi$ for some m. Hence χ is an m.i character. Thus G is an MI-group relative to L. Now the result follows from the main theorem. $\square$

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